THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 1

CONTENTS

JANUARY 1965

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

General Office: SUNY at Buffalo, Buffalo, N. Y. 14214. *Executive Director:* H. M. GEHMAN

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00. For nonmembers the subscription price is \$8.00 (\$10.00 effective August–September 1965).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

THE HELMERT MATRICES

H. O. LANCASTER, University of Sydney

1. Introductory. Helmert [5] introduced an orthogonal matrix, having a prescribed first row and a triangle of zeroes above the diagonal. Since then generalised forms of this type of matrix have found many uses in statistics, in group theory, and in the evaluation of the Jacobian of the transformation from rectangular Cartesian to polar coordinates.

2. Definitions. An $n \times n$ orthogonal matrix, $\mathbf{H}$, will be said to be a *generalized Helmertian matrix*, if it can be transformed by permutations of its rows and columns and by transposition and by change of sign of rows, to a form of a *standard Helmert matrix*, in which

$$\text{(i)} \qquad h_{ij} = 0 \quad \text{for} \quad j > i > 1.$$

In other words, in a standard Helmert matrix, the elements occupying a triangle below the first row and above the principal diagonal are all zeroes. If further

$$\text{(ii)} \qquad h_{1j} = +\sqrt{w_j}; \qquad w_j > 0; \qquad \sum_1^n w_j = 1,$$

$\mathbf{H}$ will be said to be *Helmertian in the strict sense*. In a standard Helmertian, the elements of the first row can be chosen arbitrarily subject only to the condition on its norm. Thus if one of the elements of the first row were zero, the matrix would not be Helmertian in the strict sense. In statistics, it has been customary to use Helmertians in the strict sense only; Helmert [5] used one in which $h_{1j} = \sqrt{w_j} = n^{-1/2}$. Irwin [6] used a more general set of positive w_j. For Helmertian matrices in the strict sense, it is possible to impose a third condition, that all elements below the diagonal shall be positive, namely:

$$\text{(iii)} \qquad h_{ij} > 0 \quad \text{if} \quad j < i.$$

This last condition is inconvenient for more general work and will be disregarded.

Example (i).

$$\begin{bmatrix} 0.64 & 0.48 & 0.60 \\ 0.60 & -0.80 & 0 \\ 0.48 & 0.36 & -0.80 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

are Helmertian matrices in the strict sense.

Example (ii). The unit matrix is a standard Helmert matrix but not a Helmert matrix in the strict sense.

Example (iii). For $n=2$, there can be no zero above the diagonal and below the first row, so that

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad\text{and}\quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

are to be taken by convention as standard Helmert matrices.

Example (iv). Permutation matrices are generalized Helmert matrices; and so are diagonal matrices with diagonal elements of ± 1. Transposes of standard Helmert matrices are also generalized Helmertians.

3. The Helmert matrices in the strict sense. Helmert's original matrix may be defined by

$$\text{(1)} \qquad \mathbf{H} = \mathbf{D}^{-1}\mathbf{H}_0 \equiv \mathbf{D}^{-1}\begin{bmatrix} 1 & 1 & 1 & 1\cdots & 1 \\ 1 & -1 & 0 & 0\cdots & 0 \\ 1 & 1 & -2 & 0\cdots & 0 \\ 1 & 1 & 1 & -3\cdots & 0 \\ \cdot & \cdot & \cdot & \cdot\cdots & \cdot \\ 1 & 1 & 1 & 1\cdots & -(n-1) \end{bmatrix}$$

where $\mathbf{D}^2$ is a diagonal matrix with diagonal elements, $d_{ii}^2 = n,\ 1\cdot 2,\ 2\cdot 3,\ \cdots,\ (n-1)n$. $\mathbf{H}$ is orthogonal since the rows of $\mathbf{H}_0$ are mutually orthogonal and each row of $\mathbf{H}$ has unit norm since d_{ii} is the norm of the ith row of $\mathbf{H}_0$. The matrix $\mathbf{H}$ is used in statistics because it is the simplest to exhibit or compute of all orthogonal matrices having $n^{-1/2}$ as the element in each position of the first row. Let $x_1, x_2, x_3, \cdots, x_n$ be a set of observations made on n identically distributed but mutually independent random variables and

$$\text{(2)} \qquad \mathbf{y} = \mathbf{H}\mathbf{x}.$$

$y_1 = n^{-1/2}\sum x_i = n^{1/2}\bar{x}$, where $\bar{x}$ is the mean of the x_i. y_1 is the mean of the set of the observations, $\{x_i\}$, multiplied by a standardizing factor. Similarly except for factors, y_2 is $x_1 - x_2$, y_3 is $\frac{1}{2}(x_1+x_2) - x_3, \cdots, y_{j+1}$ is $j^{-1}(x_1+x_2+\cdots+x_j) - x_{j+1}$. The origin of the convention as to sign in statistical usage is evident. In a very common situation in statistics, the null hypothesis specifies that the x_i form a sample from a population of mutually independent random variables having a common mean and a common variance. By using the rule of invariance under orthogonal transformation, the sum of squares, $\sum x_i^2 = \sum y_i^2$, can be partitioned into two parts, y_1^2 and $(y_2^2+y_3^2+\cdots+y_n^2)$. If further the distribution of each x_i is normal, the same holds for the y_i and the $\{y_i\}$ form a mutually independent set. The parent mean appears, moreover, only in the distribution of y_1. For general n, the Helmert matrices provide the simplest transformation.

Very often the y_i can be associated with comparisons of interest; for if $i > 1$, y_i compares the mean of the first $(i-1)$ observations with the ith observation.

In an example, given by Lancaster (1948), the bacterial counts, a_i, developing in successive plates exposed to increasing concentration of disinfectant were given, namely 427,440,494,422,409,310,302. The first count can be compared with the second, then the mean of the first two with the third and so on. The χ^2's, each distributed with one degree of freedom, were 0.195, 5.379, 1.687, 2.465, 32.947 and 28.299. Neglecting the second value, which is somewhat raised, a highly significant fall in the counts between the sixth and earlier counts is evident. Similar physical applications may be obtained with, say, the effect of decreasing or increasing doses of chemical fertiliser on crop yields. Alternative sets of linear forms in the observations x_i may not have the same meaningful or easily apparent interpretation. R. A. Fisher (see [3] and [4]) did much to make such orthogonal transformations of sets of normal variables familiar to statisticians.

The matrix $\mathbf{H}$ of (1) is a special case, with equal weights, of a matrix considered by Irwin [6] with possibly unequal weights w_j. He solved explicitly the equations for the h_{ij}. The solution is obtained by writing

$$W_j = w_1 + w_2 + \cdots + w_j, \qquad W_n = 1 \tag{3}$$

and the defining conditions yield

$$\begin{aligned} &h_{1j} = +\sqrt{w_j}; \qquad h_{ij} = 0 && \text{for } j > i > 1, \\ &h_{ii} = -(W_{i-1}/W_i)^{1/2}; \qquad h_{ij} = (w_i w_j / W_i W_{i-1})^{1/2} && \text{for } i > j,\ i > 1. \end{aligned} \tag{4}$$

In (4) the formula is correct for $i=2$ but there are some cancellations so that $h_{21}=\sqrt{w_2}/\sqrt{W_2}$ and $h_{22}=-\sqrt{w_1}/\sqrt{W_2}$. In calculating the elements of the kth row of $\mathbf{H}$, it can be observed that the vector of the first $(k-1)$ elements is orthogonal to the vector formed by the first $(k-1)$ elements of the 2nd, 3rd, $\cdots$ $(k-1)$th rows. The first $(k-1)$ elements of the kth row thus form a vector parallel to that of those of the first row,

$$h_{k1} = \lambda h_{11},\ h_{k2} = \lambda h_{12}, \cdots, h_{k,k-1} = \lambda h_{1k-1}. \tag{5}$$

Further the kth row is to be orthogonal to the first so that

$$\sum_1^{k-1} \lambda h_{1j}^2 + h_{1k}h_{kk} = 0; \qquad \lambda W_{k-1} + h_{kk}\sqrt{w_k} = 0. \tag{6}$$

The normalising condition is

$$\sum_1^{k-1} \lambda^2 h_{1j}^2 + h_{kk}^2 = 1; \qquad \lambda^2 W_{k-1} + h_{kk}^2 = 1. \tag{7}$$

Substituting from (6) into (7), $\lambda^2 W_{k-1} + \lambda^2 W_{k-1}/w_k = 1$, or $\lambda^2 W_{k-1} W_k / w_k = 1$, and λ is to be taken with the positive sign.

A probabilistic interpretation is possible for matrices which are Helmertian in the strict sense. If $\{w_j\}$ is a set of probabilities that a random variable Z takes values $j=1, 2, \cdots, n$, then the Helmert matrix yields an orthonormal basis for all finite functions on the probability distribution or measure $\{w_j\}$. The first row yields a constant and the other orthonormal functions may be defined by

$$z_j^{(k-1)} = h_{kj}/h_{1j}, \qquad k = 2, \cdots, n; \qquad j = 1, 2, \cdots, n. \tag{8}$$

The set is orthonormal, for

$$\sum_{j=1}^{n} z_j^{(k-1)} z_j^{(k'-1)} w_j = \sum h_{kj}h_{k'j} = \delta_{kk'}, \tag{9}$$

since w_j can be replaced by h_{1j}^2.

This is the basis of an application made by Irwin [7] to a partition of χ^2 by Lancaster [8] in the multinomial distribution. Let N independent observations be made of a variable which can take values $1, 2, 3, \cdots, n$ with probabilities $p_1, p_2, p_3, \cdots, p_n$, and suppose that the variable has taken the value, j, a_j times. With

$$x_j = (a_j - Np_j)/\sqrt{(Np_j)}, \tag{10}$$

the usual Pearsonian χ^2 with $(n-1)$ degrees of freedom is given by

$$\chi^2 = \sum_1^n (a_j - Np_j)^2/(Np_j) = \sum x_j^2. \tag{11}$$

If the transformation,

$$\mathbf{y} = \mathbf{Hx}, \quad \text{with} \quad w_j = p_j, \tag{12}$$

is made, y_1 is identically zero and the remaining variables have zero expectation and unit variance and are asymptotically normal and mutually independent; in fact each y_k other than y_1 is a standardized difference between Poisson variables.

Using a variation of the Bernstein [1] multivariate central limit theorem, we can give a proof for the (asymptotically true) distribution of χ^2. For if we have N mutually independent observations on a random variable Z, taking values, $1, 2, \cdots, n$, then we can define a set of $(n-1)$ orthonormal variables, $\{z^{(k-1)}\}$ for $k=2, 3, \cdots, n$ with values for $Z=j$ defined by (8). Then Bernstein's theorem enables us, defining

$$y_k = N^{-1/2} \sum z^{(k-1)}, \tag{13}$$

with summation over the N independent observations, to assert that $\{y_2, y_3, \cdots, y_n\}$ are mutually uncorrelated and normal (0, 1) and hence that $(y_2^2+y_3^2+ \cdots +y_n^2)$ is distributed as χ^2 with $(n-1)$ degrees of freedom. But this sum is equal to $\sum x_j^2$ of (11) and so Bernstein's theorem enables us to prove

Pearson's proposition. In the proof of K. Pearson [9], an assumption is made that if the marginal variables of a set are normal, the joint distribution has the special form usually referred to as joint normal. We may note in passing that the covariance matrix of the first $(n-1)$ variables, x_j, of (10) is given by

$$\mathbf{V} = (p_i\delta_{ij} - p_ip_j). \tag{14}$$

Pearson [9] gave a long discussion on its inversion, which may be considerably shortened. Let $\mathbf{p}$ be the unit vector with elements $p_i^{\frac{1}{2}}(1-p_n)^{-\frac{1}{2}}$, $i=1, 2, \cdots, n-1$, and let $\mathbf{D}$ be the diagonal matrix, diag $(p_i^{\frac{1}{2}})$. Then

$$\mathbf{D}^{-1}\mathbf{V}\mathbf{D}^{-1} = \mathbf{1}_{n-1} - (1 - p_n)\mathbf{p}\mathbf{p}^T. \tag{15}$$

Now the matrix on the right can be brought to canonical diagonal form by transformation by a Helmert matrix with first column equal to $\mathbf{p}$, in which case the first element is p_n and all the other diagonal elements are units. We have

$$\mathbf{H}^T\mathbf{D}^{-1}\mathbf{V}\mathbf{D}^{-1}\mathbf{H} = \operatorname{diag}(p_n, 1^{n-2}). \tag{16}$$

Taking inverses, we get

$$\mathbf{H}^T\mathbf{D}\mathbf{V}^{-1}\mathbf{D}\mathbf{H} = \operatorname{diag}(p_n^{-1}, 1^{n-2}). \tag{17}$$

Hence

$$\mathbf{D}\mathbf{V}^{-1}\mathbf{D} = (\mathbf{1} + (1 - p_n)p_n^{-1}\mathbf{p}\mathbf{p}^T) \tag{18}$$

and

$$\mathbf{V}^{-1} = (p_i^{-1}\delta_{ij} + p_n^{-1}). \tag{19}$$

$\mathbf{x}^T\mathbf{V}^{-1}\mathbf{x}$ is then easily evaluated to be equal to the usual χ^2 of (11). This suggests a shorter method; if $\mathbf{U}$ is idempotent, the inverse of $\mathbf{1}_{n-1}+a\mathbf{U}$ is $\mathbf{1}_{n-1}-a(a+1)^{-1}\mathbf{U}$, as can be confirmed by a straightforward multiplication. (19) then follows from (15) without the explicit introduction of the orthogonal matrix.

4. Factorization of Helmert matrices.

THEOREM 1. *Every (standard) Helmert matrix of size n depends on $(n-1)$ parameters which can be taken to be the angles of certain rotations, namely in the plane of the* 1*st and j-th coordinate axes, j taking successively the values,* 2, 3, $\cdots$, n.

Proof. The proof is by induction on n, the size of the matrix. Matrices representing rotations in the plane of two coordinate axes are written

$$\mathbf{R}_{k+1} = \begin{bmatrix} \cos\theta_k & \mathbf{0}^T & \sin\theta_k \\ \mathbf{0} & \mathbf{1}_{k-2} & \mathbf{0} \\ -\sin\theta_k & \mathbf{0}^T & \cos\theta_k \end{bmatrix} + \mathbf{1}_{n-k}, \tag{20}$$

where if $n=k$, $\mathbf{1}_0$ is to be deleted. Choose $\mathbf{R}_n$, so that $\cos\theta_{n-1}=h_{nn}$ and $\sin\theta_{n-1} = -h_{1n}$, which is possible since every other h_{in} is zero. $\mathbf{R}_n\mathbf{H}_n$ is the direct sum of a Helmert matrix of size $(n-1)$ and unity. For $\mathbf{R}_n$ is an orthogonal matrix and

so $\mathbf{R}_n\mathbf{H}_n$ is orthogonal but the element at the (n, n) position is $h_{1n}^2+h_{nn}^2=1$ and so every other element in the last row or in the last column is zero. Premultiplication by $\mathbf{R}_n$ does not change the elements of the rows 2 to $(n-1)$ and so, as implied by the notation, $\mathbf{H}_{n-1}$ is also Helmertian and every element of its first row is positive. We have finally, by repeating the process

$$\mathbf{R}_2\mathbf{R}_3 \cdots \mathbf{R}_n\mathbf{H}_n = \mathbf{1}_n, \tag{21}$$

or

$$\mathbf{H}_n = \mathbf{R}_n^T \cdots \mathbf{R}_3^T\mathbf{R}_2^T. \tag{22}$$

It may be remarked that a Helmert matrix is uniquely determined by the $(n-1)$ parameters $\theta_1, \theta_2 \cdots \theta_{n-1}$ of the matrices, $\mathbf{R}_2, \mathbf{R}_3 \cdots \mathbf{R}_n$, and determines them uniquely; they are equal to the parameters of the vector $\boldsymbol{\phi}_1$ introduced in (26) below. A geometrical interpretation can be given to the theorem. From any rectangular coordinate system, a rotation can be found to a new rectangular coordinate system, where one of the new axes has a prescribed set of coordinates in the old system and the transformation can be made in $(n-1)$ steps. At the first step, a coordinate axis is chosen in the plane of the first and second coordinate axes of the original space, all other axes remaining the same in the two systems. At the second step, a coordinate axis is chosen in the plane of the new first and the old third axis, the second, fourth and all other axes remaining fixed; and generally the ith step takes a new axis in the plane of the 1st axis of the $(i-1)$th step and the $(1+i)$th axis in the original set; at the ith step, the 2nd, 3rd $\cdots$ ith axis remains fixed if $i>1$ and also the $(i+2)$th, $(i+3)$th $\cdots$ nth if $i<n-1$. The simplest manner to compute an orthogonal matrix given the first row is to use the method suggested by (21).

5. The Helmert matrices and the Euler factorization. It is of interest to find a representation of the general $n\times n$ orthogonal matrix whereby the $\frac{1}{2}n(n-1)$ parameters can be displayed. Schur [10] gave such a representation for the real orthogonal matrices and Brauer [2] one for the unitary matrices. The proof here is to exhibit as a factor of the $n\times n$ orthogonal matrix a standard Helmertian and then use the theorem of Section 4.

LEMMA. *Any orthogonal matrix, $\mathbf{A}_n$, of size $n\times n$ can be represented as the product of the direct sum of unity and an orthogonal matrix of size $(n-1)$, and a Helmertian having elements of the first row, a_{1j}.*

Proof. Form a standard Helmertian $\mathbf{H}_n$ with elements a_{1j} in the first row. Then

$$\mathbf{A}\mathbf{H}_n^T = 1 \dotplus \mathbf{A}_{n-1}, \tag{23}$$

where $\mathbf{A}_{n-1}$ is orthogonal, for $\mathbf{A}$ and $\mathbf{H}_n^T$ are both orthogonal and the leading element of $\mathbf{A}\mathbf{H}_n^T$ is $\sum a_{1j}^2=1$. All other elements in the first row and column are

therefore zero. From (17), there follows

$$\mathbf{A} = (1 \dotplus \mathbf{A}_{n-1})\mathbf{H}_n = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & \mathbf{A}_{n-1} \end{bmatrix} \mathbf{H}_n. \tag{24}$$

THEOREM 2. *Any general orthogonal matrix of size n can be factored into the product*

$$\mathbf{A} = (\mathbf{1}_{n-2} \dotplus \mathbf{H}_2)(\mathbf{1}_{n-3} \dotplus \mathbf{H}_3) \cdots (1 \dotplus \mathbf{H}_{n-1})\mathbf{H}_n, \tag{25}$$

where $\mathbf{H}_2$, $\mathbf{H}_3 \cdots \mathbf{H}_{n-1}$, $\mathbf{H}_n$ are standard Helmertians of size $2, 3 \cdots (n-1), n$.

Proof. The proof is by induction. The theorem is true for $k=2$, for then $\mathbf{A}$ is Helmertian. If it is true for $k=2, 3, \cdots (n-1)$ then the lemma shows that it is true for $k=n$, and hence that it is true for any finite n.

THEOREM 3. *Any general orthogonal matrix $\mathbf{A}$, of size n, can be factored into the product of $\frac{1}{2}n(n-1)$ matrices, each of which is a rotation in the plane of a pair of coordinate axes.*

Proof. This follows from an application of Theorem 1 to the matrices $\mathbf{H}_k$ of Theorem 2.

The number of these elementary rotation matrices is $(n-1)+(n-2)+ \cdots +1$ or $\frac{1}{2}n(n-1)$ and each can be represented by a single parameter.

COROLLARY. *Any orthogonal matrix of size n can be represented by $\frac{1}{2}n(n-1)$ parameters. It has $\frac{1}{2}n(n-1)$ degrees of freedom.*

Such a factorization was first given by Euler for the 3×3 orthogonal matrix.

The factorization also can be used to find a general orthogonal matrix with rational elements. It suffices to find rational values for the sines and cosines.

6. The transformation from Cartesian to polar coordinates. A Helmert matrix appears in the evaluation of the Jacobian of the transformation from rectangular Cartesian to polar coordinates. Let us write the transformation as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} = r\boldsymbol{\phi}_1 \equiv r \begin{bmatrix} \sin\phi_{n-1} \ \sin\phi_{n-2} \cdots \sin\phi_2 \ \sin\phi_1 \\ \sin\phi_{n-1} \ \sin\phi_{n-2} \cdots \sin\phi_2 \ \cos\phi_1 \\ \cdots\cdots\cdots\cdots\cdots \\ \sin\phi_{n-1} \ \sin\phi_{n-2} \ \cos\phi_{n-3} \\ \sin\phi_{n-1} \ \cos\phi_{n-2} \\ \cos\phi_{n-1} \end{bmatrix} \tag{26}$$

$$0 \leqq \phi_1 < 2\pi,$$

$$0 \leqq \phi_i < \pi, \qquad i = 2, 3, \cdots n-1.$$

This can be rewritten as

$$\mathbf{x}^T = r\boldsymbol{\phi}_1^T. \tag{27}$$

We have to determine J, the Jacobian of the transformation,

$$J = |\mathbf{J}|. \tag{28}$$

The rows of $\mathbf{J}$ are given by

$$\frac{\partial}{\partial r}(r\boldsymbol{\phi}_1^T),\ \frac{\partial}{\partial \phi_1}(r\boldsymbol{\phi}_1^T),\ \frac{\partial}{\partial \phi_2}(r\boldsymbol{\phi}_1^T)\cdots;$$

$$\mathbf{J} = \mathbf{GH}, \tag{29}$$

where the diagonal elements of the diagonal matrix $\mathbf{G}$ are

$$\begin{aligned} g_{11} &= 1 \\ g_{22} &= r\prod_2^{n-1}\sin\phi_k \\ g_{33} &= r\prod_3^{n-1}\sin\phi_k \\ &\cdots \\ g_{nn} &= r \end{aligned} \tag{30}$$

$\mathbf{H}$ has $\boldsymbol{\phi}_1^T$ as the first row; $h_{21}=\cos\phi_1$, $h_{22}=-\sin\phi_1$, all other $h_{2j}=0$; $h_{31}=\cos\phi_2\sin\phi_1$, $h_{32}=\cos\phi_2\cos\phi_1$, $h_{33}=-\sin\phi_2$, all other $h_{3j}=0$. And so on.

$$\begin{aligned} J &= |\mathbf{J}| = |\mathbf{G}|\,|\mathbf{H}| \\ &= r^{n-1}\sin^{n-2}\phi_{n-1}\sin^{n-3}\phi_{n-2}\cdots\sin\phi_2. \end{aligned} \tag{31}$$

the neatest derivation of this result given in the texts is by van der Waerden [11], who proves it by a recursive scheme, which is equivalent to premultiplying $\mathbf{J}$ by a matrix of the same form as $\mathbf{R}_n$ of Theorem 1, with $\theta_{n-1}=\phi_{n-1}$.

LEMMA. *If the rows of a matrix are mutually orthogonal, its determinant is equal to the product of the norms of the rows.*

Proof. The proof is obtained by dividing each row by its norm to obtain an orthogonal matrix.

Now the rows of $\mathbf{J}$ are mutually orthogonal. First neglecting the multiplier r and differentiating the equation $\boldsymbol{\phi}_1^T\cdot\boldsymbol{\phi}_1=1$ partially with respect to ϕ_j, we obtain

$$\frac{\partial\boldsymbol{\phi}_1^T}{\partial\phi_j}\boldsymbol{\phi}_1 = 0, \qquad (j = 1, 2, \cdots, n-1). \tag{32}$$

Further, it is easily verified that the vector obtained by differentiating $\boldsymbol{\phi}_1^T$ with respect to ϕ_j and then with respect to ϕ_k, $k>j$, is $\partial\boldsymbol{\phi}_1^T/\partial\phi_j$ except for a factor com-

mon to all nonzero elements. Differentiating (26) with respect to ϕ_k therefore gives

$$\frac{\partial \phi_1^T}{\partial \phi_j} \frac{\partial \phi_1}{\partial \phi_k} = 0, \qquad j \neq k. \tag{33}$$

The rows of **J** are thus mutually orthogonal and their norms are easily found to be $g_{11}, g_{22} \cdots g_{nn}$ of (30).

References

1. S. Bernstein, Sur l'extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes, Math. Ann., 97 (1926) 1–59.

2. R. Brauer, Die stetigen Darstellungen der komplexen orthogonalen Gruppe, S.-B. Preuss. Akad. Wiss., 1929, 626–638.

3. R. A. Fisher, Frequency distribution of the values of the correlation coefficient in samples from an indefinitely large population, Biometrika, 10 (1915) 507–521.

4. ———, On the probable error of a coefficient of correlation deduced from a small sample, Metron, 1 (4), (1921) 1–32.

5. F. R. Helmert, Die Genauigkeit der Formel von Peters zur Berechnung des wahrscheinlichen Beobachtungsfehlers directer Beobachtungen gleicher Genauigkeit, Astronom. Nachr., 88 (1876) 115–132.

6. J. O. Irwin, On the distribution of a weighted estimate of variance and on analysis of variance in certain cases of unequal weighting, J. Roy. Statist. Soc. Ser., 105 (1942) 115–118.

7. ———, A note on the subdivision of χ^2 into components, Biometrika, 36 (1949) 130–134.

8. H. O. Lancaster, The derivation and partition of χ^2 in certain discrete distributions, Biometrika, 36 (1949) 117–129.

9. K. Pearson, On a criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling, Philos. Mag., (5) 50 (1900b) 157–175.

10. I. Schur, Neue Anwendungen der Integralrechnung auf Probleme der Invariantentheorie, S.-B. Preuss. Akad. Wiss., 1924, 297–321.

11. B. L. van der Waerden, Mathematische Statistik, Die Grundlehren math. Wiss. in Einzeldarstellungen, Band 87, Springer-Verlag, Berlin, 1957.

THE TRIANGLE REINVESTIGATED

J. GARFUNKEL, Forest Hills H.S., AND S. STAHL, University of California, Berkeley

Of all the figures of plane geometry, the triangle is the most interesting, and the most prolific in producing theorems. Furthermore, of all the triangles, the equilateral seems to stand out as perfection personified. Thus, Morley attained mathematical immortality by discovering equilateral triangles in the midst of a maze of angle trisectors. Similarly, in the theorems that follow, equilateral triangles seem to emerge out of nowhere and everywhere. With some exceptions, the theorems are original, their proofs are simple, and the results are provocative.

THEOREM 1. *If in the middle third of each side of any scalene triangle equilateral triangles are constructed inwards, then the join of their vertices forms an equilateral triangle.*

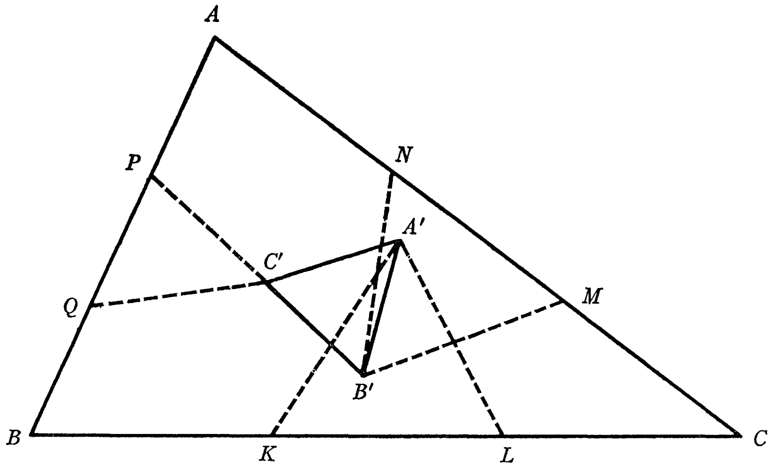


FIG. 1

Restatement: If K, L, M, N, P, Q are points of trisection of the sides of any triangle ABC and if $A'KL$, $B'MN$, $C'PQ$ are equilateral, then $A'B'C'$ is an equilateral triangle. (See Fig. 1.)

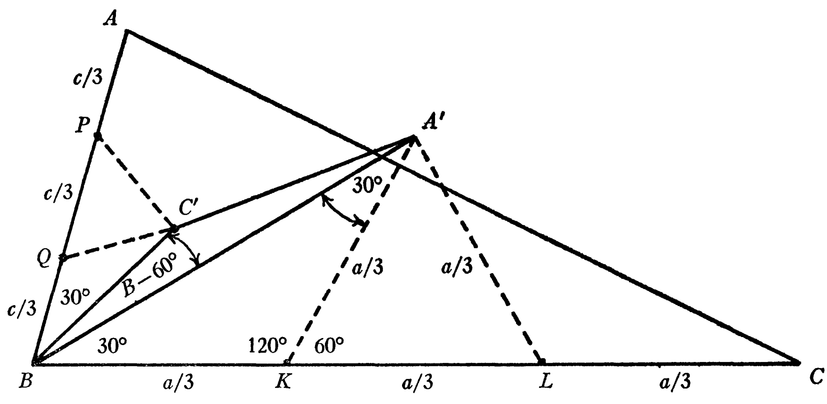


FIG. 2

Proof of Theorem 1. Notation as in Fig. 2. Since triangles $A'BL$ and $C'PB$ are right triangles,

$$(BA')^2 = (2a/3)^2 - (a/3)^2 = a^2/3$$

$$(BC')^2 = (2c/3)^2 - (c/3)^2 = c^2/3$$

$$\begin{aligned}(A'C')^2 &= a^2/3 + c^2/3 - (2ac/3)\cos(B - 60^\circ)\\ &= a^2/3 + c^2/3 - (2ac/3)\cos B\cos 60^\circ - (2ac/3)\sin B\sin 60^\circ\\ &= a^2/3 + c^2/3 - (2ac/6)\cos B - 2\sqrt{3}(1/2)(ac\sin B)\\ &= a^2/3 + c^2/3 - (a^2 + c^2 - b^2)/6 - (2/\sqrt{3})S,\end{aligned}$$

where S is the area of triangle $ABC.(A'C')^2 = (a^2+b^2+c^2)/6-(2/\sqrt{3})S$. Hence $A'C'$, the side of triangle $C'B'A'$ depends symmetrically on the sides of triangle ABC, and $A'B' = B'C' = A'C'$ and triangle $C'B'A'$ is equilateral. This theorem has been proved in another way by I. M. Yaglom, see [3], page 93, also [1] pages 105–107.

Note. Triangle $A'B'C'$ is retrograde, i.e., the sense of $A'B'C'$ is clockwise if ABC is counterclockwise.

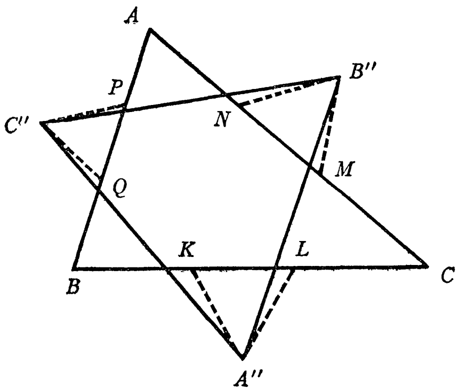


FIG. 3

THEOREM 1′. *If K, L, M, N, P, Q are points of trisection of the sides of any scalene triangle ABC, and if triangles $A''KL$, $B''MN$, $C''PQ$ are equilateral, then triangle $A''B''C''$ is equilateral.*

Since the proof of this theorem follows identically along the same line as that of Theorem 1, we omit the proof. Since Theorem 1′ has been credited ("Mathesis" 1938, p. 293 (footnote)) to Napoleon, we shall now refer to the two equilateral triangles of Theorem 1 and 1′ as the "inner and outer Napoleon triangles."

Note. The sense of $A''B''C''$ is the same as that of ABC.

THEOREM 2. *The area of the original triangle is equal to the difference between the areas of the outer and inner Napoleon triangles.*

Proof. Within the proof of Theorem 1, we deduced the following results:

$$(A'B')^2 = (B'C')^2 = (C'A')^2 = \frac{a^2 + b^2 + c^2}{6} - \frac{2S}{\sqrt{3}}$$

$$(A''B'')^2 = (B''C'')^2 = (C''A'')^2 = \frac{a^2 + b^2 + c^2}{6} + \frac{2S}{\sqrt{3}}\cdot$$

Let S' be the area of triangle $C'B'A'$ and let S'' be the area of triangle $A''B''C,''$ then

$$S'' = (\sqrt{3})\frac{a^2 + b^2 + c^2}{24} + S/2,\ S' = (\sqrt{3})\frac{a^2 + b^2 + c^2}{24} - S/2 \text{ and } S'' - S' = S.$$

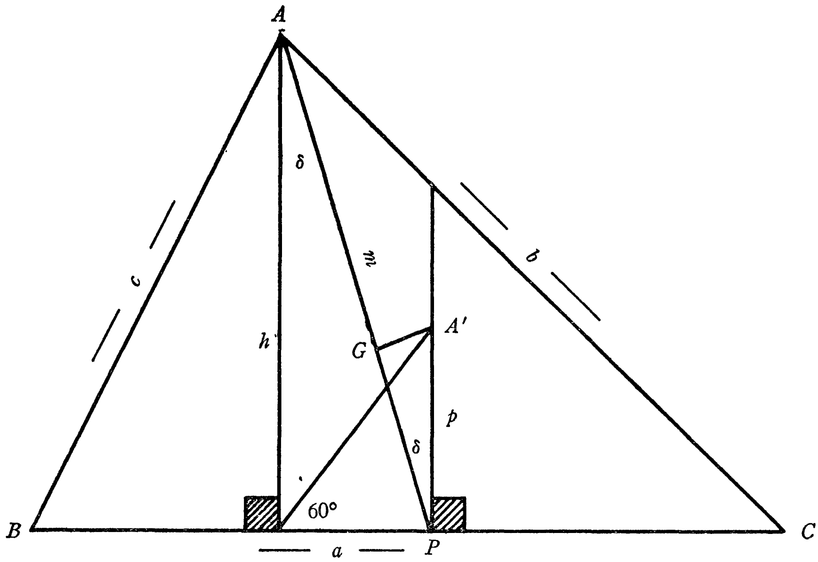


FIG. 4

THEOREM 3. *The centroids of the inner and outer Napoleon triangles coincide with the centroid of the original triangle.*

Proof. It is sufficient to show that the points A', B', C' are equidistant from G, the centroid of triangle ABC. Let m be the length of the median from A to side BC. Let p be the length of the perpendicular bisector to BC from BC to AC. Let h be the altitude to BC, then

$$\begin{aligned}(GA')^2 &= (GP)^2 + (A'P)^2 - 2(GP)(PA')\cos\delta \\ &= (m/3)^2 + (a/2\sqrt{3})^2 - 2(m/3)(a/2\sqrt{3})(h/m) \\ &= m^2/9 + a^2/12 - (2/3\sqrt{3})(ah/2).\end{aligned}$$

But $m = \frac{1}{2}\sqrt{(2b^2+2c^2-a^2)}$ and $ah/2 = S$, the area of triangle ABC. Therefore

$$(GA')^2 = \frac{2b^2 + 2c^2 - a^2}{36} + a^2/12 - 2S/3\sqrt{3} = \frac{a^2 + b^2 + c^2}{18} - (2/3\sqrt{3})S.$$

From the nature of the result, it follows that $GA' = GB' = GC'$. By similar reasoning we can deduce that $GA'' = GB'' = GC''$, hence the theorem is established. This theorem is actually a special case of a more general theorem proved in [2] p. 39.

THEOREM 4. *If K, L, M, N, P, Q are points of trisection, then the vertices of the inner Napoleon triangle $A'B'C'$ form with T', V', S', the vertices of equilateral triangles erected on LM, NP, QK, a regular hexagon.*

Proof. It is easy to show that triangle $DEF \cong$ triangle ABC and that triangle $S'T'V'$ is an inner Napoleon triangle of triangle DEF, (see Fig. 5). Moreover, triangle DEF is the image of triangle ABC under a 180° rotation about its centroid. Hence, triangle $S'T'V'$ is the image of triangle $A'B'C'$ under the same mapping, and consequently the points A', V', C', S', B', T' form a regular hexagon.

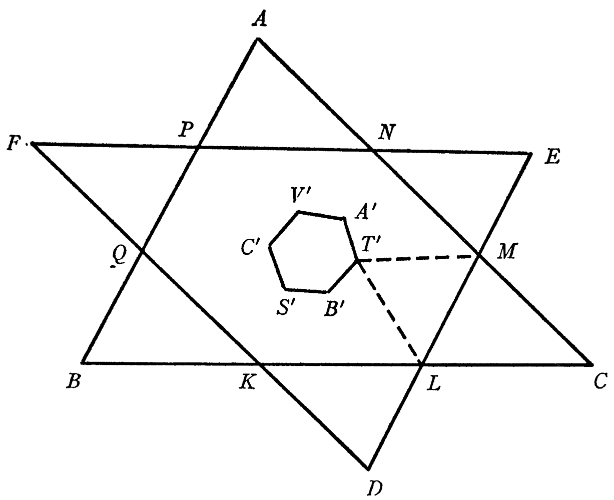


FIG. 5

THEOREM 5. *The inner and outer Napoleon triangles of any scalene triangle are perspective with each other and with the original triangle.*

Proof. (a) In the case where the triangles considered are the inner and outer Napoleon triangles, the proof is immediate. The lines $A'A''$, $B'B''$, $C'C''$ are, by construction, the perpendicular bisectors of the sides of triangle ABC, and hence they concur.

(b) Consider the case of the original triangle and the outer Napoleon triangle (see Fig. 6). Construct the lines $DA''F$, $FB''H$, $IC''J$ parallel to BC, CA, AB respectively. Then

$$BK/KC = DA''/A''E, \quad CL/LA = FB''/B''H, \quad AM/MB = IC''/C''J,$$

$$(BK/KC)(CL/LA)(AM/MB) = (DA''/A''E)(FB''/B''H)(IC''/C''J)$$
$$= (DA''/C''J)(IC''/B''H)(FB''/A''E).$$

But angle $CA''E$ = angle $CB''F = 30°$ and angle CEA'' = angle CFB'' = angle C; therefore triangle $CEA'' \sim$ triangle CFB'' and $FB''/A''E = B''C/CA''$. Quite

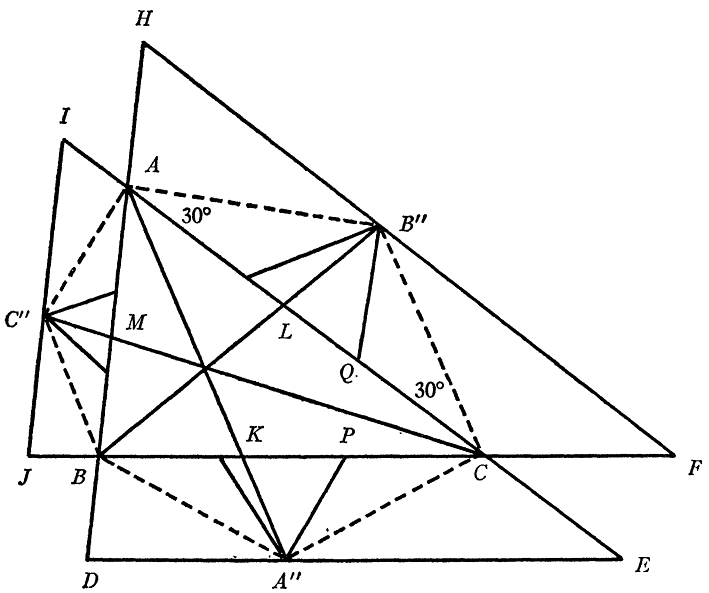


FIG. 6

obviously, triangle $PA''C \sim$ triangle $QB''C$, therefore, $B''C/CA'' = CQ/PC = CA/BC$ and therefore $FB''/A''E = CA/BC$; similarly

$$IC''/B''H = AB/CA \quad \text{and} \quad DA''/C''J = BC/AB.$$

Hence,

$$(BK/KC)(CL/LA)(AM/MB) = (BC/AB)(AB/CA)(CA/BC) = 1.$$

Therefore, by Ceva's theorem, the lines AK, BL, CM concur. The proof of the third possibility is the same, and is therefore omitted.

THEOREM 6. *The lines $C''B'$, $B''C'$, $A''A'$ form an equilateral triangle whose centroid coincides with the centroid of triangle ABC.*

Proof. Since the equilateral triangles $A'C'B'$ and $A''B''C''$ are concentric and similarly oriented (see Fig. 7), the rotation through 120° about their common center must cyclically permute their vertices, and consequently also cyclically permute the joins $A'A''$, $C'B''$, $B'C''$ (and likewise $A'B''$, $C'C''$, $B'A''$,

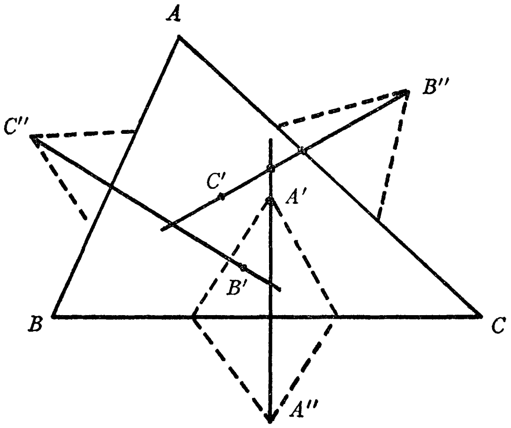


FIG. 7

$A'C''$, $C'A''$, $B'B''$). Since ABC is scalene, the line $A'A''$, containing the circumcenter, does also contain the centroid G. Therefore the three lines that are cyclically permuted are not concurrent, but form an equilateral triangle. (We are indebted to the referee for this elegant proof.)

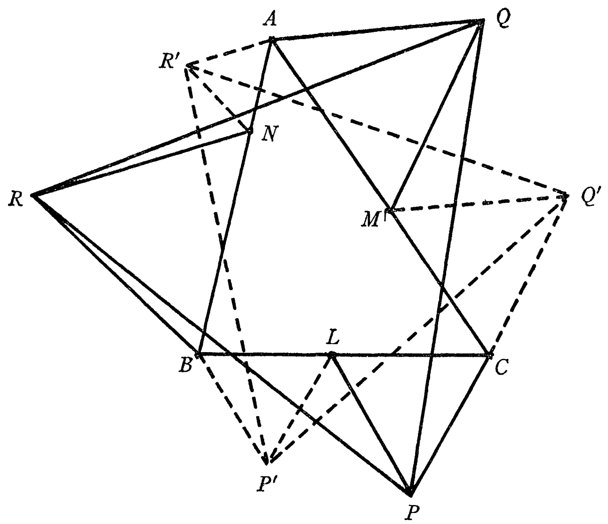


FIG. 8

THEOREM 7. *If equilateral triangles are erected on the alternate segments LC, MA, NB of the sides of triangle ABC, and the join of their vertices P, Q, R form an equilateral triangle, then the join of the vertices of equilateral triangles erected on the other alternate segments also form an equilateral triangle.*

Since the proof of this theorem requires the construction of quite a few additional lines, it was deemed advisable to make a separate sketch for the proof (see Fig. 8).

Proof. Let $r=(X, +, 120°)$ mean the rotation r about a point X, counterclockwise, 120°.

Draw the lines as in the Figure 9. $AR=BR'$ since triangle $ANR\cong$triangle BNR' $(N, +, 60°)$. $AR=MP$ since triangle $AQR\cong$triangle MQP $(Q, +, 60°)$. $MP=LQ'$ since triangle $MPC\cong$triangle LCQ' $(C, +, 60°)$. Therefore $AR=LQ'=BR'=MP$.

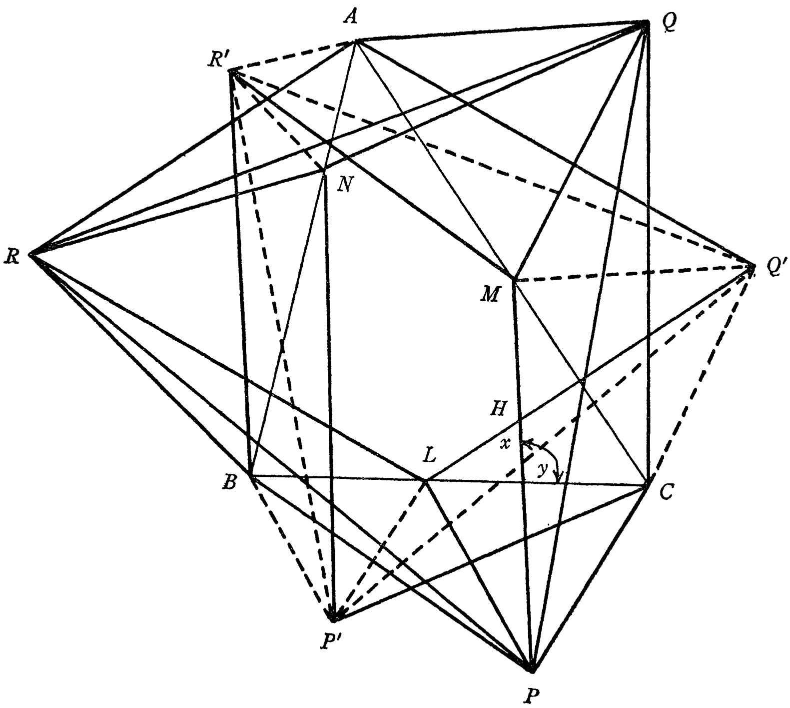


Fig. 9

By similar arguments, $MR'=BP$, $P'C=NQ$, $NP'=QC$, $AQ'=RL$. Therefore $MR'BP$, $LQ'AR$, $NP'CQ$ are parallelograms. $\angle Q'LC=\angle MPC$ since triangle $Q'LC\cong$triangle MPC (see above). Therefore $LPCH$ is a cyclic quadrilateral and

$$\angle x = \angle LCP = 60° \quad \text{and} \quad \angle y = \angle Q'LC + 60°.$$

But $\angle y = \angle R'BC$ (corr. angles of parallel lines $R'B$ and MP) therefore

$$\angle R'BC = 60° + \angle Q'LC.$$

Therefore

$$\angle R'BP' = \angle R'BC + \angle LBP' = 60° + \angle Q'LC + 60°,$$

$$\angle Q'LC + \angle CLP + \angle PLP' = \angle Q'LP'.$$

Triangle $R'BP' \cong$ triangle $Q'LP'$ ($BR' = LQ'$, $BP' = P'L$, $\angle R'BP' = \angle Q'LP'$). Therefore $R'P' = P'Q'$ and by similar arguments $P'Q' = Q'R'$. Hence, $P'Q'R'$ is equilateral.

References

1. N. A. Court, College geometry, 1st ed., Johnson, Richmond, Va.
2. H. G. Forder, Calculus of extension, Chelsea, New York, 1960.
3. I. M. Yaglom, Geometric transformations, New Math. Library, Random House, New York, 1962.

ON THE PRODUCT OF REAL VALUED UNIFORMLY CONTINUOUS FUNCTIONS IN METRIC SPACES

NORMAN LEVINE, Ohio State University, and
NICHOLAS J. SABER, University of Pittsburgh

I. Introduction. It is well known that the composite map of two continuous functions is continuous. It is also well known that the product map of two real valued continuous functions is continuous. Although the composite map of two uniformly continuous functions is uniformly continuous, however, the product of two real valued uniformly continuous functions is not necessarily uniformly continuous, as for example, x and $\sin x$ on the line. This paper is concerned with the product of real valued uniformly continuous functions defined on metric spaces and is a generalization and extension of [1].

The paper is divided into three sections including the introduction. Section II consists of preliminary lemmas, definitions, and examples that serve as background to the central discussion. Section III characterizes classes (K_n^*) of uniformly continuous functions. A uniformly continuous real valued function will belong to the class K_n^* if and only if the multiple product map of this function with any n uniformly continuous maps is uniformly continuous.

II. Preliminary definitions, lemmas, and examples. We present first two simple results.

DEFINITION 1. *Let $\{M, d\}$ be a metric space. $\{M, d\}$ will be said to have property B if and only if every bounded subset of M is totally bounded.*

For example, Euclidean n-space has property B for all n, whereas Hilbert space does not. Property B is not a topological property; consider the following:

Example 1. Let $\{R, d\}$ be the reals with the usual metric. Let $\{T, d_T\}$ be the reals with the bounded metric $d_T(x, z) = d(x, y)/[1+d(x, y)]$. Let $f: R \to T$ be the identity map, that is, $f(x) = x$. Now f is a homeomorphism, $\{R, d\}$ has property B, but $\{T, d_T\}$ does not since relative to d_T, T is bounded but not totally bounded (see [2], Ex. 6.5, p. 212).

LEMMA 1. *Let $f: M \to [1, \infty)$ be uniformly continuous, where $\{M, d\}$ is a metric space with property B. Let $A \subset M$ be bounded. Then $f(A)$ is bounded.*

Proof. Let $\epsilon = 1$. There exists then a $\delta > 0$ such that $d(x, y) < \delta$ implies that $|f(x) - f(y)| < 1$. Since A is bounded and M has property B, A is totally bounded. There then exists a sequence $\{p_i\}_{i=1}^n$ such that $A \subset \bigcup_{i=1}^n S_\delta(p_i)$. Now $x \in A$ implies that there exists i_* such that $x \in S_\delta(p_{i^*})$ and hence $f(x) < f(p_{i^*}) + 1$. Thus $x \in A$ implies that $f(x) < f(p_1) + \cdots + f(p_n) + 1$ and therefore $f(A)$ is bounded.

LEMMA 2. *Let $\{M, d\}$ be a metric space. Let $f: \{M, d\} \to [1, \infty)$ and $h: \{M, d\} \to [1, \infty)$ be uniformly continuous and bounded. Then $f(x) \cdot h(x)$ is uniformly continuous.*

Proof. $f(x)$ and $h(x)$ bounded implies there exist R_1 and R_2, real, such that $f(x) < R_1$ and $h(x) < R_2$. Let $\epsilon > 0$ be given; there exist δ_1 and δ_2 such that $|f(x) - f(y)| < \epsilon/2R_2$ for $d(x, y) < \delta_1$ and $|h(x) - h(y)| < \epsilon/2R_1$ for $d(x, y) < \delta_2$. It can easily be seen that $|f(x)h(x) - f(y)h(y)| < \epsilon$ for $d(x, y) < \delta = \min(\delta_1, \delta_2)$. A straightforward generalization follows.

LEMMA 3. *Let $\{M, d\}$ be a metric space with property B. If $A \subset M$ is bounded, then the product of two real valued uniformly continuous functions on A is uniformly continuous.*

Proof. Use Lemma 1 and Lemma 2.

We proceed to a few more preliminary results.

LEMMA 4. *Let $\{M, d\}$ be an unbounded metric space. Let $f: M \to [1, \infty)$ be single valued (continuity not assumed), where f is unbounded on M but bounded on every bounded subset of M. Then there exists a sequence of points $\{x_k\}_{k=1}^\infty$ where*

$$(1) \quad f(x_{k+1}) > f(x_k) + 1, \qquad (3) \quad \lim f(x_k) = \infty,$$

$$(2) \quad d(x_k, x_{k+1}) > 2, \qquad (4) \quad d(x_k, x_1) \geqq 2k - 2.$$

Proof. Let $x_1 \in M$; consider $S_3(x_1)$. $S_3(x_1)$ is bounded and hence f is bounded on $S_3(x_1)$. Therefore there exists an x_2 such that $x_2 \notin S_3(x_1)$ and $f(x_2) > f(x_1) + 1$. Otherwise f is bounded on M, a contradiction. Now if x_2 is not a member of $S_3(x_1)$ then $d(x_2, x_1) \geqq 3 > 2$. Suppose now that $x_1, x_2, \cdots, x_m$ have been chosen so that (1), (2), and (4) above are satisfied and $f(x_i) \geqq i$ for $i = 1, 2, \cdots, m$. Then $S_{d(x_1, x_m)+3}(x_1)$ is bounded and hence again f is bounded on it. Furthermore, f is unbounded on M and thus there exists an x_{m+1} such that x_{m+1} is not a member of $S_{d(x_1,x_m)+3}(x_1)$ and $f(x_{m+1}) > f(x_m) + 1 \geqq m+1$. Otherwise as above f would be bounded on M, a contradiction. Now

$$x_{m+1} \notin S_{d(x_1,x_m)+3}(x_1)$$

and hence $d(x_{m+1}, x_m) > 2$ since $d(x_{m+1}, x_m) \leqq 2$ and

$$d(x_{m+1}, x_1) \leqq d(x_m, x_1) + d(x_{m+1}, x_m) \leqq d(x_m, x_1) + 2$$

would be a contradiction. Thus we can choose $\{x_k\}_{k=1}^{\infty}$ such that $d(x_{k+1}, x_k) > 2$ and $f(x_k) \geqq k$ for all k. Therefore $\lim_{k\to\infty} f(x_k) = \infty$. Further the $\{x_k\}_{k=1}^{\infty}$ have been chosen so that

$$d(x_{k+1}, x_1) \geqq d(x_1, x_k) + 3 > d(x_1, x_k) + 2 \quad \text{and} \quad d(x_k, x_1) \geqq 2k - 2.$$

LEMMA 5. *Let $\{M, d\}$ be an unbounded metric space. Then there exists a function g: $\{M, d\} \to [1, \infty)$ which is unbounded and uniformly continuous on M.*

Proof. Let x_0 be a point in M; let g: $\{M, d\} \to [1, \infty)$ be defined as follows:

$$g(x) \equiv \begin{cases} 1 & \text{if } x \in S_1(x_0) \\ d(x, x_0) & \text{if } x \notin S_1(x_0). \end{cases}$$

Since M is unbounded, g is clearly unbounded. Furthermore g is uniformly continuous on M. This can be seen directly. Assume $\epsilon > 0$ and given. Select $\delta = \epsilon$ and consider $|g(x) - g(y)|$ with $d(x, y) < \delta$.

DEFINITION 2. *Let $\{M, d\}$ be a metric space, $z \in M$, and $A \subset M$. Let f: $\{M, d\} \to [1, \infty)$. Now f: $\{M, d\} \to [1, \infty)$ is defined to be monotone increasing relative to z on $A \subset M$ if, and only if, for $x, y \in A$, $f(x) \geqq f(y)$ whenever $d(x, z) \geqq d(y, z)$.*

DEFINITION 3. *Let $\{M, d\}$ be a metric space. Let $A \subset M$ and g: $\{M, d\} \to [1, \infty)$. We define g to be noncontracting on $A \subset M$ if, and only if, $d(x, y) \leqq |g(x) - g(y)|$ for all $x, y \in A$.*

We observe that if g: $\{M, d\} \to [1, \infty)$ is uniformly continuous and noncontracting on M, then g is unbounded on M if M is unbounded; furthermore, if $\{M, d\}$ has property B, then g will be bounded on any bounded subset of M but unbounded on any unbounded subset of M by Lemma 1.

Example 2. On the line g: $[1, \infty) \to [1, \infty)$ is defined as follows: $g(x) = x$. Then g is unbounded, uniformly continuous, and noncontracting on $[1, \infty)$.

DEFINITION 4. *Let $\{M, d\}$ be a metric space. Let $z \in M$ and $A \subset M$. A is said to have property C on M relative to z if and only if, for $x, y \in A$ with $x \neq y$, $d(y, z) \geqq d(x, z)$, and δ real such that $0 < \delta < d(x, y)$, then there exists $\{z_i\}_{i=1}^{m}$ such that*

$$z_1 = x, \quad z_m = y, \quad \delta/2 < d(z_i, z_{i+1}) < \delta,$$

and $d(z_{i+1}, z) \geqq d(z_i, z)$ for all $i = 1, 2, \cdots, m-1$.

DEFINITION 5. *Let $\{M, d\}$ be an unbounded metric space and let $z \in M$. $\{M, d\}$ is said to be a C-B space relative to z if and only if the following hold:*

(1) *M has property B and*

(2) *there exists $R < \infty$ such that $S_R^c(z)$, the complement of $S_R(z)$, has property C on M relative to z.*

We note that $[1, \infty)$ has property C relative to $z=1$ and property B and is therefore a C-B space relative to $z=1$. E^n for $n \geqq 2$ is a C-B space relative to the origin in the Euclidean space.

We present now the main preliminary lemma.

LEMMA 6. *Let $\{M, d\}$ be an unbounded metric space and $z \in M$. Let $\{M, d\}$ be a C-B space relative to z and let g: $\{M, d\} \rightarrow [1, \infty)$ be uniformly continuous. Further let there exist a real number $\alpha < \infty$ such that g is noncontracting on $S_\alpha^c(z)$ and a real number $\beta < \infty$ such that g is monotone increasing on $S_\beta^c(z)$ relative to z. If f: $\{M, d\} \rightarrow [1, \infty)$ is uniformly continuous, then $f(x)/g(x)$ is bounded.*

Proof. We first remark that on any bounded subset of M, by Lemma 1, both f and g are bounded. Now $g \geqq 1$ implies $f(x)/g(x)$ is bounded on any bounded subset of M. Hence, if M were bounded and had property B, the lemma would hold automatically. The proof is done by denial. Then by Lemma 4 there exists a sequence $\{x_k\}_{k=1}^\infty$ such that, if $C_k \equiv f(x_k)/g(x_k)$, then $C_{k+1} > C_k$ and $C_k \rightarrow \infty$ as $k \rightarrow \infty$. Further we can choose the x_k so that $x_1 = z$, $d(x_{k+1}, z) > d(x_k, z) + 2$ and $d(x_k, z) \geqq 2k-2$ for all k. Now there exists k_* such that $d(x_k, z) > \alpha + \beta + R$ for $k > k_*$, where R is such that $S_R^c(z)$ has property C relative to z. We can write

$$f(x_{k+1}) - f(x_k) = C_{k+1}g(x_{k+1}) - C_k g(x_k) > C_k(g(x_{k+1}) - g(x_k)) > 0.$$

Now let $\epsilon = 1$, let δ be arbitrary except that $0 < \delta < 1$. There exists k_0 such that $k > k_0$ implies $C_k > 2/\delta$. Let $k_\#$ be $> \max(k_0, k_*)$ and let $\{z_i\}_{i=1}^m$ be such that $z_1 = x_{k_\#}$, $z_m = x_{k_\#+1}$, $d(z_{i+1}, z) \geqq d(z_i, z)$ and $\delta/2 < d(z_{i+1}, z_i) < \delta$. We assert for some j that

$$|f(z_{j+1}) - f(z_j)| \geqq C_{k_\#}(g(z_{j+1}) - g(z_j)).$$

Otherwise,

$$\begin{aligned} |f(x_{k_\#+1}) - f(x_{k_\#})| &= f(x_{k_\#+1}) - f(x_{k_\#}) \\ &\leqq \sum_{j=1}^{m-1} |f(z_{j+1}) - f(z_j)| < C_{k_\#}(g(x_{k_\#+1}) - g(x_{k_\#})) \end{aligned}$$

which is a contradiction. Now

$$\begin{aligned} |f(z_{j+1}) - f(z_j)| &\geqq C_{k_\#}(g(z_{j+1}) - g(z_j)) = C_{k_\#}|g(z_{j+1}) - g(z_j)| \\ &\geqq C_{k_\#}d(z_{j+1}, z_j) > (2/\delta)(\delta/2) = 1. \end{aligned}$$

But $d(z_j, z_{j+1}) < \delta$ with $|f(z_{j+1}) - f(z_j)| > 1 = \epsilon$ which contradicts f being uniformly continuous.

We conclude this section with the following definition and remark.

DEFINITION 6. *Let $\{M, d\}$ be a metric space with $z \in M$. Let g: $\{M, d\} \rightarrow [1, \infty)$. We define g to be of special character on M relative to z if and only if the following hold:* (1) *g is uniformly continuous,* (2) *there exists $\alpha < \infty$ such that g is monotone increasing relative to z on $S_\alpha^c(z)$, and* (3) *there exists $\beta < \infty$ such that g is noncontracting on $S_\beta^c(z)$.*

The above Lemma can now be restated as follows: *Let* $\{M, d\}$ *be a C-B space relative to* $z\in M$. *Let* g: $\{M, d\}\to[1, \infty)$ *be of special character on* M *relative to* z. *If* f: $\{M, d\}\to[1, \infty)$ *is uniformly continuous, then* $f(x)/g(x)$ *is bounded.*

III. The classes K, K^*, and K_n^*.

DEFINITION 7. *Let* $\{M, d\}$ *be a metric space and let* L *be the subset of the line, together with the usual metric, denoted by* $[1, \infty)$. *Let* $K(M)$ *denote the class of functions* $\{f: M\to L\}$ *such that* f *is uniformly continuous from* M *to* L.

Lemma 6 has the following corollary.

COROLLARY 1. *If* $f\in K(L)$, *then* $f(x)/x$ *is bounded.*

Proof. Let $M=L$, $z=1$, and $g(x)=x$ in Lemma 6. A proof of this corollary is given in [1].

LEMMA 7. *Let* $\{M, d\}$ *be a C-B space relative to* $z\in M$. *Let* g: $M\to L$ *be of special character relative to* z. *Let* $f(x)\in K(M)$ *and* $f\cdot g\in K(M)$. *Then for* $\epsilon>0$, *there exists a* $\delta>0$ *such that for* $d(x, y)<\delta$, *then* $g(x)|f(x)-f(y)|<\epsilon$.

Proof. $gf\in K(M)$ implies by Lemma 6 that there exists an $R>0$ such that $f(x)<R$ for all $x\in M$. Now $\epsilon>0$ given implies there exists a $\delta>0$ such that $|g(x)-g(y)|<\epsilon/2R$ and $|g(x)f(x)-g(y)f(y)|<\epsilon/2$ whenever $d(x, y)<\delta$. Now

$$\begin{aligned} g(x)\,|f(x)-f(y)| &\leqq |g(x)f(x)-g(y)f(y)| + f(y)\,|g(y)-g(x)| \\ &< \epsilon/2 + R\cdot\epsilon/2R = \epsilon. \end{aligned}$$

COROLLARY 2. *If* $f\in K(L)$ *and* $xf(x)\in K(L)$, *then for* $\epsilon>0$, *there exists a* $\delta>0$ *such that* $x|f(x)-f(y)|<\epsilon$ *whenever* $|x-y|<\delta$.

Proof. Let $z=1$, $M=L$, and $g(x)=x$ in Lemma 7. A proof of this corollary is given in [1].

DEFINITION 7. *Let* $\{M, d\}$ *be a metric space.* $K^*(M)$ *is defined to be the class of all uniformly continuous functions* $\{f: M\to L\}$ *such that for* $h\in K(M)$, *then* $h\cdot f\in K(M)$.

In general, $K^*(M)$ is a proper subset of $K(M)$. For let f: $L\to L$ be the identity map. Then $f\in K(L)$ but $f\notin K^*(L)$ since $x^2\notin K(L)$. It is obvious that $K^*(M)=K(M)$ when M is compact. It can also be easily seen by Lemma 1 and 2 that $K^*(M)=K(M)$ when M is totally bounded.

We present now the two main theorems.

THEOREM 1. *Let* $\{M, d\}$ *be a C-B space relative to* $z\in M$. *Let* g: $\{M, d\}\to L$ *be of special character relative to* z *on* M. *Let* $f\in K(M)$. *Then* $f\in K^*(M)$ *if and only if* $f\cdot g\in K(M)$.

Proof. Since $g(x)\in K(M)$ the necessity is trivial. Conversely suppose that $f\cdot g\in K(M)$. Let $\epsilon>0$ and let $h\in K(M)$. By Lemma 6 there exist constants R_1 and R_2 such that $f(x)<R_1$ and $h(x)/g(x)<R_2$ for all $x\in M$. By Lemma 7 there

exists a $\delta>0$ such that $|h(x)-h(y)|<\epsilon/2R_1$ and $g(y)|f(x)-f(y)|<\epsilon/2R_2$ whenever $d(x, y)<\delta$. Hence,

$$|f(x)h(x)-f(y)h(y)| \leqq f(x)|h(x)-h(y)| + (h(y)/g(y))\cdot g(y)\cdot |f(y)-f(x)|$$
$$< R_1\cdot\epsilon/2R_1 + R_2\cdot\epsilon/2R_2 < \epsilon/2+\epsilon/2 = \epsilon,$$

whenever $d(x, y)<\delta$. Since h is arbitrary in $K(M)$, it follows that $f\in K^*(M)$ and the theorem is proved.

COROLLARY 3. *If $f(x)\in K(L)$, then $f(x)\in K^*(L)$ if and only if $x\cdot f(x)\in K(L)$.*

Proof. Let $z=1$, $M=L$, and $g(x)=x$ in Theorem 1. A proof of this corollary is given in [1].

At this point the above results can be extended and generalized by induction.

DEFINITION 8. *Defining $K_0^*=K$ and $K_1^*=K^*$ and assuming that $K^*_{(n-1)}(M)$ has been defined for a metric space $\{M, d\}$ we then define $K_n^*(M)$ as the set of all functions $f\in K^*_{(n-1)}(M)$ such that for $h\in K(M)$ then $f\cdot h\in K^*_{(n-1)}(M)$.*

LEMMA 8. *Let $\{M, d\}$ be a C-B space relative to $z\in M$ and let $g\in K(M)$ be of special character relative to z. Then, if $(g(x))^n\cdot f(x)\in K(M)$ and $f(x)\in K(M)$ and $\epsilon>0$, there exists $\delta>0$ such that $(g(x))^n|f(x)-f(y)|<\epsilon$ whenever $d(x, y)<\delta$.*

Proof. If $g^nf\in K(M)$ then there exists an $R>0$ such that $g(x)^{n-1}\cdot f(x)<R$ for all $x\in M$ by Lemma 6. Now

$$g(x)^n(f(x)-f(y)) = (g(x))^nf(x) - (g(y))^nf(y) + (g(y))^nf(y) - (g(x))^nf(y)$$
$$= (g(x))^nf(x) - (g(y))^nf(y) + (g(y))^{n-1}f(y)\,\frac{(g(y))^n-(g(x))^n}{(g(y))^{n-1}}\cdot$$

Now

$$\frac{(g(y))^n-(g(x))^n}{(g(y))^{n-1}} = \frac{(g(y)-g(x))[g(y)^{n-1}+g(x)g(y)^{n-2}+\cdots+g(x)^{n-1}]}{(g(y))^{n-1}}$$
$$= (g(y)-g(x))\left(1+g(x)/g(y)+\cdots+\frac{g(x)^{n-1}}{g(y)^{n-1}}\right).$$

Choose $\delta_1>0$ such that $|g(x)-g(y)|<1$ whenever $d(x, y)<\delta_1$. Pick $\delta_2>0$ such that $|g(y)-g(x)|<\epsilon/2R(2^n-1)$ whenever $d(x, y)<\delta_2$. Let δ_3 be chosen such that $|g(x)^nf(x)-g(y)^nf(y)|<\epsilon/2$ whenever $d(x, y)<\delta_3$. Define $\delta=\min(\delta_1, \delta_2, \delta_3)$. Now $d(x,y)<\delta$ implies that $|g(x)/g(y)|<(g(y)+1)/g(y)=1+1/g(y)\leqq 2$. Further, if $d(x, y)<\delta$,

$$\left|\frac{g(y)^n-g(x)^n}{g(y)^{n-1}}\right| < |g(y)-g(x)|\cdot(1+2+\cdots+2^{n-1})$$
$$= |g(y)-g(x)|\cdot(2^n-1).$$

Thus

$$\begin{aligned}(g(x))^n\,|f(x)-f(y)| &\leqq |g(x)^nf(x)-g(y)^nf(y)| + |g(y)^{n-1}\cdot f(y)|\cdot\left|\frac{g(y)^n-g(x)^n}{g(y)^{n-1}}\right| \\ &< \epsilon/2 + [R\cdot(\epsilon/2R(2^n-1))]\cdot 2^n - 1 \\ &< \epsilon/2+\epsilon/2 = \epsilon,\end{aligned}$$

whenever $d(x, y) < \delta$.

COROLLARY 4. *If $f\in K(L)$ and $x^nf(x)\in K(L)$ then for $\epsilon>0$, there exists $\delta>0$ such that $x^n|f(x)-f(y)|<\epsilon$ whenever $|x-y|<\delta$.*

THEOREM 2. *Let $\{M, d\}$ be a C-B space relative to $z\in M$. Let $g\in K(M)$ be of special character relative to z. Then for $f\in K(M)$, $f\in K_n^*(M)$ if and only if $f\cdot g^n \in K(M)$.*

Proof. We make the following induction hypothesis: $f\in K^*_{(n-1)}(M)$ if and only if $g^{n-1}\cdot f\in K(M)$.

Let $f\in K_n^*(M)$. Then $f\in K^*_{(n-1)}(M)$ and hence $gf\in K^*_{(n-1)}(M)$. Therefore by the induction hypothesis $g^{n-1}(gf)\in K(M)$ and thus $g^n\cdot f\in K(M)$.

Conversely, let $g^nf\in K(M)$; we assert first that $f\in K^*_{(n-1)}(M)$. $g^nf\in K(M)$ implies that $g(g^{n-1}f)\in K(M)$ and hence $g^{n-1}f\in K^*(M)$ and therefore $\in K(M)$. Then, by the inductive hypothesis $f\in K^*_{(n-1)}(M)$. It follows that if $f\cdot h_1h_2\cdots h_n \in K(M)$ for an arbitrary selection of h_i, where $h_1, h_2, \cdots, h_n\in K(M)$, then $f\in K_n^*(M)$. We now show that this is the case. We know $h_i\in K(M)$ implies that there exists an $R_i>0$ such that $h_i/g<R_i$ for $i=1, 2, \cdots, n$. Also $g^nf\in K(M)$ implies that there exists an $R>0$ such that $g^{n-1}f<R$ by Lemma 6. Then

$$\begin{aligned}f(x)\prod_{i=1}^n h_i(x) - f(y)\prod_{i=1}^n h_i(y) &= (g(x))^{n-1}f(x)\left\{\frac{\prod_{i=1}^n h_i(x)}{(g(x))^{n-1}} - \frac{\prod_{i=1}^n h_i(y)}{(g(y))^{n-1}}\right\} \\ &\quad + (g(x))^{n-1}f(x)\left\{\frac{\prod_{i=1}^n h_i(y)}{(g(y))^{n-1}} - \frac{\prod_{i=1}^n h_i(y)}{(g(x))^{n-1}}\right\} \\ &\quad + \prod_{i=1}^n h_i(y)\cdot(f(x)-f(y))\,\frac{(g(y))^n}{(g(y))^n}\,.\end{aligned}$$

Then

$$f(x)\prod_{i=1}^n h_i(x) - f(y)\prod_{i=1}^n h_i(y) = I_1+I_2+I_3,$$

where the I_i have the obvious meaning. Now

$$I_2 = (g(x))^{n-1}f(x)\,\frac{\prod_{i=1}^{n} h_i(y)}{(g(y))^n}\,g(y)\left\{1-\frac{(g(y))^{n-1}}{(g(x))^{n-1}}\right\}.$$

But

$$1-\frac{(g(y))^{n-1}}{(g(x))^{n-1}}=\frac{g(x)-g(y)}{g(x)}\left\{1+\frac{g(y)}{g(x)}+\cdots+\frac{(g(y))^{n-2}}{(g(x))^{n-2}}\right\}$$

and

$$g(y)\left\{1-\frac{(g(y))^{n-1}}{(g(x))^{n-1}}\right\}=\{g(x)-g(y)\}\left\{\frac{g(y)}{g(x)}+\cdots+\frac{(g(y))^{n-1}}{(g(x))^{n-1}}\right\}.$$

Choose $\delta_1>0$ such that $|g(x)-g(y)|<1$ when $d(x, y)<\delta_1$. Then if $d(x, y)<\delta_1$,

$$\left|g(y)\left\{1-\frac{(g(y))^{n-1}}{(g(x))^{n-1}}\right\}\right| \leqq |g(x)-g(y)|\,(2^n-2).$$

Furthermore, since $g^nf\in K(M)$ for $\epsilon>0$, there exists a $\delta_2>0$ such that if $d(x,y)<\delta_2$, then $(g(y))^n|f(x)-f(y)|<\epsilon/3R_1R_2\cdots R_n$ by Lemma 8.

Consider I_1; let $\delta_3>0$ be chosen such that

$$\left|\frac{\prod_{i=1}^{n} h_i(x)}{(g(x))^{n-1}}-\frac{\prod_{i=1}^{n} h_i(y)}{(g(y))^{n-1}}\right|<\epsilon/3R,$$

whenever $d(x, y)<\delta_3$. This is possible since $h_n\in K$ and therefore $g^{n-1}h_n/g^{n-1}\in K$ and by the induction hypothesis $h_n/g^{n-1}\in K^*_{(n-1)}$. Hence $h_nh_{n-1}/g^{n-1}\in K^*_{(n-2)}$ and then it follows $h_nh_{n-1}\cdots h_1/g^{n-1}\in K$. Pick $\delta_4>0$ such that $|g(x)-g(y)|<\epsilon/3R_1R_2\cdots R_n\cdot R\cdot(2^n-2)$ whenever $d(x, y)<\delta_4$. Define δ to be $\min(\delta_1, \delta_2, \delta_3, \delta_4)$. Now δ depends only on ϵ and for $d(x, y)<\delta$,

$$\begin{aligned}
&\left|f(x)\prod_{i=1}^{n} h_i(x)-f(y)\prod_{i=1}^{n} h_i(y)\right|\\
&\leqq I_1+I_2+I_3\\
&< R\cdot\epsilon/3R+\frac{R\cdot R_1R_2\cdots R_n(2^n-2)}{3R_1\cdot R_2\cdots R_n\cdot R(2^n-2)}+\frac{\epsilon R_1R_2\cdots R_n}{3R_1R_2\cdots R_n}\\
&< \epsilon/3+\epsilon/3+\epsilon/3=\epsilon.
\end{aligned}$$

Hence, for $f\in K(M)$ and $h_i(x)$ arbitrary in $K(M)$ for $i=1, 2, \cdots, n$ then $f(x)\prod_{i=1}^{n} h_i(x)\in K(M)$ and the converse is proved.

THEOREM 3. *If* $f \in K_n^*(M)$ *and* $h \in K_n^*(M)$, *then* $f+h \in K_n^*(M)$.

THEOREM 4. *If* f *and* $h \in K_n^*(M)$, *then* $f \cdot h \in K_n^*(M)$.

Theorems 3 and 4 follow directly from the above definitions and discussions.

Thanks are due to the referee for suggestions which were found helpful and worthwhile.

References

1. E. Elyash, G. Laush, and N. Levine, On the Product of Two Uniformly Continuous Functions on the Line, this MONTHLY, 67 (1960) 265–267.
2. Dick Wick Hall and G. L. Spencer, Elementary Topology, Wiley, New York, 1955.
3. John L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

A PROPERTY OF ALGEBRAIC HOMOGENEITY FOR LINEAR FUNCTION-SPACES

ANDREW SOBCZYK, University of Miami

For different points t_1, t_2 of a set T, a function $x(t)$ *separates* t_1, t_2 in case $x(t_1) \neq x(t_2)$. As in the hypothesis of the well-known Stone-Weierstrass theorem, we say that a linear space $D(T)$ of functions $x(t)$ *separates the points* of T if for each pair t_1, t_2 of points of T, $t_1 \neq t_2$, there exists $x \in D(T)$ such that $x(t_1) \neq x(t_2)$. By defining $x(t) = t(x)$, we may regard any linear space X, for each choice of a total or separating set T of linear functionals t on X ([3], p. 153), as a function-space $D(T)$; for such T, $D(T) = X$ separates the points of T.

If the linear space $C(S, \mathfrak{I})$ of all bounded real continuous functions on an infinite topological space $(S, \mathfrak{I})$ separates the points of S, then it may be seen by use of the theorem on p. 41 of [1] that $\mathfrak{I}$ contains a completely regular topology $\mathcal{S}$. For each finite subset $\{s_1, \cdots, s_n\}$ of S, by complete regularity there exist functions $x_1, \cdots, x_n$ in $C(S, \mathcal{S})$ such that $x_i(s_j) = \delta_{ij}$. Therefore arbitrary real values $a_1, \cdots, a_n$ at respectively $s_1, \cdots, s_n$ are fitted by some $x \in C(S, \mathcal{S})$. (Since $\mathfrak{I} \supset \mathcal{S}$, we know that $C(S, \mathfrak{I}) \supset C(S, \mathcal{S})$; by Theorem 3.9 of [1], $C(S, \mathfrak{I}) = C(S, \mathcal{S})$.) Let us describe this situation by saying that for every n, each set of n different points of S is *n-separated* by the linear space $C(S, \mathfrak{I})$, and that the linear space is *algebraically homogeneous* (a.h.). Clearly any larger linear function-space than an a.h. space is an a.h. space.

A one-dimensional linear space $D(T)$ of real functions may separate the

points of T if the cardinality of T is not greater than that of the set of real numbers. It is impossible for an n-dimensional $D(T)$ to m-separate any set of m points of T if $m>n\geqq 1$. Following is an example of an infinite dimensional linear space $D(T)$ which separates the points of T, but which is not a.h. Set T is the set of all integers; $x_1(t)$ assumes a different real value for each $t\in T$, so that it separates the points of T; $x_2(t)=\delta_{2t}, \cdots, x_j(t)=\delta_{jt}, \cdots$; $D(T)$ is the set of all finite linear combinations of $x_1, x_2, \cdots$. The linear space $D(T)$ is not a.h., since for $n\geqq 2$ each set of n negative values of t is not n-separated.

If T is a Hamel basis for a linear space L, then L may be regarded as the space of all functions x on T which are such that the subset of T, where $x(t)\neq 0$ is finite; call such functions *finitely-many valued* functions. Then L is a.h. and evidently for $D(T)$ to be not a.h., it is necessary that $D(T)$ not contain all finitely-many valued functions on T.

THEOREM 1. *Suppose that $D(T)$ is any linear space of real functions on an arbitrary set T. Let N be any n-dimensional linear subspace of $D(T)$. Then there always exist some n points $t_1, \cdots, t_n$ of T which are n-separated by N.*

Proof. By hypothesis, there is a set of n functions $x_1, \cdots, x_n$ which are a basis for N. Denote the n-tuple $(0, \cdots, 0)$ by θ. For any real n-tuple $(c_{11}, \cdots, c_{1n})\neq\theta$, by the hypotheses that N is n-dimensional and $x_1, \cdots, x_n$ is a basis, there exists t_1 such that $c_{11}x_1(t_1)+\cdots+c_{1n}x_n(t_1)\neq 0$. Choose a second n-tuple $(c_{21}, \cdots, c_{2n})\neq\theta$, orthogonal to $(x_1(t_1), \cdots, x_n(t_1))$. By the hypotheses, there exists t_2 such that $c_{21}x_1(t_2)+\cdots+c_{2n}x_n(t_2)\neq 0$. Continue this procedure up to and including $(c_{n1}, \cdots, c_{nn})$ and t_n. The matrix product

$$\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ c_{n1} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} x_1(t_1) & \cdots & x_1(t_n) \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ x_n(t_1) & \cdots & x_n(t_n) \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ 0 & d_{22} & \cdots & d_{2n} \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

is nonsingular, since by construction $d_{11}\neq 0$, $d_{22}\neq 0, \cdots, d_{nn}\neq 0$, and the determinant of the triangular matrix has the value $d_{11}d_{22}\cdots d_{nn}$. Therefore each matrix on the left is nonsingular, and for arbitrary values $a_1, \cdots, a_n$, there exist $b_1, \cdots, b_n$ such that $b_1x_1+\cdots+b_nx_n$ fits the values $a_1, \cdots, a_n$ at $t_1, \cdots, t_n$; i.e., $t_1, \cdots, t_n$ are n-separated by N, as required.

If for each n, each n-dimensional subspace $N\subset D(T)$ n-separates every set of n points of T, then $D(T)$ is an a.h. space; but the converse is not necessarily true. In case $D(T)$ is of finite dimension m, $D(T)$ would be described as being a.h. in case all sets of n points are n-separated by $D(T)$ for each $n\leqq m$. The space of all polynomials of degree $\leqq(m-1)$ in a real variable t is an m-dimensional a.h. space. The three-dimensional space of polynomials of degree $\leqq 1$ in two real variables s, t, however, does not 3-separate all triples of points in the s, t-plane, and thus this space is not a.h.

For any subset R of T, denote by $D_R(T)$ the subspace of all those functions of $D(T)$ which vanish on R. A pair M, N of linear subspaces of $D(T)$ are *algebraic*

complements in case $D(T)=M\oplus N$; i.e., each $z\in D(T)$ has a unique expression $z=x+y$, $x\in M$, $y\in N$.

THEOREM 2. *For any set R of n points which are n-separated by $D(T)$, $D_R(T)$ and an n-dimensional subspace $N\subset D(T)$ are algebraic complements iff the n points are n-separated by N.*

Proof. Let $R=\{t_1, \cdots, t_n\}$, and suppose that the points of R are n-separated by N. Then if $y_1, \cdots, y_n$ are a basis for N, the sets of values $(1, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, 1)$ may be fitted in succession by linear combinations of $y_1, \cdots, y_n$. Therefore there exist functions $x_1, \cdots, x_n$, which also form a basis for N, such that $x_i(t_j)=\delta_{ij}$. For any $z\in D(T)$, we have $z=z(t_1)x_1+\cdots+z(t_n)x_n+y=x+y$, $x\in N$, $y\in D_R(T)$, as required.

Suppose conversely that N and $D_R(T)$ are algebraic complements. For any t_i, t_j of R, $i\neq j$, by hypothesis there is a $z\in D(T)$ such that $z(t_i)\neq z(t_j)$. Since $z=x+y$, $y\in D_R(T)$, we must have $x(t_i)\neq x(t_j)$; therefore the points of R are n-separated by N.

THEOREM 3. *Given any space $D(T)$ which separates points, there always exists a completely regular topology $\mathfrak{J}$ for T, such that $D(T)\subset C(T, \mathfrak{J})$. Thus in particular there always is an a.h. function-space which contains $D(T)$.*

Proof. The topology is the weak topology $\mathfrak{J}$([1], pp. 38–39) for T which is induced by $D(T)$. The topology $\mathfrak{J}$ is Hausdorff, because if x separates t_1, t_2, the inverse images of disjoint open intervals respectively containing $x(t_1)$, $x(t_2)$ are disjoint and open. Therefore by Theorem 3.7 on p. 40 of [1], the topology $\mathfrak{J}$ is completely regular.

In [2], Hewitt gives examples of regular Hausdorff spaces $(T, \mathcal{R})$ which are such that $C(T, \mathcal{R})$ contains only constants; thus of course $C(T, \mathcal{R})$ separates no pair of points. For any topological space $(V, \mathcal{V})$, for $v, v'\in V$, define $v\alpha v'$ in case $x(v)=x(v')$ for all $x\in C(V, \mathcal{V})$. The relation α is an equivalence relation; if the quotient space of $(V, \mathcal{V})$ by the α-subsets is $(T, \mathcal{U})$, then $(T, \mathcal{U})$ is Hausdorff and $C(T, \mathcal{U})$ separates the points (α-subsets) of T.

It was remarked at the beginning that for any topology $\mathcal{U}$ such that $C(T, \mathcal{U})$ separate points, there exists a completely regular topology $\mathcal{S}$, $\mathcal{S}\subset\mathcal{U}$, with the same continuous functions. If $D(T)$ is a larger linear space of functions on T than $C(T, \mathcal{U})$, then by Theorem 3 there is a completely regular topology $\mathfrak{J}$ for T which is larger than $\mathcal{U}$. Thus for any set T and completely regular topology $\mathcal{S}$ which is smaller than the discrete topology for T, there is an ascending sequence of completely regular topologies up to and including the discrete topology. There is a corresponding ascending sequence of a.h. spaces of bounded functions, up to and including the space $m(T)$ of all bounded functions on T. (By Theorem 3.6 of [1], the spaces of all continuous functions on $(T, \mathcal{U})$, and of all bounded continuous functions on $(T, \mathcal{U})$, induce the same completely regular topology.)

This work was supported by the Air Force Office of Scientific Research, under Contract No. AF-49-638-1055.

References

1. L. Gillman and M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, N. J., 1960.
2. E. Hewitt, On two problems of Urysohn, Annals of Math., vol. 47, 1946, pp. 503–509.
3. A. E. Taylor, Introduction to Functional Analysis, Wiley, New York, 1958.

RANK-SETS AND RANK-SPACES IN LINEAR FUNCTION-SPACES

ANDREW SOBCZYK, University of Miami

Let S be a set which contains at least two points, and denote by $D(S)$ an arbitrary linear space of real-valued functions on S. For any linear subspace $L \subset D(S)$, the *variety* A determined by L is the maximal subset of S on which all functions of L vanish. If L separates points of S, then A either is empty or contains a single point; in the latter case the point may be removed from S without changing the linear space $D(S)$. The *nullity* of a subset $B \subset S$ is the Hamel cardinal dimension of the subspace $D_B(S)$ of all functions which vanish on B. Varieties A which contain B may be determined by various proper subspaces L of $D_B(S)$.

For a subset B of S, if a linear subspace $K \subset D(S)$ has the property that for each function z of $D(S)$, there is a function w of K which coincides with z throughout B, we shall say that K *fits* arbitrary functions of $D(S)$ on B. Call the subset B a *rank-set associated with* K, *relative to* $D(S)$, in case K fits arbitrary functions of $D(S)$ on B, and if furthermore there is no proper linear subspace K_1 of K which has the same property. In this case call K a *rank-space* associated with B, relative to $D(S)$.

THEOREM 1. *Iff a subset $B \subset S$ has nullity zero, it is a rank-set associated with the entire space $D(S)$.*

Proof. Suppose that B is a rank-set associated with $D(S)$. Then if the nullity of B were not zero, there would be a function $u \in D(S)$ which vanishes on B and which is not the zero-function θ. Let H be any algebraic complement in $D(S)$ of the one-dimensional subspace of scalar multiples of u; then on B, H fits the values of arbitrary functions $v \in D(S)$, contrary to the hypothesis that there is no proper subspace of $D(S)$ which has this property. Therefore there is no function $u \neq \theta$ in $D(S)$ which vanishes on B; i.e. the nullity of B is zero.

Suppose that B has nullity zero. Then for any function $v \in D(S)$, of course v itself fits the values of v on B. Therefore $D(S)$ is the rank-space associated with B provided that it has no proper subspace which on B fits the values of each $v \in D(S)$. If there were such a subspace, there would be some $v \in D(S)$ and $u \in D(S)$, u not identical with v on S, with $u = v$ on B. But then $(u - v)$ would be a function which vanishes on B, and which on S is different from the zero-func-

tion θ, contrary to the hypothesis that B has zero nullity. Thus the rank-space associated with B is the entire space $D(S)$.

As an example of Theorem 1, if S is a closed region of the x, y-plane, bounded by a simple closed curve B, and $D(S)$ is any linear space of functions which are continuous on S and harmonic in the open region bounded by B, then since by the maximum principle B has nullity zero for harmonic functions, the boundary B is a rank-set associated with the entire space $D(S)$. By Theorem 1 of [2], for any n-dimensional space $D(S)$ of harmonic functions on S, there exist rank-sets of n points on B. Further examples and applications to curve-fitting are suggested by reference [1] (brought to the author's attention by J. H. Curtiss).

THEOREM 2. *A subspace K of $D(S)$ is a rank-space, with B as an associated rank-set, iff K is an algebraic complement* (see [2]) *of $D_B(S)$.*

Proof. Suppose that B is a rank-set with K as an associated rank-space. Then for any $z \in D(S)$, by hypothesis there is a function x which fits z on B, so the difference $z-x=y$ is in $D_B(S)$. Assume that $v \in D(S)$ is not the zero-function θ, and $v \in D_B(S) \cap K$. Then a hyperplane K_1 in K which is an algebraic complement of the one-dimensional space $\{cv\}$ fits the functions of $D(S)$ on B as well as K does, contrary to the requirement of the definition of K as a rank-space. Therefore $D_B(S) \cap K = \{\theta\}$, and $D_B(S)$ and K are algebraic complements.

For the converse, if $D_B(S)$ and K are algebraic complements in $D(S)$, then for each $z \in D(S)$, we have uniquely $z=x+y$, $x \in K$, $y \in D_B(S)$. Therefore x and z coincide on B. If each z may be fitted on B by $x_1 \in K_1 \subset K$, then $z-x_1=y_1$ belongs to $D_B(S)$, and since by hypothesis all of K is required together with $D_B(S)$ to span $D(S)$, by uniqueness it follows that $x_1=x$, $y_1=y$, $K_1=K$. Therefore B is a rank-set with K as an associated rank-space.

COROLLARY 3. *Two rank-spaces K, L have a common associated rank-set A iff they are both algebraic complements of $D_A(S)$. Two rank-sets A, B have a common associated rank-space K iff $D_A(S)$ and $D_B(S)$ are both algebraic complements of K.*

THEOREM 4. *For a rank-space K and each associated rank-set A, there is a unique maximal rank-set $B \supset A$, which has the same associated rank-spaces as A (including K).*

Proof. By Theorem 2, $D_A(S)$ is an algebraic complement of K. Let B be the variety which is determined by the subspace $D_A(S)$. Then $D_B(S)=D_A(S)$. For each s in the complement of B, there is a $y \in D_B(S)$ with $y(s) \neq 0$, so that if $z=x+y$, x cannot fit z at s; therefore B is maximal. By Theorem 2, any complement L of $D_B(S)$ also is a rank-space, with B as a maximal associated rank-set.

THEOREM 5. *Two rank-sets, A and C, associated with the same rank-space K, are contained in the same maximal rank-set B iff the functions of K fit the functions of $D(S)$ on $A \cup C$.* (The alternative possibility is that the functions $z \in D(S)$ are fitted both on A and on C by K, but that there are functions z which require a different function from K for fitting on A than for fitting on C.)

Proof. If $A \cup C \subset B$, where B is a rank-set associated with K, then the functions of K in particular fit the functons of $D(S)$ on $A \cup C$. Conversely, if K fits the functions of $D(S)$ on $A \cup C$, then K is a rank-space associated with $A \cup C$, since no smaller linear subspace will fit the functions of $D(S)$ on either A or C. The variety B which is determined by the subspace $D_{A \cup C}(S)$ is the maximal rank-set containing $A \cup C$.

THEOREM 6. *If A and B are rank-sets, and M is the associated rank-space of $A \cup B$, then M contains the linear span $K+L$ of K and L, where K, L are suitable rank-spaces associated respectively with A, B. It is not always necessary that $M=K+L$.*

Proof. Since M fits the functions of $D(S)$ on $A \cup B$, in particular it fits them on A, and so M contains a K. Similarly M contains an L. Therefore $M \supset K+L$. Abbreviate $D_A(S)$ to D_A, etc. By Theorem 2, M is an algebraic complement of $D_A \cap D_B = D_{A \cup B}$. In case $D_{A \cup B}$ is a proper subspace of D_A and of D_B, and K is a common algebraic complement of D_A and D_B, then $L=K$, $K+L=K$, and $K+L$ is a proper subset of M. In case $L \cap D_A$ complements $D_A \cap D_B$ in D_A, or $K \cap D_B$ complements $D_A \cap D_B$ in D_B, then $M=K+L$.

COROLLARY 7. *Each subset C of S is a rank-set. The associated rank-spaces are the algebraic complements of the subspace $D_C(S)$. A subset C for which the maximal rank-set is all of S has $D(S)$ as associated rank-space.* For example, in case of any linear space of continuous functions on a topological space S, the entire space S is the maximal rank-set which contains each dense subset. (A subset B at which all the functions of $D(S)$ vanish of course has $\{\theta\}$ as its associated rank-space, and conversely all functions of $D(S)$ vanish on each rank-set which is associated with $\{\theta\}$.)

Proof. By Theorem 2, any algebraic complement K of $D_C(S)$ is a rank-space with C as an associated rank-subset. If $\{s \in S: y(s)=0 \text{ for all } y \in D_C(S)\}=S$, then $D_C(S)=D_S(S)=\{\theta\}$, so in this case the algebraic complement is unique; it is $D(S)$. The only other case in which the associated rank-space is unique is when $D_C(S)=D(S)$: for such a rank-set C, the associated rank-space is $\{\theta\}$.

THEOREM 8. *Any finite-dimensional linear subspace K of $D(S)$ is a rank-space. If K is of dimension k, there is at least one associated rank set in S which contains exactly k points.*

Proof. If K is of dimension k, then by Theorem 1 of [2], there is at least one rank-set of k points, which has K as associated rank-space. (Also by Theorem 1 of [2], the various maximal rank-sets associated with K are all varieties determined by subspaces $D_A(S)$, where A is some set $\{s_1, \cdots, s_k\}$ of k different points of S.)

For an infinite dimensional $D(S)$, it need not be true that every linear subspace L of $D(S)$ is a possible rank-space. If L is not a rank-space, then by Theo-

rem 2, L is not the algebraic complement of $D_A(S)$ for any $A \subset S$. Each $D_A(S)$ either contains a nonzero function of L, or is insufficient with L to span $D(S)$, or both. For example, if $D(S)$ is the space of all bounded continuous functions on the open unit interval $S=(0, 1)$, and L is the subspace of all bounded functions on $(0, 1)$ which have an extension to be continuous on the closed interval $[0, 1]$, then although L fits the functions of $D(S)$ on every closed subinterval of $(0, 1)$, it is not the complement of $D_A(S)$ for any $A \subset S$. Therefore L is not a rank-space.

This work was supported, under Contract No. AF 49(638)-1055, by the Air Force Office of Scientific Research.

References

1. L. V. Kantorovitch and V. I. Krylov, Approximate Methods of Higher Analysis, translated by C. D. Benster, Interscience, New York, 1958, pp. 487–488.

2. A. Sobczyk, A property of algebraic homogeneity for linear function-spaces, this MONTHLY, the preceding paper.

ON FINITE DIMENSIONAL LINEAR TOPOLOGICAL SPACES

E. F. STEINER, University of New Mexico

By a linear space X, we will mean a vector space of finite dimension n over the real numbers R; X' will denote the algebraic complement of X. A linear topology is a topology on X with respect to which addition and scalar multiplication are continuous.

It is known (see [2]) that X admits exactly one Hausdorff linear topology. The purpose of this paper is to describe the lattice of linear topologies on X.

Let T be a linear topology on X, $\mathfrak{U}$ be a base for neighborhoods of zero, and S be the closure of the set whose only element is zero. It follows that S is a subspace of X and that every element U of $\mathfrak{U}$ contains S.

Consider the quotient space $Y=X/S$. Since S is closed, the canonical linear topology T' on Y is Hausdorff.

Let $\{e_1, e_2, \cdots, e_m\}$ be a basis for S and extend it to X by the addition of $\{e_{m+1}, \cdots, e_n\}$. Then the images $\{f_{m+1}, \cdots, f_n\}$ of $\{e_{m+1}, \cdots, e_n\}$ under the canonical mapping constitute a basis for Y.

Since there is only one Hausdorff linear topology on Y, the set $B=\{y: y\in Y, y=\sum_{m+1}^n \alpha_i f_i, |\alpha_i| <1\}$ is open in T'. Thus the pre-image

$$A = \left\{x: x \in X, x = s + \sum_{m+1}^{n} \alpha_i e_i, s \in S, |\alpha_i| < 1\right\} \text{ of } B$$

is an open set in T. Hence every set $\lambda A, \lambda\neq 0$ contains a member of $\mathfrak{U}$. From the properties of neighborhoods of zero, it follows that every element U of $\mathfrak{U}$ contains a set $\lambda A, \lambda\neq 0$. Therefore, the λA's, $\lambda\neq 0$, form a base for neighborhoods at zero. Examination of these sets allows us to state:

THEOREM 1. *Let R_i denote the real line with the topology T_i. Every linear topological space of dimension n is linearly homeomorphic to the linear space $R_1+R_2+\cdots+R_n$ with the product topology $T_1\times T_2\times\cdots\times T_n$, where T_i is either the trivial or the natural topology on the real line.*

Now let F be the family of linear functions on X defined as:

$$f_i(e_j) = 0 \qquad j = 1, \cdots, m; \qquad i = m+1, \cdots, n$$
$$f_i(e_j) = \delta_{ij} \qquad j = m+1, \cdots, n; \qquad i = m+1, \cdots, n.$$

The weak topology $\mathfrak{I}(X, F)$ on X associated with F has as a neighborhood base at zero the sets λA, $\lambda\neq 0$. The subspace of X^f generated by F is merely the annihilator S^0 of S. Thus $T=\mathfrak{I}(X, S^0)$.

THEOREM 2. *Every linear topology on X is the weak topology associated with some subspace of X^f.*

The trivial topology corresponds to the zero subspace and the Hausdorff linear topology corresponds to X^f.

COROLLARY. *Every linear topological space of finite dimension is locally convex.*

It is known (see [1]) that under topological ordering the family of linear topologies on a linear space is a lattice. Since the only linear topologies on a space of finite dimension are weak topologies and the dimension of X^f is the same as X, we may state:

THEOREM 3. *The lattice of linear topologies on X is isomorphic to the lattice of subspaces of X.*

References

1. H. Nakano, Topology and linear topological spaces, Tokyo Mathematical Book Series Vol. III, Maruzen Co., Tokyo, 1951.

2. A. Tychonoff, Ein Fixpunktsatz, Math. Ann., 111 (1935) 767–776.

ON THE ZEROS OF POLYNOMIALS

Q. G. MOHAMMAD, Jammu and Kashmir University, India

The following theorem is due to Landau and Markovitch ([1], p. 99).

THEOREM A. *For any given $t>0$ let $M=\mathrm{Max}|a_k|t^k$, $k=1, 2, 3, \cdots, n$. Then all the zeros of $p(z)=\sum_0^n a_kz^k$ lie in $|z|\geqq(|a_0|t)/(|a_0|+M)$ i.e. all the zeros of $P(z)=z^np(1/z)$ lie in $|z|<(|a_0|+M)/(|a_0|t)$.*

We prove

THEOREM 1. *For any given $t>0$ all the zeros of $P(z)$ lie in*

$$|z| < \frac{1}{t}\left(1+\left(\frac{D_p}{|a_0|}\right)^q\right)^{1/q}, \tag{1}$$

where $D_p = (\sum_1^n (|a_k| t^k)^p)^{1/p}$, $p>1$, $q>1$, $1/p+1/q=1$.

Proof. If $|z|t>1$, we have

$$|P(z)| \geqq |a_0|\,|z|^n\left(1-\frac{1}{|a_0|}\sum_1^n \frac{|a_k|\,t^k}{|z|^k t^k}\right).$$

Now

$$\sum_1^n \frac{|a_k|\,t^k}{|z|^k t^k} \leqq \left(\sum_1^n (|a_k|\,t^k)^p\right)^{1/p}\left(\sum_1^n \frac{1}{(|z|^k t^k)^q}\right)^{1/q} = D_p\left(\sum_1^n \frac{1}{(|z|^k t^k)^q}\right)^{1/q}$$

$$< D_p\left(\sum_1^\infty \frac{1}{(|z|^k t^k)^q}\right)^{1/q} = \frac{D_p}{((|z|\,t)^q - 1)^{1/q}}.$$

Hence $|P(z)| > 0$ if $1 - [D_p/(|a_0|((|z|t)^q - 1)^{1/q})] \geqq 0$, i.e. if $|z| \geqq (1+(D_p/|a_0|)^q)^{1/q}/t$. Hence those zeros of $P(z)$ with $|z|>1/t$ satisfy (1). Also those zeros of $P(z)$ with $|z| \leqq 1/t$ already satisfy (1). This proves the theorem.

COROLLARY. *Letting* $p\to\infty$ *in Theorem* 1 *we obtain Theorem* A.

REMARK. Consider $P(z) = -nz^n+z^{n-1}+z^{n-2}+\cdots+z+1$ and $0<t\leqq 1$. Then $M=t$. Hence by Theorem *a* all the zeros of $P(z)$ lie in $|z|<(n+t)/nt$. By Theorem 1 all the zeros of $P(z)$ lie in

$$|z| < \frac{1}{t}\left\{1+\frac{t^q}{n^q}(1+t^p+t^{2p}+\cdots+t^{(n-1)p})^{q/p}\right\}^{1/q} \leqq \frac{1}{t}\left\{1+\frac{t^q n^{q/p}}{n^q}\right\}^{1/q}$$

$$= \frac{1}{t}\left\{1+\frac{t^q}{n}\right\}^{1/q} \leqq \frac{n+t}{nt} \qquad \text{for } 0<t\leqq 1 \text{ and } q>1.$$

THEOREM 2. *Let* $G = \operatorname{Max}|a_k|^{1/k}$, $k=1, 2, \cdots, n$. *Then all the zeros of* $P(z)$ *lie in*

$$(2) \qquad |z| < G\left(1+\frac{1}{|a_0|}\right).$$

Proof. If $|z|>G$, we have

$$|P(z)| \geqq |a_0|\,|z|^n\left\{1-\frac{1}{|a_0|}\sum_1^n\frac{|a_k|}{|z|^k}\right\} \geqq |a_0|\,|z|^n\left\{1-\frac{1}{|a_0|}\sum_1^n\frac{G^k}{|z|^k}\right\}$$

$$> |a_0|\,|z|^n\left\{1-\frac{1}{|a_0|}\sum_1^\infty\frac{G^k}{|z|^k}\right\}$$

$$= |a_0|\,|z|^n\left\{1-\frac{G}{|a_0|(|z|-G)}\right\}.$$

Hence, $|P(z)|>0$ if $1-G/(|a_0|(|z|-G))\geqq 0$, i.e. if $|z|\geqq G(1+(1/|a_0|))$. Hence, those zeros of $P(z)$ with $|z|>G$ satisfy (2). Also those zeros of $P(z)$ with $|z|\leqq G$ already satisfy (2). This proves the theorem.

THEOREM 3. *Let G be defined as in Theorem 2. Then all the zeros of $P(z)$ lie in*

$$(3)\qquad |z| < \left\{1+\left(\frac{\delta_p}{|a_0|}\right)^q\right\}^{1/q}, \quad \textit{where} \quad \delta_p = \left(\sum_1^n G^{kp}\right)^{1/p}.$$

Proof. If $|z|>1$, we have $|P(z)|\geqq |a_0|\ |z|^n\{1-(1/|a_0|)\sum_1^n G^k/|z|^k\}$. Now

$$\sum_1^n \frac{G^k}{|z|^k} \leqq \left(\sum_1^n G^{kp}\right)^{1/p}\left(\sum_1^n \frac{1}{|z|^{kq}}\right)^{1/q} = \delta_p\left(\sum_1^n \frac{1}{|z|^{kq}}\right)^{1/q}$$
$$< \delta_p\left(\sum_1^\infty \frac{1}{|z|^{kq}}\right)^{1/q} = \frac{\delta_p}{(|z|^q-1)^{1/q}}.$$

Hence, $|P(z)|>0$ if $1-\delta_p/(|a_0|(|z|^q-1)^{1/q})\geqq 0$, i.e., if $|z|\geqq\{1+(\delta_p/|a_0|)^q\}^{1/q}$. Hence as before all the zeros of $P(z)$ lie in the region defined by (3).

REMARK. Consider $P(z)=-nz^n+z^{n-1}+\cdots+z+1$.

By Theorem 2 all the zeros of $P(z)$ lie in $|z|<1+1/n$.

By Theorem 3 all the zeros of $P(z)$ lie in

$$|z| < \left(1+\frac{n^{q/p}}{n^q}\right)^{1/q} = \left(1+\frac{1}{n}\right)^{1/q} < \left(1+\frac{1}{n}\right) \quad \text{as} \quad q>1.$$

Letting $p\to 1$, we get the exact bound 1 as $[1+(1/n)]^{1/q}\to 1$.

THEOREM 4. *Let $S=\operatorname{Max}|a_k|/k$, $k=1,2,\cdots,n$. Then all the zeros of $F(z)=Sz^n+a_1z^{n-1}+\cdots+a_n$ lie in*

$$(4)\qquad |z| < \frac{3+5^{1/2}}{2}.$$

Proof. If $|z|>1$, we have

$$|F(z)| \geqq S|z|^n\left\{1-\frac{1}{S}\sum_1^n\frac{|a_k|}{|z|^k}\right\} \geqq S|z|^n\left\{1-\sum_1^n\frac{k}{|z|^k}\right\}$$
$$> S|z|^n\left\{1-\sum_1^\infty\frac{k}{|z|^k}\right\} = S|z|^n\left\{1-\frac{|z|}{(|z|-1)^2}\right\}.$$

Hence $|F(z)|>0$ if $1-|z|/(|z|-1)^2\geqq 0$, i.e. if $|z|\geqq(3+5^{1/2})/2$. This proves the theorem.

THEOREM 5. *All the zeros of $P(z)$ lie in*

$$(5)\qquad |z| \leqq 2r.$$

where $r = \text{Max } |a_{k+1}|/|a_k|$, $a_k \neq 0$, $k = 0, 1, 2, \cdots, n$.

Proof. If $|z| > r$, we have

$$|P(z)| \geqq |a_0| \, |z|^n \left\{1 - \frac{1}{|a_0|} \sum_1^n \frac{|a_k|}{|z|^k}\right\} \geqq |a_0| \, |z|^n \left\{1 - \sum_1^n \frac{r^k}{|z|^k}\right\}$$

$$> |a_0| \, |z|^n \left\{1 - \sum_1^\infty \frac{r^k}{|z|^k}\right\} = |a_0| \, |z|^n \left\{1 - \frac{r}{(|z| - r)}\right\},$$

hence $|P(z)| > 0$ if $1 - r/(|z| - r) \geqq 0$, i.e. if $|z| \geqq 2r$. Hence, those zeros of $P(z)$ with $|z| > r$ satisfy (5). Also those zeros of $P(z)$ with $|z| \leqq r$ already satisfy (5). This proves the theorem.

Reference

1. M. Marden, The geometry of zeros, Amer. Math. Soc., Math. Surveys, No. 3, 1963.

A CONGRUENCE PROPERTY OF $C_k(n)$

A. M. Vaidya, Texas Technological College

1. Let $\prod_{m=1}^\infty (1 - x^m)^r = \sum_{n=0}^\infty p_r(n) x^n$. Gupta has shown in [1] that

$$(-1)^n p_r(n) = \sum_{k=0}^\infty C_k(n) \binom{r}{k},$$

where $C_k(n)$ is the coefficient of $(-x)^n$ in y^k and

$$y = \sum_{j=1}^\infty (-1)^j \left\{x^{j(3j-1)/2} + x^{j(3j+1)/2}\right\}.$$

We show here that if p is a prime, then $C_p(n) \equiv 0 \pmod p$ unless $n = \frac{1}{2}j(3j \pm 1)p$, and for $n = \frac{1}{2}j(3j \pm 1)p$, we have $C_p(n) \equiv (-1)^{\frac{1}{2}j(j \pm 1)} \pmod p$.

2. In what follows, the modulus for all the congruences shall be an odd prime p. We shall prove the results by induction. We have

$$y^p = (-x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - \cdots)^p$$
$$= -x^p(1 + x - x^4 - x^6 + x^{11} + x^{14} - x^{21} - x^{25} + x^{34} + \cdots)^p.$$

Expanding $\{1 + (x - x^4 - x^6 + x^{11} + x^{14} - x^{21} - x^{25} + x^{34} + \cdots)\}^p$ by the binomial theorem and using the fact that $\binom{p}{r} \equiv 0 \pmod p$, for $1 \leqq r \leqq p-1$, we get successively

$$\begin{aligned} y^p &\equiv -x^p - x^{2p}(1 - x^3 - x^5 + x^{10} + x^{13} - x^{20} - x^{24} + x^{33} + \cdots)^p \\ &\equiv -x^p - x^{2p} + x^{5p}(1 + x^2 - x^7 - x^{10} + x^{17} + x^{21} - x^{30} - \cdots)^p \\ &\equiv -x^p - x^{2p} + x^{5p} + x^{7p}(1 - x^5 - x^8 + x^{15} + x^{19} - x^{28} - \cdots)^p \\ &\equiv -x^p - x^{2p} + x^{5p} + x^{7p} - x^{12p}(1 + x^3 - x^{10} - x^{14} + x^{23} + \cdots)^p. \end{aligned}$$

We can continue in this fashion. This shows that the results which we wish to prove are true up to $n=7$. Let us assume that if we continue the above procedure we shall obtain at the $(2j-2)$-th step,

$$\begin{aligned} y^p \equiv -x^p - x^{2p} + x^{5p} + x^{7p} - \cdots + (-1)^{j-1}x^{\frac{1}{2}(j-1)(3j-2)p} \\ + (-1)^j x^{\frac{1}{2}j(3j-1)p}(1 + x^j - x^{3j+1} - x^{4j+2} + \cdots)^p. \end{aligned}$$

This assumption is certainly true for $j=2$. Again expanding

$$(1 + x^j - x^{3j+1} - x^{4j+2} + \cdots)^p,$$

we get

$$\begin{aligned} y^p \equiv -x^p - x^{2p} + x^{5p} + x^{7p} - \cdots + (-1)^j x^{\frac{1}{2}j(3j-1)p}, \\ + (-1)^j x^{\frac{1}{2}j(3j+1)p}(1 - x^{2j+1} - x^{3j+2} + \cdots)^p, \end{aligned}$$

and at the next step, we obtain

$$\begin{aligned} y^p \equiv -x^p - x^{2p} + x^{5p} + x^{7p} - \cdots + (-1)^j x^{\frac{1}{2}j(3j+1)p} \\ + (-1)^{j+1}x^{\frac{1}{2}(j+1)(3j+2)p}(1 + x^{j+1} - x^{3j+4} - x^{4j+6} + \cdots)^p. \end{aligned}$$

This proves our assertion regarding the $(2j-2)$-th step. This proof also shows that at the $(2j-1)$-th step, we have

$$\begin{aligned} y^p \equiv -x^p - x^{2p} + x^{5p} + x^{7p} - \cdots + (-1)^j x^{\frac{1}{2}j(3j-1)p} \\ + (-1)^j x^{\frac{1}{2}j(3j+1)p}(1 - x^{2j+1} - x^{3j+2} + \cdots)^p. \end{aligned}$$

It follows that the coefficient of $(-x)^n$ in y^p is 0 unless $n=\frac{1}{2}j(3j\pm 1)p$, and if $n=\frac{1}{2}j(3j\pm 1)p$, then it is

$$(-1)^{j+\frac{1}{2}j(3j\pm 1)p} = (-1)^{j+\frac{1}{2}j(3j\pm 1)} = (-1)^{\frac{1}{2}j(j\pm 1)}.$$

We thus have $C_p(n)\equiv 0 \pmod p$ if $n\neq\frac{1}{2}j(3j\pm 1)p$ and $C_p(n)\equiv(-1)^{\frac{1}{2}j(j\pm 1)} \pmod p$ if $n=\frac{1}{2}j(3j\pm 1)p$. For $p=2$, the results are immediate.

3. By the same method, one can prove the following general result:

$$C_{p^k}(n) \equiv 0 \pmod p \text{ if } n \neq \tfrac{1}{2}j(3j \pm 1)p,$$

and

$$C_{p^k}(n) \equiv (-1)^{\frac{1}{2}j(j\pm 1)} \pmod p \text{ if } n = \tfrac{1}{2}j(3j \pm 1)p,$$

where k is a positive integer.

This research was supported by the NSF Grant No. G 19015.

Reference

1. H. Gupta, On the coefficients of the powers of Dedekind's modular form. To appear in the J. London Math. Soc.

A NOTE ON FUNCTIONS WITH A NATURAL BOUNDARY

WILLIAM J. GRAY, University of Alabama

We consider a series of complex functions

$$\sum_{k=0}^{\infty} f_n(z), \tag{1}$$

where the $f_n(z)$ are defined on some subset of the plane. In the cases usually studied, where (1) is a power series, Dirichlet series, etc., the series (1) has a region R of uniform convergence, and the $f_n(z)$ are regular on R, so that (1) defines a regular function $F(z)$ on R. If R is not the plane, and the boundary of R is a natural boundary for $F(z)$, the series (1) defines a complete analytic function. There are examples of series of the form (1) which define two complete analytic functions. We wish to give an example of an extreme case where infinitely many complete analytic functions are defined by one series.

LEMMA 1. *If a and b are positive integers, the series*

$$\sum_{n=1}^{\infty} \frac{z^{an}}{1 - z^{bn}} \tag{2}$$

defines a regular function for $|z| < 1$ which has $|z| = 1$ for a natural boundary.

We do not prove this lemma since the proof is not essentially different from that given in [1], page 423, for a less general series. The conclusion is that if p and q are integers such that $\exp 2\pi i(p/q) \neq 1$, then the sum of the series (2) tends to infinity as $|z| \to 1$ along the ray $z = r \exp 2\pi i(p/q)$, $0 \leqq r \leqq 1$.

If the series (2) converges at z, let $L(a, b, z)$ denote its sum. If $b > a$ and $|z| > 1$, we have the relation $-L(b-a, b, 1/z) = L(a, b, z)$, and since $-L(b-a, b, 1/z)$ defines a regular function for $|z| > 1$, $L(a, b, z)$ must have the same property. Hence, series (2) defines two complete analytic functions which have $|z| = 1$ for a common natural boundary whenever $b > a$.

LEMMA 2. *If $b > a$, then for $|z| \geqq r > 1$, $|L(a, b, z)| \leqq [r/(r-1)]^2 (1/r)^{b-a}$.*

Proof. For each n,

$$\begin{aligned} \big|\, |z|^{bn} - 1 \,\big| &= (|z| - 1)(|z|^{bn-1} + \cdots + 1) \geqq (|z| - 1)|z|^{bn-1} \\ &\geqq (r-1)|z|^{bn-1} \end{aligned}$$

and hence

$$\begin{aligned} |L(a, b, z)| &\leqq \sum_{k=1}^{\infty} |z|^{an}/(r-1)|z|^{bn-1} = [1/(r-1)] \sum_{n=1}^{\infty} 1/|z|^{(b-a)n-1} \\ &\leqq [r/(r-1)] \sum_{n=1}^{\infty} 1/r^{(b-a)n} \leqq [r/(r-1)](1/r)^{b-a} \sum_{n=0}^{\infty} 1/r^{(b-a)n} \\ &\leqq [r/(r-1)]^2 (1/r)^{b-a}. \end{aligned}$$

Now give the set of complex lattice points $2r+2mi$, where r and m are any integers, an arbitrary enumeration $z_1, z_2, \cdots$. The set of circles $|z-z_k|=1$ partition the plane into a sequence of mutually disjoint regions, $\{R_j\}$, consisting of the open discs $|z-z_k|<1$ and the star regions which are exterior to all the circles. We assert that the series

$$\sum_{k=1}^{\infty} L(k, 2k, z-z_k) \tag{3}$$

defines a regular function $f_j(z)$ which has R_j for a natural region, $j=1, 2, \cdots$.

First let $|z-z_j| \leqq s < 1$. For $k \neq j$, $|z-z_k| \geqq 2-s > 1$. By Lemma 2, we have $|L(k, 2k, z-z_k)| \leqq [(2-s)/(1-s)]^2[1/(2-s)]^k$. It follows that the series (3) converges uniformly for $|z-z_j| \leqq s < 1$, and since every term is regular, series (3) defines a regular function $F(z)$ on the disc $|z-z_j|<1$.

Let p, q be integers such that $\exp 2\pi i(p/q)$ is not $1, -1, i$, or $-i$. As $|z-z_j| \to 1$ along the ray $z-z_j = r \exp 2\pi i(p/q)$, $0 \leqq r \leqq 1$, we have for $k \neq j$, $|z-z_k| \geqq s > 1$ for some s, and hence the series (3), with the term $L(j, 2j, z-z_j)$ deleted, converges uniformly, hence is bounded. On the other hand, $L(j, 2j, z-z_j) \to \infty$ as $|z-z_j| \to 1$ along the prescribed ray. Hence $F(z)$ does not continue beyond $|z-z_j|=1$.

A similar argument can be carried through for an open region exterior to all the circles $|z-z_k|=1$.

Reference

1. G. Sansone and J. Gerretsen, Lectures on the theory of functions of a complex variable, P. Noordhoff, Groningen, 1960.

WHEN ARE COMPACT AND CLOSED EQUIVALENT?

NORMAN LEVINE, Ohio State University

It is well known that in a compact topological space, every closed set is compact, but not conversely. It is equally well known that in a Hausdorff space, every compact set is closed, but not conversely. There are, however, spaces in which the compact sets coincide with the closed sets—compact Hausdorff spaces, for example. It is the intent of this note to give several characterizations of such spaces and to list some of their properties.

DEFINITION 1. *A topological space* $(X, \mathfrak{I})$ *will be called a C-C space iff the closed sets in X coincide with the compact sets in X.*

The following definition will be useful.

DEFINITION 2. *Suppose that* $(X, \mathfrak{I})$ *is a topological space. Then* $\mathfrak{I}$ *is said to be M.R.C. (maximal relative to compactness) iff* (1) $(X, \mathfrak{I})$ *is compact and* (2) *if* $\mathfrak{I}$ *is a proper subset of* $\mathfrak{I}^*$, *then* $(X, \mathfrak{I}^*)$ *is not compact.*

The following lemma plays a fundamental role in describing *C-C* spaces.

LEMMA 1. *Let* $(X, \mathfrak{I})$ *be a compact topological space and let A be a subset of X. If* $\mathfrak{I}^* \equiv \{U \cup (V \cap A) \mid U, V \in \mathfrak{I}\}$, *then* $(X, \mathfrak{I}^*)$ *is compact iff* $\mathcal{C}A$ *is compact in* $(X, \mathfrak{I})$, $\mathcal{C}$ *denoting the complement operator.*

Proof: Necessity. It is clear that $A \in \mathfrak{I}^*$ and thus $\mathcal{C}A$ is closed in $(X, \mathfrak{I}^*)$. Then $\mathcal{C}A$ is compact in $(X, \mathfrak{I}^*)$ and hence compact in $(X, \mathfrak{I})$ since $\mathfrak{I} \subset \mathfrak{I}^*$.

Sufficiency. Let $X = \bigcup \{U_\alpha \cup (V_\alpha \cap A) : \alpha \in \Delta\}$. Then $\mathcal{C}A \subset \bigcup \{U_\alpha : \alpha \in \Delta\}$ and since $\mathcal{C}A$ is compact in $(X, \mathfrak{I})$, it follows that $\mathcal{C}A \subset \bigcup \{U_\alpha : \alpha \in \Gamma_1\}$, where Γ_1 is a finite subset of Δ. Now $X = \bigcup \{U_\alpha \cup V_\alpha : \alpha \in \Delta\}$ and since $(X, \mathfrak{I})$ is compact, $X = \bigcup \{U_\alpha \cup V_\alpha : \alpha \in \Gamma_2\}$, where Γ_2 is a finite subset of Δ. It is clear that $A \subset \bigcup \{U_\alpha \cup (V_\alpha \cap A) : \alpha \in \Gamma_2\}$. Then $X = \mathcal{C}A \cup A = \bigcup \{U_\alpha \cup (V_\alpha \cap A) : \alpha \in \Gamma_1 \cup \Gamma_2\}$ and thus $(X, \mathfrak{I}^*)$ is compact.

THEOREM 1. *If* $(X, \mathfrak{I})$ *is a compact Hausdorff space, then* $\mathfrak{I}$ *is M.R.C.*

Proof. Suppose that the theorem is false. Then there exists a $\mathfrak{I}^\#$ such that (1) $\mathfrak{I} \subset \mathfrak{I}^\#$ properly and (2) $(X, \mathfrak{I}^\#)$ is compact. Take $A \in \mathfrak{I}^\# - \mathfrak{I}$ and let $\mathfrak{I}^*$ be the topology for X generated by $\mathfrak{I}$ and A as in Lemma 1. Then $\mathfrak{I} \subset \mathfrak{I}^* \subset \mathfrak{I}^\#$ and thus $(X, \mathfrak{I}^*)$ is compact. By Lemma 1, $\mathcal{C}A$ is compact in $(X, \mathfrak{I})$ and hence closed since $(X, \mathfrak{I})$ is Hausdorff. Then A is in $\mathfrak{I}$ which yields a contradiction.

COROLLARY 1. *Let* $(X, \mathfrak{I})$ *be a compact Hausdorff space and* $\mathfrak{I}' \subset \mathfrak{I}$ *properly. Then* $(X, \mathfrak{I}')$ *is not Hausdorff.*

THEOREM 2. *Suppose that* $(X, \mathfrak{I})$ *is a topological space. Then* $(X, \mathfrak{I})$ *is C-C iff* $\mathfrak{I}$ *is M.R.C.*

Proof: Sufficiency. It suffices to show that every compact set is closed. Let A be a compact subset of X and suppose that A is not closed. Then $\mathcal{C}A \notin \mathfrak{I}$. Let $\mathfrak{I}^*$ be the topology for X generated by $\mathfrak{I}$ and $\mathcal{C}A$ as in Lemma 1. Since $\mathcal{C}\mathcal{C}A$ is compact in $(X, \mathfrak{I})$, then by Lemma 1, $(X, \mathfrak{I}^*)$ is compact and $\mathfrak{I} \subset \mathfrak{I}^*$ properly which contradicts the fact that $\mathfrak{I}$ is M.R.C.

Necessity. Suppose that there exists a topology $\mathfrak{I}^\#$ for X such that $\mathfrak{I} \subset \mathfrak{I}^\#$ properly and $(X, \mathfrak{I}^\#)$ is compact. Take $A \in \mathfrak{I}^\# - \mathfrak{I}$. Then $\mathcal{C}A$ is closed in $(X, \mathfrak{I}^\#)$ and hence is compact in $(X, \mathfrak{I}^\#)$. Then $\mathcal{C}A$ is compact in $(X, \mathfrak{I})$ and therefore closed in $(X, \mathfrak{I})$ by condition *C-C*. Hence $A \in \mathfrak{I}$ which is a contradiction.

COROLLARY 2. *Let* $(X, \mathfrak{I})$ *be a topological space. If* $(X, \mathfrak{I})$ *is compact and Hausdorff, then* $(X, \mathfrak{I})$ *is C-C.*

Proof. This follows from Theorems 1 and 2.

THEOREM 3. *Let $(X, \mathfrak{I})$ be a topological space. If $(X, \mathfrak{I})$ is C-C, then it is compact and T_1.*

Proof. Singleton sets are compact.

REMARK 1. The converses of Corollary 2 and Theorem 3 are false as the following two examples show.

Example 1. Let $(R, \mathfrak{I})$ be the space of rationals with the relative topology and let $(X, \mathfrak{I}^*)$ be the one point compactification of $(R, \mathfrak{I})$. Since $(R, \mathfrak{I})$ is not locally compact, it follows that $(X, \mathfrak{I}^*)$ is not Hausdorff. We show now that $(X, \mathfrak{I}^*)$ is *C-C*. It suffices to show that every compact subset of X is closed. Let A be a compact subset of X.

Case 1. Suppose that $\infty \notin A$. Then $A \subset R$ and thus A is compact in $(R, \mathfrak{I})$. But $(R, \mathfrak{I})$ is Hausdorff and hence A is both closed and compact in $(R, \mathfrak{I})$. Then by definition of the one point compactification, $\mathcal{C}_X A \in \mathfrak{I}^*$ and hence A is closed in $(X, \mathfrak{I}^*)$.

Case 2. Suppose that $\infty \in A$ and that A is not closed in $(X, \mathfrak{I}^*)$. Take $x \in A' \cap \mathcal{C}A$, A' being the derived set of A in $(X, \mathfrak{I}^*)$. Then $x \in R$ and there exists for each positive integer i, an $a_i \in A - \{\infty\}$ such that $\lim a_i = x$ in both $(R, \mathfrak{I})$ and $(X, \mathfrak{I}^*)$. Without loss of generality we may assume that in the natural order, $a_1 > a_2 > \cdots > a_n > \cdots > x$. Let $F = \{x\} \cup \{a_i \mid i = 1, 2, \cdots\}$. Clearly F is closed and compact in $(R, \mathfrak{I})$ and thus $\mathcal{C}_X F \in \mathfrak{I}^*$. Let $O_i = \{r \mid r > a_{i+1}, r \in R\}$ for each positive integer i. Then O_i is open in both R and X for each i. Now $A \subset \mathcal{C}_X F \cup \bigcup \{O_i \mid i = 1, 2, \cdots\}$ which is an open cover of A from $\mathfrak{I}^*$. It is clear that this open cover of A contains no finite subcover. This is a contradiction.

Example 2. Let X be an infinite set and let $\mathfrak{I}$ be the cofinite topology. Clearly $(X, \mathfrak{I})$ is compact and T_1, but not *C-C*.

REMARK 2. From Corollary 2, Theorem 3, examples 1 and 2, we observe that compact and Hausdorff is stronger than *C-C* which in turn is stronger than compact and T_1.

THEOREM 4. *Let $(X, \mathfrak{I})$ be a topological space which satisfies the first axiom of countability. Then $(X, \mathfrak{I})$ is C-C iff $(X, \mathfrak{I})$ is compact and Hausdorff.*

Proof: *Sufficiency.* This is merely Corollary 2.

Necessity. It suffices to prove that $(X, \mathfrak{I})$ is Hausdorff. Let $x \neq y$ and suppose that x and y cannot be separated by disjoint open sets. Let $\{U_i\}$ and $\{V_i\}$ be monotone decreasing open bases at x and y respectively. Since *C-C* implies T_1, we may assume that $x \notin V_i$ for all i and that $y \notin U_i$ for all i. Take $x_i \in U_i \cap V_i$ for each positive integer i. It is clear that $x = \lim x_i$. Let $F = \{x\} \cup \{x_i \mid i = 1, 2, \cdots\}$. F is compact and thus closed by condition *C-C*. But $y \notin F$ and $y = \lim x_i$. This is a contradiction.

LEMMA 2. *Let* $\{(X_\alpha, \mathfrak{J}_\alpha)\}_{\alpha\in\Delta}$ *be a nonempty family of nonempty compact spaces and let* $(X, \mathfrak{J})$ *be the product space. If* C_{α^*} *is a compact subset of* X_{α^*}, *then* $P_{\alpha^*}^{-1}[C_{\alpha^*}]$ *is compact in* $(X, \mathfrak{J})$.

Proof. For $\alpha\neq\alpha^*$, let $(Y_\alpha, \mathfrak{J}_\alpha^*)=(X_\alpha, \mathfrak{J}_\alpha)$ and $(Y_{\alpha^*}, \mathfrak{J}_{\alpha^*}^*)=(C_{\alpha^*}, C_{\alpha^*}\cap\mathfrak{J}_{\alpha^*})$. Then $P_{\alpha^*}^{-1}[C_{\alpha^*}]=\Pi_{\alpha\in\Delta}Y_\alpha$ and hence is compact.

THEOREM 5. *Let* $\{(X_\alpha, \mathfrak{J}_\alpha)\}_{\alpha\in\Delta}$ *be a nonempty family of nonempty spaces and let* $(X, \mathfrak{J})$ *be the product space. If* $(X, \mathfrak{J})$ *is C-C, then* $(X_\alpha, \mathfrak{J}_\alpha)$ *is C-C for each* $\alpha\in\Delta$.

Proof. If C_α is closed in X_α, then C_α is compact since X_α is compact. Conversely let C_α be compact in X_α. Then $P_\alpha^{-1}[C_\alpha]$ is compact in X by Lemma 2 and thus is closed by condition *C-C*. Then $\mathcal{C}C_\alpha=P_\alpha P_\alpha^{-1}\mathcal{C}C_\alpha=P_\alpha\mathcal{C}P_\alpha^{-1}C_\alpha$ and hence $\mathcal{C}C_\alpha$ is open in X_α since projection maps are open. Thus C_α is closed in X_α.

REMARK 3. The converse of Theorem 5 is false as the next theorem implies.

THEOREM 6. *Let* $(X, \mathfrak{J})$ *be a topological space and let* $(X\times X, \mathfrak{J}^*)$ *be the Cartesian product of* $(X, \mathfrak{J})$ *with itself. Then* $(X\times X, \mathfrak{J}^*)$ *is C-C iff* $(X, \mathfrak{J})$ *is C-C and Hausdorff.*

Proof. Sufficiency: If $(X, \mathfrak{J})$ is *C-C* and Hausdorff, then $(X\times X, \mathfrak{J}^*)$ is compact and Hausdorff and hence is *C-C* by Corollary 2.

Necessity: Suppose $(X\times X, \mathfrak{J}^*)$ is *C-C*. Then $(X, \mathfrak{J})$ is *C-C* by Theorem 5. To show that $(X, \mathfrak{J})$ is Hausdorff let D be the diagonal in $X\times X$. Now D is homeomorphic to X and hence is compact. Since $(X\times X, \mathfrak{J}^*)$ is *C-C*, it follows that D is closed and hence $(X, \mathfrak{J})$ is Hausdorff.

Example 3. Let $(X, \mathfrak{J})$ be the one point compactification of the rationals. Then $(X, \mathfrak{J})$ is *C-C*, but not Hausdorff (see example 1). Then $X\times X$ is not *C-C* by the above theorem.

THEOREM 7. *Let* $(X, \mathfrak{J})$ *be C-C and let* $(Y, \mathfrak{J}')$ *be a subspace. Then* $(Y, \mathfrak{J}')$ *is C-C iff* Y *is a closed subset of* X.

THEOREM 8. *Let* $(X, \mathfrak{J})$ *be a topological space and let* $X=Y_1\cup\cdots\cup Y_n$, *where* $(Y_i, \mathfrak{J}_i)$ *are C-C subspaces of* $(X, \mathfrak{J})$. *Then* $(X, \mathfrak{J})$ *is C-C iff* Y_i *is closed in* X *for each* i.

The proofs of Theorems 7 and 8 are easy and will be omitted.

COROLLARY 3. *Let* $(X, \mathfrak{J})$ *be a topological space and let* $(Y_i, \mathfrak{J}_i)$ *be closed subspaces for* $i=1, \cdots, n$. *Let* $X=Y_1\cup\cdots\cup Y_n$. *Then* $(X, \mathfrak{J})$ *is C-C iff* $(Y_i, \mathfrak{J}_i)$ *is C-C for each* i.

Proof. The necessity follows from Theorem 7 and the sufficiency follows from Theorem 8.

Finally, the author acknowledges gratefully the referee's helpful suggestions for improving the exposition of this paper.

A GOLDBACH THEOREM FOR POLYNOMIALS WITH INTEGRAL COEFFICIENTS

D. R. Hayes, University of Tennessee

The Goldbach conjecture asserts that every even integer greater than 2 is the sum of two primes. Let $\boldsymbol{Z}$ denote the domain of all integers. The polynomial domain $\boldsymbol{Z}[x]$ is, like $\boldsymbol{Z}$, a unique factorization domain with the irreducible polynomials playing the part of the prime numbers. Our aim in this note is to prove the following analog of the Goldbach conjecture for the domain $\boldsymbol{Z}[x]$:

Theorem 1. *For every polynomial M in $\boldsymbol{Z}[x]$ of degree $n \geqq 1$, there are irreducible polynomials A and B, each of degree n, such that $A+B=M$.*

We require the following lemma.

Lemma. *Let p and q be distinct odd primes. Then there are integers c and d such that $p \nmid c$ and $q \nmid d$ and $pc+qd=1$.*

Proof. From elementary number theory, we know that there are integers c_0 and d_0 such that $pc_0+qd_0=1$. If $p \nmid c_0$ and $q \nmid d_0$, then there is nothing to prove. Suppose, therefore, as we may, that $p \mid c_0$. If $c_r=c_0+qr$ and $d_r=d_0-pr$, then clearly, for all integers r, $pc_r+qd_r=1$. Now neither of c_1 and c_2 is divisible by p. Otherwise by considering the differences c_1-c_0 and c_2-c_0, we could deduce that $p \mid q$. Also, one of d_1 and d_2 is not divisible by q. Otherwise, q would divide $d_1-d_2=p$. One of the pairs (c_1, d_1) and (c_2, d_2), therefore, satisfies the conditions of the lemma. This completes the proof.

Proof of Theorem 1. Suppose that $M=m_0x^n+m_1x^{n-1}+\cdots+m_n$. Choose distinct odd primes p and q which do not divide either of m_0 and m_n, and choose a_0' and b_0' so that $qa_0'+pb_0'=m_0$. Set $a_0=qa_0'$ and $b_0=pb_0'$. Then

$$m_0 = a_0 + b_0, \quad p \nmid a_0, \quad q \nmid b_0. \tag{1}$$

For every $0<i<n$, choose a_i' and b_i' such that $pa_i'+qb_i'=m_i$. Set $a_i=pa_i'$ and $b_i=qb_i'$. Then, for $0<i<n$,

$$m_i = a_i + b_i, \quad p \mid a_i, \quad q \mid b_i. \tag{2}$$

By the lemma, choose a_n' and b_n' such that $pa_n'+qb_n'=m_n$ but $p \nmid a_n'$ and $q \nmid b_n'$. Set $a_n=pa_n'$ and $b_n=qb_n'$. Then

$$m_n = a_n + b_n, \quad p \mid a_n, \quad p^2 \nmid a_n, \quad q \mid b_n, \quad q^2 \nmid b_n. \tag{3}$$

If $A=a_0x^n+a_1x^{n-1}+\cdots+a_n$ and if $B=b_0x^n+b_1x^{n-1}+\cdots+b_n$, then (1), (2) and (3) show that $A+B=M$ and that A and B are Eisenstein polynomials. Since such polynomials are irreducible ([2], p. 74), the proof is complete.

Theorem 1 is a special case of the following more general theorem:

Theorem 2. *Let $\mathbf{R}$ be a principal ideal domain (see [1], p. 151) which contains infinitely many prime elements. For every polynomial M in $\mathbf{R}[x]$ of degree $n \geqq 1$, there are irreducible polynomials A and B, each of degree n, such that $A+B=M$.*

The proof of Theorem 1 can be adapted to prove this more general result. It follows from Theorem 2, for example, that the analog of the Goldbach conjecture is true for the domain of polynomials in two indeterminates over an arbitrary field.

Supported in part by NSF Grant G 16485.

References

1. C. C. MacDuffee, Introduction to Modern Algebra, Wiley, London, 1940.
2. B. L. van der Waerden, Modern Algebra, Ungar, New York, 1949.

A NOTE ON A CERTAIN CLASS OF PRIME RINGS

KWANGIL KOH, University of North Carolina

1. Introduction. It is known that a primitive ring is a prime ring [1] but the converse is not necessarily true, for the ring of integers is a prime ring but not a primitive ring. In this note we will investigate the prime rings which may be embedded in a simple ring with unity, and prove that a prime ring is a primitive ring with nonzero socle if and only if it has a maximal right ideal which is an annihilator right ideal.

In the sequel, unless stated otherwise, R will denote a prime ring and M a right R-module. We note here that if S is a subset of an R-module (or a subset of R), then S_r will denote the annihilator (right annihilator) in R of the set S.

2. A right ideal I of R is said to be *prime* (see [3]) if $J\cdot K\subseteq I$, where J, K are right ideals of R, and $K\neq(0)$ imply that $J\subseteq I$.

THEOREM 2.1. *R is primitive if and only if it contains a maximal right ideal which is prime.*

Proof. If R is a primitive ring, it has a faithful irreducible module M and if m is a nonzero element of M then m_r is a maximal right ideal [4]. Now m_r is a prime right ideal, for if J and $K\neq(0)$ are right ideals of R such that $J\cdot K\subseteq m_r$, then $mJ\cdot K=0$ and $mJ=0$. Otherwise $mJ=M$ and M_r would not be zero, which is impossible since M is faithful. Conversely, if R contains a maximal right ideal which is prime, say I, then R/I is an irreducible R-module and it is faithful, for if a in R is such that $Ra\subseteq I$, then a must be zero. Otherwise the set $T=\{a \text{ in } R \mid Ra\subseteq I\}$ is a nonzero right ideal and $RT\subseteq I$ implies that $R\subseteq I$, which is a contradiction.

A right ideal I of R is said to be *large* if it has nonzero intersection with each nonzero right ideal of R.

THEOREM 2.2. *R is a primitive ring with nonzero socle if it has a maximal right ideal which is not large.*

Proof. If a maximal right ideal K is not large, then $K\cap I=0$ for some nonzero right ideal I. Since $K\oplus I=R$, I is a minimal right ideal, hence R is a primitive ring with nonzero socle.

An element m of M is said to be *singular* if m_r is a large right ideal of R. It is easy to see that the collection of all singular elements of M forms a submodule, called the *singular* submodule. In case $M = R^+$, the additive group of R, the singular submodule of R^+ is a two sided ideal (see [2]). A two sided ideal Z of a ring is said to be *semi-singular* if for each element x of Z, $x_r \neq 0$. Thus a singular ideal of R is semi-singular but a semi-singular ideal is not necessarily singular. For example, the ring of 2×2 triangular matrices over the ring of integers has a nonzero nilpotent ideal which is evidently semi-singular, but it has zero singular ideal. The class of rings for which 0 is the only semi-singular ideal is a subclass of prime rings and it contains the class of integral domains (rings with no zero divisors) properly.

THEOREM 2.3. *Let R be a ring with 0 as its only semi-singular ideal. Then R can be embedded in a right quotient ring Q of R (in the sense of [2]) which is regular, simple, and primitive.*

Proof. Since R is a ring with 0 as its only semi-singular ideal, the singular ideal of R is zero. Hence R can be embedded in a right quotient ring Q which is regular (by [2], p. 894, Th. 2). Suppose that S is a nonzero two sided ideal of Q. Then $T = S \cap R \neq 0$, for if s is a nonzero element of S, then there exist elements a and b in R such that $sa = b \neq 0$. Note that T is a two sided ideal of R. Hence, there must exist an element t in T such that $t_r \cap R = 0$ and t_r must be zero. Otherwise t_r would be a nonzero submodule of the R-module Q; hence $t_r \cap R \neq 0$. Since Q is regular we may choose x in Q such that $txt - t = t(xt - 1) = 0$. Thus 1 is contained in S. Since a simple ring with unity is a primitive ring, the theorem is proven.

COROLLARY 2.4. *An integral domain can be embedded in a regular, simple, primitive ring with unity.*

A module M is said to be *uniform* if each pair of nonzero submodules of M has nonzero intersection.

THEOREM 2.5. *If R is an arbitrary nonzero ring with a uniform module M which is not singular, then R contains a uniform right ideal.*

Proof. Since M is not singular there is an $m \neq 0$ in M and a nonzero right ideal I in R such that $m_r \cap I = 0$. If J_1 and J_2 are nonzero right ideals of R contained in I, then $J_1 \cap J_2 \neq 0$. For $m_r \cap I = 0$ implies that $mJ_1 \neq 0$ and $mJ_2 \neq 0$. Since M is uniform, $mJ_1 \cap mJ_2 \neq 0$. Thus $m(J_1 \cap J_2) = mJ_1 \cap mJ_2$, since $m(J_1 \cap J_2) \subseteq mJ_1 \cap mJ_2$, and if $x \neq 0$ is in $mJ_1 \cap mJ_2$, then $x = mj_1 = mj_2$ for some j_1 in J_1 and j_2 in J_2, and $m(j_1 - j_2) = 0$ implies that $j_1 - j_2 = 0$.

THEOREM 2.6. *Let R be a primitive ring and M be its faithful irreducible module. Then R contains a minimal right ideal if and only if M is not singular.*

Proof. Let U be a nonzero right ideal of R such that $U \cap m_r = 0$ for some m in M. Then U is a minimal right ideal. For if I is a right ideal such that $I \subsetneq U$,

then there is u_0 in U such that u_0 is not contained in I. But $0 \neq mu_0 \in mI = M$ implies that $mu_0 = mi$, for some i in I, and $m(u_0 - i) = 0$ implies that $u_0 = i$, which is a contradiction. Conversely if R is a primitive ring with a minimal right ideal, say I, and the singular submodule of M is not zero, then M is itself the singular module since M is irreducible. Now note that for each m in M, m_r contains I, hence $M \cdot I = 0$, which is impossible.

THEOREM 2.7. *R is a primitive ring with a minimal right ideal if and only if it has a maximal right ideal which is an annihilator right ideal.*

Proof. Select $a \in R$ so that a_r is a maximal right ideal of R. Since R is prime $aRa \neq 0$. Thus there exists a t in R such that $ata \neq 0$. Then $taR \cap a_r = 0$, for otherwise $(ata)_r \supsetneq a_r$. Note that taR is a minimal right ideal of R since $a_r \oplus taR = R$. Conversely if R is a primitive ring with a minimal right ideal I, then if $i \in I$ and $i \neq 0$, then i_r is a maximal right ideal which is an annihilator right ideal.

Acknowledgement: I am deeply indebted to the referee who read the original version of this note and suggested to me the short and elegant proofs of 2.2. and a part of 2.7., to my teacher Dr. A. C. Mewborn of the University of North Carolina, and to Dr. Allan Davis at the Summer NSF Institute at the University of Oklahoma, for many helpful suggestions.

References

1. Neal H. McCoy, Prime ideals in general rings, Amer. J. Math., 71 (1949) 823–833.
2. R. E. Johnson, The extended centralizer of a ring over a module, Proc. Amer. Math. Soc., 2 (1951) 891–895.
3. ———, Prime rings, Duke Math. J., 18 (1951) 799–809.
4. N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. J. Math., 67 (1945) 300–319.

ISOMORPHISM OF FINITE ABELIAN GROUPS

RONALD MCHAFFEY, University of Massachusetts, Amherst

Isomorphic finite groups must possess the property that for any positive integer k dividing their common order, the same number of elements of order k occur in each group. This condition is also sufficient for finite abelian groups. That the result cannot be extended to wider classes of groups is shown by the existence (see [1]) of two groups of order p^3, of which only one is abelian, but both have all elements except the identity of order p.

Let n_k denote the number of elements of order k, $o(G)$ the order of a group G. Then we have

THEOREM. *Two finite abelian groups G and G' are isomorphic $\Leftrightarrow o(G) = o(G')$ and for each positive integer $k \ni k \mid o(G)$, $n_k = n_k'$.*

Proof. The necessity is clear. Since two finite abelian groups of equal order are isomorphic if their Sylow subgroups are isomorphic we may restrict our consideration to p-groups P, P' satisfying the condition.

Let P have type $[e_1, e_2, \cdots, e_r]$ and let P' have type $[e_1', e_2', \cdots, e_s']$, where $e_i \geqq e_{i+1}$, $e_j' \geqq e_{j+1}'$; here $i=1, 2, \cdots, r$ and $j=1, 2, \cdots, s$. The elements of order p in P form a vector space over J_p of dimension r containing p^r-1 elements of order p. Thus our hypothesis requires that $r=s$. Furthermore, since p^{e_1} is the largest order possible for any element of P, it is clear that $e_1=e_1'$. The proof may now be completed by induction. The theorem is obviously true for groups of order p or p^2. Let $o(P)=p^u$ and assume the theorem to hold for p groups of order less than p^u.

We next show that the number of cyclic summands of order p^{e_1} is the same for P and P'. One may compute the number of elements of order p^{e_1} by the following formula from [2]. Assume that $e_1=e_2= \cdots =e_k$, and let $m_1=r$ represent the number of invariants $\geqq p$, m_2 the number of those $\geqq p^2, \cdots, m_{e_1}=k$ the number of those $=p^{e_1}$. Then the number of elements of order p^{e_1} in P equals

$$(p^k-1)p^\alpha, \qquad \alpha = m_1 + m_2 + \cdots + m_{e_1-1}.$$

Similarly, if $e_1=e_1'=e_2'= \cdots =e_l'$, then the number of elements of order p^{e_1} in P' equals

$$(p^l-1)p^\beta, \qquad \beta = m_1' + m_2' + \cdots + m_{e_1-1}'.$$

Our assumption then yields $k=l$.

We now construct the quotient groups of P and P' over their common subgroup P^* of type $[e_1, e_1, \cdots, e_1]$ (k terms). $\overline{P}$ and $\overline{P'}$ will have types $[e_{k+1}, \cdots, e_r]$, $[e_{k+1}', \cdots, e_r']$ respectively. The proof will be complete if we show that $\overline{P}$ and $\overline{P'}$ satisfy the condition of the theorem. $o(\overline{P})=o(P)(p^{e_1})^{-k}=o(\overline{P'})$. We now compute $\bar{n}_pt$.

One obtains an element of order p^t in P by either choosing an element of order p^t from P^* and one of order $<p^t$ from $\overline{P}$, or by choosing an element of order $\leqq p^t$ from P^* and one of order p^t from $\overline{P}$. Thus

$$n_pt = n_p^*t\left(\sum_{i=1}^{t-1} \bar{n}_pi\right) + \bar{n}_pt\left(\sum_{j=1}^{t} n_p^*j\right),$$

whence

$$\bar{n}_pt = \left[n_pt - n_p^*t\left(\sum_{i=1}^{t-1} \bar{n}_pi\right)\right]\left[\sum_{j=1}^{t} n_p^*j\right]^{-1}.$$

Similarly,

$$\overline{n_{pt}'} = \left[n_{pt}' - n_{pt}^*\left(\sum_{i=1}^{t-1} \overline{n_{pi}'}\right)\right]\left[\sum_{j=1}^{t} n_{pj}^*\right]^{-1}.$$

Since $n_{pt}'=n_{pt}$ for all t, one may establish easily by induction using the above formulas that $\bar{n}_{pt}=\overline{n_{pt}'}$ for all $t \leqq u$.

REMARK. The condition of the theorem may be restricted only to the powers of the various primes dividing $o(G)$.

References

1. M. Hall, Jr., The Theory of Groups, Macmillan, New York, 1959, p. 52.
2. G. A. Miller, H. F. Blichfeldt, L. E. Dickson, Theory and Applications of Finite Groups, Wiley, New York, 1916, p. 93.
3. A. G. Kurosh, The Theory of Groups, Chelsea, New York, 1960, Vol. 1, Chap. 6.

Editorial Note: W. J. Pervin has pointed out that the example given in the paper by Paul Slepian, "A Non-Hausdorff Topology such that each Convergent Sequence has Exactly One Limit," this MONTHLY, 71 (1964) 776–778, can be found in the literature. It appears explicitly in *General Topology*, by W. Sierpinski, University of Toronto Press, Toronto, 1952, p. 75. It also appeared in Pervin's recent book on topology, which was published last April and was therefore unavailable to Slepian and the referees when the paper was under consideration for the MONTHLY. In *Elementary Topology* by D. W. Hall and G. L. Spencer, John Wiley and Sons, New York, 1955, there appears on page 100 a discussion of the example, but the authors do not point out explicitly the property discussed by Slepian.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on problems and topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

THE CAUCHY INTEGRAL THEOREM

OSWALD WYLER, University of New Mexico

Proving the Cauchy Integral Theorem for simply closed curves has several disadvantages, among them the following: (i) One relies heavily on the Jordan Curve Theorem, usually without proving it. The undergraduate or first-year graduate student cannot be expected to acquire a proof of this theorem by himself. (ii) Rather delicate arguments are needed in order to conclude from the Cauchy Integral Theorem, for simply closed curves, that $\int f(z)dz$ is independent of the path. (iii) The same remark applies to the standard extensions of the Theorem.

In this note, we sketch a proof of the Cauchy Integral Theorem for arbitrary closed curves. The proof is essentially a version of the proof given in [1]. We avoid, however, the homological machinery presented in [1]. In fact, we use no topology beyond the elements which must be used in any case, and the simplest facts about the index of a point with respect to a curve. These facts can easily be developed within the framework of a complex variables course, as we shall show.

For simplicity, we make the following assumptions throughout: D is a domain of the complex plane, C a piecewise smooth curve in D, and f a function defined in D, usually holomorphic. We shall use the fact that triangles, rectangles and circles are Jordan curves, and these are all the Jordan curves we need.

One proves by the usual elementary arguments the equivalence of the following three statements. (A) $\int_C f(z)dz=0$ for any closed curve C in D. (B) $\int_a^b f(z)dz$ is independent of the path from a to b. (C) f is in D the derivative of a function F holomorphic in D.

The *index* $n(C, p)$ of a closed curve C with respect to a point p can be defined, in the complex plane, by

$$n(C, p) = \frac{1}{2\pi i}\int_C \frac{dz}{z-p}\,. \tag{1}$$

For a curve $z=z(t)$, $a\leqq t\leqq b$, one proves that the integral $\int_a^b(z'(t)dt)/(z(t)-p)$ is a value of $\log(z(b)-p)/(z(a)-p)$ by setting

$$F(t) = \int_a^t \frac{z'(\tau)d\tau}{z(\tau)-p}$$

and observing that

$$\frac{d}{dt}\left(e^{-F(t)}(z(t)-p)\right) = 0,$$

so that $e^{F(t)}=(z(t)-p)/(z(a)-p)$. It follows in particular that $n(C, p)$ is an integer for a closed curve C and a point p not on C.

We need the following properties of the index. (i) $n(C, p)=n(C, q)$ if the line segment pq does not meet C. For this one observes that $\log(z-p)/(z-q)$ is holomorphic outside this line segment, with derivative $1/(z-p)-1/(z-q)$. (ii) $n(C, p)$ is constant in a domain which does not meet C. This follows directly from (i). (iii) $n(C, p)=0$ for p in the exterior of C (i.e. in an unbounded domain disjoint with C). For this we use (ii) and the fact that a holomorphic branch of $\log(z-p)$ is defined on C for p such that $\operatorname{Re} z>\operatorname{Re} p$ for all points z on C. (iv) $n(R, p)=1$ for a rectangle R, with sides parallel to the coordinate axes, and a point p inside R.

The next step is to prove the Cauchy Integral Theorem for triangles. We refer to [2] and [3] for excellent presentations of this.

As corollaries of the Cauchy Integral Theorem for triangles, we have: (i) the Cauchy Integral Theorem for rectangles, and (ii) the Cauchy Integral Theorem for a convex domain D. The first of these is obvious. To prove (ii), let $q \in D$ be fixed and set $F(z)=\int_q^z f(u)du$, integrating along the straight line segment. Then

$$F(z+h)-F(z)=\int_z^{z+h} f(u)du,$$

the integral being taken along the straight line segment, by the Cauchy Theorem for triangles. It follows in the usual way that $F'(z)=f(z)$.

The third step of the proof consists of replacing C by a special polygonal path S, without changing $\int_C f(z)dz$. Let $z=z(t)$, $a \leqq t \leqq b$, be the equation of C, and let $a=t_0<t_1< \cdots <t_n=b$ be a subdivision of $[a, b]$ such that $|z(t)-z(t_k)| <d$ for $t_{k-1} \leqq t \leqq t_k$, where $d>0$ is the distance of C from the complement of D. We replace C, from $z(t_{k-1})$ to $z(t_k)$, by a path S_k from $z(t_{k-1})$ to $z(t_k)$ which is composed of a segment parallel to the real axis and a segment parallel to the imaginary axis. Both paths lie in the disk $|z-z(t_k)|<d$ which is contained in D, and thus $\int f(z)dz$ is the same for both paths, by the Cauchy Integral Theorem for convex domains. Let S be the path composed of all paths S_k, $1 \leqq k \leqq n$. Then S is a path from $z(a)$ to $z(b)$, composed of segments parallel to the coordinate axes, and such that $\int_S f(z)dz=\int_C f(z)dz$ for any function f holomorphic in D. In particular, $n(C, p)=n(S, p)$ for a point p not in D.

Now we come to the fourth and essential step in our proof. Let us put $I(C) =\int_C f(z)dz$ for convenience. We need a handy formula for $I(S)$, where S is now a closed polygon composed of segments parallel to the coordinate axes.

In order to obtain this formula, we draw parallels to the coordinate axes through every vertex of S. We obtain in this way a grid covering the complex plane and consisting of a finite number of rectangles and parallel half-strips. We use the name "grid segments" for the sides of grid rectangles. Any segment of S is either a grid segment or composed of a finite number of grid segments. Thus we may assume that S is a closed polygon path composed of grid segments.

Now let $R_1, \cdots, R_m$ be the grid rectangles, and let a_i be the center of R_i. We assume that $I(R_i)$ is defined whenever $n(S, a_i) \neq 0$. Then

$$I(S)=\sum_{i=1}^{m} n(S, a_i)I(R_i), \tag{2}$$

where we put $n(S, a_i)I(R_i)=0$ if $I(R_i)$ is not defined.

In order to prove (2), we consider an arbitrary grid segment L. Let L be a side of the grid rectangle R_h, and orient L so that R_h is to the left of L. Suppose that L occurs c times in S with the given orientation and d times with the opposite orientation. We must show that $I(L)$ is counted $c-d$ times on the right-hand side of (2).

We replace S by a path S' which does not contain L, by passing around the other three sides of R_h in clockwise direction every time S passes through L

with the given orientation, and in counterclockwise direction every time S passes through L with the opposite orientation. Then we have, for any point p not on S nor on R_h,

$$n(S', p) = n(S, p) - cn(R_h, p) + dn(R_h, p).$$

In particular, we have

$$\begin{aligned} n(S', a_h) &= n(S, a_h) - c + d, \\ n(S', p) &= n(S, p) \qquad \text{for } p \text{ outside } R_h. \end{aligned}$$

If R is the grid area on the other side of L and q a point inside R, then the segment a_hq meets L, but no other grid segment, so that $n(S', q) = n(S', a_h)$. Using this with the formulas displayed above, we obtain

$$n(S, q) = n(S, a_h) - c + d.$$

If R is a parallel half-strip, then $I(L)$ is counted $n(S, a_h)$ times in (2). But then $n(S, q) = 0$, and hence $n(S, a_h) = c - d$, as it should be. If R is a grid rectangle R_k, then $I(L)$ is counted $n(S, a_h) - n(S, a_k)$ times in (2). Putting $q = a_k$ in this case, we obtain $n(S, a_h) - n(S, a_k) = c - d$, again as it should be. This proves (2).

We prove now the following version of the Cauchy Integral Theorem.

THEOREM. *Let C be a closed curve in a domain D such that $n(p, C) = 0$ for any point p not in D. Then $\int_C f(z)dz = 0$ for any function f which is holomorphic in D.*

Proof. We replace C by S, with $I(C) = I(S)$ in the notation used above. We note that $n(S, p) = n(C, p) = 0$ for $p \notin D$. If there is a point p not in D on or inside the grid rectangle R_i, then $n(S, a_i) = n(S, p) = 0$ since the segment a_ip does not meet S. If R_i and its interior are contained in D, then $I(R_i) = 0$ by the Cauchy Theorem for rectangles. Thus all terms of (2) vanish, and hence $I(S) = 0$, which was to be proved.

References

1. L. Ahlfors, Complex Analysis, McGraw-Hill, New York, Toronto, London, 1953.
2. E. Hille, Analytic Function Theory, vol. 1, Ginn, Boston, 1959.
3. K. Knopp, Theory of Functions, vol. 1, Dover, New York, 1945.

A HISTORICAL NOTE ON A PROBLEM IN THIS MONTHLY

JOY B. EASTON, West Virginia University

Problem E 1560, January 1963, asks for the locus of the third vertex P of an equilateral triangle with the other two vertices Q and R remaining on two mutually perpendicular lines. It is easy to set up a rectangular or polar coordinate system and find the equations of the ellipses in the given problem. It is a generalization of the well-known locus of a point *on* a fixed line segment whose endpoints remain on two perpendicular axes. The general problem is also frequently referred to: if the two axes are perpendicular, the locus of any point

fixed with relation to the segment will be an ellipse. In modern times this seems to have been given first by Francis van Schooten (1615–1660) in his *Exercitationum Mathematicarum*, Leiden, 1657, and probably dates back to Nasir Eddin (13th century).

A less familiar problem is the generalization of the eccentric angle construction of the ellipse, i.e., when the given segment rotates about a fixed point rather than slides along two fixed lines.

First, let C be a point on AB, or AB extended. As AB rotates about point B, the ellipse is generated in the following manner: A lies on a circle with diameter DD' and C on a circle with diameter EE' (Fig. 1). Draw parallels from A and C to a pair of fixed perpendicular diameters. The locus P, of the intersections of these lines will be an ellipse.

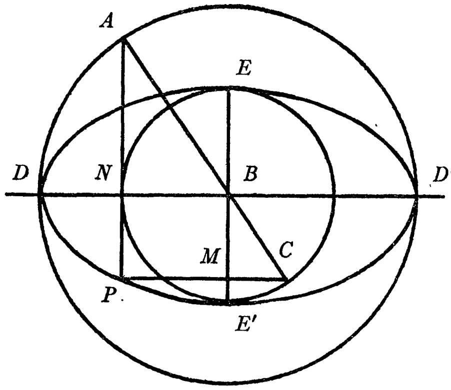


FIG. 1

Proof. $\triangle BCM \sim \triangle ABN$ giving

$$\frac{BN^2}{AB^2} = \frac{CM^2}{BC^2}$$

or, choosing a coordinate system, $BN = x$, $BM = y$, $AB = a$, $BC = b$:

$$\frac{x^2}{a^2} = \frac{b^2 - y^2}{b^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

an ellipse with center at B, semi-major axis $a = DB$, semi-minor axis $b = BE$. If C is taken between A and B, $\angle ABD = \phi$ is the eccentric angle and the parametric equations of the ellipse follow in the usual manner.

Now let ABC form a constant angle, obtuse or acute (Fig. 2). Draw $DE \perp AB$ at A and let BC lie on DF, with $DF = 2DB$. A perpendicular to DF at C inter-

sects AB at G. DE and DF determine an oblique coordinate system with origin at D.

As ABC rotates about B, A and C describe circles, and, for any position of ABC, say HBI, $HK \| DE$ and $IK \| DF$ determine a point K. The locus of K will be an ellipse with semi-major axis $DB=a$ and semi-minor axis BP equal in magnitude to $BG=b$.

Proof. Draw perpendiculars through I and K intersecting DF in L and M respectively. HK intersects AB in O and DF in N.

(1) Right triangles BCG, BAD, BON and KMN are similar. Also, since $\angle ABC = \angle HBI$, $\angle HBO = \angle IBL$ and

(2) right triangles BOH and BLI are similar.

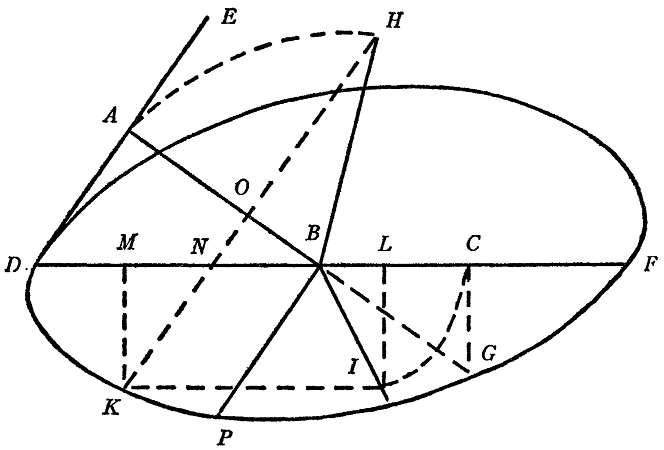


FIG. 2

From (1)

$$\frac{DB^2}{NB^2} = \frac{AB^2}{OB^2} = \frac{HB^2}{OB^2}$$

since $AB=HB$. By "conversion" of the proportion (taking the antecedent in relation to the excess by which the antecedent exceeds the consequent)

$$\frac{DB^2}{DB^2 - NB^2} = \frac{HB^2}{HB^2 - OB^2}.$$

But in $\triangle BOH$ $HB^2 - OB^2 = HO^2$ and by Euclid II, 5

$$DB^2 - NB^2 = DN \cdot NF,$$

giving

$$\frac{DB^2}{DN \cdot NF} = \frac{HB^2}{HO^2}. \tag{3}$$

Now by (2) $(HB^2/HO^2)=(BI^2/IL^2)$. But $BI=BC$ and $IL=KM$, so that $(HB^2/HO^2)=(BC^2/KM^2)$. By (1) $(BC^2/KM^2)=(BG^2/KN^2)$ and (3) becomes

$$\frac{DB^2}{DN\cdot NF}=\frac{BG^2}{KN^2}$$

or

$$KN^2=\frac{DG^2}{DB^2}\,DN\cdot NF, \tag{4}$$

the Apollonian symptom of an ellipse referred to a diameter and the tangent at the vertex as axes.

With $DB=a$, $BG=b$, $DN=x$, $KN=y$ we have

$$y^2=\frac{b^2}{a^2}\,x(2a-x). \tag{5}$$

Note that with the aid of our coordinate system the equality $DB^2-NB^2=DN\cdot NF$ becomes $a^2-(a-x)^2=x(2a-x)$, an algebraic identity.

During the late 16th and 17th centuries there was an awakening of interest in the Greek theory of the conics. Many plane locus constructions were developed by Kepler, van Schooten, de Witt and others. These differed from the Greek approach in that the curves were defined in the plane rather than as sections of cones.

The particular problems discussed above are taken from Johan de Witt (1625–1672), *Elementa Curvarum Linearum*, pp. 238–241 of the 1683 edition. De Witt was a friend and pupil of Francis van Schooten the younger, and his treatise appears as an appendix to van Schooten's 2nd Latin edition of the *Geometria* of René Descartes, Leiden, 1659–1661. I do not know whether the second problem is original with him. Nearly all of our familiar locus problems involving conics were developed during this period. I should like to point out that the synthetic approach based on the Greek theory of proportions is still the best way to handle them. Analytic geometry is a powerful tool, but, in problems of this type can lead to formidable algebraic manipulations.

THE ADJOINT OF A LINEAR TRANSFORMATION RELATIVE TO A NON-DEGENERATE BILINEAR FORM

JIANG LUH AND W. E. COPPAGE, Indiana State College

Let V and W be left and right vector spaces, respectively, over a division ring Δ. A mapping g of the product set $V\times W$ into Δ is called a bilinear form on V and W if for all $x, x_1, x_2\in V$, $y, y_1, y_2\in W$, and $\alpha, \beta\in\Delta$ we have

$$g(\alpha x_1+\beta x_2, y) = \alpha g(x_1, y)+\beta g(x_2, y), \tag{i}$$

$$g(x, y_1\alpha+y_2\beta) = g(x, y_1)\alpha+g(x, y_2)\beta. \tag{ii}$$

The bilinear form g is called nondegenerate if $g(x, y)=0$ for all $x \in V$ implies $y=0$ and $g(x, y)=0$ for all $y \in W$ implies $x=0$. If V and W both are finite dimensional over Δ and if g is nondegenerate, it is well known (e.g. [1]) that V and W have the same dimension and any linear transformation $A(B)$ on $V(W)$ has a unique right (left) adjoint $A^*(^*B)$ which is a linear transformation on $W(V)$ such that, for all $x \in V$, $y \in W$, $g(xA, y)=g(x, yA^*)(g(x, yB)=g(x^*B, y))$. It is implicit in [1] page 144 that $^*(A^*)=A$ and $(^*B)^*=B$.

If Φ is a field, then any left vector space V over Φ can also be regarded as a right vector space over Φ. Hence in this case, it is possible to define bilinear forms connecting the space with itself. The right (left) adjoint $A^{**}(^{**}A)$ of the right (left) adjoint $A^*(^*A)$ of a linear transformation A on V therefore makes sense. However $A^{**} \neq A$ and $^{**}A \neq A$ in general. It is true that $A^{**}=A$ and $^{**}A=A$ if the bilinear form g is symmetric in the sense that $g(x, y)=g(y, x)$ for all $x, y \in V$ or is skew-symmetric in the sense that $g(x, y)=-g(y, x)$ for all $x, y \in V$. Proofs can be found in most linear algebra books.

The question of finding necessary and sufficient conditions to ensure $A^{**}=A$ and/or $^{**}A=A$ seems not to have been widely recognized.

The purpose of this note is to establish two necessary and sufficient conditions for the validity of the identity $A^{**}=A$, and likewise of the identity $^{**}A=A$.

THEOREM. *Let V be a finite dimensional vector space over a field Φ, and let g be a nondegenerate bilinear form on V. Then the following conditions are equivalent*:

(1) *$A^{**}=A$ for every linear transformation A on V,*

(2) *g is either symmetric or skew-symmetric,*

(3) *$g(x, y)=g(u, v)$ implies $g(y, x)=g(v, u)$.*

Let $\mathcal{B}$ be a basis of the vector space V over Φ. For convenience, we will denote by x, y the row matrices of the vectors x, y relative to the basis $\mathcal{B}$, by A the matrix of the linear transformation A on V relative to the basis $\mathcal{B}$, by G the matrix of the bilinear form g relative to the basis $\mathcal{B}$. Then $g(x, y)=xGy^T$, where the superscript T designates the transpose.

Proof of the theorem. (1) implies (2): We have $g(xA, y)=xAGy^T$ and $g(x, yA^*)=xG(yA^*)^T=xGA^{*T}y^T$. By the identity $g(xA, y)=g(x, yA^*)$, we conclude that $AG=GA^{*T}$. Since g is nondegenerate, G is nonsingular and

$$A^* = G^T A^T (G^{-1})^T.$$

By (1), $A=A^{**}=G^T(A^*)^T(G^{-1})^T=G^TG^{-1}AG(G^{-1})^T$. But $((G^{-1})^T)^{-1}=G^T$ so $G^TG^{-1}A=AG^TG^{-1}$, and G^TG^{-1} commutes with every matrix A. Hence G^TG^{-1} is a scalar matrix, say, $G^TG^{-1}=\delta I$, $\delta \in \Phi$, where I is the identity matrix. Thus $G^T=\delta G$. It follows that $G=G^{TT}=(\delta G)^T=\delta G^T=\delta^2 G$ so $\delta^2=1$, or $\delta=\pm 1$. Hence G is either symmetric or skew-symmetric, and the same is true for g.

(2) implies (3): The proof is left to the reader.

(3) implies (1): Let A be an arbitrary linear transformation on V. Then, by the identity $g(yA, x)=g(y, xA^*)$ as well as (3), we have $g(xA^*, y)=g(x, yA)$. But

$g(xA^*, y)=g(x, yA^{**})$, so $g(x, yA)=g(x, yA^{**})$ for all $x, y\in V$. Hence $A=A^{**}$.

We note in passing that the Theorem remains true with essentially the same proof if (1) is replaced by

(1′) $^{**}A=A$ for every linear transformation A.

Reference

1. N. Jacobson, Lectures in Abstract Algebra, vol. II. Linear Algebra, Van Nostrand, New York, 1953.

CHAINS OF FACTORABLE BINOMIALS

LINCOLN LAPAZ, University of New Mexico

L. M. Court recently published in this MONTHLY an interesting divisibility theorem relating to a binomial of the form $w(k)=u^{p^k}+v^{p^k}$. Under the hypotheses that $k>0$, u, v are integers and that p is an odd prime dividing $w(k)$, Court showed in [1] that actually $w(k)\equiv 0 \bmod p^{k+1}$.

Validation of the congruence $w(k)\equiv 0 \bmod p$, basic to proof of the result just quoted, would become prohibitively costly in time and labor for any sizeable prime p, unless the integral parameter k were kept quite small. Since Court's theorem provides no means of escalating from a small value of k, convenient for validation of the basic hypothesis $w(k)\equiv 0 \bmod p$, to values of k large enough to lead to novel and interesting results, the present note constitutes a desirable supplement to his work since it makes clear that, starting from any admissible value of k, say k_0, one is enabled by use of an extension of Fermat's Simple Theorem to escalate the parameter k indefinitely. Furthermore, this extension of Court's result is obtained without loss of the escalation of modulus which constitutes the most significant feature of his own contribution.

We propose to show that under the hypotheses of Court's theorem, the following theorem is true:

THEOREM. *If the binomial* $w(k)=u^{p^k}+v^{p^k}\equiv 0 \bmod p$, then $w(k+1)\equiv 0 \bmod p^{k+2}$.

The hypotheses here adopted permit derivation of the following familiar relations in which the symbol ϕ denotes the totient function:

$$u^{\phi(p^\alpha)}\equiv 1, \qquad v^{\phi(p^\alpha)}\equiv 1, \bmod p^\alpha. \tag{1}$$

Since $\phi(p^\alpha)=(p-1)p^{\alpha-1}$, the choice of $\alpha=k+1$ leads to

$$u^{p^{k+1}-p^k}\equiv 1, \qquad v^{p^{k+1}-p^k}\equiv 1, \bmod p^{k+1}. \tag{2}$$

If we multiply the first and second relations in (2) by u^{p^k} and v^{p^k}, respectively, and add the resulting congruences, we obtain the congruence

$$u^{p^{k+1}}+v^{p^{k+1}}\equiv w(k) \bmod p^{k+1}. \tag{3}$$

From (3), in view of Court's discovery concerning $w(k)$, we have

$$u^{p^{k+1}} + v^{p^{k+1}} \equiv 0 \bmod p^{k+1}. \tag{4}$$

Since (4) implies that $w(k+1) \equiv 0 \bmod p$, we now have Court's basic congruence, with k replaced by $k+1$. From his Theorem 1 it then follows that

$$w(k+1) \equiv 0 \bmod p^{k+2} \tag{5}$$

which completes the proof of our theorem.

Since the theorem just proved permits indefinite escalation of k, starting from any admissible value k_0 of this parameter, we are in a position to produce an infinite chain of related, factorable binomials

$$w(h) = u^{p^{k_0+h}} + v^{p^{k_0+h}}, \qquad h = 0, 1, 2, \cdots.$$

Reference

1. L. M. Court, A divisibility theorem, this MONTHLY, 71 (1964) 300.

MORE ABOUT QUOTIENTS OF MONOTONE FUNCTIONS

R. P. BOAS, JR., Northwestern University

In a recent note [2] T. E. Mott showed, in effect, that if F and G are continuous and respectively convex and concave on $[a, b]$ then

$$\frac{F(x) - F(a)}{G(x) - G(a)} \tag{1}$$

increases (in the weak sense) on the subset of $[a, b]$, where $G(x) \neq G(a)$. In Mott's case F and G are given, respectively, as integrals of a nonnegative increasing, and a nonnegative decreasing, function and $F(a) = G(a) = 0$.

The special case when $G(x) = x - a$ is a known property of continuous convex functions (cf., e.g., [1], p. 146). It is geometrically obvious since it says that the slope of a chord increases if either end is fixed and the other end is moved to the right (either shortening or lengthening the chord). I observe first that this special case implies the general case. For, if

$$\frac{F(x) - F(a)}{x - a} \tag{2}$$

increases for all convex F, then

$$\frac{G(x) - G(a)}{x - a}$$

decreases for concave G, and so

$$\frac{x-a}{G(x)-G(a)} \tag{3}$$

increases where it is defined. Hence (1), as the product of (2) and (3), increases.

A proof by differential calculus that (2) increases is not quite easy since it requires us to know that a continuous convex function is absolutely continuous, a fact that is usually proved by using the very theorem that we are discussing. It is easy, however, when we know that $F(x) = \int_a^x f(t)dt$ with f increasing. Indeed, if $y > x$ we have the identity

$$\frac{F(y)-F(a)}{y-a} - \frac{F(x)-F(a)}{x-a} = \left\{\frac{F(y)-F(x)}{y-x} - \frac{F(x)-F(a)}{x-a}\right\}\frac{y-x}{y-a}. \tag{4}$$

Since the first difference quotient on the right is between $f(x+)$ and $f(y-)$ (not necessarily equal to a value of $f(t)$, $x<t<y$, however, unless f is continuous), and the second difference quotient is between $f(a+)$ and $f(x-)$, the right-hand side of (4) is nonnegative.

References

1. R. P. Boas, Jr., A primer of real functions, Carus Mathematical Monograph no. 13 (1963).
2. T. E. Mott, On the quotient of monotone functions, this MONTHLY, 70 (1963) 195–196.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

MATHEMATICS FOR LIBERAL ARTS STUDENTS

HENRY L. ALDER, University of California, Davis

The Committee on the Undergraduate Program in Mathematics (CUPM) has now made recommendations for appropriate mathematics courses for teachers at all levels, for those desiring to become research mathematicians, for the students in the physical sciences and engineering and for those in the biological, management and social sciences. CUPM therefore has—and I believe rightly—devoted its initial efforts to making recommendations for all who obviously require at least *some* mathematics as part of their college training.

Clearly, however, this set of people, that is, the beneficiaries of the recommendations made so far, comprise only part—and very likely less than half—of the total population of college students. There is much evidence that something should also be done for the large mass of students, such as those majoring in the

humanities and some of the social sciences, who do not require a mathematics course as part of their training in the major and will not take one unless a suitable course is available.

Since these remarks are made by the Secretary of the Association, there is considerable danger that the suggestions which I am about to make might be considered official recommendations of CUPM or the MAA. This is not at all the case.

On the other hand, what will be recommended here is the result of rather careful study, experimentation and evaluation by a group on the Davis campus of the University of California. It has been tried out in many classes over the last eight years, has resulted in a text book and has now become the model for similar courses throughout the country and, as I have just learned, also overseas.

First of all, let me make clear for what kinds of students this course is designed: It is intended primarily for students who do not need a mathematics course as part of their major in college and, consequently, such a course can only assume a minimum amount of mathematical preparation from high school, certainly no more than one year of algebra and some geometry. The course also includes students who do not have the time (or do not believe they have the time) to take more than a single course in mathematics in college and this—in spite of the recommendations of CUPM—includes many prospective elementary teachers. Many states—and here I must confess with sadness that California is one of them—still require only one college mathematics course for prospective elementary teachers. The course is also suitable for prospective secondary teachers with a minor in mathematics (this, of course, being only part of their mathematical training) and for prospective mathematics majors who want to get an early exposure to a greater variety of mathematics than is presented in the traditional calculus course. Finally, the course is helpful also to practicing teachers who wish to enrich their mathematical background.

It may be asked why a special course in mathematics should be offered for liberal arts students. To suggest that liberal arts students take one of the courses required for mathematics majors would clearly be inappropriate and could be done only by adhering rigidly to a belief that, if the beginning course in a field is appropriate for the majors in that field, it must also be suitable for the outsiders, a belief, incidentally, which is rapidly being abandoned in practically all the sciences. The mathematical needs of the liberal arts students should be of interest to the mathematical community: it is of interest to everyone—and to the mathematical community in particular—to have a mathematically educated public. When these students become adults it is certainly desirable that they have some understanding of the role of mathematics in our culture today. They can evaluate then in a more informed fashion the needs of mathematics as members of school boards, as government officials, and in many other capacities. Indeed utopia might return: when these students become parents, they might once more be able to communicate with their children on the mathematics they

learn in school and thus develop a positive attitude toward mathematics in these children. A course like this would be particularly beneficial to the many liberal arts students who eventually become teachers although they may have no such intentions as undergraduates.

The objectives of such a course would seem fairly obvious: It should give the students an appreciation of mathematics as an *exciting*, *valuable*, *vital* and *human* creation that also can possess the beauty that many associate only with music and poetry, or, to use the analogy made by Professors Rademacher and Toeplitz in their book "The Enjoyment of Mathematics" [5], who likened such a course to one in music appreciation: "Only very few are musically gifted in that they are able to compose music. Nevertheless there are many who can understand and perhaps reproduce music, or who at least enjoy it. We believe that the number of people who can understand simple mathematical ideas is not relatively smaller than the number of those who are commonly called musical, and that their interest will be stimulated if only we can eliminate the aversion toward mathematics that so many have acquired from childhood experiences."

For this reason, the course should present only mathematics which is interesting to both the student and instructor. This can be achieved by maximizing the ratio of ideas and theorems to definitions. The course should also enable the student to solve some real mathematical problems, specifically, to let him make discoveries, formulate conjectures, and prove theorems. Finally, it should make the student aware that mathematics is not a static field, but indeed is a fruitful field for research. Consequently unsolved problems should be mentioned wherever possible.

The topics in such a course can be quite variable, but not the spirit and approach. Indeed in teaching the course for several years, I have never offered it twice with exactly the same topics although there are certain ones which I have included every time and which I believe to be desirable to include always. Texts written for such a course include [1], [2], [3], [5], and [6]. To illustrate what such a course might contain, it might be best to list the topics and time spent on them as I taught it this past spring using a text particularly appropriate for our purposes: "Mathematics: the Man-Made Universe," by S. K. Stein [6].

The Complete Triangle: combinatorial variations of Sperner's lemma; a chapter designed to enable students to make simple discoveries, prove theorems, and find generalizations.	3 lectures
*The Primes: proof of the infinity of primes, statement of some unsolved problems.	3 lectures
*The Fundamental Theorem of Arithmetic: discussion of the Euclidian Algorithm, solution of the equation $(a, b)=ax+by$.	4 lectures
*Rationals and Irrationals: proof of the irrationality of $\sqrt{2}$, also of $\sqrt{A}$ in general, where A is not a square, generalization to cube roots, fourth roots, etc.	3 lectures
Decimal Expansions: discussion of repeating and nonrepeating decimals (includes a review of finite and infinite geometric progressions).	1 lecture
*Congruence: addition, multiplication, and division of congruences, proof of divisibility rules, casting out nines, residue classes, addition and multiplication of residue classes, construction of appropriate tables.	6 lectures

Strange Algebras: algebras as defined by tables with finitely many rows and columns, idempotent tables, commutative tables, isomorphic tables, associative tables (or groups).	5 lectures
Map Coloring: criterion for a map which can be colored with two colors, proof that $V-E+C=2$, proof of the five-color problem.	5 lectures
*The Representation of Numbers: binary system, other systems, including the representation of a rational as a sum of distinct unit fractions.	6 lectures
Infinite Sets: discussion of sets in general, proof that the rationals are denumerable and the reals are not, discussion of the continuum hypothesis.	4 lectures
	40 lectures
	+5 examinations
	45 hours

The starred topics are those which I believe should always be included in the course, the rest of the topics being chosen at the discretion of the instructor. Considerable variation in topics is possible since the text [6] contains 17 chapters each of which is suitable for inclusion in such a course and each of which contains a large number of problems to be assigned as homework exercises. Clearly more important, however, than the topics in the course are its flavor and presentation. They can perhaps best be illustrated by giving four important warnings.

1. *Avoid mathematics requiring a lot of manipulative skills.* A certain amount of manipulation, however, is necessary. To avoid it completely would give an erroneous impression of mathematics. Manipulative material, however, should be introduced only as necessary parts of problems involving ideas and theorems. Thus, for example, students might be asked early in the semester to prove that the square of an odd number minus one is always divisible by 8 which leads to the squaring of a binomial, factoring of a polynomial, etc.

2. *Don't require the student to prove something which he considers—rightly or wrongly—to be obvious.* Either have him prove something new to him or, if familiar to him, first *create doubt.* For example, don't ask a student to prove that of three points on a straight line, one is between the other two. As an example where doubt needs to be created consider the fundamental theorem of arithmetic. To create doubt we consider first another number system made up of all the numbers of the form $3n+1$; call it the Lagado System. A Lagado prime is a number which has only two divisors in the Lagado System. Let the students factor all Lagado numbers up to 100 into Lagado primes. For 100, he will find that it can be written as the product of Lagado primes in two ways, namely as $10\cdot 10$ and $4\cdot 25$, so that clearly the fundamental theorem of arithmetic fails to be true in the Lagado System. In this way, the student has become much more impressed with the fact that the fundamental theorem of arithmetic is true for our ordinary number system.

3. *Don't introduce more terminology than absolutely necessary, that is, minimize*

the number of definitions. To illustrate this point let me discuss what we do in the chapter on "Strange Algebras" referred to in the outline above. We define an algebra simply as a table, that is, a square arrangement of symbols without duplication in any row or column with a guide row and a guide column containing the same set of symbols written in the same order and such that each box of the table will contain one of the letters that appear in the guide row.

Example:

	A	B	C	D
A	A	C	D	B
B	D	B	A	C
C	B	D	C	A
D	C	A	B	D

With this and the definitions of idempotent, isomorphic, commutative, and associative, and calling an associative table a group, we prove the following theorems after first letting the students themselves discover many of them by suggesting that they construct tables of small order satisfying certain given properties:

A. There are commutative tables of all orders.
B. There is no commutative and idempotent table of order 2. (In fact, there is no idempotent table of order 2.)
C. There are idempotent commutative tables of all odd orders.
D. There is no idempotent commutative table of even order.
E. Any table that satisfies the rule $X \circ (X \circ Y) = Y \circ X$ for all X and Y is idempotent.
F. Any table that satisfies the rule $X \circ (X \circ Y) = Y \circ X$ has the property that distinct letters do not commute. That is, if X is different from Y, then $X \circ Y$ is different from $Y \circ X$.

The students are then asked to prove similar theorems as homework exercises.

4. *Don't introduce a topic merely because it is fashionable.* The decision as to whether or not to include a particular topic should be made primarily on the basis of whether or not it is consistent with the objectives of the course outlined earlier. Thus, for example, in the outline above, the topic of sets was not included among those which we feel *must* be covered in such a course. We believe—rightly or wrongly—that the popularity of sets is somewhat a matter of fashion. Obviously sets are in high fashion now. We are not really sure that there is much justification for it. We feel sure, however, that it is wrong to introduce sets without motivation. To talk about sets by merely giving the many defini-

tions usually offered in a discussion of sets, seems to us to serve no useful purpose.

If sets are to be offered in such a course (and, as you will note, this was done when I offered the course this Spring) the discussion should be started with the statement of some problems whose solution requires the introduction of the notion of sets. Thus, for example, one way to start a discussion of sets would be to ask the students the following three questions:

1. Are there more natural numbers than primes?
2. Consider a square. Are there more points on the inside of the square than on one of its sides?
3. Are there more irrational numbers than rational numbers?

An alternate way to motivate sets is their use in elementary probability theory. This has the disadvantage, however, that it takes considerably more time and might duplicate part of a course on probability which some of the students might wish to take later.

A few remarks concerning results achieved in offering this course might be of interest. There is no doubt that the overwhelming majority of students are very enthusiastic about the course. They praise it in highest terms and, what is particularly encouraging, in many instances indicate that this course has completely changed their former negative attitude towards mathematics. Significantly, such favorable comments are received even from students who find it hard or even fail it (and in this connection let me emphasize that we maintain strict standards in the course and usually fail about 5 to 10%). The Department of Education on our campus is now requiring the course of all prospective elementary teachers.

I wish to stress that I attach considerable importance to the offering of an appropriate mathematics course for liberal arts students, primarily as a small contribution to raise the level of mathematical literacy of college graduates which at the present is still shockingly low. This serious deficiency is most eloquently pointed out in Professor Kemeny's recent book "Random Essays on Mathematics, Education and Computers" [4], in which he finds it unforgivable that "most of the intellectuals in our country are almost wholly ignorant of the nature and promise of mathematics." What he condemns more than the ignorance is the attitude: "Ignorance of the identity of Brahms, Renoir, or Voltaire would surely mean ostracism for the intellectual seeking cultural companions, but the names of Euler or Gauss are likely to be unknown by those who set our cultural standards." Significantly, and really as a proof of this statement, the reviewer of this text in the New York Times was—as he admitted—caught in red-handed ignorance and had to look up a reference work concerning the identity of Gauss and Euler. Let us hope that by offering more appropriate mathematics courses for liberal arts students, future reviewers of the New York Times—and hopefully even those of papers of lesser renown—will not have to admit such embarrassing mathematical ignorance.

Condensed version of a talk given at the meeting of the MAA on August 24, 1964, at the University of Massachusetts.

References

1. R. Courant and H. Robbins, What is Mathematics?, 4th ed., Oxford University Press, London, New York, Toronto, 1960.
2. B. W. Jones, Elementary Concepts of Mathematics, 2nd ed., Macmillan, New York, 1963.
3. M. L. Keedy, A Modern Introduction to Basic Mathematics, Addison-Wesley, Reading, Mass., 1963.
4. J. G. Kemeny, Random Essays on Mathematics, Education and Computers, Prentice-Hall, Englewood Cliffs, N. J., 1964.
5. H. Rademacher and O. Toeplitz, The Enjoyment of Mathematics, Princeton University Press, 1957.
6. S. K. Stein, Mathematics: The Man-Made Universe, Freeman, San Francisco and London, 1963.

INSTITUTIONAL INFLUENCES IN THE UNDERGRADUATE TRAINING OF PH.D. MATHEMATICIANS

B. RICHARD SIEBRING, University of Wisconsin, Milwaukee

In recent years much has been written and spoken about the training of mathematicians. Much of this material has been based upon the personal experience of one man or a small group of men, or upon hearsay, opinions, or just plain prejudice. It is our object to present certain facts about the undergraduate training of Ph.D. mathematicians.

The information presented in this paper is based upon statistics gathered by several government agencies. The National Academy of Sciences has compiled information indicating the baccalaureate origins of Ph.D. mathematicians since 1936. While it would be futile to hope that not a single person had been omitted in an undertaking of this scope, it is our conviction that the statistics gathered by this agency are as accurate as it is possible to make them.

The institutions are ranked in Table I by the number of baccalaureate graduates who received their doctorates in mathematics during the period 1952–1962 [1]. As one would expect, the majority of the institutions near the top of this list are large universities and technical schools. Notable, however, are the high rankings of such colleges as Oberlin and Swarthmore. This is evidence of the excellence of the academic programs of these smaller institutions.

Well over half the Ph.D. mathematicians received their doctorates from the top 55 institutions. Thus, although the total number is diffused over a large number of institutions, a small group of institutions provided the undergraduate training for the majority of Ph.D. mathematicians.

One might conclude that the institutions which ranked high in Table I were there because of the large output of undergraduate mathematics majors. That is, institutions which graduated large numbers of undergraduate mathematics majors would naturally be expected to account for a large number of doctorates in mathematics.

A better evaluation of an institution's productivity, however, is probably the percentage of baccalaureate mathematics majors who later earned a Ph.D. in mathematics.

The Research Staff on Scientific Personnel of the American Council on Education has found that the typical number of years between the bachelor's and doctor's degrees for mathematicians was four [2]. Therefore, the years 1948–1958 were accepted as the years during which the majority of the 1952–1962 mathematics doctorates received their baccalaurate degrees.

The U. S. Office of Education tabulated baccalaureate graduates by majors and institutions beginning with July 1, 1947 [3]. By using this information and the National Academy of Science compilation of the baccalaureate origins of Ph.D. mathematicians, the writer ranked institutions according to the ratio of doctorates in mathematics granted each institution's baccalaureate graduates between January 1, 1952 and December 31, 1962, to the number of bachelor's degrees granted mathematics majors by the institutions between July 1, 1947 and June 30, 1958. The top institutions, each of which had three or more baccalaureate graduates who received their doctorates in mathematics for the period 1952–1962, are presented in Table II. Although this ranking is limited to institutions which provided the undergraduate training for more than three doctorates, there are some institutions which provide excellent mathematics training for a very small number of students.

Although a ranking of this nature is a meaningful evaluation of an institution's productivity, it may be criticized on the following points:

(1) It is slightly limited in scope because for a small group of institutions the number of baccalaurate graduates in mathematics was not reported by the U. S. Office of Education. The data for some of these institutions were obtained directly from the institutions themselves. After every possibility had been exhausted, however, the following group of thirteen institutions could not be included in this study: Brigham Young University, City College of the City University of New York, Davidson College, Hobart and William Smith Colleges, University of Delaware, University of Idaho, Louisiana State University and Agricultural and Mechanical College, Montana State College, Park College, Rensselaer Polytechnic Institute, San Diego College for Women, Stevens Institute of Technology and Virginia Polytechnic Institute.

(2) It does not take into account the fact that baccalaureate graduates other than mathematics majors may earn the doctorate in mathematics. There are no data available to determine what fraction of mathematics doctorates did their undergraduate work in fields other than mathematics.

(3) Not all mathematics majors plan a career in mathematics, but look toward secondary school teaching or other vocations as a life work. The comparison would probably have been more reliable if it could have been based only on those people who were majoring in mathematics and planned to make it their life work.

In order that various groups of institutions might be compared, composite

Ph.D.-Bachelor ratios were calculated. For the purposes of this study, the institutions were classified according to type, location, national recognition, and the granting or non-granting of the Ph.D. in mathematics.

The geographical divisions utilized by this study are those of the U. S. Bureau of Census: Northeast, North Central, West, and South. The Ph.D.-Bachelor ratios for each of these regions are:

West	6.5	North Central	4.4
Northeast	5.6	South	2.9

This investigation differs from most other studies of this sort, for in it the Western states are found the most productive. In similar studies for chemists and physicists, it was found that the Northeastern states had the highest composite ratios [4, 5]. In all studies the North Central and Southern states were found to be third and fourth respectively. Although the South is at the bottom in the geographical divisions, it should be noted that several of the institutions in Table II are in the South.

When institutions were classified according to type, the composite Ph.D.-Bachelor ratios were found to be as follows:

Engineering and technical	15.1
Private universities	8.3
State universities	7.1
Private liberal arts colleges for men	3.9
Coeducational liberal arts colleges	2.0
State and city colleges	1.0
Private liberal arts colleges for women	0.8

It was found that the Ph.D.-granting institutions had a composite Ph.D.-Bachelor ratio of 12.0, while institutions not offering doctoral work in mathematics had a ratio of only 2.4. It is apparent that the presence of doctoral programs on campus is an important factor in influencing an undergraduate to do graduate work leading to the Ph.D. degree.

Composite Ph.D.-Bachelor ratios listed below show that institutions listed by Haveman and West as having prestige and national recognition actually are superior to those which do not enjoy a national reputation [6].

Famous Technical	40.1
Big Three	26.0
Other Ivy League	18.4
Big Ten	12.2
Famous Eastern Colleges	6.1

It would be pleasant to report that the unknown college is as productive as the well-known one. This, however, is not the case. The coeducational colleges and colleges for men listed by TIME as those which are "not nationally known but deserve to be" had a composite ratio of only 2.9, while the combined ratio for all institutions was 4.5 [7].

It is not the intention of the writer to imply that certain institutions are more productive than others simply because of superior instruction. Instruction

is certainly an important factor, but other factors may well be involved. For example, a prestige institution probably attracts a greater proportion of gifted students. About such factors as this, one can, at this point, only speculate.

TABLE I

Baccalaureate Origins of Ph.D. Mathematicians who Received their Ph.D. During 1952–1962.

Rank	*Institution*	*Number*	*Cumulative percent*
1	University of Chicago	93	
2	City College of the City University of New York	92	7.1
3	Massachusetts Institute of Technology	82	
4	Harvard University	76	
5	University of California (Berkeley)	74	16.0
6 and 7	Brooklyn College	57	
	New York University	57	
8	University of Michigan	52	
9	Columbia University	43	
10	University of California (Los Angeles)	41	
11	Cornell University	39	
12 and 13	University of Minnesota	38	
	University of Illinois	38	
14	University of Texas	34	
15	California Institute of Technology	33	32.6
16 and 17	Yale University	28	
	Stanford University	28	
18	University of Wisconsin (Madison and Milwaukee)	27	
19	Princeton University	24	
20	Swarthmore College	23	
21 and 22	Purdue University	22	
	University of Pennsylvania	22	
23	Iowa State University of Science and Technology	21	
24, 25 and	Northwestern University	20	
26	Illinois Institute of Technology	20	
	Carnegie Institute of Technology	20	42.2
27	University of Washington (Washington)	19	
28	State University of Iowa	18	
29, 30, 31	Oberlin College	17	
and 32	Ohio State University	17	
	University of Oklahoma	17	
	University of Rochester	17	
33, 34	Brown University	16	
and 35	University of Florida	16	
	University of Oregon	16	
36, 37 and	Case Institute of Technology	15	
38	Reed College	15	
	Oklahoma University of Agricultural and Applied Science	15	

TABLE I (*continued*)

Rank	*Institution*	*Number*	*Cumulative percent*
39, 40, 41	Rice University	14	
and 42	St. Louis University	14	
	University of North Carolina	14	
	Yeshiva University	14	
43, 44, 45	State University of New York at Buffalo	13	
and 46	Johns Hopkins University	13	
	Pennsylvania State University	13	
	Wayne State University	13	
47, 48 and	Haverford College	12	
49	North Texas State College	12	
	University of Utah	12	
50, 51, 52	University of Dayton	11	
53 and 54	Lehigh University	11	
	Michigan State University	11	
	Notre Dame University	11	
	U. S. Naval Academy	11	
55	Rutgers University	10	58.0
	7 Institutions	9	
	8 Institutions	8	
	13 Institutions	7	
	21 Institutions	6	
	18 Institutions	5	
	27 Institutions	4	
	59 Institutions	3	
	100 Institutions	2	
	173 Institutions	1	
	TOTAL	2603	

TABLE II

Institutions ranked by ratio of Ph.D.'s in mathematics granted their baccalaureate graduates nationally, between 1952–1962, to mathematics majors who received their bachelor's degrees at that institution between July 1, 1947 and June 30, 1958.*

Rank	*Institution*	*Ratio*
1	California Institute of Technology	84.6
2	Case Institute of Technology	68.2
3	Massachusetts Institute of Technology	48.0
4	University of Chicago	38.9
5	Columbia University[1]	34.9
6	University of Notre Dame	33.3

* Complete data for institutions designated by a superscript could not be procured. However, sufficient data were available to make the comparisons meaningful, and hence, these institutions were included in the study. The years for granting of doctorates in mathematics nationally to their baccalaureate graduates is indicated in the first column and the years for granting of bachelor degrees to their undergraduate majors is listed in the second column.

Rank	*Institution*	*Number*	*Cumulative percent*
7	Princeton University		32.4
8	Rice University[2]		31.3
9	Lehigh University		29.7
10	Carnegie Institute of Technology		27.8
11	Johns Hopkins University		27.1
12	Harvard University		27.0
13	Stanford University		26.4
14	Haverford College		25.5
15	Polytechnic Institute of Brooklyn[3]		25.0
16	University of Dayton		23.9
17	Drew University		23.5
18	University of Pennsylvania[4]		22.0
19	Emory University		21.7
20	Illinois Institute of Technology		21.3
21 and 22	Antioch College		20.7
	Kenyon College		20.7
23	Reed College		20.5
24	Yale University		20.4
25	University of Rochester		18.7
26 and 27	Cornell University		18.5
	Union College and University		18.5
28	Swarthmore College		17.4
29	Oberlin College		16.0
30	Catholic University of America		15.4
31	Pacific Union College[5]		15.0
32 and 33	Pennsylvania State University		14.8
	Rockhurst College		14.8
34	Bethel College		14.3
35	Nebraska Wesleyan University		13.8
36	Georgetown University		13.6
37	Georgia Institute of Technology[6]		13.3
38, 39, 40	University of Cincinnati		13.0
41 and 42	Muhlenberg College[7]		13.0
	Seattle University		13.0
	University of Wisconsin (Madison and Milwaukee)[8]		13.0
	Yeshiva University		13.0
43 and 44	Williams College		12.7
	University of Illinois		12.7
45	University of California (all campuses)		12.6
46	Lafayette College		12.0
47 and 48	University of Florida[9]		11.9
	University of Oregon		11.9
49	University of Utah		11.7
50, 51, 52	Grinnell College		11.5
and 53	Ohio State University		11.5
	Oklahoma State University of Agricultural and Applied Science		11.5
	St. Louis University		11.5

Remarks	*Ph.D.*'s	*B.A.*'s
1	1958–62	1954–58
2	1955–62	1951–58
3	1956–62	1952–58
4	1956–62	1952–58
5	1954–62	1950–58
6	1958–62	1954–58
7	1955–62	1951–58
8	1953–62	1949–58
9	all except 1952	all except 1952

References

1. National Research Council, Office of Scientific Personnel (private communication).

2. American Council on Education Research Staff on Scientific Personnel. The Production of Doctorates in Science, 1936–1948, Amer. Council on Educ., Washington, D. C., 1951, p. 95.

3. U. S. Office of Education, Earned Degrees Conferred by Higher Educational Institutions, 1947–1948 Circular No. 247, pp. 277–289; 1948–49 Circular No. 262, pp. 73–78; 1949–50 Circular No. 282, pp. 35–37; 1950–51 Circular No. 333, pp. 80–84; 1951–52 Circular No. 360, pp. 69–73; 1952–53 Circular No. 380, pp. 63–65; 1953–54 Circular No. 418, pp. 61–63; 1954–55 Circular No. 461, pp. 96–100; 1955–56 Circular No. 499, pp. 124–129; 1956–57 Circular No. 527, pp. 137–142; 1957–58 Circular No. 570, pp. 150–55.

4. B. R. Siebring and Duane H. Schwahn, Amer. J. Phys., 27 (1959) 577–579.

5. B. R. Siebring, to be published in Science Education.

6. Ernest Haveman and Patricia West, They Went to College, Harcourt, Brace and Co., 1952, p. 178.

7. Time, No. 23, 76 (1960) 46–47.

RECENT ACTIVITIES OF THE MADISON PROJECT

ROBERT B. DAVIS, Syracuse University and Webster College

Recent activities of the Madison Project which may be of interest to mathematicians include:

(i) Development of a ninth-grade mathematics course that includes axiomatic algebra, some matrix algebra, some applications to physical and social science, a study of bounded monotonic sequences, a treatment of complex numbers via matrices, and a treatment of irrational numbers based upon bounded monotonic sequences.

(ii) Work at the elementary and junior high school level, on a large scale, with culturally deprived children especially in the cities of St. Louis and Chicago. Although the original Madison Project work was done with culturally deprived children in Syracuse, New York, the more sophisticated Project work of the past 6 years has been done primarily with bright students in culturally privileged areas. It comes as a distinct surprise to most of us to discover that *a large part of this "sophisticated" work is also feasible, and apparently appropriate, on a much wider scale, with culturally deprived urban children.* Experimentation in this area has been done in cooperation with Dr. Samuel Sheppard of the

St. Louis Public Schools, and Dr. Evelyn Carlson of the Chicago Public Schools.

(iii) Participation in a two-location conference on cognition, at Cornell University and at the University of California at Berkeley, featuring Professor Jean Piaget, Professor of Experimental Psychology of the University of Geneva, Geneva, Switzerland.

(iv) Although the Madison Project is not directly involved, mathematicians may be interested in a half-hour video-taped television program, produced by Educational Services, Incorporated and WGBH-TV, in cooperation with M.I.T. and Harvard University, featuring commentary by Professor Andrew Gleason of the Harvard Department of Mathematics. This program discusses the report *Goals for School Mathematics*, and uses excerpts of Madison Project films (showing actual classroom lessons). It has been shown on educational television stations in 85 cities, and is available on request for repeat showings over NET educational television stations anywhere in the United States. It is program number 44 in the Science Reporter series, and is entitled (regrettably) *Mathematics for Moppets.*

Financial support for the Madison Project is provided by the National Science Foundation, the Division of Cooperative Research of the Office of Education of the United States Department of Health, Education, and Welfare, and by other agencies.

References

1. R. B. Davis, The evolution of school mathematics, J. Res. Sci. Teaching, 1 (1963) 260–264.

2. ———, Report on the Syracuse University—Webster College Madison Project, this MONTHLY, 71 (1964) 306–308.

3. ———, Experimental Course Report—Grade Nine (Report no. 1, June 1964), available from The Madison Project, Webster College, St. Louis, Missouri 63119.

4. ———, Discovery in Mathematics: A text for teachers, Addison-Wesley, Reading, Mass., 1964.

5. ———, The Madison Project's approach to a theory of instruction, The Madison Project, 1964.

6. ———, Matrices, Logic, and Other Topics, Addison-Wesley, Reading, Mass., (in prdparation).

7. Goals for School Mathematics (the Report of the Cambridge Conference on School Mathematics), Houghton Mifflin, Boston, Mass., 1963.

8. Ronald Gross, Two-year-olds are very smart, New York Times Sunday Magazine Section, September 6, 1964.

9. Judith Murphy and Ronald Gross, There's no limit to learning, Parents' Magazine, February 1964, p. 63 ff.

COMMISSION ON COLLEGE PHYSICS

New members selected and appointed to the Commission on College Physics are:

Four-year terms—Herman Branson, Howard University; R. B. Leighton, California Institute of Technology; W. C. Michels, Bryn Mawr College; Philip Morrison, Cornell University; Melba Phillips, University of Chicago; R. V. Pound, Harvard University; and Robert Resnick, Rensselaer Polytechnic Institute.

Two-year terms—R. I. Hulsizer, University of Illinois, W. D. Knight, University of California, Berkeley, and E. D. Lambe, State University of New York-Stony Brook.

The Commission on College Physics, a national organization supported by the NSF, is concerned with the improvement of undergraduate physics at all levels from the freshman to the senior year. Its major function is that of acting as a "nerve center" for curriculum development, in which capacity it collects and distributes information and encourages new developments in all aspects of college physics. The Commission also sponsors the new paperback book series of Momentum Books designed for college students.

DEGREES AND ENROLLMENTS IN ENGINEERING 1962–63

Advanced degrees granted in engineering in the United States during 1962–63 showed gains of 14.2 percent at the doctorate level and 8.1 percent at the master's level over 1961–62, but the number of bachelor's degrees granted declined 3.7 percent during the same period. Coupled with increases in degree granting were also increases in enrollments in all engineering degree programs. The increase was greatest at the doctoral level (19%) and lowest at the bachelor's level (.6%). These statistics would seem to indicate there exists a trend among engineers to continue work beyond the bachelor's level toward advanced degrees.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. *Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.*

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before May 31, 1965.

E 1748. *Proposed by Michael Gemignani, University of Notre Dame*

Let $S=\{P_1(x), \cdots, P_n(x)\}$ be any finite set of nonconstant polynomials with positive integral coefficients. Show that for some natural number m, the n integers $P_1(m), \cdots, P_n(m)$ are all composite.

E 1749. *Proposed by Stanton Philipp, Long Beach State College, California*

Prove if $n>1$ then $\sigma(n)/\tau(n) \leqq 3n/4$, where $\tau(n)$ is the number of divisors of n, and $\sigma(n)$ is the sum of divisors of n.

E 1750. *Proposed by L. J. Bakanowsky, N. Greenleaf, and M. Walter, Harvard University*

If any two rays, emanating from the origin, are removed from the plane, the plane is cut into two components. What analogous statement(s) can be made with respect to the removal from three dimensional space of closed half-planes whose boundary lines pass through the origin?

E 1751. *Proposed by R. E. Mikhel, Tri-State College, Angola, Indiana*

Let A be an involutory matrix $(A^{-1}=A)$. Prove that, if every element of A is replaced by its cofactor, then the resulting matrix is an involutory matrix.

E 1752. *Proposed by Arthur Engel, Stuttgart, Germany*

A bug is crawling on the edges of a regular dodecahedron. Each time it comes to a vertex it chooses with equal probability one of the three edges which end in that vertex. What is the expected distance it covers in order to get from a vertex A to any other vertex B? ($A=B$ is not excluded.)

E 1753. *Proposed by J. O. Herzog and L. J. Simonoff, Idaho State University*

Let β be a family of subsets of positive integers such that if $A, B \in \beta$ then $A \cap B$ has at most n elements in common. Prove that β is a countable family.

E 1754. *Proposed by V. R. Rao Uppuluri, Oak Ridge National Laboratory*

Let $p, q>0$ and $p+q=1$, and let a_m denote the value of the mth order determinant

$$\begin{vmatrix} 1 & -q & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -p & 1 & -q & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -p & 1 & -q & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -p & 1 & -q \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -p & 1 \end{vmatrix}$$

with $a_0=1(=a_1)$. Show that $a_m=(p^{m+1}-q^{m+1})/(p-q)$.

E 1755. *Proposed by A. M. Vaidya, Pennsylvania State University*

Prove that 6 is the only squarefree perfect number.

E 1756. *Proposed by J. P. Ballantine, University of Washington*

Maps consist of lines, vertices and regions on a sphere. Let there be precisely 3 lines at each vertex. A region is called odd (even) if the number of lines in its boundary is odd (even). If the regions of a map are each assigned one of the colors, 1, 2, 3, 4, with no two adjacent regions of the same color, then the map is said to be colored. Prove: If a map is colored, the number of odd regions colored any two colors, say 3 and 4, is even. (Cf. solution E 1667 in this issue.)

SOLUTIONS TO ELEMENTARY PROBLEMS

Another Accident in a Calculus Class

E1661 [1964, 204]. *Proposed by K. S. Williams, University of Toronto*

A student thought that the formula for differentiating a product was

$$d\{u(x)v(x)\}/dx = d\{u(x)\}/dx \cdot d\{v(x)\}/dx.$$

He used this formula with $u(x)=(2-x)^{-2}$ and $v(x)=x^2$ and obtained the correct result! Find a general class of functions $u(x)$ and $v(x)$ satisfying the above formula.

Solution by Cornelius Groenewoud, 255 Roycroft Blvd., Buffalo, New York. It is easily seen that

$$u = 0, \qquad v = \text{const.} \times e^x$$

(or the same equations with u and v interchanged) satisfies the given formula. Furthermore, the pair

$$u = \text{const.} \times \exp\left(\int P(x)dx\right), \qquad v = \text{const.} \times \exp\left(\int Q(x)dx\right)$$

satisfy the formula if P and Q are continuous and $P(x)+Q(x)=P(x)Q(x)$. Hence, another general class of solutions is obtained by choosing $P(x)=Q(x)/[Q(x)-1]$ for Q continuous and $Q(x)\neq 1$. The illustrative example cited in the problem is obtained by choosing $Q(x)=2/x$.

Also solved by Abe Aarons, A. N. Aheart, Shair Ahmad, G. D. Antonio, Joseph Arkin, S. D. Beck and M. J. Pascual (jointly), A. Behr, W. J. Blundon, A. W. Brunson, John Burslem, L. P. Bush, P. R. Chernoff, D. I. A. Cohen, M. J. Cohen, Peter Day, J. A. Faucher, Murray Geller, Michael Goldberg, Newcomb Greenleaf, Emil Grosswald, H. S. Hahn, J. H. Halton, J. O. Herzog, Stephen Hoffman, R. F. Jackson, Erwin Just and Norman Schaumberger (jointly), G. A. Kemper, Robert Kopp, E. S. Langford, J. C. Lazzara, Tadeusz Leser, Lawrence Lessner and Rory Thompson (jointly), H. R. Lewis, D. C. B. Marsh, Morris Morduchow, C. R. Nicolaysen, J. M. O'Neil, C. B. A. Peck, Stanton Philipp, L. A. Ringenberg, Nathan Rubinstein, Bela Rul, Patricia Shaw, P. A. Scheinok, Bro. R. F. Schnepp, Robin Sibson, Jr., Phyllis Sinman, Sister Mary G. Love, R. P. Soni, E. C. Stopher, John Stout, Maynard Tomer, Simon Vatriquant, W. K. Viertel, R. J. Weinshenk, William Wernick, R. E. Whitney, K. L. Yocom, and the proposer.

J. W. Moon comments: This problem is the subject of G. Labelle, *On particular products of functions*, MATHEMATICS MAGAZINE, 35 (1962) 214.

Number of Prime Divisors of a Perfect Number

E1662 [1964, 204]. *Proposed by C. A. Nicol, University of South Carolina*

Prove that if p is the smallest prime divisor of a perfect number n, then n has at least p distinct prime divisors.

I. *Solution by A. M. Vaidya, Pennsylvania State University.* The assertion is trivially true if n is an even perfect number, since then n is of the form $2^r(2^{r+1}-1)$, where $2^{r+1}-1$ is a prime. So let n be an odd perfect number, $n=p^k p_2^{k_2}\cdots p_m^{k_m}$, where the p's are odd primes and $p<p_2<\cdots<p_m$. Now since n is perfect, we have on the one hand

$$
\begin{aligned}
2p^k p_2^{k_2}\cdots p_m^{k_m} &= \frac{(p^{k+1}-1)(p_2^{k_2+1}-1)\cdots(p_m^{k_m+1}-1)}{(p-1)(p_2-1)\cdots(p_m-1)}\\
&< \frac{p^{k+1}p_2^{k_2+1}\cdots p_m^{k_m+1}}{(p-1)(p_2-1)\cdots(p_m-1)},
\end{aligned}
$$

i.e.

$$2<\frac{pp_2\cdots p_m}{(p-1)(p_2-1)\cdots(p_m-1)},$$

while on the other hand,

$$\frac{p_2}{p_2-1}<\frac{p+1}{p},\qquad \frac{p_3}{p_3-1}<\frac{p+2}{p+1},\ \ldots,\ \frac{p_m}{p_m-1}<\frac{p+m-1}{p+m-2},$$

so that

$$\frac{p\,p_2\cdots p_m}{(p-1)(p_2-1)\cdots(p_m-1)}<\frac{p+m-1}{p-1}.$$

Thus we have $2<(p+m-1)/(p-1)$, giving $p<m+1$.

This problem gives a lower bound for the number of primes dividing a perfect number. On the other hand, we can use problem E1625 to get an upper bound. For, from E1625, we have $\sigma(n)/t(n)>n^{\frac{1}{2}}$, where $\sigma(n)$ is the sum and $t(n)$ the number of divisors of n. For a perfect number n, $\sigma(n)=2n$, so that $t(n)<2n^{\frac{1}{2}}$. Thus if the number of distinct prime divisors of n be m, then $2^m\leqq t(n)\leqq 2n^{\frac{1}{2}}$, giving $m\leqq 1+(\log n/\log 4)$. Combining this with the present problem, we see that the least prime p dividing a perfect number n satisfies $p\leqq 1+(\log n/\log 4)$.

II. *Solution by Karl K. Norton, 706 W. Main Street, Urbana, Illinois.* This result was established by C. Servais (*Mathesis*, 8 (1888) 92–93) and was later slightly improved by Grun (*Math. Zeit.*, 55 (1952) 353–354). In a recent paper (*Acta Arith.*, 6 (1961) 365–374), I obtained results which are much stronger for large p. For example, if N is a perfect number with smallest prime factor p,

then N has at least

$$\int_2^{p^2} \frac{dt}{\log t} + O(p^2 e^{-\log^b p})$$

distinct prime factors, and N has a prime factor at least as large as

$$p^2 + O(p^2 e^{-\log^b p}),$$

where b is any number $<4/7$.

Explicit inequalities can also be obtained using recent work of Rosser and Schoenfeld (*Ill. Jour. Math.*, 6 (1962) 64–94).

Also solved by J. C. Abad, Joseph Arkin, L. Carlitz, M. J. Cohen, Carl Evans and J. T. Fleck (jointly), W. D. Fryer, J. E. Jean, Jr., Erwin Just and Norman Schaumberger (jointly), E. S. Langford, D. C. B. Marsh, Leo Moser, M. G. Murdeshwar, Stanton Philipp, George Purdy, L. Y. L. Tong, Guy Torchinelli, S. Venkatramaiah, and the proposer.

Langford supplied the reference: D. Suryanarayana, *On odd perfect numbers*, II Proc. A. M. S., 14 (1963) 896–904.

Partitioning Space with Spheres

E1663 [1964, 204]. *Proposed by H. D. Ruderman, Hunter College High School*

What is the maximum number of regions into which n spheres can partition space?

I. *Partial solution by Michael Goldberg, 5823 Potomac Avenue, N.W., Washington, D. C.* We show that the maximum number R of regions is at least $n(n-2)(n+5)/6+4$ for $n>1$.

The maximum number of regions into which $n-1$ planes can partition three-space is $n+\binom{n}{3}$ [M. Brückner, *Vielecke und Vielfläche*, (1900) 46–47, or R. C. Buck, *Partition of Space*, this MONTHLY, 50 (1943) 541–544]. Of these regions, $(n-1)(n-2)+2(n>1)$ are infinite. If, now, we add a sphere which includes all of the points bounded by three or more planes, then each of the infinite regions is cut into two regions. Now, the total number of regions r is given by

$$\begin{aligned} r &= n + \binom{n}{3} + (n-1)(n-2) + 2 \\ &= n + n(n-1)(n-2)/6 + (n-1)(n-2) + 2 \\ &= (n+6)(n-1)(n-2)/6 + n + 2 = n(n+5)(n-2)/6 + 4. \end{aligned}$$

An inversion with respect to a sphere converts all the planes and the sphere into a collection of n spheres, $n-1$ of which have a point in common. The number of regions is not changed, and R is clearly as great as r.

II. *Partial solution by J. D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.* It is known that n circles, each pair of which intersect, divide the plane (and also a sphere) into n^2-n+2 regions [G. Polya, *Induction and Analogy in Mathematics*, (1954) 224–225]. Let P_n be the number of parts three-space is divided into by n spheres, each pair of which intersect. If there are $n+1$ such spheres and the $(n+1)$-st sphere intersects the other n spheres in n circles, each pair of which intersect, then the surface of the $(n+1)$-st sphere is partitioned into n^2-n+2 regions and space then into at most $P_n+(n^2-n+2)$ parts. Thus $P_{n+1} \leqq P_n+n^2-n+2$. Solving the difference equation

$$Q_{n+1} = Q_n + n^2 - n + 2,$$

with initial condition $Q_1=2$, we find that

$$Q_n = n(n^2 - 3n + 8)/3$$

and, hence, that $P_n \leqq n(n^2-3n+8)/3$.

Hyperplanes and Simplices

E1664 [1964, 204]. *Proposed by Barry Wolk, Cornell University*

It is easily shown that $n+2$ hyperplanes in general position in Euclidean n-space determine $n+2$ (closed) simplices. Show that each simplex is contained in the union of the others.

Solution by the proposer. The result follows from the following stronger theorem: *Every point of the space that is not on any of the hyperplanes is contained in an even number of simplices.*

Since some points are contained in none of the simplices, it is sufficient to prove that the parity of the number of simplices containing a given point does not change as the point moves through one hyperplane P. P is an $(n-1)$-space, and the traces on P of the other $n+1$ hyperplanes are the required hyperplanes of P that give the configuration of the theorem in P, with n reduced by one. By induction on n, the point p at which P is crossed lies in the interior of an even number of these $(n-1)$-dimensional simplices.

Each n-dimensional simplex S with a face on P is determined by P and n other hyperplanes, which are uniquely determined by the trace of S on P. Thus each $(n-1)$-dimensional simplex of P containing p determines an n-dimensional simplex whose boundary contains p, and conversely. Thus the number of n-simplices whose boundary contains p is even.

Since these simplices are the only ones entered or left as the moving point passes through P at p, the parity of the number of simplices containing the moving point is unchanged.

A Commutative Ring with No Annihilators

E1665 [1964, 205]. *Proposed by Azriel Rosenfeld, Yeshiva University*

Construct a commutative ring in which the square of every element is zero but not every product is zero. Prove that such a ring must have at least eight elements.

Solution by Leonard Carlitz, Duke University. The set of polynomials $2a+bx$, mod $(4, x^2)$, where a and b are integers, furnishes an example of such a ring. Clearly $(2a+bx)^2=4a^2+4abx+x^2\equiv 0$ but $(2+x)x\equiv 2x\not\equiv 0$. This ring is evidently of order 8.

Now let R be a ring with the stated property. Then there exist elements $a, b\in R$ such that $ab\neq 0$. Clearly $a+b\neq 0$; also $ab\neq a$ or b, since $ab=a$ implies $0=ab^2=ab$. Similarly $a+b\neq ab$. Thus the elements $0, a, b, ab, a+b$ are distinct. Next $ab+a$, $ab+b$, $ab+a+b$ are also distinct from these. Thus R contains at least eight distinct elements.

Also solved by W. J. Blundon, P. R. Chernoff, H. S. Hahn, Colonel Johnson, Jr., E. S. Langford, C. C. Lindner, D. C. B. Marsh, M. G. Murdeshwar, Robin Sibson, Jr., E. W. Wallace, J. E. Wilkins, Jr., and the proposer.

Hahn and Langford point out that the commutativity is not needed to show that a ring with the other requisite properties must have at least eight elements.

Polygons Determined by Centroids of Associated Triangles

E1666 [1964, 205]. *Proposed by Arthur Engel, Stuttgart, Germany*

Let $P_1P_2 \cdots P_n$ be any planar polygon and let G_i, $i=1, \cdots, n$ be the centroid of triangle $P_iP_{i+1}P_{i+2}$, where $P_{n+i}=P_i$. Is the polygon determined if the G_i are given?

Solution by T. I. Kolodner, University of New Mexico. The answer is yes if n is not divisible by 3, while the polygon need not be planar. Denote by P_i the position vectors of the vertices, and by G_i the position vectors of centroids of triplets of consecutive vertices. The P_i's must then satisfy the set of equations, $P_i+P_{i+1}+P_{i+2}=3G_i$, $i=1, \cdots, n$, and the matter is decided by showing that the circulant matrix A with first row $(a_0, a_1, \cdots, a_{n-1})=(1, 1, 1, 0, \cdots, 0)$ is nonsingular. Here, all subscripts are written mod n. If $n=3$, A is obviously singular.

By a known formula (derived below) for circulant matrices,

$$\det A = \prod_{j=1}^{n}\left(\sum_{\sigma=0}^{n-1} a_\sigma\omega^{\sigma j}\right) = \prod_{j=1}^{n}(1+\omega^j+\omega^{2j}),$$

where $\omega=\exp(2\pi i/n)$. We show that $\det A\neq 0$ if $3\nmid n$. If $\det A=0$, then for some $1\leq j<n$, $1+\omega^j+\omega^{2j}=0$ implying that $\omega=\exp(2\pi i[\alpha/3+k]/j)$, where $\alpha=1$ or 2, and $k=0, 1, \cdots,$ or $j-1$. Thus $(\alpha/3+k)/j=1/n$ (mod 1) and, since $\alpha/3+k<j$, we must have $(\alpha/3+k)/j=1/n$. This equality can hold only with $k=0$ since

$j<n$. Thus we must have $\alpha n=3j$, implying that $3|n$. Conversely, if $3|n$, then $1+\omega^{n/3}+\omega^{2n/3}=0$ and thus $\det A=0$.

The following generalization is obvious: if p is a prime, $p\geqq n$ and G_i is the centroid of the p-gon formed by p consecutive vertices of the polygon, beginning with P_i, then the polygon is determined by the G_i's if $p\nmid n$.

For the sake of completeness we supply a simple derivation of the formula for the determinant of a circulant matrix. For any $n\times n$ matrix B, let B_{ij} denote its i-j entry. Then $A_{ij}=a_{j-1}$ (mod n). Let ω be a primitive nth root of unity and form the Vandermonde matrix Ω with entries $\Omega_{ij}=\omega^{ij}$. Since the first n powers of ω are distinct, $\det\Omega\neq 0$. Now

$$(A\Omega)_{ij} = \sum_{k=1}^{n} A_{ik}\Omega_{kj} \sum_{k=1}^{n} a_{k-i}\omega^{kj} = \sum_{\sigma=0}^{n-1} a_\sigma\omega^{(\sigma+i)j} = b_j\omega^{ij}$$

where $b_j=\sum_{\sigma=0}^{n-1} a_\sigma\omega^{\sigma i}$. Thus the jth column of $A\Omega$ is a scalar multiple of the jth column of Ω with multiplier b_j. It now follows that

$$(\det A)(\det\Omega) = \det A\Omega = \left(\prod_{j=1}^{n} b_j\right)\det\Omega,$$

and we get the desired result on dividing by $\det\Omega\neq 0$.

Also solved by J. P. Ballantine, P. R. Chernoff, Michael Goldberg, Cornelius Groenewoud, H. S. Hahn, Ned Harrell, R. F. Jackson, E. S. Langford, Michael McGuirk, D. C. B. Marsh, Gus Mavrigian, Walter Meyer, M. G. Murdeshwar, Stanton Philipp, Steven Reyner, Bernard Rossner, P. A. Scheinok, Simon Vatriquant, Julius Vogel, and the proposer.

Meyer calls attention to E. Kasner, E. Jones, K. Way, and G. Comenetz, *Centroidal polygons and groups*, Scripta Mathematica, 4 (1936) 37–49, where this problem and others are derived from a more geometric point of view.

An Impossible Triangulation of a Sphere

E1667 [1964, 205]. *Proposed by G. J. Minty, University of Michigan*

Suppose that the surface of a sphere is divided into triangular "countries," where *triangular* means that each country touches exactly three others. A vertex of the graph formed by the boundary lines of the countries is called *even* or *odd* according as an even or an odd number of boundary lines run into it. Is there such a triangulation having exactly two odd vertices in which these vertices are adjacent?

Solution by J. W. Moon, University of Alberta. Suppose that G is such a triangulation. If the boundary line joining the two odd vertices is removed, then every vertex in the resulting graph G^* is even. Hence, we can colour the countries of G^* with two colours, say red and black, so that adjoining countries have different colours. [See E. B. Dynkin and W. A. Uspenski, *Mathematische Unterhaltungen. I. Mehrfarbenprobleme*, Berlin, 1955, p. 70]. Let r and b denote the number of countries of G^* of colours red and black, respectively. We may sup-

pose that the one country with four sides is coloured red. Since all the remaining countries have three sides it follows that

$$b = \tfrac{1}{3}(4 + (r - 1)3).$$

But this last expression is not an integer, an impossibility. Hence no such triangulation exists.

Also solved by Robert Bowen, Stephen Fisk, H. S. Hahn, Bro. R. F. Schnepp, Robin Sibson, Jr., and the proposer.

Fisk points out that the result of this problem is stated as a known fact in Sainte-Laguë, *Géométrie de Situation et Jeux,* Mémorial des Sciences Mathématiques, Fascicule XII.

A Concatenation of Catenaries

E1668 [1964, 205]. *Proposed by D. G. Wilson, IBM Corporation, Bethesda, Maryland*

Farmer Jones has a cart with square wheels. However, it suits his needs since with it he is able to travel the washboard road without any bumping. Assuming no slipping of the wheels, describe the washboard road.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* Since we want the edge of the turning wheel to remain in tangential contact with the road surface, we have $(\frac{1}{2}s-\sigma)/\frac{1}{2}s=\tan\theta=y'$, where θ is the angle of inclination of the tangent to the road surface at the point of contact, s is the length of a side of the wheel, and σ is the arc length in the direction of motion (the "x-direction") from the cusp (0, 0) of the road surface to the point of contact. Substituting for σ, we find that $s(1-y')=2\int_0^x\sqrt{(1+y'^2)}dx$ for the first arch of our road profile. We differentiate with respect to x and solve routinely to obtain $y = C_2 = \frac{1}{2}s\cosh(C_1-(2x/s))$. With the initial conditions, $x=y=0$, $y'=1$, we find the road surface to be periodic with period s arcsinh 1 (s the length of the wheel's edge) and for $0\leqq x\leqq s$ arcsinh 1,

$$y = \frac{\sqrt{2}}{2}s - \frac{\sqrt{2}}{2}s\cosh\frac{2x}{s} + \frac{1}{2}s\sinh\frac{2x}{s}\cdot$$

II. *Solution by Michael Goldberg, 5823 Potomac Ave., N.W., Washington, D. C.* Let the edge of the square be $2c$. Then, draw the catenary whose equation is $y=c\cosh(x/c)$. Take an arc of this catenary between the tangents which make an angle of 45° with the axis. Make the washboard road of these arcs with the convex sides up.

Place the square so that the midpoint of an edge coincides with the vertex of an arc. Then, when the square is rolled over the arc, the midpoint of this edge traces a tractrix which is the evolute of a catenary. But this tractrix has the property that the locus of the ends of tangent segments of length c lie on a horizontal straight line. This is the path of the center of the square as the square rolls, without slipping, over the washboard road.

For descriptions of the properties of a catenary and the associated tractrix, see the following books: E. H. Lockwood, *A Book of Curves*, 1961, pp. 118–124; R. C. Yates, *Curves*, 1946, pp. 12–14; and C. Zwikker, *Advanced Plane Geometry*, 1963, pp. 140–142.

Also solved by R. M. Anderson, Merrill Barnebey, A. Behr, C. L. Dotton, P. R. Fitall, H. S. Hahn, J. H. Halton, F. W. Herlihy, R. T. Hood, Dennis Hurst, R. F. Jackson, C. D. La Budde, E. S. Langford, Winton Laubach, H. R. Lewis, J. S. Muldowney, P. F. Multquist, Stanton Philipp, J. R. Porter, G. B. Robinson, D. J. Samuelson, J. S. Scandale, Jr., J. K. Stewart, L. Y. L. Tong and the proposer.

Multquist comments: Farmer Jones should be careful not to build a self-propelled vehicle for this road. With square wheels it will not jiggle up and down, but it will suffer from variable speed if the wheels are driven at constant angular velocity.

Robinson points out that this problem is essentially the same as E1033 whose solution was published in the April, 1953, issue of the MONTHLY. For a general approach to problems of this sort, see G. B. Robinson, *Rockers and rollers*, MATHEMATICS MAGAZINE, January-February, 1960.

Growth of Iterates of Euler's Totient Function

E1669 [1964, 205]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Let $\phi_k(n)=\phi\{\phi_{k-1}(n)\}$ and $\phi_0(n)=\phi(n)=$ Euler's ϕ-function. Show that for any k, $\phi_k(n)>1$ for all sufficiently large n.

Solution by A. E. Neumann, Madison, New Jersey. If $n=q_1^{\alpha_1}\cdots q_r^{\alpha_r}$, $\alpha_k\geqq 1$, is a prime factorization of n, then we have

$$q_1^{\alpha_1/2}\cdots q_r^{\alpha_r/2} > 2^{r-1} \geqq \frac{1}{2}\,\frac{q_1}{q_1-1}\cdots\frac{q_r}{q_r-1}.$$

$$\text{Thus } \phi(n) = \frac{q_1-1}{q_1}\cdots\frac{q_r-1}{q_r}\cdot q_1^{\alpha_1}\cdots q_r^{\alpha_r} \geqq \frac{1}{2}\,\frac{q_1^{\alpha_1}\cdots q_r^{\alpha_r}}{q_1^{\alpha_1/2}\cdots q_r^{\alpha_r/2}} = \frac{1}{2}\sqrt{n},$$

and so

$$\phi_1(n) > \frac{1}{2}\sqrt{\phi(n)} > \frac{1}{2}\sqrt{\frac{1}{4}\sqrt{n}} = \frac{1}{4}n^{1/4}.$$

In general, then, $\phi_k(n)>\frac{1}{4}n^{2^{-k-1}}$. Therefore $n\geqq 2^{2^{k+2}}$ implies $\phi_k(n)>1$; in fact, $\phi_k(n)\to\infty$ as $n\to\infty$.

Also solved by Leonard Carlitz, P. R. Chernoff, M. J. Cohen, R. B. Eggleton, Emil Grosswald, H. S. Hahn, Leroy Junker (2 solutions), Erwin Just and Norman Schaumberger (jointly), E. S. Langford, Andrzej Makowski, D. C. B. Marsh, Leo Moser, Stanton Philipp, George Purdy, Robin Sibson, Jr., A. M. Vaidya, S. Venkatramaiah, and the proposer.

Two solutions were based on the inequality $\phi(n)>$const.$\times n/\log\log n$. [See, for example, Hardy and Wright, *Theory of Numbers*, 3rd ed., p. 267] and several others on the relation $\phi(n)>A(\epsilon)n^{1-\epsilon}$, $\epsilon>0$. [See, for example, Landau, *Vorlesungen über Zahlentheorie*, vol. 1, Theorem 245.] Makowski cites H. Shapiro, *An arithmetic function arising from the ϕ function*, this MONTHLY 50 (1943), 18–30, wherein it is proved that the greatest integer n for which $\phi_k(n)>1$ and $\phi_{k+1}(n)=1$ is $n=2\cdot 3^k$.

The Differential Operator xd/dx

E1670 [1964, 205]. *Proposed by Tai-ichi Kitamura, Ibaraki University, Japan*

If $\mathfrak{D}$ is the differential operator xd/dx, prove that $e^{\mathfrak{D}}P(x)=P(ex)$, where $P(x)$ is any polynomial in x.

Solution by M. G. Murdeshwar and V. K. Rohatgi, University of Alberta, Edmonton. Because of the linearity of the operator $e^{\mathfrak{D}}$, it suffices to prove the result in the case $P(x)=x^k$, where k is any nonnegative integer.

It is easily proved by induction that $\mathfrak{D}^n(x^k)=k^nx^k$ (k and n nonnegative integers). Therefore,

$$e^{\mathfrak{D}}(x^k) = \left(\sum_{n=0}^{\infty} \frac{\mathfrak{D}^n}{n!}\right)x^k = \sum_{n=0}^{\infty} \frac{k^nx^k}{n!} = e^kx^k.$$

Also solved by M. T. L. Bizley, Brother R. F. Schnepp, Jim Campbell, P. R. Chernoff, M. J. Cohen, R. M. Conklin, Michael Goldberg, D. M. Good, Arthur Greenspan, Cornelius Groenewoud, Emil Grosswald, J. E. Hafstrom, Stephen Hoffman, R. F. Jackson, Leroy Junkers, P. G. Kirmser, E. S. Langford, J. C. Lazzara, D. C. B. Marsh, Jim Morrow, Mary P. Moseley, J. S. Muldowney, D. E. Myers, C. R. Nicolaysen, M. J. Pascual, Stanton Philipp, George Purdy, Steven Reyner, Robin Sibson, Jr., R. E. Whitney, S. Zamoscianyk, and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed, before July 31, 1965, to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903.

5253. *Proposed by Joseph Hammer, University of Sydney, Australia*

"Addition" of plane convex figures ϕ_1 and ϕ_2 is the following operation: Take any point R_1 in ϕ_1, and any point R_2 in ϕ_2, with an arbitrary point in the plane taken as origin O. The point R which is defined as the vector-sum

$$\overrightarrow{OR} = \overrightarrow{OR_1} + \overrightarrow{OR_2}$$

is considered as the sum of the points R_1 and R_2. If for every possible R_1 in ϕ_1 and R_2 in ϕ_2 we construct R as the sum of R_1 and R_2, then the locus of R is called the "sum" of ϕ_1 and ϕ_2.

Let us denote the boundaries of ϕ_1, ϕ_2 and $\phi=\phi_1+\phi_2$ by K_1, K_2 and K respectively. Prove that the area enclosed by K is greater than the sum of the areas enclosed by K_1 and K_2. Has this area an upper bound?

5254. *Proposed by A. M. Gleason, Harvard University*

Let f be a weakly increasing function from the positive integers to the positive real numbers. Suppose that $f(mn)=f(m)f(n)$ whenever m and n are relatively prime integers. Show that there is a nonnegative number α such that $f(n)=n^\alpha$ for all n.

5255. *Proposed by J. H. E. Cohn, Bedford College, London, England*

Prove (or disprove): The equation in positive integers $2y^2=3x^4-1$ has solutions only if $x=1$ or $x=3$.

5256. *Proposed by K. D. Taylor, George Washington University*

Show that the transformation $A:(x_1, \cdots, x_n)\to(\sigma_1, \cdots, \sigma_n)$, where the σ's are the elementary symmetric functions in n variables, has the Jacobian equal to the Vandermonde determinant $\Pi(x_i-x_j)$, where i runs from 1 to $n-1$ and j runs through all values greater than i.

5257. *Proposed by H. S. Shapiro, New York University*

Show that there exists an infinite set N of positive integers, and a sequence of complex numbers z_n, $|z_n|<1$, such that $\lim |z_n|=1$ and, for every $f(z)=\sum_{n=0}^{\infty} a_n z^n$ analytic in $|z|<1$, if (i) $a_n=0, n\notin N$, and (ii) $f(z_n)=0, n=1, 2, \cdots$, then $f\equiv 0$.

5258. *Proposed by David Singmaster, University of California, Berkeley*

In any ring, a divisor of zero cannot have an inverse. Show that, in a finite ring, the existence of an element which is not a divisor of zero implies the existence of a unit and implies that any noninvertible element is a divisor of zero. (We consider 0 as a divisor of zero.)

5259. *Proposed by K. Mahler, The Australian National University, Canberra*

Let $f(x_1, \cdots, x_n)=\sum_{h=1}^{n}\sum_{k=1}^{n} a_{hk}x_hx_k$ be a positive definite quadratic form. Assume that, as $(x_{11}, \cdots, x_{1n}), \cdots, (x_{n1}, \cdots, x_{nn})$ run over all sets of n points with integral coordinates,

$$f(x_{11}, \cdots, x_{1n}) \cdots f(x_{n1}, \cdots, x_{nn}) \geqq a_{11}a_{22}\cdots a_{nn}\begin{vmatrix} x_{11} & \cdots & x_{1n} \\ \cdot & \cdots & \cdot \\ x_{n1} & \cdots & x_{nn}\end{vmatrix}^2.$$

Then f is a diagonal form, i.e., $a_{hk}=0$ if $h\neq k$.

5260. *Proposed by M. Rajagopalan and A. Wilansky, Lehigh University*

A topological algebra is called a Q-algebra if its set of invertible elements is open. Must a normed Q-algebra be complete?

SOLUTIONS OF ADVANCED PROBLEMS

Non-Archimedean Field

5112 [1963, 672; 1964, 691]. *Proposed by N. R. Riesenberg, University of Wisconsin.*

In Dieudonné, *Foundations of Modern Analysis* (Academic Press, N. Y.,

1960), a real number system is defined as a field which (1) is Archimedean ordered, and (2) possesses the nested interval property. It is well known that neither (1) nor (2) alone suffices to give a real number system and many examples of Archimedean ordered fields which are not real number systems are in the literature. Give an example of a field which is non-Archimedean ordered but which possesses the nested interval property.

I. *Editorial Note.* When the example in the proffered solution [1964, 691] is amended to include power series with a finite number of negative exponents, we have only a partial ordering of the resulting field. For a complete solution Dean has provided a reference to *Rings of Continuous Functions*, by L. Gillman and M. Jerison. It is proved there (p. 179) that every hyper-real residue class field of $C(X)$ has the nested interval property. Such fields are also ordered and non-Archimedean.

II. *Solution by Roy O. Davies, Leicester, England.* Our construction of a field with the requisite properties depends on the following four observations:

1. Given any collection of ordered fields such that of any two one is an extension of the other, their union can be made into a (unique) ordered field which serves as a common extension of them all.

2. It follows from Lagrange's interpolation formula that a polynomial of degree k with coefficients in an ordered field changes sign at most k times.

3. If $a_1 < a_2 < \cdots$ are elements of an ordered field $(\mathfrak{F}, <)$, then there is an extension $(\mathfrak{F}', <)$ which contains an element x such that

(i) $a_n < x$ for all n,

(ii) if $a_n < b$ for all n, $b \in \mathfrak{F}$, then $x < b$. An example of such an extension is the field $(\mathfrak{F}', <)$ of rational forms $R(x)$ with coefficients in $\mathfrak{F}$. We order the elements in $\mathfrak{F}'$ by letting $0 < R(x)$ if $0 < R(a_n)$ for all sufficiently large n. We note that by observation 2 the order relation is established for every pair of elements in $\mathfrak{F}'$.

4. Using transfinite induction and observations 1 and 3, we have the result: if $(\mathfrak{F}, <)$ is any ordered field, an extension $(\mathfrak{F}^*, <)$ of $(\mathfrak{F}, <)$ exists such that for each increasing sequence $a_1 < a_2 < \cdots$ of elements in $\mathfrak{F}$, there is at least one element $x \in \mathfrak{F}^*$ with the properties (i), (ii), above.

Our construction now proceeds by defining a transfinite sequence of ordered fields $\mathfrak{F}_\alpha$, $1 \leqq \alpha \leqq \omega_1$. Let $\mathfrak{F}_1$ be the ordered field of rational numbers, $\mathfrak{F}_2 = \mathfrak{F}_1^*$ (observation 4) and, in general

$$\mathfrak{F}_\alpha = \overset{*}{\mathfrak{F}}_{\alpha-1} \quad \text{if } \alpha \text{ is not a limit ordinal,}$$

$$\mathfrak{F}_\alpha = \bigcup_{\beta<\alpha} \mathfrak{F}_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

By taking $a_n = n$ in observation 4, we see that $\mathfrak{F}_2$ and, a fortiori, all succeeding extensions are non-Archimedean.

Now $\mathfrak{F}_{\omega_1}$ has the nested interval property. For if $a_1 < a_2 < \cdots < b_2 < b_1$ are elements in $\mathfrak{F}_{\omega_1}$, where $a_n \in \mathfrak{F}_{\alpha_n}$, $b_n \in \mathfrak{F}_{\beta_n}$; $n = 1, 2, \cdots$; α_n, $\beta_n < \omega_1$, let

$\gamma=\sup\{\alpha_1, \alpha_2, \cdots, \beta_1, \beta_2, \cdots\}$. Now γ must be an ordinal corresponding to a countable set and cannot be a limit ordinal. (See, e.g., Kamke, *Theory of Sets*, Chapter 5.) Hence, by the definition of $\mathfrak{F}_\gamma^*$, we have an element x such that

$$x \in \mathfrak{F}_\gamma^* = \mathfrak{F}_{\gamma+1} \subset \mathfrak{F}_{\omega_1} \quad \text{and} \quad a_n < x < b_n \quad \text{for all } n.$$

This completes the proof. We note finally that $\mathfrak{F}_{\omega_1}$ has the cardinality of the continuum.

Rank of a Sum of Orthogonal Matrices

5160 [1963, 1107]. *Proposed by Alvin Hausner, The City College of New York*

Let A and B be $n\times n$ (real) orthogonal matrices. Prove that the rank of $A+B$ is $n-2k$, $0\leq k\leq\frac{1}{2}n$, if A, B are both proper-orthogonal or both improper-orthogonal. Further, prove that the rank of $A+B$ is $n-(2k+1)$, $0\leq k\leq\frac{1}{2}(n-1)$, if A is proper and B is improper.

Solution by R. F. Rinehart, Case Institute of Technology. Let R be rank $(A+B)$. Then $R=\text{rank } A(I+A^{-1}B)=\text{rank}(I+A^{-1}B)$. The orthogonality of A and B implies the orthogonality of A^{-1} and of $A^{-1}B$. Therefore there exists a matrix U (unitary) diagonalizing $A^{-1}B$, and $R = \text{rank } U^{-1}(I + A^{-1}B)U$ $=\text{rank}(I+U^{-1}A^{-1}BU)=\text{rank}(I+\text{diag}[\lambda_1, \lambda_2, \cdots, \lambda_n])$. Since $A^{-1}B$ is orthogonal, each λ_i has absolute value one, and the imaginary λ_i occur in conjugate pairs. Let p be the number of the λ_i which are equal to -1. Then $R=n-p$. Further, it is evident that $\det(\text{diag}[\lambda_1, \cdots, \lambda_n]) = \lambda_1\lambda_2\cdots\lambda_n = (-1)^p$ $=\det(A^{-1}B)=(\det B)/(\det A)$. Hence p is even if A and B are both proper or both improper, and p is odd if one of A, B is proper and the other improper. In the first instance $p=2k$, $0\leq k\leq\frac{1}{2}n$, in the second $p=2k+1$, $0\leq k<\frac{1}{2}n$.

Also solved by L. Carlitz, A. S. Householder, John B. Kelly, M. G. Murdeshwar, and the proposer.

Subfields in Finite Fields

5161 [1964, 98]. *Proposed by Seth Warner, Duke University*

Let K be a finite field and K^* the multiplicative group of all nonzero elements of K. Show that $H\cup\{0\}$ is a subfield of K for every subgroup H of K^* if and only if the order of K^* is a Mersenne prime.

Solution by Harlan Stevens, The Pennsylvania State University. If the order of K is p^n (p a prime) and the order of H is q, we recall that $q\mid p^n-1$, and that $q+1\mid p^n$ if $H\cup\{0\}$ is to be a subfield. For $p>2$ this is impossible when $H=\{1\}$, since $2\nmid p^n$. But then p^n-1 is never a Mersenne prime, either. Let $p=2$; if 2^n-1 is a prime, then H must be $\{1\}$ or K^*, so that $H\cup\{0\}$ is in either case a subfield. Conversely, if $2^n-1=s\cdot t(s>1, t>1)$, there exist subgroups H_1, H_2 of

order s and t. If $H_1 \cup \{0\}$ and $H_2 \cup \{0\}$ are to be subfields it follows that $s=2^e-1$, $t=2^g-1$. This is impossible, however, since $st\equiv 1 \pmod 4$, whereas $2^n-1\equiv -1 \pmod 4$.

Also solved by Robert Breusch, David Colton, Ralph Greenberg, Hwa S. Hahn, J. E. Humphreys, C. C. Lindner, J. W. Mades, R. C. Mullin, M. G. Murdeshwar, Veselin Perić, A. Radhakrishna (India), W. R. Scott, Robert Lee Wilson, and by Joseph Shoenfield and the proposer.

Asymptotic Formula for $\sum \sigma^2(n)/n^2$

5162 [1964, 98]. *Proposed by W. E. Briggs, University of Colorado*

Show that $\sum_{n\leq x} \sigma^2(n)/n^2$ is asymptotic to Ax, and find the constant A.

Solution by J. Koekoek, Technological University, Eindhoven, Netherlands. We use the following thorems:

(1) *Suppose that $c_1, c_2, \cdots$ is an arbitrary sequence of numbers, and that $f(x)$ has a continuous derivative for $x\geqq 1$. Put $C(x)=\sum_{n\leq x} c_n$. Then, for $x\geqq 1$,*

$$\sum_{n\leq x} c_n f(n) = C(x)f(x) - \int_1^x C(t)f'(t)dt.$$

(See W. J. LeVeque: Topics in Number Theory, vol. I, p. 103.)

$$\sum_{n\leq x} \sigma^2(n) = \tfrac{5}{6}x^3\zeta(3) + O\{x^2(\log x)^2\}. \tag{2}$$

(A. E. Ingham, Some asymptotic formulae in the theory of numbers, J. London Math. Soc., vol. II (1927) 202–208.)

In theorem (1) we take $c_n=\sigma^2(n)$ and $f(x)=x^{-2}$; by using formula (2) we then get

$$\begin{aligned}\sum_{n\leq x} n^{-2}\sigma^2(n) &= \tfrac{5}{6}x\zeta(3) + O\{(\log x)^2\} + 2\int_1^x [\tfrac{5}{6}\zeta(3) + O\{t^{-1}(\log t)^2\}]dt \\ &= \tfrac{5}{2}\zeta(3)\cdot x + O\{(\log x)^3\}.\end{aligned}$$

Also solved by Robert Breusch, R. G. Buschman, L. Carlitz, Martin J. Cohen, Ralph Greenberg, Emil Grosswald, O. P. Lossers (Netherlands), R. A. Smith, and the proposer.

Editorial Note. Buschman's solution starts with Ramanujan's formula

$$\sum n^{-s}\sigma_a(n)\sigma_b(n) = \zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)/\zeta(2s-a-b).$$

(E. C. Titchmarsh, The Theory of the Riemann Zeta Function) and, using methods in his paper in the Pacific Journal (1959) obtains the formula

$$\sum_{n\leq x} n^{d-1-a-b}\sigma_a(n)\sigma_b(n) \sim \frac{\zeta(1+a+b)\zeta(1+b)\zeta(1+a)}{d\zeta(2+a+b)}\, x^d, \qquad d>0.$$

The case $a=b=d=1$ is the proposed problem. Cohen also obtains this formula in the case $d=1$.

Measure Preserving Transformation

5163 [1964, 99]. *Proposed by S. D. Chatterji, University of New South Wales, Australia*

Let (S, Σ, m) be a totally-finite positive measure space and let $f(s)$ be a real-valued integrable function on S such that $f(s)>0$ almost everywhere with respect to the measure m. Let T be a one-to-one measure-preserving transformation of S onto itself. Prove that

$$\int_S \frac{f(s)}{f(Ts)}\, dm(s) \geqq m(S).$$

I. *Solution by P. R. Chernoff, Princeton University.* For every integer k, T^k is measure-preserving, so that, substituting T^kx for x, we have

$$\int_S \frac{f(x)}{f(Tx)}\, dm(x) = \int_S \frac{f(T^kx)}{f(T^{k+1}x)}\, dm(x).$$

Therefore, by the arithmetic-geometric inequality,

$$\begin{aligned}\int_S \frac{f(x)}{f(Tx)}\, dm(x) &= \int_S \frac{1}{n}\left(\frac{f(x)}{f(Tx)} + \frac{f(Tx)}{f(T^2x)} + \cdots + \frac{f(T^{n-1}x)}{f(T^nx)}\right) dm(x)\\ &\geqq \int_S \left(\frac{f(x)}{f(T^nx)}\right)^{1/n} dm(x).\end{aligned}$$

Let $\epsilon>0$ be given. For $0<a<b<\infty$ define

$$E = \{x \in S \mid a \leqq f(x) \leqq b\}.$$

The hypotheses on S and f imply that we can find a, b so that $m(S-E)<\epsilon$. Then if $F=T^{-n}(E)\cap E$,

$$m(F) = m(E) - m(E - T^{-n}E) \geqq m(E) - m(S - T^{-n}E) > m(S) - 2\epsilon.$$

Now choose n so large that $(a/b)^{1/n}>1-\epsilon$. Then

$$\int_S \frac{f(x)}{f(Tx)}\, dm(x) \geqq \int_S \left(\frac{f(x)}{f(T^nx)}\right)^{1/n} dm(x) \geqq \int_F \left(\frac{f(x)}{f(T^nx)}\right)^{1/n} dm(x).$$

Now $x\in F$ implies $T^nx\in E$ and $x\in E$, so that

$$\left(\frac{f(x)}{f(T^nx)}\right)^{1/n} \geqq \left(\frac{a}{b}\right)^{1/n} > 1 - \epsilon.$$

Then

$$\int_F \left(\frac{f(x)}{f(T^nx)}\right)^{1/n} dm(x) > (1 - \epsilon)m(F) > (1 - \epsilon)(m(S) - 2\epsilon).$$

Since ϵ was arbitrary, the theorem is proved.

II. *Additional result by N. J. Fine, Pennsylvania State University.* In the stated inequality, we observe that equality is obtained if and only if f is invariant under T. For, applying the result just proved to $\sqrt{f(s)}$ and assuming that

$$\int \frac{f(s)}{f(Ts)}\, dm = m(s),$$

we obtain

$$m(s) \leqq \int_S \left[\frac{f(s)}{f(Ts)}\right]^{1/2} dm(s) \leqq \left[\int_S \frac{f(s)}{f(Ts)}\, dm\right]^{1/2} \left[\int_S 1^2 dm\right]^{1/2} = m(s).$$

Therefore the second inequality must be equality and this implies the linear dependence of $[f(s)/f(Ts)]^{1/2}$ and 1. Thus $f(s) = f(Ts)$ almost everywhere, as asserted.

Also solved by George Bergman, N. G. de Bruijin (Netherlands), Harley Flanders, Wallace Franck, D. G. Kabe, Brockway McMillan, John Rainwater, and the proposer.

Editorial Note. The proposer comments that this is a continuous analog of problem E 1468 [1962, 59]. Incidentally, solution III there is faulty because, after one has chosen k such that $a_k/b_k \geqq 1$, one cannot be sure that the rest of the a's and b's are permutations of each other. Hence the induction hypothesis used there does not apply.

Taylor Expansion of Bessel Functions

5164 [1964, 99]. *Proposed by Gregory Thompson and Peter Treuenfels, Minneapolis-Honeywell Regulator Co.*

Find the Taylor series expansion of $e^{-x}J_0(ix)$ about the origin.

I. *Solution by E. S. Langford, North American Aviation.* From

$$J_0(ix) = \pi^{-1}\int_0^\pi e^{-x\cos\theta} d\theta$$

(G. N. Watson, Treatise on Bessel Functions, 1958, p. 48), we have

$$\begin{aligned} e^{-x}J_0(ix) &= \pi^{-1}\int_0^\pi e^{-x(1+\cos\theta)}\, d\theta \\ &= \pi^{-1}\sum_{n=0}^\infty \frac{(-1)^n x^n 2^n}{n!}\left\{\int_0^\pi \cos^{2n}\tfrac{1}{2}\theta\, d\theta\right\} \\ &= \sum_{n=0}^\infty \frac{(-1)^n(2n)!}{2^n(n!)^3}\, x^n. \end{aligned}$$

II. *Solution by W. D. Fryer, Cornell Aeronautical Laboratory.* The function $e^{-x}J_0(ix)$ satisfies the differential equation

$$y'' + (2 + 1/x)y' + (1/x)y = 0.$$

(Jahnke-Emde, Tables of Functions, p. 147), and this function has an ordinary power series expansion $a_0+a_1x+\cdots$, valid about the origin, with $a_0=1$ known. Putting this power series formally into the differential equation, we obtain $a_1=-1$,

$$a_{k+1} = -(2k+1)(k+1)^{-2}a_k, \qquad k = 1, 2, \cdots.$$

It now follows that

$$a_k = (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{1^2 2^2\cdots k^2} = \frac{(-1)^k}{2^k}\frac{(2k)!}{(k!)^3}$$

are the coefficients in the Maclaurin expansion of $e^{-x}J_0(ix)$.

Also solved by R. W. Barbieri, Robert Breusch, F. P. Callahan, Donald Childs, J. A. Faucher, Harley Flanders, Eldon Hansen, J. M. Horner, W. C. Janes, Hajna János and Horváth Sándor (Hungary), J. T. Johnson, J. Koekoek (Netherlands), V. Linis, A. E. Livingston, Immanuel Marx, Stanton Philipp, A. L. Rabenstein, O. G. Ruehr, E. J. Schmahl, A. Somayajulu, W. F. Trench, and the proposer.

Editorial Note. Trench notes the following generalization: Letting $Q_n(x)=e^{-x}I_n(x)$, where $I_n(x)$ is the nth order Bessel function of imaginary argument, one sees that

$$Q_n(x) = (-1)^n \sum_{m=n}^{\infty} \frac{1}{m!}\binom{2m}{m-n}\left(-\frac{x}{2}\right)^m.$$

Alternant Type Determinant

5165 [1964, 99]. *Proposed by A. S. Galbraith, Army Research Office, Durham, N. C.*

Let p_k and q_k, $k=1, 2, \cdots$, be the terms of two arithmetic progressions. Evaluate the determinant whose element in the ith row and jth column is p_{i+j}/q_{i+j}.

Solution by N. J. Fine, The Pennsylvania State University. The terms of the determinant have the form $[a+c(i+j)]/[b+d(i+j)]$. If each element is multiplied by d/c, the given determinant is seen to be $(c/d)^n$ times the determinant

$$D_n(\alpha, \beta) = \det\left|\frac{\alpha+i+j}{\beta+i+j}\right| \qquad i, j = 1, 2, \cdots, n,$$

where $\alpha=a/c$, $\beta=b/d$.

In D_n subtract row n from row i ($i=1,\cdots,n-1$). Factor $(\alpha-\beta)(n-i)$ from row i ($i=1,\cdots,n-1$). Factor $(\beta+n+j)^{-1}$ from column j ($j=1,\cdots,n$). Next, subtract column n from column j ($j=1,\cdots,n-1$). Factor $(n-j)$ from column j ($j=1,\cdots,n-1$). Factor $(\beta+i+n)^{-1}$ from row i ($i=1,\cdots,n-1$). Now subtract $(\alpha+2n)^{-1}$ times row n from row i ($i=1,\cdots,n-1$). We now have

$$D_n(\alpha,\beta) = (n-1)!^2(\alpha-\beta)^{n-1}\prod_{i=1}^{n-1}(\beta+i+n)^{-1}\prod_{j=1}^{n}(\beta+n+j)^{-1}F_n,$$

where

$$F_n = \left|\begin{array}{c|c} & 0 \\ \dfrac{(\alpha+\beta+2n)+i+j}{(\alpha+2n)(\beta+i+j)} & 0 \\ & \cdot \\ & \cdot \\ & \cdot \\ & 0 \\ \hline -1\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot-1 & \alpha+2n \end{array}\right| = (\alpha+2n)^{-n+2}D_{n-1}(\alpha+\beta+2n,\beta).$$

By use of this relation and an inductive argument, we find

$$D_n(\alpha,\beta)$$
$$= [1!\,2!\cdots(n-1)!]^2(\alpha-\beta)^{n-1}\prod_{i,j=1}^{n}(\beta+i+j)^{-1}(\alpha+(n-1)\beta+n(n+1)).$$

Also solved by J. v. IJzeren (Netherlands), A. E. Livingston, W. F. Trench, and the proposer.

Spectrum of Closed, Densely Defined Operators

5166 [1964, 99]. *Proposed by Peter Rejto and Charles Conley, New York University*

Let A be a closed and densely defined operator on a Hilbert space such that the range of A is contained in its domain, and such that A^2 is the identity on the domain of A. Does it follow that the spectrum of A includes at most the points 1 and -1?

Solution by Francis P. Callahan, Blue Bell, Pennsylvania. Define A on $L_2(-\infty, +\infty)$ by

$$Af(x) = \begin{cases} f(x) + (1+x)f(-x) & x \geqq 0 \\ -f(x) & x < 0. \end{cases}$$

The domain of A is defined to be the subset of $L_2(-\infty, +\infty)$ such that

$$\int_0^\infty [(1+x)f(-x)]^2dx < \infty.$$

Using the definitions in Stone, *Linear Transformations in Hilbert Space*, we show easily that A is closed, has domain dense in the space, and has range equal to its domain. A is its own inverse and is clearly unbounded. It follows that zero belongs to its continuous spectrum, and answers the question in the negative.

Also solved by the proposers.

Unity in Subrings of Rings of Integers

5168 [1964, 99]. *Proposed by A. Himmelfarb, Fordham University*

All subrings of Z_m, the ring of integers modulo m, are of the form dZ_m, where d is a divisor of m (dZ_m denotes the set of all ring multiples of d). For a given m, determine (a) those d for which dZ_m has a unity element, (b) the unity element for such d, and (c) the number of such d.

Solution by J. R. C. Leitzel, Indiana University. For d a divisor of m, the subring $dZ_m = \{nd \mid 0 \leqq n \leqq (m/d)-1\}$ will have unity kd if and only if $kd \equiv 1 \pmod{m/d}$. The latter congruence is solvable if and only if $(d, m/d)=1$. For then $(kd)(nd) \equiv (kd-1)nd+nd \equiv (tm/d)(nd)+nd \equiv mtn+nd \equiv nd \pmod{m}$. Let $m = p_1^{e_1} \cdot p_2^{e_2} \cdots p_j^{e_j}$; then the number of divisors of m such that $(d, m/d)=1$ is the number of ways of selecting j distinct items any number at a time, that is 2^j-1, noting that the case $d=m$ is not accepted.

Also solved by Harry Gonshor, Ralph Greenberg, Hwa S. Hahn, J. Humphreys, E. S. Langford, C. R. MacCluer, M. G. Murdeshwar and V. K. Rohatgi, Veselin Perić (Yugoslavia), K. A. Post (Netherlands), Harlan Stevens, W. C. Waterhouse, R. L. Wilson, K. L. Yocom, and the proposer.

Bi-Ideals of Semigroups

5169 [1964, 99]. *Proposed by Sándor Lajos, University of Economics, Budapest, Hungary*

A subsemigroup B of a semigroup S is called bi-ideal of S if $BSB \subseteq B$ (see A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, v. I, p. 84). Prove that a subset A of a semigroup S is a bi-ideal in S if and only if A is a left [right] ideal of a right [left] ideal of S.

Solution by J. E. Humphreys, Yale University. (a) Suppose A is a bi-ideal of S, and set $B=AS$. B is automatically a right ideal in S, and the condition $ASA=BA \subseteq A$ implies A is a left ideal in B.

(b) Let B be a right ideal in S such that A is a left ideal in B. Since $AA \subseteq BA \subseteq A$, A is a subsemigroup of S. Moreover, $ASA \subseteq BSA \subseteq BA \subseteq A$; hence A is a bi-ideal of S. The bracketed case is handled similarly.

Also solved by G. M. Bergman, D. R. Brown, C. V. Heuer, E. S. Langford, C. C. Lindner, Veselin Perić, R. J. Plemmons, and the proposer.

Editorial Note. Heuer and Lindner exhibited examples of subsets A which are not subsemigroups of a semigroup S but which satisfy the condition $ASA \subseteq A$.

Hausdorff Spaces and Stone-Cech Compactification

5170 [1964, 99]. *Proposed by Stanley Franklin, University of California at Los Angeles*

For any infinite cardinal $\aleph$, there exists a compact Hausdorff space E of cardinality $\aleph$ which is not the Stone-Cech compactification of any of its proper dense subspaces.

Solution by W. C. Waterhouse, Harvard University. Let X be a discrete space of cardinality $\aleph$, and let E be its one-point compactification. Then E is compact Hausdorff, and its only proper dense subspace is X. Let C be an infinite subset of X such that $X-C$ is also infinite, and let $f(x)$ be 0 on C and 1 on $X-C$. Then $f(x)$ is a bounded continuous function on X which has no continuous extension to E, and hence E is not the Stone-Cech compactification of X.

Also solved by W. M. Lambert, Jr., Philip Nanzetta, R. L. McKinney, P. R. Meyer, Joel Pitcairn, S. M. Robinson, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, University of Calif. and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. Programmed Materials: K. O. May, University of California, Berkeley, Calif. 94720. Films: E. P. Vance, Oberlin College, Oberlin Ohio 44074.

Asymptotic Approximations. By Harold Jeffreys. Clarendon Press, Oxford, 1962. vii+144 pp. $4.80.

The theory and practice of asymptotic representation are concerned with characterizing the behavior of a function for large values of the independent variable. The present monograph is confined to the type of asymptotic representation which consists of a finite or infinite linear combination of negative powers of the variable, possibly multiplied by an exponential. The function is generally a function of a complex variable. Poincaré's definition of convergence is used: If $f(z)$ is the function and $S_n(z)$ is the nth partial sum of a series of the type $S=\sum_{k=0}^{\infty} a_k z^{-k}$, then S is an asymptotic expansion or series for $f(z)$ in a sector $\alpha \leqq \arg z \leqq \beta$ if $\lim_{z\to\infty} z^n[f(z)-S_n(z)]=0$ for each n, uniformly for $\alpha \leqq \arg z \leqq \beta$.

To one who has no proprietary interest in the subject, asymptotics might seem to be a branch of traditional analysis full of *ad hoc* tricks and weird formulas. A remark of metamathematical nature like this perhaps can best be illustrated by a burlesque. Consider the following interesting line integral which arises in dynamic psychoceramics:

$$I=\int_C \cosh\left[(\sec ze^{4t})^{3/8}\right]H_7(t)\,dt,$$

where H_7 is a Hankel function and C looks like one of the more advanced Boy Scout knots. We integrate by parts twice and then make the substitution $t=\operatorname{arc\,cosh}\,(J_4(u))^{7/2}+17\exp\,(-u)^{1/14}$, where J_4 is the Bessel function of 4th

Graphical representations: A contribution to the methods of curve tracing. By C. B. Glavas. Athens, 1964. 142 pp.

The text is in Greek. According to the one-page English abstract, it covers the traditional material on curve tracing together with some new methods based on the author's research.

Tensor analysis. Theory and applications to geometry and mechanics of continua. 2nd ed. By I. S. Sokolnikoff. Wiley, New York, 1964. xii+361 pp. $9.75.

The mathematics of physics and chemistry, vol. II. By Henry Margenau and George Murphy. Van Nostrand, Princeton, N. J , 1964. 786 pp. $15.00.

The mathematical principles of natural philosophy. By Isaac Newton. Philosophical Library, New York, 1964. 447 pp. $10.00.

Elements of astromechanics. By Peter van de Kamp. Freeman, San Francisco, 1964. 140 pp. Cloth bound $4.00, paper bound $2.00.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor J. N. Eastham, Queensborough Community College, represented the Association at the inauguration of Albert H. Bowker as Chancellor of the City University of New York on November 5, 1964.

Professor C. T. Salkind, Polytechnic Institute of Brooklyn, represented the Association at the inauguration of Ralph G. Hoxie as Chancellor of Long Island University on October 9, 1964.

Brown University: Professor Solomon Lefschetz, Princeton University, has been appointed Visiting Professor; Dr. J. P. LaSalle, RIAS, Baltimore, Maryland, has been appointed Professor and Dr. Leonard Weiss, RIAS, Baltimore, Maryland, has been appointed Assistant Professor.

California State College at Long Beach: Dr. P. B. Norman, North American Aviation, Anaheim, California, has been appointed Associate Professor; Mr. J. R. Baugh, University of California, Los Angeles, and Mr. S. C. Choi, Aerospace Corporation, El Segundo, California, have been appointed Assistant Professors; Assistant Professor Joseph Verdina has been promoted to Associate Professor.

University of California, Davis: Associate Professor W. H. Simons, University of British Columbia, has been appointed Visiting Associate Professor; Associate Professor Sherman Stein has been promoted to Professor.

Carnegie Institute of Technology: Professor I. I. Kolodner, University of New Mexico, has been appointed Professor and Head of the Department of Mathematics; Assistant

Professors H. S. Leonard and V. J. Mizel have been promoted to Associate Professors; Associate Professor B. P. Hoover retired in June 1964.

Central Washington State College: Mr. D. R. Comstock, Oregon State University, has been appointed Assistant Professor; Assistant Professor B. L. Martin has returned from a two year academic leave of absence at Oregon State University.

Clemson University: Assistant Professor S. M. Lukawecki, Texas Women's University, Dr. W. R. Hare, Duke University, and Mr. E. L. Bethel, Knox College, have been appointed Associate Professors; Associate Professor C. V. Aucoin has been promoted to Professor and appointed Chairman of the Department of Mathematics.

Dalhousie University: Associate Professor Michael Edelstein, Michigan State University, has been appointed Associate Professor; Drs. Gilbert Steiner, Reed College, and R. C. Courter, Rutgers, The State University, have been appointed Assistant Professors.

East Carolina College: Professor S. E. Pence, University of Kentucky, has been appointed Professor; Mr. R. M. Woodside, Harvard University, has been appointed Assistant Professor; Dr. T. J. Pignani, University of Kentucky, has been appointed Director of the Department of Mathematics.

East Central State College: Assistant Professor R. L. Tennison has been promoted to Associate Professor; Associate Professor J. O. Danley has been appointed Head of the Department of Mathematics.

Eastern Illinois University: Associate Professor Bernard Derwort, Western Illinois University, has been appointed Professor; Assistant Professor J. M. Laible, Western Illinois University, has been appointed Assistant Professor; Associate Professor A. J. DiPietro has been promoted to Professor; Mr. R. A. Meyerholtz has been promoted to Assistant Professor.

Fairleigh-Dickinson University: Dr. Arthur Schlissel, New York University, has been appointed Assistant Professor; Assistant Professor Maurice Machover has been appointed Chairman of the Department of Mathematics.

University of Hawaii: Assistant Professors H. G. Loomis, University of Hawaii, Hilo Campus, Zuei-Zong Yeh, California Western University, and Dr. F. B. Strauss, University of California, Los Angeles, have been appointed Assistant Professors; Professor S. B. Townes retired in June 1964.

Hobart and William Smith Colleges: Associate Professor J. S. Klein, Monmouth College, New Jersey, has been appointed Professor and Head of the Department of Mathematics; Mr. T. A. Bick has been promoted to Assistant Professor; Professor W. H. Durfee has retired with the title of Professor Emeritus.

University of Illinois: Professor L. J. Mordell, St. John's College, Cambridge, England, has been appointed Visiting Professor; Associate Professor R. W. Carroll, Rutgers, The State University, has been appointed Associate Professor; Professor F. E. Hohn is on sabbatical leave for the academic year 1964–65 at the University of Delft in The Netherlands; Professor P. T. Bateman is on leave for the academic year 1964–65 as Visiting Professor at the graduate center of the City University of New York; Assistant Professor Hiram Paley is on sabbatical leave for the academic year 1964–65 at the University of Chicago; Assistant Professor L. R. McCulloh is on leave for the academic year 1964–65 at Indiana University.

Institute for Advanced Study: The following have been appointed Members in Mathematics at the School of Mathematics: Professor S. S. Chern, University of California, Berkeley; Professor J. L. Doob, University of Illinois; Mr. Nathaniel Grossman, University of Minnesota; Dr. David Henderson, University of Wisconsin; Associate Professor J. J. Price, Purdue University; and Mr. T. F. Storer, University of Southern California.

University of Kansas: Assistant Professors R. D. Adams and L. M. Sonneborn have been promoted to Associate Professors; Professor Nachman Aronszajn will be on sab-

batical leave during the academic year 1964–65 as Visiting Professor at the Sorbonne in Paris.

Kansas State University: Drs. H. M. Farkas, Illinois Institute of Technology, and C. F. Koch, University of Minnesota, have been appointed Assistant Professors; Assistant Professor N. E. Foland has been promoted to Associate Professor.

McMaster University: Assistant Professor S. G. Mohanty, State University of New York at Buffalo, has been appointed Associate Professor; Associate Professor N. D. Lane has been promoted to Professor.

Miami University: Dr. Joseph Kennedy, Indiana State Teachers College, has been appointed Associate Professor; Associate Professor L. C. Peck has been promoted to Professor; Assistant Professor Alberta Wolfe retired in June 1964.

University of Miami: Dr. James McKnight, General Electric, Sunnyvale, California, has been appointed Associate Professor; Assistant Professor Hermann Simon, McGill University, has been appointed Assistant Professor.

University of Michigan: Assistant Professor Roger Low has been appointed Associate Professor in the Department of Engineering Mechanics; Associate Professors Frank Harary, N. D. Kazarinoff, and Allen Shields have been promoted to Professors; Assistant Professor G. G. Minty has been promoted to Associate Professor; Drs. R. G. Douglas and R. C. O'Neill have been promoted to Assistant Professors.

Monmouth College, New Jersey: Professor Walter Miller, Dickinson College, has been appointed Professor; Mr. W. B. Pitt, University of California, Berkeley, has been appointed Assistant Professor; Assistant Professor Rosemary Miller has been promoted to Associate Professor.

North Carolina State of the University of North Carolina at Raleigh: Dr. Kwangil Koh, University of North Carolina, has been appointed Assistant Professor; Assistant Professor J. W. Bishir has been promoted to Associate Professor.

University of North Carolina: Professor I. A. Barnett, University of Cincinnati, has been appointed Visiting Professor; Associate Professor W. E. Jenner has been promoted to Professor; Assistant Professors A. C. Mewborn and C. W. Patty have been promoted to Associate Professors; Dr. F. M. Cholewinski has been promoted to Assistant Professor.

Northeastern University: Professor F. T. Haimo, Washington University, has been appointed Visiting Professor; Associate Professor E. M. Cook has been promoted to Professor; Assistant Professors Shirley A. Blackett, Ellen H. Dunlap, and R. D. Klein have been promoted to Associate Professors.

Ohio University: Dr. C. A. Green, University of Wisconsin, has been appointed Assistant Professor; Associate Professor W. T. Fishback has been promoted to Professor.

Ohio Wesleyan University: Associate Professor S. E. Ganis has been promoted to Professor; Assistant Professor D. H. Staley has been promoted to Associate Professor.

University of the Pacific: Professor F. C. Gentry, University of New Mexico, has been appointed Associate Professor; Assistant Professor G. P. Speck, University of Dayton, has been appointed Assistant Professor.

Rutgers, The State University: Associate Professor Joanne Elliott, Barnard College, has been appointed Professor; Drs. J. A. Fridy, University of North Carolina, J. A. Murtha, University of Wisconsin, and Barbara Osofsky, Rutgers, The State University, have been appointed Assistant Professors.

Sacramento State College: Associate Professor T. Y. Chow, Rensselaer Polytechnic Institute, has been appointed Associate Professor; Dr. E. L. Allgower, University of Arizona, and Mr. Armand Seri, University of Illinois, have been appointed Assistant Professors.

University of South Carolina: Dr. O. L. Buchanan, Jr., University of Kansas, has been appointed Assistant Professor; Professor H. W. Smith retired on July 1, 1964.

University of Southern Mississippi: Professor E. P. Kelly, Jr. has been appointed Chairman of the Department of Mathematics and Director of the Computer Center; Associate Professor Porter Webster has been promoted to Professor.

Syracuse University: Drs. L. J. Lardy, University of Minnesota, and W. A. Schneider, University of Wisconsin, have been appointed Assistant Professors; Assistant Professor Peter Frank has been promoted to Associate Professor.

Vanderbilt University: Assistant Professor G. F. Clanton has been appointed Assistant to the Director of the Vanderbilt University Planning Study; Associate Professor B. G. Clark retired on June 30, 1964.

Virginia Military Institute: Assistant Professor E. G. Zdinak, Duquesne University, has been appointed Assistant Professor; Professor W. G. Saunders has been appointed Head of the Department of Mathematics; Professor W. E. Byrne retired June 30, 1964 as Head of the Department of Mathematics but will continue teaching under the new rank of Senior Professor of Mathematics.

Mathematics Research Center, University of Wisconsin: Professor I. J. Schoenberg, University of Pennsylvania, has been appointed Professor; Professor R. E. Langer retired June 30, 1964 with the title of Professor Emeritus.

University of Wisconsin, Milwaukee: Associate Professor W. J. Pervin, Pennsylvania State University, has been appointed Associate Professor; Dr. W. W. Babcock, Tulane University, has been appointed Assistant Professor.

Yale University: Associate Professor Walter Feit, Cornell University, has been appointed Professor; Assistant Professors D. M. Burton, University of New Hampshire, and K. A. Ross, University of Rochester, have been appointed Visiting Assistant Professors.

Dr. J. M. Adams, White Sands Missile Range, New Mexico, has been appointed Associate Professor at Texas Western College.

Mr. G. C. Anderson, Hamline University, has been promoted to Assistant Professor.

Assistant Professor D. R. Andrew, University of Southwestern Louisiana, has been promoted to Associate Professor.

Dr. P. M. Bailyn, New York University, has been appointed Assistant Professor at Cooper Union.

Associate Professor Mabel Barnes, Occidental College, has been promoted to Professor.

Professor L. C. Barrett has returned to the South Dakota School of Mines and Technology as Head of the Department of Mathematics after a year's sabbatical leave at N.O.T.S., China Lake, California.

Dr. R. B. Bennett, University of Tennessee, has been appointed Assistant Professor at Knox College.

Assistant Professor G. G. Bilodeau, Boston College, has been promoted to Associate Professor.

Associate Professor H. W. Brockman, Capital University, has been promoted to Professor.

Mr. John Brown, Montana State College, has been promoted to Assistant Professor.

Assistant Professor J. D. Buckholtz, University of North Carolina, has been appointed Assistant Professor at the University of Kentucky.

Miss Fay B. Burras, Randolph-Macon Woman's College, has been appointed Assistant Professor at Lebanon Valley College.

Mr. A. C. Calianese, St. Peter's College, has been promoted to Assistant Professor.

Mr. J. T. Chen, Central State College, has been promoted to Assistant Professor.

Associate Professor H. E. Chrestenson, Reed College, has been promoted to Professor.

Dr. L. W. Cohen, on leave from his post as Chairman of the Department of Mathematics of the University of Maryland, has been appointed Executive Secretary of the

Division of Mathematics of the National Academy of Sciences-National Research Council. He will continue as Executive Secretary of the Conference Board of the Mathematical Sciences.

Professor L. J. Deck, Muhlenberg College, retired June 1964 with the title of Professor Emeritus.

Mr. F. E. Fischer, Harvard University, has been appointed Associate Professor at State University College of New York at Oswego.

Acting Assistant Professor R. P. Goblirsch, University of Colorado, has been appointed Associate Professor at College of St. Thomas.

Dr. L. D. Graber, Iowa State University, has been appointed Assistant Professor at Florida Presbyterian College.

Assistant Professor D. B. Hinton, University of Tennessee, has been appointed Assistant Professor at the University of Georgia.

Professor C. T. Holmes, Bowdoin College, retired June 1964 with the title of Wing Professor of Mathematics Emeritus.

Mr. A. J. Jackowski, Technical High School, Springfield, Massachusetts, has been appointed Assistant Professor at Westfield State College.

Professor J. R. Lee, College of William and Mary, has been appointed Professor and Head of the Department of Mathematics at Colorado School of Mines.

Professor John McNamee, University of Alberta, has been appointed NSF Visiting Senior Foreign Scientist at the University of Oklahoma.

Associate Professor J. C. Mairhuber, University of New Hampshire, has been appointed Professor at the University of Richmond.

Professor Emeritus J. R. Musselman, Western Reserve University, has been appointed Visiting Professor at Hiram College.

Mr. David Neu, Seattle Pacific College, has been promoted to Assistant Professor.

Dr. E. A. Nordgren, University of Michigan, has been appointed Assistant Professor at the University of New Hampshire.

Mr. R. P. Osborne, Michigan State University, has been appointed Assistant Professor at the University of Idaho.

Dr. Morris Ostrofsky, Westinghouse Electric Corporation, Baltimore, Maryland, has been appointed Professor and Chairman of the Department of Mathematics at Rose Polytechnic Institute.

Associate Professor R. H. Owens, University of New Hampshire, has been appointed Professor and Chairman of the Department of Applied Mathematics at the University of Virginia.

Professor R. N. Pendergrass, Southern Illinois University, Edwardsville, has been appointed Chairman of the Faculty of Mathematical Studies.

Professor E. E. Posey, Virginia Polytechnic Institute, has been appointed Professor at the University of North Carolina at Greensboro.

Assistant Professor A. L. Rabenstein, Pennsylvania State University, has been appointed Assistant Professor at Macalester College.

Associate Professor Anthony Ralston, Stevens Institute of Technology, has been promoted to Professor.

Assistant Professor J. M. Rice, Fort Hays Kansas State College, has been promoted to Associate Professor.

Assistant Professor E. C. Riggle, Chico State College, has returned from two years' leave of study at Oregon State University and has been promoted to Associate Professor.

Assistant Professor H. L. Rolf, Vanderbilt University, has been appointed Professor at Baylor University.

Visiting Associate Professor I. L. Rose, University of Kansas, has been appointed Senior Lecturer at Newcastle University College, Australia.

Assistant Professor Ernest Schlesinger, Connecticut College, has been promoted to Associate Professor.

Sister M. Joan Patricia, O.P., Aquinas High School, Chicago, Illinois, has been appointed Instructor at St. Dominic College.

Professor George Springer, University of Kansas, has been appointed Professor at Indiana University.

Assistant Professor Mary B. Turner, Seattle University, has been promoted to Associate Professor.

Dr. W. F. Tyndall, Princeton University, has been appointed Assistant Professor at Franklin and Marshall College.

Dr. W. E. Walden, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, has been appointed Associate Professor and Director of the Computing Center at the University of Omaha.

Assistant Professor J. R. Webb, Louisiana State University, has been appointed Associate Professor at the University of Mississippi.

Assistant Professor G. K. Williams, University of Virginia, has been appointed Assistant Professor at the University of Notre Dame.

Associate Professor L. N. Zaccaro, Hiram College, has been appointed Associate Professor at Worcester Polytechnic Institute.

Professor Emeritus L. E. Dix, Norwich University, died on April 7, 1959. He was a member of the Association for 31 years.

Assistant Professor Emeritus L. J. Rouse, University of Michigan, died on December 30, 1963. He was a member of the Association for 41 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The twenty-third annual meeting of the Metropolitan New York Section of the MAA was held on April 11, 1964, at Pace College. Professor Abraham Schwartz, Chairman of the Section, presided. There were 120 persons present, of whom 101 were members of the Association.

As the officers of the Section had been elected for a two-year term in 1963, no new officers were elected. Mr. Aaron Shapiro, Treasurer, presented the Treasurer's Report.

The address of welcome was given by Dean J. F. Sinzer, Pace College.

The program was as follows:

1. *Groups and their relations to other branches of mathematics*, by Gilbert Baumslag, New York University.
2. *Weak continuity of stochastic processes*, by E. J. McShane, The Rockefeller Institute.
3. *Report of the chairman of the contest committee and presentation to winners of Metropolitan New York Section contest*, by C. T. Salkind, Polytechnic Institute of Brooklyn.
4. *Reports of the Section Governor and of the Director of the Speakers' Bureau*, by J. N. Eastham, Queensborough Community College.
5. *Panel Discussion—Goals for school mathematics: The report of the Cambridge Conference*, by Mary Dolciani, Hunter College; Laura Eads, Bureau of Curriculum Research, Board of Education of the City of New York; and Julius Hlavaty, NCTM.

ABRAHAM SCHWARTZ, *Chairman*

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 2

CONTENTS

FEBRUARY 1965

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamilar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00. For nonmembers the subscription price is \$8.00 (\$10.00 effective August–September 1965).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

PRINTED IN THE UNITED STATES OF AMERICA

WHAT CAN BE DONE ABOUT TEACHING THE APPLICATIONS OF MATHEMATICS IN COLLEGES AND UNIVERSITIES

BERNARD FRIEDMAN, University of California and HERMAN H. GOLDSTINE, Thomas J. Watson Research Center

1. Organization. The Committee on Applications of Mathematics of the Division of Mathematics, National Academy of Sciences—National Research Council, was requested by the Division in 1963, "to draw up suggestions as to how departments of mathematics should be organized to give proper attention to the applications of mathematics." The members of the Committee on Applications were Bernard Friedman, Chairman; R. A. Bradley, H. H. Goldstine, J. B. Keller, P. D. Lax, M. H. Protter, M. M. Schiffer, and S. S. Wilks (deceased). H. H. Goldstine has since succeeded to the chairmanship, and G. B. Dantzig, H. O. Pollak, J. B. Rosser, and G. S. Young have been added to the Committee as new members.

The Division of Mathematics believes that the ideas and recommendations made by the Committee on Applications and others who attended meetings of the Committee should be brought to the attention of the mathematical community. It is the Committee's hope that its recommendations can have the effect of strengthening applied mathematics in the United States, not only by suggestions directed toward augmenting the numbers of college departments offering work in applied mathematics, but also by indicating points of view which, with time, the pure mathematician and applied mathematician can grow to share.

The Committee would like to express its especial thanks to Dr. Gail S. Young who, even prior to his membership in the Committee, provided particularly valuable ideas and insights. Many of the points made in this report stem from his articulate expression of his ideas and experiences.

2. The Problem. At the present time there is a sharp dichotomy in the mathematical curriculum between pure and applied mathematics. This division has existed, to some extent, since the earliest days and will continue to do so in the future. It is not the purpose of this discussion to attempt to evaluate the merits of the issues involved here, but rather to see what can be done to strengthen the role of applied mathematics in American universities.

3. Recommendations.

a. *Enlarge the course offerings in applied mathematics.* It seems axiomatic that mathematics departments should offer a variety of courses dealing with the applications of mathematics to other sciences and to engineering. In the past this has always meant a series of courses in classical physics from a mathematical point of view. It goes without saying that such a series is always a most welcome and important part of the curriculum in applied mathematics. Since such courses usually emphasized partial differential equations, they were usually taught by analysts.

Today, however, because of the large and important new applicational areas arising from the advent of the modern computer, courses are available which can be taught by mathematicians with different fields of specialization. For example, specialists in logic could teach modern combinatorics and formal logics, more specifically logic of computing machines and theory of automata; algebraists and topologists could teach linear programming, theory of games, combinatorial analysis, and graph theory. Evidently a well-rounded curriculum should include many of these courses.

In addition to these computer-oriented subjects, the biological and social sciences are now ripe for mathematical exploitation and may not only provide the stimulus for much mathematical research, but may also open up new areas in pure mathematics.

b. *Make available existing staff to teach applied mathematics courses.* If a staff member is going to venture into teaching a course in applied mathematics, the department should be prepared to release him from some of his teaching duties so that he may prepare himself. If possible, it would be desirable for such an individual to spend a year or so in an environment relevant to his new-found interest. The National Science Foundation already encourages such stays at established centers by offering a number of visiting memberships.

c. *Solicit Government support to implement 3b.* To assist the departments in solving the financial problems raised by the released time suggested above, the Committee suggests that the Federal Government be asked to institute a system of training grants to go directly to the departments concerned. The grants could be used either to support summer salaries for faculty members interested in developing courses in the applications of mathematics, or to give them released time to prepare such courses, or to support them during their stay at an established center of applied mathematics, or simply to assist the department in securing the services of a man already versed in the applications.

d. *Encourage broadened study in various sciences by mathematics students.* All mathematics majors should be encouraged and counseled to take courses in various sciences beyond the general introductory courses. Moreover, the student should be encouraged to begin this work as early as possible. There are two reasons for this. Conditioned by the trend to rigor that prevails in mathematics, the student who begins to study other subjects late in his academic career may find it difficult to follow the motivations of other disciplines because he is too easily thwarted by non-rigorous and essentially intuitive processes. It is this inability to see the forest for the trees that may possibly be avoided by giving the mathematics student an early prolonged exposure to more empirical subjects. Secondly, scientific intuition is a valuable tool for the applied mathematician and it will quite likely never mature without a more extensive exposure to the empirical sciences than is represented by one or two introductory courses.

e. *Create suitable applied mathematics textbooks at all levels.* There is a vital need for suitable textbooks at both the undergraduate and graduate level for

applied mathematics. In general, there are better works available on the graduate level, but there is no reason why, with proper textbooks, the undergraduate could not be introduced to fluid mechanics, elasticity, electromagnetic theory, mathematical economics, automata theory, etc.

A faculty member whose entire graduate background lies in abstract mathematics and whose undergraduate training may not even include a course in physics would be a hardy soul indeed were he to attempt to each a course in fluid dynamics or in electromagnetic fields, however convinced he may be in his own mind that this is a desirable thing to do. A series of monographs setting forth the applications of mathematics to such diverse disciplines as biology, physics, and the social sciences is almost essential for the success of this plan. Feller's book on probability, the two volumes of Courant-Hilbert on mathematical physics, and Friedman's "Techniques of Applied Mathematics" are good examples of what can be done along these lines.

A good monograph giving the applications of Hilbert space to quantum mechanics with emphasis on applications to specific physical problems is badly needed, as are scholarly approaches to the roles that mathematics plays today in biology, the social sciences, and operations analysis. Such books would enlighten and enrich the mathematics curriculum and enable many mathematicians to broaden and deepen their point of view. Funds for the support of those qualified and willing to undertake this important task can be obtained; commercial publishers would be more than happy to cooperate and will pay to get good manuscripts. In addition, the Air Force Office of Scientific Research has a program under which writers of advanced scientific monographs may seek financial support.

f. *Assist mathematical librarians.* Not only are adequate textbooks needed, but libraries should have guidance in the acquisition of applied mathematics books. The CUPM (Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America) has prepared, and is now revising, a list of basic books in mathematics for the undergraduate library, listing only those that every library should have. In applied mathematics particularly, there is need for a larger list covering advanced books also so that a college or university with limited library resources (this probably includes all of them) and without applied mathematicians can be sure that their collection is adequate and that they are getting the most from their money. Reviews alone do not provide adequate guidance. But a list of 700 books, say, broken up into graduate and undergraduate and described as "essential," "desirable," and "supplementary," would be of tremendous value, particularly if it were followed by annual or biennial supplements.

At many schools, the funds available for keeping the library adequately stocked in the fields in which research work is offered are too limited. A mechanism is badly needed for getting enough money to build an adequate collection in applied mathematics in the broad sense of the word.

g. *Hold summer institutes to prepare applied mathematics instructors.* A school

regarding itself as a serious graduate center should have its own applied mathematicians. But a good undergraduate college, or a weaker graduate school, will very likely have to provide courses taught, in the first place, by people with little training in the area, and, in the second place, by people with lesser mathematical ability. There should be a program of summer institutes to prepare such people for teaching good advanced undergraduate courses in applied mathematics. For these programs to be successful, the faculty salaries should compete, not with summer school salaries, but with research-grant or consulting salaries, and probably so should the stipends for the students. The $75-plus-dependency-allowance NSF formula for the student will not attract people with a degree of competence in other areas.

h. *Help mathematicians switch into applied mathematics.* Moreover, there needs to be some way to provide support to scholars of proved ability who wish to begin work outside their present fields of competence. Existing research-grant mechanisms tend to freeze scholars into their fields. Certainly few people would submit proposals for changing their fields of research unless the evaluation was based on new standards. High-level summer institutes, perhaps on higher level than the ones mentioned above, would allow the man who is very competent in one field to understand the important research problems in a new field.

i. *Form study group.* To attack the problem of applied mathematics preparation on the secondary school level, the CUPM Teacher Training Panel, of which Dr. Gail S. Young is the chairman, is planning the introduction of a course on "Applications of Mathematics for the Secondary School Teacher" in the training of secondary school teachers of mathematics. The panel believes that a study group should meet to outline a book for such a course. Individuals would then be enlisted to write the chapters and finally a smaller group would meet in the summer of 1965 to do the final editorial work. The Committee on Applications of Mathematics was called on to make suggestions to the Panel about possible topics and possible members of the working group. The Committee on Applications applauds the aims of the Panel and hopes for their success.

4. Conclusion. In conclusion, the Committee on Applications feels that the growth of applied mathematics in the United States is not only vital for our society from the point of view of national interest and, if you will, culture, but for mathematics itself. This need for mathematics to return to "reality" was well stated by John von Neumann, who contributed so greatly to both pure and applied mathematics:

As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from 'reality,' it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art.* This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-

developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities . . . whenever this stage is reached, the only remedy seems to me to be the rejuvenating return to the source: the reinjection of more or less directly empirical ideas. I am convinced that this was a necessary condition to conserve the freshness and the vitality of the subject and that this will remain equally true in the future. [1]

Reference

1. John von Neumann, *The Mathematician.* In: "The Works of the Mind," R. B. Heywood ed., University of Chicago Press, 1 (1947) 180–196. Reprinted in the Collected Works, Vol. 1, pp. 1–9.

CROSS-EXAMINING PROPOSITIONAL CALCULUS AND SET OPERATIONS*

JOHN F. RANDOLPH, University of Rochester

1. The proposition $P=[p\rightarrow(q\vee r)]\wedge[\sim(p\leftrightarrow\sim r)]$ has the truth table given in Fig. 1

p	q	r	P
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

FIG. 1

Even without the intermediate working columns, it may not be evident that P is independent of q, and a Boolean algebraic proof of this is somewhat involved. In [2] pg. 13, a truth table for P is given with 112 T's and F's; a person experienced in constructing truth tables would probably use at least 64 entries.

The purpose of this paper is to give an operational method for obtaining the truth value column of a compound proposition without writing any other column of the truth table. The iconic notation also reveals pertinent facts about a proposition much as the graph of an equation displays properties of the equation. In addition, much of Boolean algebra may be bypassed but, on the other hand, Boolean operations may be clarified by using the symbols. The symbols are already being used by several Electrical Engineering groups whose work involves simplifying switching circuits.

2. The truth values for the primary statement p are represented by T/F and for the secondary statement q by $F\backslash T$; that is, "T above the line, F below the line." The two segments placed together form a St. Andrews cross $\times$ which is lettered

* This trickery was a byproduct of work supported by the Air Force Office of Scientific Research —Contract Number AF-AFOSR-481-64.

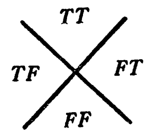
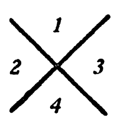


FIG. 2

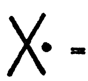
In any quadrant the first letter is T or F according as the quadrant is above or below the p-line; the second letter is T above the q-line but F below. Upon numbering these quadrants as shown in Fig. 3, the quadrant i corresponds to row i of the truth table listing p first and q second in the usual way



	p	q
1	T	T
2	T	F
3	F	T
4	F	F

FIG. 3

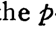
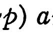
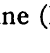
3. A St. Andrews cross with a dot in one quadrant, and the remaining quadrants left blank, will be used to represent a compound statement whose truth column has T in the row corresponding to the dot and F in each row corresponding to a blank. For example,

 $=(\sim p)\wedge q$

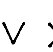
p	q	$\sim p\wedge q$
T	T	F
T	F	F
F	T	T
F	F	F

FIG. 4

Rather than thinking of the truth table, however, it is easier to consider the dot as below the p-line (hence $\sim p$) *and* above the q-line (hence q). In the same way

 $= p \wedge q$  $= p \wedge \sim q$  $= \sim p \wedge \sim q.$

A cross with two dots may, therefore, be considered as the *disjunction* of two single-dot propositions. Thus

 $=$  $\vee$  $= (\sim p \wedge q) \vee (\sim p \wedge \sim q).$

But also, in this illustration, the two dots "fill" the entire space below the p-line so that

 $= \sim p.$

Thus two readings of this single cross is equivalent to the algebraic manipulation

$$(\sim p \wedge q) \vee (\sim p \wedge \sim q) = \sim p \wedge (q \vee \sim q) = \sim p.$$

Similarly $\times = q$. Consequently $\sim p \vee q = \times \vee \times = \times$; that is

$$\times = \text{“If } p\text{, then } q\text{”} = p \rightarrow q.$$

The disjunction of two crosses is a cross having a dot where, and only where, the first **or** *the second of the given crosses has a dot.* ("Or" is used in the inclusive sense.)

Again, various readings of a cross reveal equivalencies:

$$\begin{aligned} p \rightarrow q = \times = \times \vee \times &= q \vee (\sim p \wedge \sim q) \\ = \times \vee \times &= (p \wedge q) \vee \sim p \\ = \times \vee \times \vee \times &= (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q) \end{aligned}$$

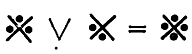
The conjunction of two crosses is a cross having a dot where, and only where, the first **and** *the second of the given crosses has a dot.*

Thus $(p \rightarrow q) \wedge (q \rightarrow p) = \times \wedge \times = \times = p \leftrightarrow q$.

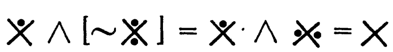
To "not" a cross, erase all dots and dot where there were blanks.

is de Morgan's rule $\sim(p \vee q) = \sim p \wedge \sim q$.

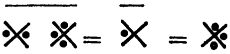
A completely filled cross $\times$ *is a tautology while a completely blank cross* $\times$ *is a contradiction.* For example $(\sim p \rightarrow q) \vee \sim q$ is a tautology:



That $(p \wedge q) \wedge [\sim(p \leftrightarrow q)]$ is a contradiction is established by



We shall henceforth use "$\wedge$" or merely juxtaposition whichever is convenient; and also for "not" we shall use either "$\sim$" or an over bar. Thus



shows that $\sim[(p \underline{\vee} q) \wedge (q \rightarrow p)] = p \rightarrow q$ where "$\underline{\vee}$" is the "exclusive or."

4. Remember the old chestnut about a prisoner in a cell with two unlocked doors; one to freedom, the other to death? (See [2] pg. 16 or [8] pg. 48.) There are two guards and the prisoner may ask either of them one question. The trouble is one guard is always honest but the other always lies; the prisoner knows this but does not know which is which. The prisoner may ask "Does Door 1 lead to freedom and you tell the truth or does Door 2 lead to freedom and you lie?" A guard who can understand the question should give up guarding and take up logic.

To be practical, the prisoner tells the guards to go to separate corners and

not let anyone see what they draw. He instructs the guards to draw / and then a second line to form X . Now he says "I want you to put one dot in one of the four spaces so that the dot is above the first line if Door 1 leads to freedom, below the first line if Door 2 leads to freedom; but wait a minute, also the dot is to be above the second line if you are honest, below the second line if you lie." Even a guard should be able to do this.

The captive knows the possibilities of dots are as given in Fig. 5a (since the liar is honest with himself, for otherwise etc.).

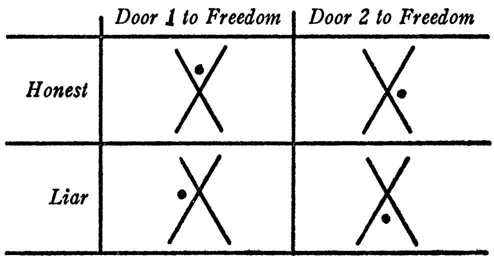


FIG. 5a

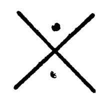


Fig. 5b

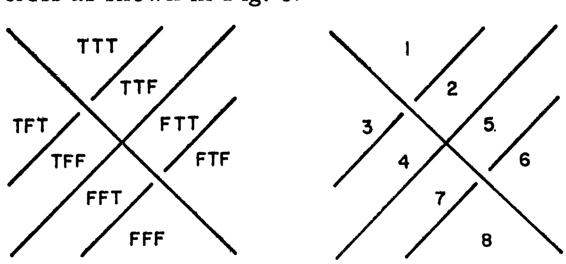
The prisoner now draws Fig. 5b points to either guard and asks his one question, "Does your dot agree with either of these?"

Write the answer on Fig. 5a for each possibility and see why the prisoner goes free.

Could the prisoner have asked, "Is your dot in one of my blank spaces?"

5. W. S. McCulloch [3] introduced the use of crosses and dots. The extension to three or more statements, as given below, differs from McCulloch's.

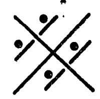
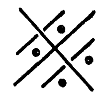
With p and q as before and r a third statement, we now use a small slant / in *each* quadrant of the (p, q)-cross. Each quadrant is then divided into T- and F-portions for r; T above and F below / in that quadrant. Both portions of any quadrant now have three letters; the first two are the same as they were before, but the third has T above the new slant and F below. Thus the eight rows of a (p, q, r)-truth table correspond, respectively, to the numbered regions of the (p, q, r)-cross as shown in Fig. 6.



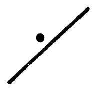
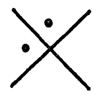
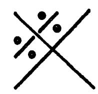
	p	q	r
1	T	T	T
2	T	T	F
3	T	F	T
4	T	F	F
5	F	T	T
6	F	T	F
7	F	F	T
8	F	F	F

FIG. 6

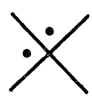
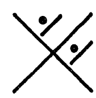
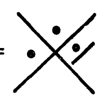
It now takes four dots to represent the tertiary r alone:

$r =$  while $\sim r =$ 

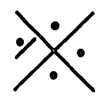
Notice that p alone (where q and r are also to be considered) may be represented as

 or  or 

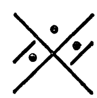
For example,

$p \vee q r =$  $\vee$  $=$ 

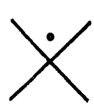
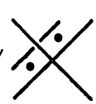
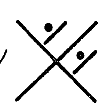
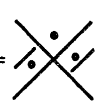
illustrates how, in disjunctions, once a secondary region receives a dot, then that region need not be further encumbered with anything tertiary. Again, for $(\sim p) \vee q \vee r$ we first fill in two dots below the p-line for $\sim p$, then above the q-line place one further dot (since one dot is already there) for q, and finally, in the only remaining quadrant, insert ·/ to obtain

$(\sim p) \vee q \vee r =$ 

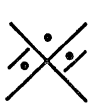
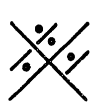
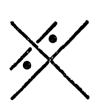
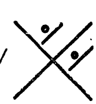
With very little practice it is easy to dot a cross for a proposition formed by disjunctions of conjunctions. As an example, write directly

$pq \vee p\bar{r} \vee qr =$ 

rather than

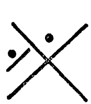
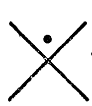
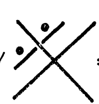
 $\vee$  $\vee$  $=$ 

On the other hand a dot in a secondary region may be expanded to ·/. in that region:

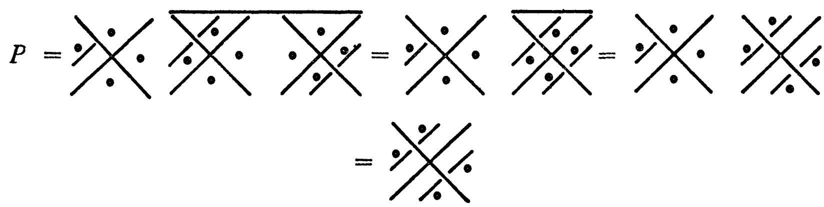
 $=$  $=$  $\vee$  $= p\bar{r} \vee qr,$

thus showing that $pq \vee p\bar{r} \vee qr = p\bar{r} \vee qr$. Thus the whole "clause" pq on the left is superfluous.

Also in $pq \vee p\bar{q}r$ the $\bar{q}$ is superfluous ([7] shows why such information is useful) as shown by

$p q \vee p \bar{q} r =$  $=$  $\vee$  $= p q \vee p r.$

6. The proposition $P = [p \rightarrow (q \vee r)][\sim(p \leftrightarrow \sim r)]$, given at the beginning of this article, is cross-examined as



(1)

By following the enumeration as given in Fig. 6, the truth column (written as a row) for P is

T	F	T	F	F	T	F	T
1	2	3	4	5	6	7	8

Hence the detailed construction of the truth table has been avoided.

Also the crosses reveal facts which would be hard to glean from the truth table:

1. The final cross (1) has the same configuration above as below the q-line thus showing that P is independent of q.
2. In (1) the designation for r is in both quadrants *above* the p-line while the designation for $\sim r$ is in both $\sim p$-quadrants thus revealing that P is T provided p and r are either both T or else are both F; i.e.

$$P = (p \leftrightarrow r).$$

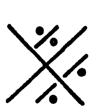
3. The cross in (1) is also the last cross in the line above (1) thus showing that P is in fact equal to its second factor. In other words, whenever the second factor is T the first is also T; that is, $\sim(p \leftrightarrow \sim q)$ strictly implies $p \rightarrow (q \vee r)$:

$$\sim(p \leftrightarrow \sim q) \Rightarrow p \rightarrow (q \vee r).$$

The adjective "strict" is used since some authors use "p implies q" and "If p then q" interchangeably. Further discussion of strict implication is given in section 9.

7. The above shows how the truth values of a proposition may be obtained. Conversely, propositions having given truth values are easily made. For example, to find a proposition having truth values

$$T\ T \quad F\ F \quad F\ T \quad F\ T$$

first dot a cross 

If each dot is read individually the result is

$$p\, q\, r \vee p\, q\, \bar{r} \vee \bar{p}\, q\, \bar{r} \vee \bar{p}\, \bar{q}\, \bar{r}$$

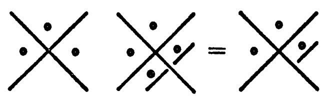
In the top quadrant, however, ˙/. may be condensed to a dot, while the two lower right quadrants have /. so that the simpler proposition

$$p\,q \vee \bar{p}\,\bar{r}$$

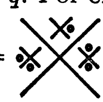
may be obtained with no algebraic manipulation. In fact, such obvious contractions and groupings on a cross lend meaning to such factoring and reduction as

$$p\,q\,r \vee p\,q\,\bar{r} = p\,q\,(r \vee \bar{r}) = p\,q.$$

Also the distributive law $(p\vee q)(p\vee r)=p\vee qr$ may be "proved" by noting that



8. When four statements p, q, r, s are involved, then use a St. Andrews cross for r, s in each of the quadrants of the cross for p, q. For example

$p\,q\,\bar{r}\,s \vee p\,\bar{q}\,\bar{s} \vee \bar{p}\,q(r \leftrightarrow s) =$ 

has truth values

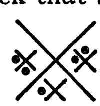
$$F\ F\ T\ F \quad F\ T\ F\ T \quad T\ F\ F\ T \quad F\ F\ F\ F$$

since the small crosses follow the established order and the quadrants of the large cross are picked up in the same order.

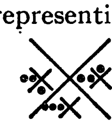
As another example, the equality

$$p\,\bar{q}\,\bar{s} \vee \bar{p}\,q\,r \vee \bar{p}\,\bar{q}\,r\,\bar{s} = p\,\bar{q}\,\bar{s} \vee \bar{p}\,q\,r \vee \bar{p}\,r\,\bar{s} \vee \bar{q}\,r\,\bar{s} \tag{2}$$

will be established. Merely check that both sides have the same cross



Once a region receives a dot it is, of course, redundant to insert other dots in the same region. The use of multiple dots does, however, reveal hidden duplicity in a proposition. Thus, cross-representing the right side of (2) could be

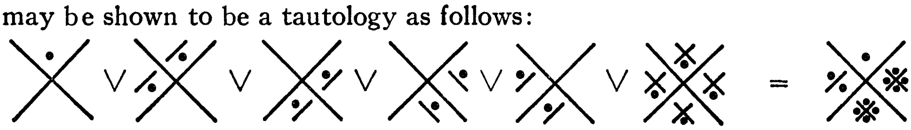


showing double inclusion of both $p\,\bar{q}\,r\,\bar{s}$ and $\bar{p}\,q\,r\,\bar{s}$ and triple inclusion of $\bar{p}\,\bar{q}\,r\,\bar{s}$.

If all 16 spaces have, or can be seen to have, a dot then the proposition is a tautology. Thus (see [6] pg. 30)

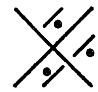
$$p\,q \vee p\,\bar{r} \vee \bar{p}\,r \vee \bar{p}\,s \vee \bar{q}\,r \vee \bar{r}\,\bar{s}$$

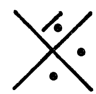
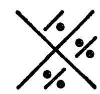
may be shown to be a tautology as follows:



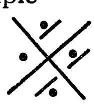
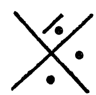
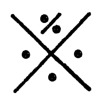
Conversely, it is easy to make up complicated tautologies.

9. As mentioned earlier, strict implication $P \Rightarrow Q$, meaning "Q is true whenever P is true," is easily visualized. For example, given

$$P = (q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) =$$ 

$$Q = (p \wedge q \wedge \sim r) \vee \sim p =$$  $=$ 

then $P \Rightarrow Q$, since every dot for P has a corresponding dot for Q (and Q has additional dots). For this example

$$\overline{P} \vee Q =$$  $\vee$  $=$ 

is a tautology.

Visually, in any case, $\overline{P} \vee Q$ is a tautology iff each blank for Q has a dot for $\overline{P}$, therefore a blank for P. Hence $\overline{P} \vee Q$ is a tautology iff each dot for P has a corresponding dot for Q. This proves the theorem

(3) $P \Rightarrow Q$ *iff* $\overline{P} \vee Q$ *is a tautology*; *that is*, "$P \Rightarrow Q$ *iff* $P \rightarrow Q$ *is a tautology*."

From my point of view, it is most unfortunate that "$P \supset Q$" is sometimes used for the conditional "If P then Q." For note that by considering the dots on the crosses for P and Q as point sets P and Q, then the set relation "$P \subset Q$" means "The proposition P strictly implies the proposition Q." Then instead of $P \supset Q$ it would be more revealing to use $P \subset Q$ since, for P to imply Q strictly, P must be the weaker (if anything) in truth values. Also the theorem in (3) would be written

(3′) $P \subset Q$ *iff* $\overline{P} \vee Q$ *is a tautology*

thus resembling the set theoretic analogue

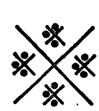
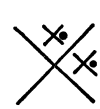
$P \subset Q$ *iff* $\overline{P} \cup Q$ *is the universal set.*

Instead of (3), it may be easier to use the equivalent theorem

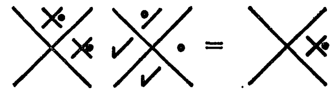
(4) $P \Rightarrow Q$ *iff* $\overline{Q} \wedge P$ *is a contradiction,*

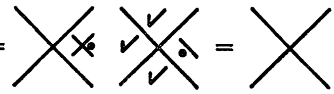
as shown in establishing $[p \rightarrow (q \rightarrow r)][\sim s \vee p](q) \Rightarrow (s \rightarrow r)$, taken from [**9**], pg. 83.

Since

$$s \rightarrow r =$$  $$[\sim(s \rightarrow r)]q =$$ 

Henceforth we merely check large spaces already known to be blanks. Thus, since $p \rightarrow (q \rightarrow r) = \bar{p} \vee \bar{q} \vee r$,

$[\sim(s \to r)](q)[p \to (q \to r)] =$ 

$[\sim(s \to r)](q)[p \to (q \to r)](\sim s \vee p) =$ 

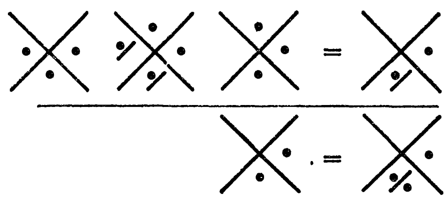
A test for validity of an argument is quickly made without the necessity of truth tables, Boolean algebra, *modus ponens* [10], etc.

$\sim p \vee \sim q$

$\sim q \to r$

$p \to q$

$\therefore \sim p$

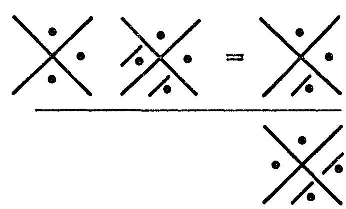
 valid.

The argument is valid since each dot in the final cross above the line has a corresponding dot in the cross below the line.

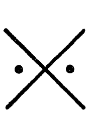
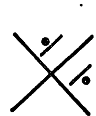
$p \to q$

$\sim q \to \sim r$

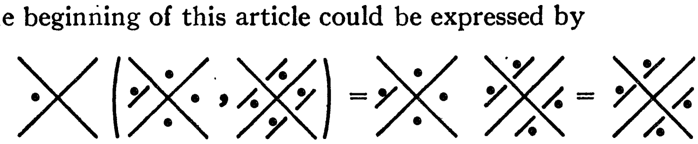
$r \to p$

 not valid.

10. Having established the order of the lines on a cross, a cross may be used as a function. The arguments are then freed from restriction to $p, q, r, \cdots$. Thus

 $(a, b) = a\bar{b} \vee \bar{a}b,$  $(u, v, w) = v(uw \vee \bar{u}\bar{w}).$

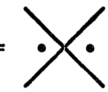
Now, with the ultimate arguments understood, crosses may themselves be used as arguments in a "function of functions" mode. For example, the proposition P at the beginning of this article could be expressed by

$P =$  $= (p \leftrightarrow r).$

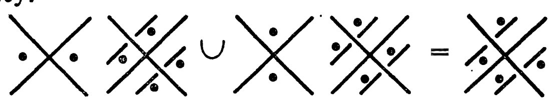
For a discussion of the resulting "Function Algebra" using two arguments, see [4].

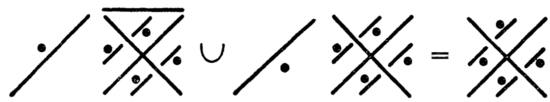
11. It is redundant to point out that the cross notation may also be used for set operations, but as an example we prove the associativity of the symmetric

difference $A+B$ defined by

$$A + B = (A - B) \cup (B - A) = A\bar{B} \cup \bar{A}B =$$ 

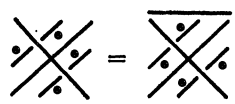
Proof of associativity.

$$(A + B) + C =$$ 

$$A + (B + C) =$$ 

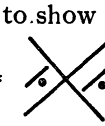
and both final crosses are the same.

Note that the cross for $A+B+\bar{C}$ is the above final cross with each dot moved across its (and only its) tertiary line so that

$$A + B + \bar{C} =$$  $$= \overline{A + B + C},$$

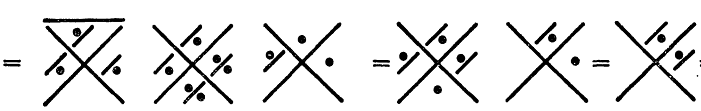
a relation which is fairly long to prove by using Boolean algebra.

The distributive law need not be used before dotting a cross for such an expression as $(A\bar{B}\cup\bar{A}B)\bar{C}$. First put two dots to represent $A\bar{B}\cup\bar{A}B$ and then merely insert a tertiary line above each dot to show the intersection with $\bar{C}$:

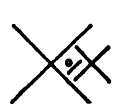
$$(A\,\bar{B} \cup \bar{A}\,B)\bar{C} =$$ 

A spectacular saving, over the use of the distributive law and de Morgan's rules, is illustrated in:

$$\overline{\bar{A}\,B\,C \cup (A\,\bar{B} \cup \bar{A}\,B)\bar{C}}\;(\bar{A}\,C \cup \bar{C})(B \cup A\,C)$$

 $$= B(A\bar{C} \cup \bar{A}C).$$

12. Many diagrammatic methods for the propositional calculus or set theory have been proposed. A fairly complete account is given in [1]; also see [5]. It seems, however, that no one of the other schemes combines the operational features, iconic appeal, and systematic extension (theoretically, at least) that crosses do. Even the venerable Venn diagrams become messy for more than three components, and they are not operational (who ever put a "$\vee$" or "$\cup$" between two partially shaded Venn diagrams?). Moreover, the whole Venn diagram must be drawn before the search for the representation of a given set is started. For example, on Fig. 7 locate the set

$$\bar{A}\;B\;C\;\bar{D}\;E =$$ 

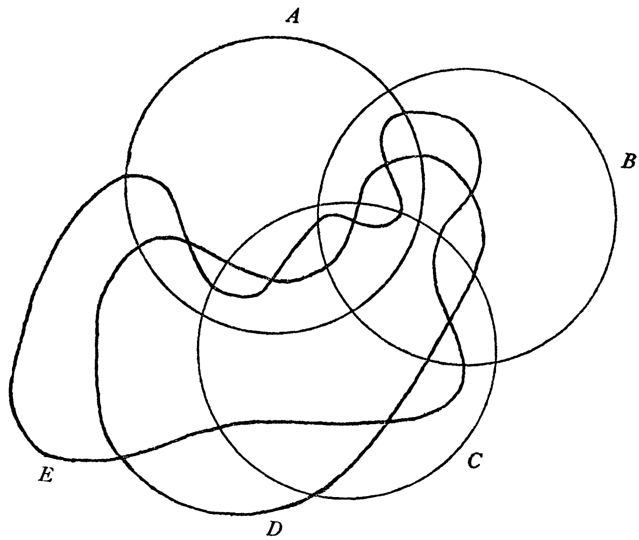


FIG. 7

References

1. Martin Gardner, Logic Machines and Diagrams, McGraw-Hill, New York, 1958.

2. John G. Kemeny, J. Laurie Snell and Gerald Thompson, Finite Mathematics, Prentice Hall, Englewood Cliffs, N. J., 1962.

3. W. S. McCulloch, Agatha Tyche of Nervous Nets—The Lucky Recomers. Natl. Phys. Lab. Symposium No. 10. Her Majesty's Stationer's Office, London, 1959.

4. Karl Menger, Function Algebra and Propositional Calculus. Self-Organizing Systems, Spartan Books, 1962, pp. 525–532.

5. Tuthill Parry, A New Symbolism for the Propositional Calculus, J. of Symbolic Logic, 19 (1954) 161–169. Also Gerald B. Standley, Ideographic Computation in the Propositional Calculus, *ibid.*, pp. 169–171.

6. W. V. Quine, Methods of Logic (1950). Harvard.

7. ———, The Problem of Simplifying Truth Functions, this MONTHLY, 59 (1952) 521–531.

8. Francis J. Scheid, Elements of Finite Mathematics, Addison Wesley, Reading, Mass., 1962.

9. Robert R. Stoll, Sets, Logic, and Axiomatic Theories, Freeman, San Francisco, 1961.

10. Irving M. Copi, Symbolic Logic, Macmillan, New York, 1959, pp. 22 and 41.

THE DISTRIBUTION FUNCTIONS OF RANDOM VARIABLES IN ARITHMETIC DOMAINS MODULO a

P. SCHEINOK, Radio Corporation of America

1. Introduction. Consider a point P on the circumference of a roulette wheel with initial angular position zero. Let the wheel be spun in a fixed direction, and suppose that a known probability distribution were defined on the distance traveled by the point P. What is the corresponding probability distribution of the final angular position ($[0, 2\pi)$) of P?

In addition, if the wheel were spun twice, one might then ask the following question. What is the probability distribution of the sum (modulo 2π), of the respective final angular positions of P? Similar questions can naturally be asked for the quotient or product.

In this presentation we shall formulate the corresponding mathematical problem, and compute the distribution functions of random variables resulting from arithmetic modulo a finite number a. Only the basic operations of addition, multiplication and division, all modulo a, will be considered. This may be useful pedagogically, as it might aid the reader in familiarizing himself with the proper calculation of distribution functions. Mathematically, it might lead to seeing how various classical theorems in the theory of probability are modified in this arithmetic domain. For some interesting results dealing with distribution functions of quotients and products of random variables, see Craig [1].

The language and notation of Parzen [2] will be used throughout the paper.

2. The reduction of a random variable mod a. One assumes the existence of a probability space $(R, \mathfrak{A}, p)$, defined on the space of points $R = \{x: -\infty < x < \infty\}$, with a collection $\mathfrak{A}$ of Borel sets defined on R, and a probability measure p defined on the sets of $\mathfrak{A}$. Let X be a random variable defined on the sets of $\mathfrak{A}$ having distribution function $F_X(\cdot)$. Let X_a be congruent to X modulo a, and restrict X_a to the interval $[0, a)$. Then

$$\begin{aligned} \Pr\{X_a \leqq u\} &= \sum_{k=-\infty}^{\infty} \Pr\{ka \leqq X \leqq ka + u\} \\ &= \sum_{k=-\infty}^{\infty} \{F_X(ka + u) - F_X(ka) + \Pr\{X = ka\}\}. \end{aligned} \tag{2.1}$$

Assume that $F_X(\cdot)$ is absolutely continuous, and thus $\Pr\{X = ka\}$ is zero for all k. Let $f_X(\cdot)$ and $\bar{f}_{X_a}(\cdot)$ denote the probability density functions of X and X_a respectively. Then if $\sum_{k=-\infty}^{\infty} f_X(ka+u)$ converges uniformly in $0 \leqq u \leqq a$, one knows that

$$f_{X_a}(u) = \begin{cases} \sum_{k=-\infty}^{\infty} f_X(ka + u) & 0 \leqq u \leqq a \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Example. Let X have the exponential distribution on $[0, \infty)$, and consider X_a. One has

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 \leqq x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\tilde{f}_{X_a}(x) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \lambda e^{-\lambda(ka+x)} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda a}} & 0 \leqq x \leqq a \\ 0 & \text{otherwise.} \end{cases}$$

3. The addition of random variables mod a. Let X and Y be independent random variables defined on $[0, a]$ having distribution functions $F_X(\cdot)$ and $G_Y(\cdot)$ respectively. Define $Z = X + Y$ with distribution function $H_Z(\cdot)$. Then

$$H_Z(z) = \Pr\{X + Y \leqq z\}$$

$$(3.1) \qquad = \begin{cases} \displaystyle\int_0^z G_Y(z - y)dF_X(y) & 0 \leqq z \leqq a \\ F_X(z - a) + \displaystyle\int_{z-a}^a G_Y(z - y)dF_X(y), & a \leqq z \leqq 2a. \end{cases}$$

If Z_a is congruent to Z modulo a, having distribution function $\tilde{H}_{Z_a}(\cdot)$, then

$$(3.2) \qquad \tilde{H}_{Z_a}(u) = \Pr\{Z_a \leqq u\} = H_Z(u) + H_Z(a + u) - H_Z(a) + \Pr\{Z_a = a\}.$$

Again, in the absolutely continuous case, $\Pr\{Z_a = a\}$ is zero. If one denotes the corresponding probability density functions by $f_X(\cdot)$, $g_Y(\cdot)$, $h_Z(\cdot)$ and $\tilde{h}_{Z_a}(\cdot)$, then one sees that

$$(3.3) \qquad \tilde{h}_{Z_a}(u) = \int_0^u f_X(y)g_Y(u - y)dy + \int_u^a f_X(y)g_Y(a + u - y)dy.$$

It is clear from (3.3) that if X and Y are uniformly distributed on $[0, a]$, then Z_a is also uniformly distributed on $[0, a]$. Thus, the uniform distribution is one which reproduces itself under addition mod a. The converse question is whether this reproducibility property is one peculiar to the uniform distribution. In the language of the theory of integral equations, this is equivalent to proving that the nonlinear integral equation for $0 \leqq u \leqq a$

$$(3.4) \qquad f(u) = \int_0^u f(y)f(u - y)dy + \int_u^a f(y)f(a + u - y)dy$$

has a unique solution $f(u)$ in the interval $0 \leqq u \leqq a$, under suitable conditions on f. Actually one can prove the following:

THEOREM. *Let $f(u)$ be nonnegative and continuous on $[0, a]$, and let $\int_0^a f(u)du = 1$. If $f(u)$ satisfies* (3.4) *then $f(u) \equiv 1/a$, $0 \leqq u \leqq a$.*

Proof (due to L. Flatto): Assume that f is not a constant in the interval of definition. Then, by the continuity assumption, f assumes a maximum in $0 \leqq u \leqq a$. Let this maximum be at $u = u_0$, where u_0 is not an end-point of the interval $[0, a]$. Then

$$\int_0^{u_0} f(y)f(u_0 - y)dy < f(u_0) \int_0^{u_0} f(y)dy, \tag{3.5a}$$

and

$$\int_{u_0}^{a} f(y)f(a + u_0 - y)dy < f(u_0) \int_{u_0}^{a} f(y)dy. \tag{3.5b}$$

By addition,

$$\begin{aligned} f(u_0) &= \int_0^{u_0} f(y)f(u_0 - y)dy + \int_{u_0}^{a} f(y)f(a + u_0 - y)dy \\ &< f(u_0)\left[\int_0^{u_0} f(y)dy + \int_{u_0}^{a} f(y)dy\right]. \end{aligned} \tag{3.6}$$

Since $\int_0^a f(y)dy = 1$, one reaches the contradiction that $f(u_0) < f(u_0)$. Thus $f(u)$ is a constant, and must be $1/a$.

If the maximum were at the end-points 0, a, one has the strict inequality holding in either (3.5a) or (3.5b), while the other inequality becomes an equality with zero on both sides. The strict inequality in (3.6) holds nevertheless, proving the theorem.

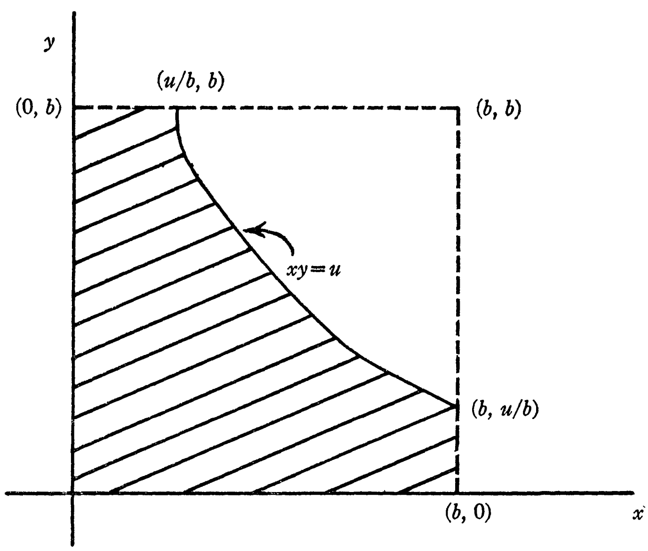


FIG. 1

4. Products and quotients of random variables, mod a. Let X and Y be two independent random variables defined on the interval $[0, b]$, having distribution functions $F_X(\cdot)$ and $G_Y(\cdot)$ respectively. Let $U=XY$, with distribution function $K_U(\cdot)$. One obtains this distribution function from Fig. 1.

$$\begin{aligned} K_U(u) &= \iint_{xy\leq u} dF_X(x)dG_Y(y) \qquad\qquad 0 \leqq u \leqq b^2 \\ &= \int_{u/b}^{b} dG_Y(y)\int_0^{u/y} dF_X(x) + \int_0^{u/b} dG_Y(y)\int_0^b dF_X(x) \end{aligned} \tag{4.1}$$

or

$$K_U(u) = G_Y(u/b) + \int_{u/b}^{b} F_X(u/y)dG_Y(y) \qquad 0 \leqq u \leqq b^2. \tag{4.2}$$

Here again, if $F_X(\cdot)$ and $G_Y(\cdot)$ are absolutely continuous with corresponding densities, $f_X(\cdot)$ and $g_Y(\cdot)$, then U has a probability density $k_U(\cdot)$ which is obtained from (4.2) by differentiation.

$$k_U(u) = \int_{u/b}^{b} f_X(u/y)g_Y(y)\,\frac{dy}{y} \qquad 0 \leqq u \leqq b^2. \tag{4.3}$$

Consider the random variable U_a, where $a \leqq b^2$. Then $b^2=ma+r$, where m is the largest integral multiple of a in b^2, and $r \geqq 0$ is the remainder. Now consider $\Pr\{U_a \leqq c\}$, where $0 \leqq c < a$. One obtains that

$$\Pr\{U_a \leqq c\} = \sum_{n=0}^{m-1} \Pr\{na \leqq U \leqq na + c\} + R(c, r), \tag{4.4}$$

$$R(c, r) = \begin{cases} \Pr\{ma \leqq U \leqq ma + c\} & 0 \leqq c \leqq r, \\ \Pr\{ma \leqq U \leqq b^2\} & r \leqq c \leqq a. \end{cases} \tag{4.5}$$

The combination of (4.2), (4.4) and (4.5) now allows one to write down the distribution function of U_a explicitly in terms of $F_X(\cdot)$ and $G_Y(\cdot)$.

For quotients, let $V=Y/X$, with distribution function $M_V(\cdot)$. By definition,

$$M_V(v) = \iint_{y/x\leq v} dF_X(x)dG_Y(y). \tag{4.6}$$

From Fig. 2, we see that the distribution function $M_V(v)$ can be written as follows,

$$M_V(v) = \begin{cases} \displaystyle\int_0^b dF_X(x)\int_0^{vx} dG_Y(y) & 0 \leqq v \leqq 1, \\ \displaystyle 1 - \int_0^{b/v} dF_X(x)\int_{vx}^{b} dG_Y(y) & 1 \leqq v < \infty. \end{cases} \tag{4.7}$$

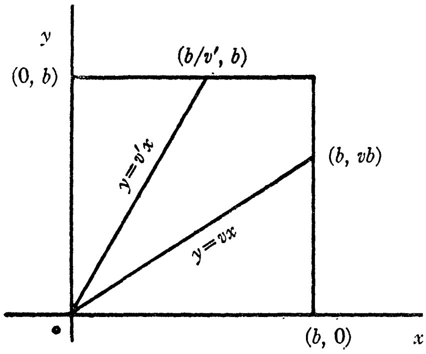


FIG. 2

Integrating with respect to y, one gets the following expression for $M_V(v)$:

$$(4.8)\qquad M_V(v) = \begin{cases} \int_0^b G_Y(vx)dF_X(x) & 0 \leqq v \leqq 1, \\ 1 - F_X(b/v) + \int_0^{b/v} G_Y(vx)dF_X(x) & 1 \leqq v < \infty. \end{cases}$$

Again, if $F_X(\cdot)$ and $G_Y(\cdot)$ were absolutely continuous, one obtains the density function $m_V(v)$ by differentiation as follows:

$$(4.9)\qquad m_V(v) = \begin{cases} \int_0^b xg_Y(vx)f_X(x)dx & 0 \leqq v \leqq 1, \\ \int_0^{b/v} xg_Y(vx)f_X(x)dx & 1 \leqq v < \infty. \end{cases}$$

Now consider what happens to the random variable V_a congruent to V mod a.

Case 1. $a \geqq 1$. Then, for $0 \leqq v < a$,

$$(4.10)\qquad \Pr\{V_a \leqq v\} = M_V(v, a) + S_1(v),$$

where $M_V(v, a)$ is defined only for $0 \leqq v < a$ and is identical to $M_V(v)$ there, and where for $m \geqq 1$,

$$(4.11)\qquad \begin{aligned} S_m(v) = \sum_{k=m}^{\infty}\left[F_X\left(\frac{b}{ka}\right) - F_X\left(\frac{b}{ka+v}\right)\right] \\ + \sum_{k=m}^{\infty}\int_0^{b/(ka+v)} G_Y((ka+v)x)dF_X(x) \\ - \sum_{k=m}^{\infty}\int_0^{b/ka} G_Y(kax)dF_X(x). \end{aligned}$$

Case 2. $a<1$. Then there exists an integer $m>1$, such that $ma \geqq 1 > (m-1)a$. We again recognize two sub-cases.

A) If v is such that $(m-1)\,a+v \leqq 1 < ma$, then

$$\Pr\{V_a \leqq v\} = \sum_{k=0}^{m-1} \int_0^b [G_Y((ka+v)x) - G_Y(kax)]dF_X(x) + S_m(v). \tag{4.12}$$

B) On the other hand if $(m-1)\,a<1 \leqq (m-1)a+v<ma$, then

$$\begin{aligned} \Pr\{V_a \leqq v\} &= \sum_{k=0}^{m-2} \int_0^b [G_Y((ka+v)x) - G_Y(kax)]dF_X(x) \\ &\quad + 1 - F_X\left(\frac{b}{(m-1)a+v}\right) - \int_0^b G_Y((m-1)ax)dF_X(x) \\ &\quad + \int_0^{b/[(m-1)a+v]} G_Y(((m-1)a+v)x)dF_X(x) + S_m(v), \end{aligned} \tag{4.13}$$

where, in both (4.12) and (4.13), $S_m(v)$ is defined by (4.11).

5. Examples. Let X and Y be uniformly distributed on the interval $[0, b]$. Let $U=XY$ and $V=Y/X$. We shall be interested in the distribution functions of U_a and V_a.

For the case $U=XY$, one has that

$$\Pr\{U \leqq u\} = \frac{u}{b^2} + \frac{1}{b^2}\int_{u/b}^b \frac{u}{x}\,dx = \frac{u}{b^2}\left(1 - \ln\frac{u}{b^2}\right) \qquad 0 \leqq u \leqq b^2 \tag{5.1}$$

while the density $k_U(u)$ is expressed as

$$k_U(u) = -\frac{1}{b^2}\ln\frac{u}{b^2} \qquad 0 \leqq u \leqq b^2. \tag{5.2}$$

Now consider the distribution function of U_a. One sees that for $b^2=ma+r$, $0 \leqq c<a$

$$\Pr\{U_a \leqq c\} = \frac{(m-1)c}{b^2} + \sum_{n=0}^{m-1} \frac{na+c}{b^2}\ln\frac{b^2}{na+c} + \sum_{n=0}^{m-1}\frac{na}{b^2}\ln\frac{na}{b^2} + R(c,r), \tag{5.3}$$

where

$$R(c,r) = \begin{cases} \dfrac{c}{b^2} + \left(\dfrac{ma+c}{b^2}\right)\ln\dfrac{b^2}{ma+c} + \dfrac{ma}{b^2}\ln\dfrac{ma}{b^2} & 0 \leqq c \leqq r \\[2ex] 1 - \dfrac{ma}{b^2}\left(1 - \ln\dfrac{ma}{b^2}\right) & r \leqq c < a. \end{cases} \tag{5.4}$$

Now consider $V=Y/X$

$$(5.5)\quad \Pr\{V \leqq v\} = \begin{cases} \dfrac{1}{b^2}\displaystyle\int_0^b v\,x\,dx = \dfrac{v}{2} & 0 \leqq v \leqq 1 \\ 1 - \dfrac{1}{v} + \dfrac{1}{b^2}\displaystyle\int_0^{b/v} v\,x\,dx = 1 - \dfrac{1}{2v} & 1 \leqq v < \infty. \end{cases}$$

The density function $m_V(v)$ is then

$$(5.6)\qquad m_V(v) = \begin{cases} \dfrac{1}{2} & 0 \leqq v \leqq 1 \\ \dfrac{1}{2v^2} & 1 \leqq v < \infty. \end{cases}$$

For V_a, applying the results of section 4, one sees that for $0 \leqq v \leqq a \leqq 1$, and $(m-1)a+v \leqq 1 \leqq ma$

$$(5.7a)\qquad \Pr\{V_a \leqq v\} = \frac{mv}{2} + \frac{1}{2}\sum_{k=m}^{\infty} \frac{v}{ka(ka+v)}$$

while if $(m-1)a < 1 \leqq (m-1)a+v \leqq ma$, then

$$(5.7b)\quad \Pr\{V_a \leqq v\} = 1 + \frac{(m-1)(v-a)}{2} - \frac{1}{2[(m-1)a+v]} + \frac{1}{2}\sum_{k=m}^{\infty} \frac{v}{ka(ka+v)}.$$

If $a>1$, then

$$(5.8)\qquad \Pr\{V_a \leqq v\} = M_V(v, a) + \frac{1}{2}\sum_{k=1}^{\infty} \frac{v}{ka(ka+v)},$$

where

$$(5.9)\qquad M_V(v, a) = \begin{cases} \dfrac{v}{2} & 0 \leqq v \leqq 1 \\ 1 - \dfrac{1}{2v} & 1 \leqq v < a. \end{cases}$$

The author is now at the Hahnemann Medical College, Philadelphia. He would like to thank the referee for his careful reading of this paper, and his most helpful suggestions.

References

1. C. C. Craig, On frequency distributions of the quotient and of the product of two statistical variables, this MONTHLY, 49 (1942) 24–32.
2. E. Parzen, Modern Probability Theory and its Applications, Wiley, New York, 1960.

THE BRAUER-RADEMACHER IDENTITY

M. V. SUBBARAO, University of Alberta

1. Introduction. Let n and r be integers, $r>0$ and let $\phi(r)$ and $\mu(r)$ denote the well known arithmetic functions of Euler and Möbius. The so called Brauer-Rademacher identity [1] says that

$$(1) \qquad \phi(r) \sum_{\substack{d \mid r \\ (d,n)=1}} d\,\mu\left(\frac{r}{d}\right) \Big/ \phi(d) = \mu(r) \sum_{d \mid (r,n)} d\,\mu\left(\frac{r}{d}\right).$$

Eckford Cohen gave new proofs of this identity in [2], [3] and [4] with some generalizations, in two of which ([2], [3]) he used the principle of inversion for even arithmetic functions, and in the other, the theory of expansion of Cauchy products of such functions. For another proof of (1) and generalization we refer to McCarthy [6].

Now it is well known that

$$\sum_{d \mid (r,n)} d\mu(r/d) = \phi(r)\mu(r/g)/\phi(r/g), \qquad g = (r, n)$$

and in fact the expressions on both sides of the equation are identical with the Ramanujan sum $c(n, r) = \sum \exp(2\pi inx/r)$, summed for all x varying over a reduced residue system modulo r. Hence (1) becomes

$$(2) \qquad \sum_{\substack{d \mid r \\ (d,n)=1}} d\mu(r/d)/\phi(d) = \mu(r)\mu(r/g)/\phi(r/g).$$

It is important to note that the arithmetic function $f(r) = r/\phi(r)$ is multiplicative and has the additional property that, for every prime p,

$$(3) \qquad f(p) = f(p^2) = f(p^3) = \cdots.$$

We wish to show that every such multiplicative function satisfies an identity of the Brauer-Rademacher type. In deriving this identity we do not use the methods of Cohen or McCarthy, but simple properties of multiplicative arithmetic functions of two arguments.

2. Preliminaries. All functions considered in the sequel are assumed to be complex-valued arithmetic functions. The letter p always denotes a prime.

A function $f(r)$ is said to be a multiplicative function of r (or, briefly, multiplicative) if $f(1) = 1$ and $f(r_1r_2) = f(r_1)f(r_2)$ whenever $(r_1, r_2) = 1$. This definition can be extended for functions of more than one argument. Thus the function $f(r, s)$ is said to be a multiplicative function of both arguments r and s if $f(1, 1) = 1$ and

$$f(r_1r_2, s_1s_2) = f(r_1, s_1)f(r_2, s_2)$$

whenever $(r_1s_1, r_2s_2)=1$. The function $f(r, s)$ is said to be multiplicative in r if, by keeping s arbitrarily fixed, $f(r, s)$ is a multiplicative function of r. A similar definition holds for $f(r, s)$ to be multiplicative in s.

If $f(r, s)$ is a multiplicative function of the arguments r and s, it need not be multiplicative in r or s, but $f(r, 1)$ and $f(1, s)$ are multiplicative in r and s respectively. It has also to be noted that a multiplicative function $f(r, s)$ is uniquely determined whenever, for every prime p and all nonnegative integers a and b, the values of $f(p^a, p^b)$ are known.

The following result is useful:

LEMMA. *If $f(r)$ and $g(r)$ are multiplicative and*

$$h(n, r) = \sum_{\substack{d\mid r\\(d,n)=1}} f(d)g(r/d),$$

then $h(n, r)$ is a multiplicative function of both arguments and is also multiplicative in r.

Proof. If $r=r_1r_2$, $n=n_1n_2$ and $(r_1n_1, r_2n_2)=1$, then any divisor d of r for which $(d, n_1n_2)=1$ can be uniquely written as $d=d_1d_2$ where

$$d_1\mid r_1,\ d_2\mid r_2,\ (d_1, n_1) = (d_2, n_2) = 1. \tag{4}$$

Conversely, whenever d_1 and d_2 satisfy (4), $d_1d_2\mid r$ and $(d_1d_2, n)=1$. Thus

$$h(n, r) = \sum_{\substack{d_1\mid r_1\\(d_1,n_1)=1}} f(d_1)g(r_1/d_1)\cdot \sum_{\substack{d_2\mid r_2\\(d_2,n_2)=1}} f(d_2)g(r_2/d_2) = h(n_1, r_1)h(n_2, r_2),$$

showing that $h(n, r)$ is multiplicative in its two arguments.

A similar proof can be given to show that $h(n, r)$ is multiplicative in r.

We note that $h(n, r)$ need not be multiplicative in n.

3. THEOREM. *Let $f(r)$ be a multiplicative function such that $f(p^a)=f(p^{a+1})$ for all $a\geqq 1$ and every prime p. Let $h(r)$ be any multiplicative function for which $h(p)=f(p)-1$ for every prime p. Then for $r\geqq 1$, $n\geqq 1$,*

$$\sum_{\substack{d\mid r\\(d,n)=1}} f(d)\mu(r/d) = \mu(r)\mu(r/t)h(r/t), \tag{5}$$

where $t=(n, r)$.

Proof. It follows from the lemma that the left side of (5) is a multiplicative function of both the arguments r and n; also it is easily verified that the same result holds for the right side of (5). Thus the theorem follows if it is shown that (5) holds for $r=p^a$, $n=p^b$ for every prime p and all nonnegative integers a and b. The result is trivially true for $a=b=0$. Denoting the left and right sides of (5) by $F(n, r)$ and $H(n, r)$ respectively, we have for the case $b>0$,

$$F(p^b, p^a) = \sum_{\substack{0 \leq k \leq a \\ \min(k,b)=0}} f(p^k)\mu(p^{a-k}) = f(1)\mu(p^a) = \mu(p^a).$$

If $b=0$ then

$$F(p^b, p^a) = \sum_{d|p^a} f(d)\mu(p^a/d); \tag{6}$$

for $a=1$ this becomes $\mu(1)f(p)+\mu(p)f(1)=f(p)-1$, while for $a \geqq 2$ (6) gives $\mu(1)f(p^a)+\mu(p)f(p^{a-1})=f(p^a)-f(p^{a-1})=0$. Thus

$$F(p^b, p^a) = \begin{cases} 1, & a = b = 0 \\ f(p) - 1, & a = 1, b = 0 \\ 0, & a \geqq 2, b = 0 \\ \mu(p^a) & b > 0. \end{cases} \tag{7}$$

Similarly, it can be verified directly that the function $H(p^b, p^a) = \mu(r)\mu(r/t)h(r/t)$ has the same values as on the right side of (7) for different values of a and b. This completes the proof of the theorem.

REMARK. In the statement of the theorem, for a given $f(r)$, we can take for $h(r)$ any multiplicative function subject to the only restriction $h(p)=f(p)-1$ for every prime p. Thus the values of $h(p^a)$ for $a>1$ can be arbitrarily chosen and therefore $h(r)$ can be chosen in an infinite number of ways.

4. Applications. (1) Let $f(r)=1$ for all r and $h(r)=[1/r]=1$ or 0 according as $r=1$ or >1. The theorem gives

$$\sum_{\substack{d|r \\ (d,n)=1}} \mu(r/d) = [1/r].$$

This result, of course, can easily be seen otherwise. (2) Let $f(r)=2^{w(r)}$, where $w(r)$ is the number of distinct prime divisors of r. Then $f(p^k)=2$ for all $k>0$. We should then have $h(p)=1$ and thus we can define either $h(r)=1$ for all r, or $h(r)=|\mu(r)|$. The corresponding identities will be

$$\sum_{\substack{d|r \\ (d,n)=1}} 2^{w(d)}\mu(r/d) = \mu(r)\mu(r/t) = \mu(r)\mu(r/t)\,|\mu(r/t)|\,.$$

(3) Taking $f(r)=r/\phi(r)$ so that $f(p^k)=p/(p-1)$, for all $k>0$, we can choose $h(r)=1/\phi(r)$ since $h(p)=1/(p-1)=f(p)-1$. This gives the Brauer-Rademacher identity.

We can, in fact, get a generalization of this identity by setting

$$f(r) = (r/\phi(r))^a,$$

where a is any real number. In particular, for $a=-1$ we should have $h(p) = -1/p$ so that we can choose, for example,

$$h(r) = \mu(r)/r \quad \text{or} \quad h(r) = \lambda(r)/r;$$

here $\lambda(r)$ is Liouville's function given by $\lambda(r) = (-1)^{\alpha(r)}$, where $\alpha(r)$ denotes the number of prime divisors of r, the multiplicity of each such divisor being taken into account. Thus we obtain

$$\sum_{\substack{d \mid r \\ (d,n)=1}} \phi(d)\mu(r/d)/d = \mu(r)t/r = t\mu(r)\mu(r/t)\lambda(r/t)/r.$$

(4) Let $\phi_k(r)$ denote Jordan's totient function representing the number of ordered sets $\{a_1, \cdots, a_k\}$, each a_i being chosen from a complete residue system mod r in such a way that the g.c.d. of $r, a_1, \cdots, a_k$ is unity.

Choosing $f(r) = \phi_k(r)/r^k$ and $f(r) = r^k/\phi_k(r)$ we can take either $h(r) = \mu(r)/r^k$ or $h(r) = \lambda(r)/r^k$ for the first choice of $f(r)$ and $h(r) = 1/\phi_k(r)$ for the second. In the latter case we get

$$\sum_{\substack{d \mid r \\ (d,n)=1}} d^k\mu(r/d)/\phi_k(d) = \mu(r)\mu(r/t)/\phi_k(r/t).$$

This generalization of the Brauer-Rademacher identity has been obtained by Cohen and is given, in a slightly different form, in formula (3.4) of his paper [5].

Research done under NSF Grant GP1222.

References

1. A. Brauer and H. Rademacher, Aufgabe 31, Jber. Deutsch. Math. Verein., 35 (1926) 94–95 (supplement).

2. Eckford Cohen, The Brauer-Rademacher identity, this MONTHLY, 67 (1960) 30–33.

3. ———, Arithmetical inversion formulas, Canadian J. of Math., 12 (1960) 399–409.

4. ———, Representations of even functions (mod r): Cauchy products, Duke Math. J., 26 (1959) 165–182.

5. ———, A trigonometric sum, The Mathematics Student, 28 (1960) 29–32.

6. P. J. McCarthy, Some remarks on arithmetical identities, this MONTHLY, 67 (1960) 539–548.

MATHEMATICAL SWIFTIES

"As $x \to 0$, the function $(1/x)(\sin 1/x)$ oscillates," Tom said wildly.

"The function takes on only the values 0 and 1," said Tom with characteristic modesty.

"Rearrangement of this series does not affect its convergence," Tom said unconditionally.

"It's easy to construct the reals, given the rationals," Tom said cuttingly.

PRIME NUMBERS IN ARITHMETIC PROGRESSIONS WITH DIFFERENCE 24

PAUL T. BATEMAN AND MARC E. LOW, University of Illinois

Dirichlet proved that if l and k are coprime positive integers, then there are infinitely many prime numbers p such that

$$p \equiv l \pmod{k}.$$

For proofs of this general result see [2]–[7], and [9]–[12].

In this note we give a simple proof of Dirichlet's Theorem for the special case $k=24$, in which case l can be assumed to be one of the numbers 1, 5, 7, 11, 13, 17, 19, 23. The proof which we give is of the same general nature as Euclid's familiar proof that there are infinitely many primes in toto. Namely, for each value of l we consider a certain polynomial in P, where P is the product of some finite set S of primes greater than 24, and infer from the divisibility properties of the polynomial that S cannot possibly include all primes greater than 24 which are congruent to l modulo k. For general k a proof of this kind can be given only when

$$l^2 \equiv 1 \pmod{k}. \tag{1}$$

Since the only values of k for which (1) holds for *every* l relatively prime to k are $k=1, 2, 3, 4, 6, 8, 12$, and 24, the modulus 24 is the largest which can be settled completely by the present method. Proofs of the present kind occur in the literature for $k=3, 4, 6, 8, 12$, for example, on pages 28–29 of [1]. As far as we are aware, however, no one has bothered to give a complete proof of the present type for $k=24$ (although the values $l=7, 13, 17, 19$ are treated on pages 51, 45, 47, 51 respectively of [1]).

Since $k=24$ is the natural boundary of the present method, it seems worthwhile to give a systematic and complete account of this case. The proof uses nothing beyond the simplest cases of the quadratic reciprocity law and so is readily accessible to students in a course in elementary number theory.

Let us make some more specific remarks about the nature of the proof, particularly for those familiar with the elementary theory of algebraic number fields. For any k the case $l=1$ of Dirichlet's Theorem can be established by using the kth cyclotomic polynomial $\Phi_k(x)$, the polynomial with leading coefficient 1 whose zeros are the primitive kth roots of unity. In fact, aside from the primes dividing k, it can be shown that any prime dividing $\Phi_k(n)$ for some integer n must be congruent to 1 modulo k. If $l^2 \equiv 1 \pmod{k}$ but $l \not\equiv 1 \pmod{k}$, there exists a polynomial with integral coefficients $f_{l,k}(x)$ such that, aside from a finite number of possible exceptional primes, any prime dividing $f_{l,k}(n)$ for some

integer n must be congruent to either 1 or l modulo k. Further elementary arguments with congruences will generally show that primes congruent to l modulo k occur infinitely often. Actually $f_{l,k}(x)$ is simply any polynomial of degree $\frac{1}{2}\phi(k)$ whose zeros generate that subfield of the field of the kth roots of unity which is invariant under the automorphism $\zeta \to \zeta^l$, where ζ is a primitive kth root of unity. For a more detailed discussion see [**8**].

We now proceed with our specific argument for $k=24$. We shall not assume the general results quoted in the preceding paragraph. In fact the following lemma about the Legendre symbol is all we need.

LEMMA 1. *Suppose that p is a prime greater than* 3. *Then*

$$\begin{aligned}
(-1\,|\,p) &= 1 \text{ if and only if } p \equiv 1 \pmod 4,\\
(2\,|\,p) &= 1 \text{ if and only if } p \equiv 1,\,7 \pmod 8,\\
(-2\,|\,p) &= 1 \text{ if and only if } p \equiv 1,\,3 \pmod 8,\\
(3\,|\,p) &= 1 \text{ if and only if } p \equiv 1,\,11 \pmod{12},\\
(-3\,|\,p) &= 1 \text{ if and only if } p \equiv 1 \pmod 6,\\
(6\,|\,p) &= 1 \text{ if and only if } p \equiv 1,\,5,\,19,\,23 \pmod{24},\\
(-6\,|\,p) &= 1 \text{ if and only if } p \equiv 1,\,5,\,7,\,11 \pmod{24}.
\end{aligned}$$

Proof. These assertions follow directly from the quadratic reciprocity law and its two supplements. Alternatively they can be deduced directly from Gauss's lemma to the quadratic reciprocity law.

LEMMA 2. *If n is any integer, then any prime factor of n^8-n^4+1 is congruent to* 1 *modulo* 24.

Proof. If n has one of the values -1, 0, or 1, then n^8-n^4+1 takes the value 1 and so there are no prime factors at all. So suppose that $|n|>1$ and p is a prime such that

$$n^8 - n^4 + 1 \equiv 0 \pmod p.$$

Clearly $p>3$. Since

$$n^2(n^3+n)(n^3-n) \equiv n^8 - n^4 \equiv -1 \pmod p,$$

we must have $p\nmid n^2$, $p\nmid(n^3+n)$, and $p\nmid(n^3-n)$. Thus there exist integers a, b, and c such that

$$an^2 \equiv 1 \pmod p,\quad b(n^3+n) \equiv 1 \pmod p,\quad c(n^3-n) \equiv 1 \pmod p.$$

In view of the identity

$$x^8 - x^4 + 1 = (x^4-1)^2 + (x^2)^2,$$

we have

$$(an^4-a)^2 + 1 \equiv (an^4-a)^2 + (an^2)^2 \equiv a^2(n^8-n^4+1) \equiv 0 \pmod p.$$

Similarly from the identities

$$\begin{aligned} x^8 - x^4 + 1 &= (x^4 + x^2 + 1)^2 - 2(x^3 + x)^2 = (x^4 - x^2 + 1)^2 + 2(x^3 - x)^2 \\ &= (x^4 + 1)^2 \qquad\quad - 3(x^2)^2 \quad\; = (x^4 - \tfrac{1}{2})^2 \qquad + 3(\tfrac{1}{2})^2 \\ &= (x^4 + 3x^2 + 1)^2 - 6(x^3 + x)^2 = (x^4 - 3x^2 + 1)^2 + 6(x^3 - x)^2, \end{aligned}$$

we readily infer that

$$\begin{aligned} (bn^4 + bn^2 + b)^2 - 2 &\equiv 0 \pmod{p}, & (cn^4 - cn^2 + c)^2 + 2 &\equiv 0 \pmod{p}, \\ (an^4 + a)^2 - 3 &\equiv 0 \pmod{p}, & (2n^4 - 1)^2 + 3 &\equiv 0 \pmod{p}, \\ (bn^4 + 3bn^2 + b)^2 - 6 &\equiv 0 \pmod{p}, & (cn^4 - 3cn^2 + c)^2 + 6 &\equiv 0 \pmod{p}. \end{aligned}$$

Thus

$$\left(\frac{-1}{p}\right) = \left(\frac{2}{p}\right) = \left(\frac{-2}{p}\right) = \left(\frac{3}{p}\right) = \left(\frac{-3}{p}\right) = \left(\frac{6}{p}\right) = \left(\frac{-6}{p}\right) = 1.$$

Hence by Lemma 1 we have $p \equiv 1 \pmod{24}$.

The roots of the polynomial $x^8 - x^4 + 1$ are the primitive 24th roots of unity. The identities used in the preceding proof show that the field of the 24th roots of unity contains the seven quadratic fields generated by the square roots of the numbers -1, ± 2, ± 3, ± 6. However, we do not use this fact as such.

In what follows we shall use the following seven polynomials:

$$\begin{aligned} f_5(x) &= x^4 \qquad\quad + 9 = (x^2)^2 \qquad + 3^2 \;= (x^2 + 3)^2 - 6x^2 &&= (x^2 - 3)^2 + 6x^2 \\ f_7(x) &= x^4 + 2x^2 + 4 = (x^2 + 2)^2 - 2x^2 = (x^2 + 1)^2 + 3 &&= (x^2 - 2)^2 + 6x^2 \\ f_{11}(x) &= x^4 + 4x^2 + 1 = (x^2 + 1)^2 + 2x^2 = (x^2 + 2)^2 - 3 &&= (x^2 - 1)^2 + 6x^2 \\ f_{13}(x) &= x^4 - x^2 + 1 = (x^2 - 1)^2 + x^2 = (x^2 - \tfrac{1}{2})^2 + 3(\tfrac{1}{2})^2 &&= (x^2 + 1)^2 - 3x^2 \\ f_{17}(x) &= x^4 \qquad\quad + 1 = (x^2)^2 \qquad + 1 \;= (x^2 + 1)^2 - 2x^2 &&= (x^2 - 1)^2 + 2x^2 \\ f_{19}(x) &= x^4 - 2x^2 + 4 = (x^2 - 2)^2 + 2x^2 = (x^2 - 1)^2 + 3 &&= (x^2 + 2)^2 - 6x^2 \\ f_{23}(x) &= x^4 - 4x^2 + 1 = (x^2 - 1)^2 - 2x^2 = (x^2 - 2)^2 - 3 &&= (x^2 + 1)^2 - 6x^2 \end{aligned}$$

Those interested can easily verify that the zeros of these polynomials generate the seven quartic subfields of the field of the 24th roots of unity.

LEMMA 3. *Suppose l is one of the numbers* 5, 7, 11, 13, 17, 19, 23. *Suppose n is an integer and p is a prime greater than* 3. *Then if $f_l(n) \equiv 0 \pmod{p}$, it follows that $p \equiv 1, l \pmod{24}$.*

Proof. The assertion is a consequence of Lemma 1 and the above identities for $f_l(x)$. Let us illustrate the argument with the case $l = 11$. If $f_{11}(n) \equiv 0 \pmod{p}$, where $p > 3$, then

$$(n^2 + 1) + 2n^2 \equiv (n^2 + 2)^2 - 3 \equiv (n^2 - 1)^2 + 6n^2 \equiv 0 \pmod{p}.$$

Since n cannot be divisible by p, we may conclude as in the proof of Lemma 2 that

$$\left(\frac{-2}{p}\right) = \left(\frac{3}{p}\right) = \left(\frac{-6}{p}\right) = 1.$$

By Lemma 1 this implies that p is congruent to either 1 or 11 modulo 24. A similar argument can be given for each of the other values of l.

Now let us introduce the polynomials:

$$\begin{aligned}
g_5(x) &= \tfrac{1}{2}f_5(12x+1) = 24(432x^4 + 144x^3 + 18x^2 + x) + 5\\
g_7(x) &= f_7(6x+1) = 24(54x^4 + 36x^3 + 12x^2 + 2x) + 7\\
g_{11}(x) &= \tfrac{1}{3}f_{11}(6x+2) = 24(18x^4 + 24x^3 + 14x^2 + 4x) + 11\\
g_{13}(x) &= f_{13}(12x+2) = 24(864x^4 + 576x^3 + 138x^2 + 14x) + 13\\
g_{17}(x) &= f_{17}(6x+2) = 24(54x^4 + 72x^3 + 36x^2 + 8x) + 17\\
g_{19}(x) &= \tfrac{1}{12}f_{19}(12x+4) = 24(72x^4 + 96x^3 + 47x^2 + 10x) + 19\\
g_{23}(x) &= \tfrac{1}{2}f_{23}(12x+3) = 24(432x^4 + 432x^3 + 150x^2 + 21x) + 23
\end{aligned}$$

The actual values of the coefficients here are not really required, except that we need to know the values of the constant terms and the fact that the other coefficients are integers divisible by 24.

LEMMA 4. *Suppose l is one of the numbers* 5, 7, 11, 13, 17, 19, 23 *and n is any integer. Then $g_l(n)$ has at least one prime factor congruent to l modulo* 24.

Proof. From the definition of $g_l(x)$ we see that $g_l(n)$ is a positive integer congruent to l modulo 24. In particular $g_l(n)$ cannot be divisible by either 2 or 3. Therefore by Lemma 3 and the definition of $g_l(x)$ we conclude that all prime factors of $g_l(n)$ are congruent to either 1 or l modulo 24. Since $g_l(n) \equiv l \pmod{24}$, it would be impossible for all the prime factors of $g_l(n)$ to be congruent to 1 modulo 24. Thus the assertion of the lemma follows.

THEOREM. *If l is an integer relatively prime to* 24, *then there are infinitely many primes p such that $p \equiv l$* (mod 24).

Proof. We may suppose that l is one of the numbers 1, 5, 7, 11, 13, 17, 19, 23. Let P be the product of some finite set S of primes each greater than 24. In view of Lemma 1, each prime factor of $P^8 - P^4 + 1$ must be congruent to 1 modulo 24 but different from any of the primes in S. Thus S cannot include all primes congruent to 1 modulo 24. (We may assume S to be nonempty, so that $P > 1$.)

If l is one of the numbers 5, 7, 11, 13, 17, 19, 23, choose a positive integer c_l such that

$$g_l(c_l) \not\equiv 0 \pmod{l}.$$

Since l is a prime and the leading coefficient of $g_l(x)$ is not divisible by l, the existence of c_l follows from the theorem that an algebraic congruence in one unknown modulo a prime cannot have more roots than its degree. However, we can choose c_l numerically, for example, as follows:

$$c_5 = 2, \quad c_7 = 1, \quad c_{11} = 1, \quad c_{13} = 1, \quad c_{17} = 2, \quad c_{19} = 1, \quad c_{23} = 2.$$

Now consider the positive integer $g_l(c_lP^{l-1})$. By Lemma 4 we know that $g_l(c_lP^{l-1})$ has at least one prime factor congruent to l modulo 24. Since the constant term of $g_l(x)$ is l and $(l, P) = 1$, we see that $g_l(c_lP^{l-1})$ is not divisible by any of the primes in S. By Fermat's Theorem

$$g_l(c_lP^{l-1}) \equiv g_l(c_l) \not\equiv 0 \pmod{l}$$

and so $g_l(c_lP^{l-1})$ is not divisible by l. Thus $g_l(c_lP^{l-1})$ has a prime factor congruent to l modulo 24 which is different from l and different from any prime in S. Hence S cannot include all those primes greater than 24 which are congruent to l modulo 24.

Since S can be taken to include any preassigned finite set of primes greater than 24, our theorem is established.

This work was supported (in part) by the Office of Naval Research.

References

1. A. S. Bang, Elementaere Beviser for specielle Tilfaelde af Dirichlets Saetning om Differensraekker, Doctoral dissertation, H. Chr. Bakkes Boghandel, Copenhagen, 1937.

2. L. E. Dickson, Modern elementary theory of numbers, University of Chicago Press, 1939, pp. 269–305.

3. H. Hasse, Vorlesungen über Zahlentheorie, Springer-Verlag, Berlin, 1950, pp. 167–269.

4. E. Hecke, Vorlesungen über die Theorie der algebraischen Zahlen, Akademische Verlagsgesellschaft, Leipzig, 1923, pp. 155–172.

5. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Teubner, Leipzig, 1909, pp. 391–435.

6. E. Landau, Elementary number theory, Chelsea, New York, 1958 (translation of the first half of the first volume of Vorlesungen über Zahlentheorie, Hirzel, Leipzig, 1927), pp. 104–125.

7. W. J. LeVeque, Topics in number theory, vol. II, Addison-Wesley, Reading, Mass., 1956, pp. 201–228.

8. I. Schur, Über die Existenz unendlich vieler Primzahlen in einigen speziellen arithmetischen Progressionen, S-B Berlin. Math. Ges., 11 (1912) 40–50, appendix to Archiv der Math. und Phys. (3), vol. 20 (1912–1913).

9. A. Selberg, An elementary proof of Dirichlet's theorem about primes in an arithmetic progression, Ann. of Math. (2), 50 (1949) 297–304.

10. H. N. Shapiro, On primes in arithmetic progressions, II, Ann. of Math. (2), 52 (1950) 231–243.

11. E. Trost, Primzahlen, Birkhäuser, Basel, 1953, pp. 73–78.

12. H. Zassenhaus, Über die Existenz von Primzahlen in arithmetische Progressionen, Comm. Math. Helv., 22 (1949) 232–259.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146.

ON MATRICES OF COFACTORS

E. W. WALLACE, University of Leeds, England

1. Introduction. Two questions concerning matrices of cofactors, raised by W. R. Utz in [2], were recently answered by H. Schwerdtfeger in [1]. Extensions of these results are given in this note.

2. Definitions. Basic results. Unless otherwise stated, all matrices will be real and of order $n>2$. If M is such a matrix, we denote its matrix of cofactors by M^*, its transpose by M', its determinant by $|M|$, and the identity matrix of order n by I_n.

The following results are well known, but are listed here for ease of reference.

(1) $M'M^*=M^*M'=|M|I_n$

(2) $|M^*|=|M|^{n-1}$

(3) $M^{**}=(M^*)^*=|M|^{n-2}M$

(4) $(MN)^*=M^*N^*$

(5) $(M^*)'=(M')^*$

(6) $(-M)^*=(-1)^{n-1}M^*$

(7) M has rank n; rank $n-1$; rank $<n-1$; if and only if M^* has rank n; rank 1; rank 0.

We shall be concerned mainly with finding solutions to equations of the form $A^*=B$, where B is a given matrix.

To avoid repetition, we note here that any matrix A of rank $<n-1$ is a solution of the equation $A^*=(0)$.

For proofs of the following two theorems see [1] and [2].

THEOREM 1. *If B is a singular matrix, the equation $A^*=B$ has a solution A if and only if the rank of B is* 0 *or* 1.

THEOREM 2. *If B is nonsingular, the equation $A^*=B$ has* (i) *the unique solution $A=|B|^{1/(n-1)}(B')^{-1}$ if n is even*; (ii) *no solution if n is odd and $|B|<0$*; (iii) *exactly two solutions $A=\pm|B|^{1/(n-1)}(B')^{-1}$ if n is odd and $|B|>0$.*

3. The equation $A^*=kA$. Here k is any fixed nonzero real number. By taking $k=1$, results of [1] are recovered.

If A is singular then $(0)=A^{**}=(kA)^*=k^{n-1}A^*=k^nA$. Hence,

THEOREM 3. *If A is singular, and $k\neq 0$, then $A^*=kA$ if and only if $A=(0)$.*

If $|A| \neq 0$, then $|A^*| = |A|^{n-1} = k^n|A|$, so that $|A|^{n-2} = k^n$. We also have $kA'A = |A| I_n$.

If $k>0$, and λ is the positive root x of the equation $x^{n-2}=k^n$, then $|A| = \lambda$. For, if not, then

$$A'A = -k^{2/(n-2)}I_n = -\mu I_n \qquad (\mu > 0),$$

and

$$\sum_{\beta=1}^{n} a_{\beta 1}a_{\beta 1} = -\mu, \quad \text{where} \quad A = (a_{\alpha\beta}).$$

It follows that $A'A = k^{2/(n-2)}I_n$. Similar methods apply when $k<0$.

Because of the following lemma, the proof of which is obvious, our results are contained in Theorem 4 (where $|k|$ will denote the absolute value of the real number k).

Lemma. *A matrix A satisfies the equation $A'A = \alpha^2 I_n$, where α is a nonzero real number, if and only if $\alpha^{-1}A$ is orthogonal.*

Theorem 4. *If A is non-singular and $k \neq 0$, then $A^* = kA$ if and only if $|k|^{-1/(n-2)}A$ is orthogonal, $|A| = k^{n/(n-2)}$, and $|A|$ and k have the same sign.*

Proof. The necessity of the conditions for $k>0$ has been proved.

If the conditions for $k>0$ are satisfied, then $A'A^* = |A| I_n = k^{n/(n-2)}I_n$, and

$$kA'A = k. \qquad k^{2/(n-2)}I_n = k^{n/(n-2)}I_n,$$

by the lemma. Hence $A^* = kA$.

A similar proof can be given when $k<0$.

4. Conditions for a symmetric solution. Conditions are now determined for the equation $A^* = B$ to have symmetric solutions A. (The skew symmetric case was considered in [1]). It is easily verified that B must be symmetric, and when B is nonsingular, Theorem 2 can be applied with $B' = B$.

If B is singular we can suppose that it has rank 1, and put $B = (\lambda_i x_j)$, where $x_t \neq 0$ for some t. Since $x_j = \lambda_j x_t/\lambda_t$, it follows that

$$B = \frac{x_t}{\lambda_t}(\lambda_i\lambda_j)$$

and the nonzero diagonal elements of B have the same sign as x_t/λ_t. If they are all positive, put

$$\mu_i = \lambda_i\left(\frac{x_t}{\lambda_t}\right)^{1/2},$$

so that $B = (\mu_i\mu_j)$. Assuming, for example, that $\mu_1 \neq 0$, we verify easily, using the cofactor-expansion of $|A|$, that the matrix

$$A = \frac{1}{\mu_1^{(n-3)/(n-1)}} \begin{bmatrix} c_{11} & -\mu_2 & -\mu_3 & \cdots & -\mu_n \\ -\mu_2 & \mu_1 & 0 & \cdots & 0 \\ -\mu_3 & 0 & \mu_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\mu_n & 0 & 0 & \cdots & \mu_1 \end{bmatrix}$$

where $c_{11} = (\sum_{i=2}^{n} \mu_i^2)/\mu_1$, is a solution. (This value of c_{11} ensures that $|A| = 0$.)

If B has negative nonzero diagonal elements, then $B = -(\nu_i \nu_j)$, where

$$\nu_i = \lambda_i \left(\frac{-x_t}{\lambda_t} \right)^{1/2}.$$

If $\nu_1 \neq 0$, then the matrix

$$A = \frac{1}{\nu_1^{(n-3)/(n-1)}} \begin{bmatrix} d_{11} & \nu_2 & -\nu_3 & -\nu_4 & \cdots & -\nu_n \\ \nu_2 & -\nu_1 & 0 & 0 & \cdots & 0 \\ -\nu_3 & 0 & \nu_1 & 0 & \cdots & 0 \\ -\nu_4 & 0 & 0 & \nu_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ -\nu_n & 0 & 0 & 0 & \cdots & \nu_1 \end{bmatrix}$$

is a solution, where $d_{11} = (\nu_3^2 + \nu_4^2 + \cdots + \nu_n^2 - \nu_2^2)/\nu_1$. Hence,

THEOREM 5. *If B is singular, the equation $A^* = B$ has a symmetric solution A if and only if either $B = (0)$, or B is symmetric of rank 1 with its nonzero digonal elements all positive or all negative.*

5. The equation $A^* = A^r$. As a second generalisation of the equation $A^* = A$, we now consider the equation $A^* = A^r$, where r is a fixed positive integer.

When A is singular,

$$0 = A^{**} = (A^r)^* = (A^*)^r = A^{r^2},$$

so that A is nilpotent and all its latent roots are zero. A real orthogonal matrix P exists, therefore, such that $T = P'AP$ is upper triangular with zeros on the main diagonal. From $A^* = A^r$ it follows that $P^*T^*(P^*)' = PT^rP'$, and hence $T^* = T^r$ since $P^* = \pm P$ (Theorem 4). But T^r is upper triangular and T^* is lower triangular, giving

$$T^* = T^r = (0) = A^* = A^r.$$

Moreover rank $A < n-1$.

It is obvious that if $A^r = (0)$ and rank $A < n-1$ then $A^* = A^r = (0)$. When $r = 2$, $A^2 = (0)$ implies that A has rank $< n-1$. For, in this case, choose $T = (t_{ij}) = P'AP$ to be upper triangular. Since $T^2 = (0)$,

$$\sum_{\alpha=1}^{n} t_{1\alpha}t_{\alpha 3} = 0 = t_{12}t_{23}$$

and so rank $T<n-1$. This implies that rank $A<n-1$. If $r>2$, $A^r=(0)$ does not imply that rank $A<n-1$, as can be seen by choosing $r=3$, and

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have proved

THEOREM 6. *If r is a fixed positive integer, and A is singular, $A^*=A^r$ if and only if $A^r=(0)$ and A has rank $<n-1$.*

When A is nonsingular, only the case $r=2$ will be studied in detail, the general problem still being open. For any $r>0$, however,

$$|A^*| = |A|^{n-1} = |A|^r,$$

and $A^{**}=(A^*)^r=A^{r^2}=|A|^{n-2}A$. Hence, $A^{r^2-1}=|A|^{n-2}I_n=|A|^{r-1}I_n$. Three cases can arise:

(i) $n-1=r$; $A^{r^2-1}=|A|^{r-1}I_n,\ A'A^r=|A|\,I_n$;

(ii) $n-1-r$ is even; $|A|=\pm 1,\ A^{r^2-1}=\pm I_n,\ A'A^r=\pm I_n$;

(iii) $n-1-r$ is odd; $|A|=1,\ A^{r^2-1}=I_n,\ A'A^r=I_n$.

It should be noted that, for example, if $n-1-r$ is odd, the conditions $|A|=1$, $A^{r^2-1}=I_n$ alone are not sufficient to ensure that $A^*=A^r$. This is seen by taking $n=r=2$,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

We conclude by proving the following

THEOREM 7. *If A is non-singular, then $A^*=A^2$ if and only if*
(i) *$A=I_n$, when $n-3$ is odd;*
(ii) *$A=\pm I_n$, when $n-3$ is even;*
(iii) *$A=|A|^{1/3}I_3$, when $n=3$.*

Proof. We have $A^3=|A|I_n=A'A^2$ and so A is symmetric. Its latent roots $\lambda_1, \cdots, \lambda_n$ are therefore all real.

(i) If $n-3$ is odd then $|A|=1$, $A^3=I_n$, and so $\lambda_i=1$ $(i=1, \cdots, n)$. Since a real symmetric matrix is orthogonally similar to a diagonal matrix, it follows that $A=I_n$.

(ii) If $n-3$ is even, $|A|=\pm 1$ and $A^3=\pm I_n$. If $A^3=I_n$ then $A=I_n$ as in (i). If $A^3=-I_n$, then $\lambda_i=-1$ $(i=1, \cdots, n)$ and so $A=-I_n$.

(iii) Finally, when $n=3$, $A^3=A'A^2=|A|I_3$. If A has (real) latent roots $\lambda_1, \lambda_2, \lambda_3$, then $\lambda_i^3=|A|$ and $\lambda_i=|A|^{1/3}$. There is a real orthogonal matrix P such that $P'AP=|A|^{1/3}I_3$, and hence $A=|A|^{1/3}I_3$.

References

1. H. Schwerdtfeger, On the adjugate of a matrix, Portugal. Math., 20 (1961) 39–41.
2. W. R. Utz, Some theorems and questions on matrices, Portugal. Math., 18 (1959) 225–229.

EQUIVALENCE OF THE ONE-TERM RITZ APPROXIMATION AND LINEARIZATION IN A CHEBYCHEV SENSE

A. C. SOUDACK, University of British Columbia

The behavior of nonlinear, single-degree-of-freedom systems is often described by the normalized system

$$E(x) = \ddot{x} + f(x) = 0, \qquad x(0) = 1, \ \dot{x}(0) = 0, \tag{1}$$

where $f(x)$ is monotonically increasing or decreasing, is defined over the normalized region $-1 \leqq x \leqq +1$, and exhibits zero-point symmetry.

A classical, and often-used method for determining an approximate solution, $\bar{x}$, to equation (1) was formulated by W. Ritz in [1], [2], [3], and [4]. The motivation for the Ritz method is a physical one, namely, Hamilton's Principle, as in [2].

Let us first consider the case in which $f(x)$ is monotonically increasing, i.e., in the 1st and 3rd quadrants of the $x, f(x)$-plane. If we introduce the function $V(x)$ defined by

$$V(x) = \int_0^x f(x)dx,$$

equation (1) can be rewritten as $\ddot{x}+V'(x)=0$.

Multiplying by $\dot{x}$ and integrating, we have

$$\tfrac{1}{2}\dot{x}^2 + V(x) = h$$

which is the familiar energy conservation law. If we now examine $V(x)$, the potential energy, and its first and second derivatives, we find that there is an energy minimum at the singularity $x=\dot{x}=0$. The singularity is therefore a center point and the solution will be oscillatory (see [5]).

Hence, for this case, consider the one-term approximation

$$\bar{x} = \cos \omega t. \tag{2}$$

Where $f(x)$ is monotonically decreasing, the above argument leads to a potential maximum at $x=\dot{x}=0$. The singularity is therefore a saddle point and the solution will be nonoscillatory (see [5]).

For this case, consider the one-term approximation

(2a) $$\bar{x} = \cosh \omega t.$$

Minimization of the energy integral using the assumed dependent variable leads to the equation

(3) $$\int_{-\pi}^{\pi} \bar{x}E(\bar{x})d(\omega t) = 0$$

from which ω is obtained.

It is the purpose of this paper to show that an entirely mathematical (rather than physical) approach, namely that of linearizing $f(x)$ in a Chebychev sense, leads to precisely the same results as those obtained using the one-term Ritz approximation.

Substituting (2) into (3), one obtains

(4) $$\int_{-\pi}^{\pi} \cos \omega t[-\omega^2 \cos \omega t + f(\cos \omega t)]d(\omega t) = 0.$$

Integrating the first part of (4), and noting that $f(\cos \omega t)$ is an even function of t, one obtains

(5) $$\omega^2 = \frac{2}{\pi}\int_{0}^{\pi} f(\cos \omega t) \cos \omega t \, d(\omega t).$$

Since $\cos \omega t$ is the solution to $\ddot{x}+\omega^2 x=0$, $x(0)=1$, $\dot{x}(0)=0$, and $\cosh \omega t$ is the solution to $\ddot{x}-\omega^2 x=0$, $x(0)=1$, $\dot{x}(0)=0$, equation (1), by application of the Ritz method, has been "linearized" to $\ddot{x}+\omega^2 x=0$, $x(0)=1$, $\dot{x}(0)=0$.

A purely mathematical approach might be to linearize (1) right from the start, i.e., replace $f(x)$ by $b_1 x$. Here we recall that $f(0)=0$, since $f(x)$ is symmetric with respect to the origin. Evidently, there are any number of straight lines that could be chosen. Let us consider that straight line derived from a truncated series of Chebychev polynomials.

In order to be expandable in a series of Chebychev polynomials, $f(x)$ must be "absolutely integrable" (see [6]). Since the $f(x)$ under consideration is of this nature, we are guaranteed that the expansion exists. Thus, where $T_i(x)$ is the ith order Chebychev polynomial, we have $f(x) = \sum_{i=1} b_i T_i(x)$. Due to the orthogonality of these polynomials, the coefficients, b_i, may be obtained from the relation

$$b_i = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_i(x)dx}{\sqrt{(1-x^2)}}\,.$$

Since we want a straight line approximation to $f(x)$, consider

$$b_1 = \frac{2}{\pi}\int_{-1}^{1} \frac{xf(x)dx}{\sqrt{(1-x^2)}},$$

since $b_1T_1(x) = b_1x$. Making the substitution $x = \cos \omega t$, one obtains

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(\cos \omega t) \cos \omega t \, d(\omega t). \tag{6}$$

Comparing equations (5) and (6), one obtains $b_1 = \omega^2$ and hence linearizing $f(x)$ in a Chebychev sense leads to the same results as those obtained by applying the Ritz method.

When $f(x)$ is a polynomial, the method of linearizing $f(x)$ involves less labor than the Ritz method does. The reason is simply that tables of coefficients of the Chebychev polynomials are available (see [7]), and hence the necessity of carrying out an integration is eliminated.

References

1. W. Ritz, Oeuvres complètes, Paris, 1911.
2. L. V. Kantorovich and V. I. Krylov, Approximate methods of higher analysis, Interscience Publishers, New York, 1958, pp. 258–303.
3. W. J. Cunningham, Introduction to nonlinear analysis, McGraw-Hill, New York, 1958, pp. 157–159.
4. K. Klotter, Nonlinear vibration problems treated by the averaging method of W. Ritz, Part I, Tech. Report No. 17, Division of Eng. Mech., Stanford University, 1951.
5. A. Andronow and S. Chaikin, Theory of oscillations. Translated from the Russian by S. Lefschetz. Princeton University Press, New Jersey, 1949, p. 64.
6. C. Lanczos, Applied analysis, Prentice-Hall, Englewood Cliffs, N. J., 1956, pp. 210, 451.
7. ———, *ibid.*, p. 515.

SOME INEQUALITIES INVOLVING ELLIPTIC FUNCTIONS

S. Fempl, University of Beograd, Yugoslavia

In [1], P. L. Duren noted that in the literature on elliptic functions there does not seem to be an adequate source of inequalities. He derived the following two:

$$\frac{\operatorname{sn} u}{(\operatorname{dn} u + \operatorname{cn} u)^2} \leqq \frac{u}{K(1 - k^2)}, \qquad 0 \leqq u \leqq K, \tag{1}$$

$$\frac{\operatorname{sn} u}{(dn\, u + k \operatorname{cn} u)^2} \geqq \frac{2u}{\pi(1 + k)^2}, \qquad 0 \leqq u \leqq K, \tag{2}$$

where K is

$$K = \int_0^{\pi/2} (1 - k^2 \sin^2 \theta)^{-1/2} d\theta.$$

On the present paper two new inequalities are derived which follow easily from (1) and (2). They may be employed to bound the function $\operatorname{dc} u \equiv \operatorname{dn} u / \operatorname{cn} u$ from both sides.

Dividing (1) by $(\operatorname{dn} u-\operatorname{cn} u)^2$, and keeping in mind that $\operatorname{dn}^2 u-\operatorname{cn}^2 u=k'^2 \operatorname{sn}^2 u(k'^2=1-k^2)$, $\operatorname{cn}^2 u+\operatorname{dn}^2 u=2-(1+k^2)\operatorname{sn}^2 u$ [2, 121.00], we get

$$\frac{2}{\operatorname{sn}^3 u}-\frac{1+k^2}{\operatorname{sn} u}-\frac{2\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn}^3 u} \leqq \frac{uk'^2}{K}.$$

Taking the integral between the limits (u, k), following [2, 311.03 and 311.01]

$$\int \frac{du}{\operatorname{sn}^3 u}=\frac{1}{2}\left[(1+k^2)\ln\frac{\operatorname{sn} u}{\operatorname{cn} u+\operatorname{dn} u}-\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn}^2 u}\right],$$

$$\int \frac{du}{\operatorname{sn} u}=\ln\frac{\operatorname{sn} u}{\operatorname{cn} u+\operatorname{dn} u},$$

$(\operatorname{sn} u)'=\operatorname{cn} u \operatorname{dn} u$ [2, 731.01] and because of $\operatorname{sn} K=1$, $\operatorname{cn} K=0$, $\operatorname{dn} K=k'$ [1, 122.02], we get

$$\frac{\operatorname{cn} u \operatorname{dn} u}{\operatorname{sn}^2 u}+1-\frac{1}{\operatorname{sn}^2 u} \leqq \frac{(K^2-u^2)k'^2}{2K}$$

i.e.,

$$\frac{\operatorname{cn} u \operatorname{dn} u-\operatorname{cn}^2 u}{\operatorname{sn}^2 u}=\frac{\operatorname{cn} u(\operatorname{dn} u-\operatorname{cn} u)}{\operatorname{sn}^2 u} \leqq \frac{(K^2-u^2)k'^2}{2K}.$$

Multiplying by $\operatorname{dn} u+\operatorname{cn} u$, we get

$$\frac{\operatorname{cn} u}{\operatorname{dn} u+\operatorname{cn} u} \leqq \frac{K^2-u^2}{2K}. \tag{3}$$

By a similar treatment of (2), using $\operatorname{dn}^2 u-k^2 \operatorname{cn}^2 u=k'^2$, we get

$$\frac{\operatorname{sn} u(\operatorname{dn}^2 u-2k \operatorname{cn} u \operatorname{dn} u+k^2 \operatorname{cn}^2 u)}{k'^4} \geqq \frac{2u}{\pi(1+k)^2},$$

from which it follows that

$$(1+k^2)\operatorname{sn} u-2k^2\operatorname{sn}^3 u-2k\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \geqq \frac{2u}{\pi}(1-k)^2.$$

Integrating between limits (u, k) and using

$$\int \operatorname{sn} u\, du=\frac{1}{k}\ln(\operatorname{dn} u-k\operatorname{cn} u),$$

$$\int \operatorname{sn}^3 u\, du=\frac{1}{2k^3}[k\operatorname{cn} u \operatorname{dn} u+(1+k^2)\ln(\operatorname{dn} u-k\operatorname{cn} u)],$$

[2, 310.01 and 310.03] we get the inequality

$$\operatorname{cn} u \operatorname{dn} u - k(1 - \operatorname{sn}^2 u) \geqq \frac{K^2 - u^2}{\pi}(1 - k)^2;$$

i.e.,

$$\operatorname{cn} u(\operatorname{dn} u - k \operatorname{cn} u) \geqq \frac{K^2 - u^2}{\pi}(1 - k)^2 \cdot$$

After multiplying by $\operatorname{dn} u + k \operatorname{cn} u$ we get

$$\frac{\operatorname{cn} u}{\operatorname{dn} u + k \operatorname{cn} u} \geqq \frac{1-k}{1+k} \frac{K^2 - u^2}{\pi}. \tag{4}$$

Taking reciprocal values of (3) and (4) we can get left and right bounds for $\operatorname{dn} u/\operatorname{cn} u$.

Let us make the following note. Exchanging u for $K-u$ in (3), because of $\operatorname{sn}(K-u) = \operatorname{cn} u/\operatorname{dn} u$, $\operatorname{cn}(K-u) = k' \operatorname{sn} u/\operatorname{dn} u$, $\operatorname{dn}(K-u) = k'/\operatorname{dn} u$; [2, 122.03], the inequality (3) is reduced to

$$\operatorname{sn} u \leqq \frac{u(2K - u)}{2K - u(2K - u)} \cdot$$

Together with inequality $\operatorname{sn} u \geqq u/K$ [1, (4)] we get the double inequality

$$\frac{1}{K} \leqq \frac{\operatorname{sn} u}{u} \leqq \frac{2K - u}{2K - u(2K - u)} \cdot$$

References

1. P. L. Duren, Two inequalities involving elliptic functions, this MONTHLY, 70 (1963) 650–651.

2. P. F. Byrd and M. D. Friedman, Handbook of elliptic integrals for engineers and physicists Springer-Verlag, Berlin, 1954.

BROWN'S METHOD OF EXTENDING FIXED POINT THEOREMS

J. D. McKNIGHT, JR., University of Miami

While the two theorems presented by A. B. Brown in [1] to extend the Brouwer fixed point theorem are interesting as observations, their proofs are even more interesting because, being not very deep, they may readily be generalized to extend a large class of fixed point theorems. The main tool of Brown's proofs is the absolute extensor property of retracts of cubes, the fixed point property of such retracts being used only in the form of the traditional Brouwer theorem. (Here a cube is simply a topological power I^C of the closed unit interval I, where C is any set.) Several geometrical insights into the theorems themselves have been reported (cf., e.g., [2]), but the essential ingredients of Brown's analytical proofs have not previously been displayed elaborately. We propose not only to display some of these by a sequence of simple propositions, but also to illustrate the technique by a larger class of concrete examples than that class

forming the theme of [1]. To this latter end, we use two theorems of the "well-known" type, which are easily proved from their famous special cases, and one lemma of the "classroom" type. We distinguish between *compact absolute retracts*, which are retracts of cubes, each being therefore a retract of any compact Hausdorff space in which it is embedded, and *absolute extensors for a class K* of spaces, each such extensor A being a space which is such that every continuous function from a closed subspace of any $X \in K$ to A has an extension to a continuous function from X to A. Thus, the compact Hausdorff absolute extensors for the compact Hausdorff spaces or the normal spaces are the compact absolute retracts according to the

WELL-KNOWN THEOREM 1 (Tietze). *Every continuous function from a closed subset of a normal space X to a compact absolute retract A has an extension to a continuous function from X to A.*

WELL-KNOWN THEOREM 2 (Brouwer). *Every continuous function from a compact absolute retract to itself has a fixed point.*

CLASSROOM LEMMA. *A completely regular (Hausdorff) space A is an absolute extensor for compact Hausdorff spaces iff there is an embedding e of A into a compact absolute retract R so that for every compact subset of $e(A)$ there is a map of R into $e(A)$ leaving points of the compact subset fixed.*

Note. An absolute extensor for a class containing X is an absolute extensor for the possibly larger class containing also the closed subspaces of X.

We shall deal with maps of pairs of spaces. We write $f: (S, A) \to (T, B)$ if f is a continuous function from S to T and $f(A) \subset B$, where $A \subset S$ and $B \subset T$. We say that the pair (X, S) is *extensibly coupled to* (T, B) if every map $f: (S, bdyS) \to (T, B)$ has a continuous extension $g: X \to T$ for which $g: (X, X \setminus S) \to (T, B)$.

BROWN LEMMA 1. *If S is a closed subspace of a space X, if X is of class K, and if B is an absolute extensor for class K contained as a subspace in the space T, then (X, S) is extensibly coupled to (T, B).*

Proof. If $f: (S, bdyS) \to (T, B)$, then the restriction of f to $bdyS$ has an extension g that is a function from the closure of $X \setminus S$ to B. The domains of f and g intersect in the set $bdyS$, on which f and g coincide. Therefore, $f \cup g$ is an extension of f, and $f \cup g: (X, X \setminus S) \to (T, B)$.

COROLLARY. *If S is a closed subspace of a normal space X and B is a compact absolute retract that is a subspace of a space T, then (X, S) is extensibly coupled to (T, B).*

Proof. According to Well-Known Theorem 1, this is a special case of Brown Lemma 1.

COROLLARY. *If S is a closed subspace of a compact Hausdorff space X and if T is a space having a subspace B with the property that every compact subset of B is*

contained in a subset of B that is homeomorphic to an absolute retract, then (X, S) is extensibly coupled to (T, B).

Proof. According to the Classroom Lemma, this follows from Brown Lemma 1.

As usual, a space X has the *fixed point property* if every continuous function from X to itself has a fixed point. Thus, Well-Known Theorem 2 states that absolute retracts have the fixed point property.

BROWN LEMMA 2. *If (X, A) is extensibly coupled to (X, B), if $B \subset A$, and if X has the fixed point property, then every map from $(A, bdyA)$ to (X, B) has a fixed point.*

Proof. If f: $(A, bdyA) \to (X, B)$, then f has an extension over X to a function g for which $g(X \backslash A) \subset B \subset A$. The fixed point of g is therefore not in $X \backslash A$. Since g coincides with f on A, the fixed point of g must be a fixed point of f.

BROWN THEOREM. *If the subspace B of the space X is an absolute extensor for any class containing X and if X has the fixed point property, then, for every closed set A of X containing B, each map from $(A, bdyA)$ to (X, B) has a fixed point.*

Proof. According to Brown Lemma 1, (X, A) is extensibly coupled to (X, B). Therefore, Brown Lemma 2 applies.

COROLLARY. *If the subspace B of the normal space X is an absolute extensor for normal spaces, if A is a closed subset of X containing B, and if the map f: $(A, bdyA) \to (X, B)$ has the property that $A \cup f(A)$ is contained in a closed subspace of X with the fixed point property, then f has a fixed point.*

COROLLARY. *If the compact absolute retract B is a subspace of the normal space X, if A is a closed subset of X containing B, and if the map f: $(A, bdyA) \to (X, B)$ has the property that $A \cup f(A)$ is contained in a closed subset of X with the fixed point property (e.g., an absolute retract), then f has a fixed point.*

COROLLARY. *If the subspace B of the compact Hausdorff space X is an absolute extensor for compact Hausdorff spaces, if A is a closed subset of X containing B, and if the map f: $(A, bdyA) \to (X, B)$ has the property that $A \cup f(A)$ is contained in a closed subset of X with the fixed point property, then f has a fixed point.*

ABIAN-BROWN THEOREM. *If B is a subspace of the completely regular space X, if B is an absolute extensor for the compact Hausdorff spaces, and if A is a compact subset of X containing B, then any map from $(A, bdyA)$ to (X, B) has a fixed point.*

Proof. We may assume that X is a subspace of a cube I^C, so that B is also. The compact set A is then closed in I^C and a map f: $(A, bdyA) \to (X, B)$ is a map f: $(A, bdyA) \to (I^C, B)$. Since I^C has the fixed point property, the last corollary applies.

The Classroom Lemma may be applied by replacing the condition that B be an absolute extensor for compact Hausdorff spaces by the conditions which

are equivalent to that one according to the lemma. More specially:

COROLLARY. *If B is a subspace of the completely regular space X, if every compact subset of B is contained in a subspace of B that is homeomorphic to an absolute retract, and if A is a compact subset of X containing B, then any map from $(A, bdyA)$ to (X, B) has a fixed point.*

The author acknowledges the sponsorship of the Air Force Office of Scientific Research of the Office of Aerospace Research under Contract No. AF49(638)-1215 with the General Electric Computer Laboratory.

References

1. A. B. Brown, Extensions of the Brouwer fixed point theorem, this MONTHLY, 69 (1962) 643.
2. P. H. Doyle, A proof of the Abian-Brown fixed point theorem, Michigan Math. J., 8 (1961) 209–210.

THE EXTREMA OF A CERTAIN TYPE OF CONNECTIVITY FOR GRAPHS

LOUIS V. QUINTAS, St. John's University, New York

1. Introduction. Let G denote a linear graph which is not necessarily connected and such that the open edges of G are pairwise disjoint. Let D denote a subset of G no two points of which come from the same closed edge of G and let $\delta(G)$ denote the least number of points D can contain such that $G-D$ is disconnected.

We consider the problem of finding the extreme values of $\delta(G)$ when G ranges over all graphs of the above type having p vertices and q edges.

In this note we show that the type of connectivity involved is intermediate to the concepts of vertex connectivity and edge connectivity for linear graphs. This observation combined with some recent results of F. Harary [1] resolves the above problem, the solution of which is contained in the following theorem.

THEOREM. *If G is a graph having p vertices and q edges, then*

$$\operatorname{Max}\{0, q-(p-1)(p-2)/2\} \leqq \delta(G) \leqq \begin{cases} 0 & \text{if } q < p-1 \\ [2q/p] & \text{if } q \geqq p-1. \end{cases}$$

Furthermore, for each pair (p, q) the indicated extreme values for δ are realized by some graph.

2. Proof of the theorem. Let $\kappa(H)$ and $\lambda(H)$ denote the vertex and edge connectivity, respectively, of a graph H (for definitions cf. [1], where $\kappa(H)$ and $\lambda(H)$ are called the connectivity and line connectivity of H respectively).

LEMMA. *If a graph G has vertex connectivity κ and edge connectivity λ, then $\kappa \leqq \delta(G) \leqq \lambda$.*

Proof. Let D denote a subset of G as described in Section 1. Then, D contains α vertices of G and β points of G each of which is contained in a distinct open edge of G which is not incident to any of the vertices in D. Let A denote the

union of the open stars of those vertices of G which are contained in D and let B denote the union of those open edges of G which contain points of D. Then, $G-D$ is connected if and only if the subgraph $G-(A\cup B)$ of G is connected.

Since $G-A$ is a subgraph of G, we have

$$\kappa - \alpha \leqq \kappa(G - A). \tag{2.1}$$

For any graph, the vertex connectivity cannot exceed the edge connectivity (cf. [2] p. 159). This together with (2.1) implies $\kappa-\alpha\leqq\lambda(G-A)$.

Thus, if $\beta<\kappa-\alpha$, then $G-(A\cup B)$ is connected. Therefore, if D is a disconnecting set we must have $\kappa\leqq\alpha+\beta$. This implies

$$\kappa \leqq \delta(G). \tag{2.2}$$

Since G can be disconnected by the removal of λ open edges, we have

$$\delta(G) \leqq \lambda. \tag{2.3}$$

Combining (2.2) and (2.3) completes the proof of the lemma.

F. Harary [1] has shown that among all graphs having p vertices and q edges (i) the maximum edge and vertex connectivities are equal to 0 if $q<p-1$ and $[2q/p]$ if $q\geqq p-1$ and (ii) the minimum edge and vertex connectivities are equal to $q-(p-1)(p-2)/2$ or 0 whichever is larger. The proof of the theorem of this note now follows directly from these facts and the above lemma.

3. Remark and a problem. For the case $q=p(p-1)/2$ our theorem yields the solution of a specialization of a problem which appeared in the June-July 1963 issue of this MONTHLY [3]. In that problem the configuration of points and line segments considered could be thought of as a complete graph with a restricted type of self-intersection permitted. The specialization we refer to is the requirement that the open line segments be pairwise disjoint, our definition of disconnecting set being the same. These remarks suggest considering the problem of this note for a class of graphs which includes graphs having the above indicated type of self-intersection. Specifically, let S denote a subset of Euclidean space which consists precisely of the points belonging to a (rectilinear) graph G having q edges and p vertices no r of which are collinear. Let D denote a subset of S no two points of which come from the same closed edge of G and let $\Delta(S)$ denote the least number of points D can contain such that $S-D$ is disconnected.

Problem. For each pair (p, q) and each $r(r=3, 4, \cdots, p+1)$, what are the extreme values of $\Delta(S)$ as S ranges over all subsets of the above type?

References

1. F. Harary, The maximum connectivity of a graph, Proc. Nat. Acad. Sci. U.S.A., 48 (1962) 1142–1146.

2. H. Whitney, Congruent graphs and the connectivity of graphs, Amer. J. Math., 54 (1932), 150–168.

3. M. C. Gemignani, Problem E 1602, this MONTHLY, 70 (1963) 668. Solution E 1602, this MONTHLY, 71 (1964) 433.

A NOTE ON PERIODICITIES OF PREFERENTIAL ARRANGEMENT NUMBERS

PETER G. NEUMANN, Bell Telephone Laboratories, Murray Hill, N. J.

The integers f_n defined by $f_n = \sum_{r=1}^{n} A(n, r)$, where

$$A(n, r) = \sum_{j=0}^{r} (-1)^j \binom{r}{j} (r - j)^n,$$

have received considerable attention recently [1, 2, 3]. It is the purpose of this note to draw attention to a congruence of J. Touchard [4], and then to extend it by use of a little-known theorem given by R. D. Carmichael [5]. The periods of $f_n \pmod{10^k}$, given by O. A. Gross [1] for $k=1$ and by D. H. Kauffman [2] for $k=2, 3, 4$, follow as special cases.

Touchard has shown that $f_{n+\phi(m)} \equiv f_n \pmod{m}$, $n \geqq k \geqq 1$, where $\phi(m)$ is Euler's ϕ-function (the number of integers up to m which are prime to m), and where the factorization of m contains no prime to a power higher than k. This follows from the Euler-Fermat theorem $b^{\phi(m)} \equiv 1 \pmod{m}$, $(b, m)=1$. An improvement upon the Euler-Fermat theorem is given by Carmichael: $b^{\lambda(m)} \equiv 1 \pmod{m}$, $(b, m)=1$; here $\lambda(2^a)=\phi(2^a)$, $a=0, 1, 2$, $\lambda(2^a)=\frac{1}{2}\phi(2^a)$, $a>2$, $\lambda(p^a)=\phi(p^a)$, and $\lambda(2^a \cdot p_1^{a_1} \cdots p_q^{a_q}) = \text{LCM}[\lambda(2^a), \lambda(p_1^{a_1}), \cdots, \lambda(p_q^{a_q})]$, where the p_i are distinct odd primes. It then follows that $f_{n+\lambda(m)} \equiv f_n \pmod{m}$, under the same conditions as the Touchard result. For $k \leqq 4$, this leads to the periods given by Gross and Kauffman, $f_{n+4 \cdot 5^{k-1}} \equiv f_n \pmod{10^k}$, $n \geqq k$. For $n \geqq k > 4$, the corresponding periods are given by $f_{n+5 \cdot 10^{k-3}} \equiv f_n \pmod{10^k}$.

The Carmichael theorem implies that $3^n \pmod{10^k}$ has the above periods of $f_n \pmod{10^k}$, with no restrictions on n. These periods are in fact minimal for $3^n \pmod{10^k}$. Furthermore, the minimal period of $b^n \pmod{10^k}$ divides the minimal period of $3^n \pmod{10^k}$ for any b and k. Although the Carmichael theorem (when applicable) is never less powerful than the Euler-Fermat theorem (since $\lambda(m)$ divides $\phi(m)$), there may remain a gap between $\lambda(m)$ and the actual minimal periods, as for example in the case of $7^n \pmod{10^k}$. Thus it is not surprising that some of the above periods $\lambda(m)$ of $f_n \pmod{m}$ are not minimal. An example of non-minimality occurs for $m=16$, where $\lambda(16)=4$ and $f_{n+2} \equiv f_n \pmod{16}$, $n \geqq 3$.

Acknowledgment. I am extremely grateful to John Riordan for various helpful suggestions.

References

1. O. A. Gross, Preferential arrangements, this MONTHLY, 69 (1962) 4–8.

2. D. H. Kauffman, Note on preferential arrangements, this MONTHLY, 70 (1963) 62.

3. D. Zeitlin, Remarks on a formula for preferential arrangements, this MONTHLY, 70 (1963) 183–187.

4. J. Touchard, Propriétés arithmétiques de certains nombres récurrents, Ann. Soc. Sci. Bruxelles, 53 (1933) 21–31.

5. R. D. Carmichael, The Theory of Numbers, Wiley, New York, 1914.

SOME TOTALLY EQUIVALENT MATRICES, II

B. E. RHOADES, Lafayette College

The author raised in [4] some questions concerning the existence of infinite matrices that are totally equivalent, but not necessarily identical or almost identical. Brauer in [1] investigated the problem, obtained some results, and raised the following questions.

1. In order that two regular matrices which are strictly stronger than the identity matrix be totally equivalent, is it necessary or sufficient that there exist two natural numbers N_1, N_2 such that for each $n > N_1$, the nth row of one matrix appears as the mth row of the other, for some $m > N_2$?

2. Let A and B be two positive, regular matrices. What are necessary and sufficient conditions on A and B in order that each sequence x such that $A_n(x) \to \infty$ satisfies $A_n(x)/B_n(x) \to 1$?

Partial answers to both questions are contained in a paper by Miss Debi [2]. (When [4] was written the author did not have access to Miss Debi's paper.)

Two matrices are totally equivalent if each sequence evaluated by one of the matrices is evaluated by the other matrix to the same value—finite or infinite.

In [2] (Theorem 6) it is shown that the methods L and $(\overline{N}, 1/(n+1))$ are totally equivalent, where L denotes the logarithmic mean method, and $(\overline{N}, 1/(n+1))$ denotes the weighted mean method. See [3] page 59 for the definitions of these methods.

Employing the basic idea of the above-mentioned theorem, one can establish the following:

Let $\{p_k\}$ denote a sequence of nonnegative real numbers such that $p_0 > 0$, $P_n = p_0 + p_1 + \cdots + p_n$ and $P_n \to +\infty$. Let $\{Q_n\}$ denote any sequence such that $P_n/Q_n \to 1$ as $n \to \infty$. Then the sequence-to-sequence transformations defined by

$$t_n = \frac{\sum_{k=0}^{n} p_k x_k}{P_n}, \qquad u_n = \frac{\sum_{k=0}^{n} p_k x_k}{Q_n}$$

are totally equivalent.

Clearly $t_n = u_n P_n/Q_n$. Since $P_n/Q_n \to 1$, $u_n \to l$ if and only if $t_n \to l$, l finite or infinite.

The above class of matrices provides a set of sufficient conditions in answer to question 2.

To show that the condition stated in question 1 is not necessary, one may use the example of [2], Theorem 6, i.e., $p_k = (k+1)^{-1}$ and $Q_n = \log n$.

To show that the condition in question 1 is not sufficient, consider the following example. Let A denote the matrix defined by $a_{nn} = 2/3$, $a_{n,n+1} = 1/3$, $a_{nk} = 0$ otherwise. Let B be defined by $b_{nk} = a_{n+1,k}$ for $n > 1$, and $b_{1,k} = 1/k(k+1)$. Then A and B are both totally regular and strictly stronger than the identity

matrix. However $\{(-2)^n\}$ is summable by A, but not by B. Thus A and B are not equivalent, hence not totally equivalent.

References

1. G. Brauer, Remarks on a paper of Rhoades, this MONTHLY, 70 (1963) 300–301.
2. Sobha Debi, Some results on total inclusion for Nörlund summability, Bull. Calcutta Math. Soc., 47 (1955) 135–141.
3. G. H. Hardy, Divergent series, Oxford, 1949.
4. B. E. Rhoades, Some totally equivalent matrices, this MONTHLY, 69 (1962) 523–524.

SOME METABELIAN-LIKE VARIETIES OF GROUPS

I. D. MACDONALD, Newcastle University College, Australia

Important classes of groups, called varieties, may be presented by stating which equations or identities are to be valid within each group of the class.

Before considering examples of this situation we need notation. Let $x_1, x_2, \cdots, x_n, \cdots$ be elements in some group. The commutator (x_1, x_2) is defined to be $x_1^{-1}x_2^{-1}x_1x_2$ while for $n>2$ we specify $(x_1, \cdots, x_n)$ inductively as $((x_1, \cdots, x_{n-1}), x_n)$. In any group G the commutator subgroup G' is that generated by all commutators (x_1, x_2) to be found as x_1 and x_2 vary in G; and $G''=(G')'$.

Now the groups G in which $(a, b)=1$ for *all* elements a and b in G form a well-known variety, namely the abelian variety. Another important variety is that defined by the equation

$$((a, b), (c, d)) = 1, \tag{1}$$

holding for *all* choices of the elements a, b, c, d in each group of the variety. It is easily proved that an alternative equation corresponding to this variety, which is known as the metabelian variety, is

$$(a, b, c, d) = (a, b, d, c), \tag{2}$$

a fact which we shall generalise in

THEOREM 1. *The varieties corresponding to the equations*

$$((a, b), (c, d)) = (a, b, c, d)^{-n}, \tag{3}$$

$$(a, b, c, d) = (a, b, d, c)^{n+1} \tag{4}$$

respectively, where n is a fixed integer, coincide.

Then we investigate this variety and find a strong resemblance to the metabelian case:

THEOREM 2. *Let G be a group of the variety defined in Theorem 1. Then G'' is in the center of G and has exponent 2. If n is odd the variety may also be defined by the pair of equations*

$$(a, b, c, d) = (a, b, d, c),$$
$$(a, b, c, d)^n = 1.$$

In the proofs it will be convenient to write $(u, v; x, y)$ for $((u, v), (x, y))$ and $(u, v; x, y; z)$ for $((u, v), (x, y), z)$, in accordance with standard practice. By x^y we shall mean $y^{-1}xy$. The following identities, which hold in all groups, may be verified by direct calculation or by reference to standard works:

$$(x^{-1}, y)^x = (x, y^{-1})^y = (x, y)^{-1}, \tag{5}$$

$$(uv, xy) = (u, y)^v(v, y)(u, x)^{vy}(v, x)^y, \tag{6}$$

$$(x, y^{-1}, z)^y(y, z^{-1}, x)^z(z, x^{-1}, y)^x = 1. \tag{7}$$

An immediate deduction from (7) with (5) is:

$$(u, v; x, y) = (u, v, y^{-1}, x^{(u,v)})^y(u, v, x^{-1}, y^{-1})^{xy}. \tag{8}$$

Proof of Theorem 1. Assume (3). Then

$$(a, b, d; a, b) = 1, \tag{9}$$

$$((a, b)^m, c) = (a, b, c)^m \tag{10}$$

for any integer m, by (9) and (5); and

$$(a, b, c; d, c) = 1 \tag{11}$$

because

$$\begin{aligned}(a, b, c; d, c) &= (a, b, c; c, d)^{-1} && \text{by (10)}\\ &= (a, b, c, c, d)^n && \text{by (3)}\\ &= ((a, b, c, c)^n, d) && \text{by (10)}\\ &= 1 && \text{by (3).}\end{aligned}$$

On applying (6) and (9) to (8) we find

$$(a, b; c, d) = (a, b, d^{-1}, c)^d(a, b, c^{-1}, d^{-1})^{cd}, \tag{12}$$

and we simplify the right-hand side of this:

$$\begin{aligned}(a, b, d^{-1}, c)^d &= ((a, b, d)^{-1}, c(c, d)) && \text{by (5)}\\ &= (a, b, d, c(c, d))^{-1} && \text{by (10)}\\ &= (a, b, d, c)^{-(c,d)}(a, b, d; c, d)^{-1} && \text{by (6)}\\ &= (a, b, d, c)^{-(c,d)} && \text{by (11).}\end{aligned}$$

Similarly, using (5), (6), (10), and (11) we can show that

$$(a, b, c^{-1}, d^{-1})^{cd} = (a, b, c, d)^{(c,d)}.$$

We therefore deduce from (12) that

$$(a, b; c, d) = (a, b, d, c)^{-(c,d)}(a, b, c, d)^{(c,d)}.$$

On using (9) we have $(a, b; c, d) = (a, b, d, c)^{-1}(a, b, c, d)$, and (4) follows from this and (3).

Now assume (4). Then (9) and (10) may again be verified, which enables us to deduce (12) from (8). Note also that

$$(a, b, c, d)^{n(n+2)} = 1. \tag{13}$$

Now since

$$\begin{aligned}(a, b, c^{-1}, d^{-1})^{cd} &= (a, b, d^{-1}, c^{-1})^{(n+1)cd} && \text{by (4)}\\ &= (a, b, d^{-1}, c)^{-(n+1)d} && \text{by (5),}\end{aligned}$$

we find from (12) that

$$\begin{aligned}(a, b; c, d) &= (a, b, d^{-1}, c)^{-nd}\\ &= (a, b, c, d^{-1})^{-n(n+1)d} && \text{by (4)}\\ &= (a, b, c, d)^{n(n+1)} && \text{by (5)}\\ &= (a, b, c, d)^{-n} && \text{by (13).}\end{aligned}$$

Therefore we have proved (3), and completed the proof of Theorem 1.

Proof of Theorem 2. We note that combination of (3) and (13) gives

$$(a, b; c, d)^{n+2} = 1. \tag{14}$$

Further we have

$$\begin{aligned}(a, b, c, d, e)^{n} &= (d, e; a, b, c) && \text{by (3)}\\ &= (d, e, (a, b), c)^{-n} && \text{by (3)}\\ &= (a, b, (d, e), c)^{n} && \text{by (10)}\\ &= ((a, b, d, e)^{-n}, c)^{n} && \text{by (3)}\\ &= (a, b, d, e, c)^{-n^2} && \text{by (10);}\end{aligned}$$

but because also

$$\begin{aligned}(a, b, d, e, c) &= (a, b, d, c, e)^{n+1} && \text{by (4)}\\ &= ((a, b, c, d)^{n+1}, e)^{n+1} && \text{by (4)}\\ &= (a, b, c, d, e)^{(n+1)^2} && \text{by (10)}\\ &= (a, b, c, d, e) && \text{by (13),}\end{aligned}$$

we arrive at the identity $(a, b, c, d, e)^{n(n+1)} = 1$. Here another application of (3) yields $(a, b, c; d, e)^{n+1} = 1$, comparison of which with (14) gives the result

$$(a, b, c; d, e) = 1. \tag{15}$$

When we raise each side of this to the $(n+1)$st power and use (4) we obtain

$$(a, b; d, e; c) = 1, \tag{16}$$

that is, G'' is in the center of G as asserted in Theorem 2. Note that (3) and (15) together imply that

$$(a, b, c, d, e)^n = 1. \tag{17}$$

Next we show that G'' has exponent 2. By (5), (6) and (15),

$$\begin{aligned}(a, b^{-1}, c)^b &= (b, a, c(c, b)) \\ &= (b, a; c, b)(b, a, c)^{(c,b)} \\ &= (b, a; c, b)(b, a, c).\end{aligned}$$

This with further applications of (6) and (15), enables us to deduce the following identity from (7):

$$(b, a, c, d)(c, b, a, d)(a, c, b, d) = 1. \tag{18}$$

This may be raised to the nth power and simplified by means of (15) again, $(b, a, c, d)^n(c, b, a, d)^n(a, c, b, d)^n = 1$, after which use of (3) produces

$$(a, c; b, d)(c, b; a, d)(b, a; c, d) = 1. \tag{19}$$

When a and d in (19) are interchanged there results

$$(d, c; b, a)(c, b; d, a)(b, d; c, a) = 1. \tag{20}$$

But by (10)

$$\begin{aligned}(d, c; b, a) &= (b, a; c, d), \\ (c, b; d, a) &= (c, b; a, d)^{-1}, \\ (b, d; c, a) &= (a, c; b, d),\end{aligned}$$

so (19) and (20) combine to give

$$(c, b; a, d)^2 = 1. \tag{21}$$

We have thus proved, in view of (16), that G'' has exponent 2, as asserted in Theorem 2.

When n is odd (14) and (21) at once show that G is metabelian, and the remaining assertion of Theorem 2 then follows from Theorem 1. Note that (3), (21) and (4) imply that the groups of our variety satisfy the two equations

$$\begin{aligned}(a, b, c, d)^{2n} &= 1, \\ (a, b, c, d)^2 &= (a, b, d, c)^2.\end{aligned}$$

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

THE TRIPLE VECTOR PRODUCT A×(B×C)

K. E. BISSHOPP, Rensselaer Polytechnic Institute

Although triple vector products are used extensively in vector algebra, little attention is currently given to the geometrical structure of many of the earlier proofs for the fundamental identities. With the almost universal adoption of triple index e systems, formal proofs, involving only trivial manipulations, have superseded geometrical reasoning in terms of coordinate-free representations of vectors. Many of these geometrical proofs for the expansion of triple vector products are not entirely straight-forward, and even the discussion of Coe and Rainich [1] admits of some simplification which the following equivalent procedure achieves by introducing an appropriate system of base vectors and their reciprocals.

From any two nonzero vectors **B** and **C**, which are neither parallel nor collinear, it is always possible to construct the system of linearly independent base vectors $\{\mathbf{e}_i\}$, where

$$\mathbf{e}_1 = \mathbf{B}; \qquad \mathbf{e}_2 = \mathbf{C}; \qquad \mathbf{e}_3 = \mathbf{B} \times \mathbf{C}. \tag{1}$$

It is trivial to verify that the corresponding set of reciprocal base vectors is

$$\mathbf{e}^1 = \frac{\mathbf{C} \times (\mathbf{B} \times \mathbf{C})}{(\mathbf{B} \times \mathbf{C})^2}; \qquad \mathbf{e}^2 = \frac{(\mathbf{B} \times \mathbf{C}) \times \mathbf{B}}{(\mathbf{B} \times \mathbf{C})^2}; \qquad \mathbf{e}^3 = \frac{\mathbf{B} \times \mathbf{C}}{(\mathbf{B} \times \mathbf{C})^2}, \tag{2}$$

where the reciprocity of $\{\mathbf{e}_i\}$ and $\{\mathbf{e}^i\}$ is expressed by

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad (i, j = 1, 2, 3). \tag{3}$$

Since the vectors $\mathbf{e}^i$ are not all parallel to a plane, they are linearly independent, and equally valid expressions for an arbitrary vector **V**, by virtue of equation (3), are

$$\begin{aligned} \mathbf{V} &= (\mathbf{V}\cdot\mathbf{e}_1)\mathbf{e}^1 + (\mathbf{V}\cdot\mathbf{e}_2)\mathbf{e}^2 + (\mathbf{V}\cdot\mathbf{e}_3)\mathbf{e}^3 \\ &= (\mathbf{V}\cdot\mathbf{e}^1)\mathbf{e}_1 + (\mathbf{V}\cdot\mathbf{e}^2)\mathbf{e}_2 + (\mathbf{V}\cdot\mathbf{e}^3)\mathbf{e}_3 \\ &= (\mathbf{V}\cdot\mathbf{e}^1)\mathbf{B} + (\mathbf{V}\cdot\mathbf{e}^2)\mathbf{C} + (\mathbf{V}\cdot\mathbf{e}^3)(\mathbf{B} \times \mathbf{C}). \end{aligned} \tag{4}$$

The expansion of the triple vector product will be facilitated by appeal to

the identity

$$\mathbf{n} \times (\mathbf{U} \times \mathbf{n}) = \mathbf{U}, \tag{5}$$

where $\mathbf{n}$ is a unit vector that is perpendicular to $\mathbf{U}$. The proof follows by repeated application of the definition of the cross product. If $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is substituted for $\mathbf{V}$ in equation (4) the result is

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \frac{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\cdot\mathbf{C}\times(\mathbf{B}\times\mathbf{C})}{(\mathbf{B}\times\mathbf{C})^2}\mathbf{B} + \frac{\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\cdot(\mathbf{B}\times\mathbf{C})\times\mathbf{B}}{(\mathbf{B}\times\mathbf{C})^2}\mathbf{C} \\ &= \mathbf{A}\cdot\frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|}\times\left(\mathbf{C}\times\frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|}\right)\mathbf{B} \\ &\quad - \mathbf{A}\cdot\frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|}\times\left(\mathbf{B}\times\frac{\mathbf{B}\times\mathbf{C}}{|\mathbf{B}\times\mathbf{C}|}\right)\mathbf{C}.\end{aligned} \tag{6}$$

The unit vector, $\mathbf{B}\times\mathbf{C}/|\mathbf{B}\times\mathbf{C}| = \mathbf{n}$, is perpendicular to $\mathbf{B}$ and $\mathbf{C}$ by definition, hence equation (5) applies so that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{A}\cdot\mathbf{n} \times (\mathbf{C} \times \mathbf{n})\mathbf{B} - \mathbf{A}\cdot\mathbf{n} \times (\mathbf{B} \times \mathbf{n})\mathbf{C} = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \tag{7}$$

and the identity is proved without evaluating unnecessary scalar parameters.

In spite of the fact that the concept of reciprocal vectors is employed for the proof, the only operations involved are those of scalar and vector multiplication.

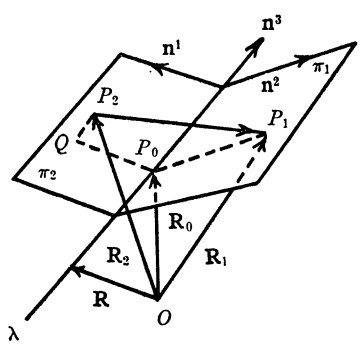


FIG. 1

The particular choice of reciprocal vectors that simplifies the expansion of $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ also is convenient for the solution of numerous problems in vector geometry. For example, let it be required to find the equation of the line of intersection of two planes without solving simultaneous equations. Figure 1 shows two planes π_1 and π_2 which are identified by points P_1, P_2 and normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ respectively.

If O is an arbitrary origin, the equation of λ, the line of intersection, is

$$\mathbf{R} = \mathbf{R}_0 + s\frac{\mathbf{n}_1 \times \mathbf{n}_2}{|\mathbf{n}_1 \times \mathbf{n}_2|}, \tag{8}$$

where P_0 is any point on λ.

A suitable set of reciprocal vectors constructed from $\mathbf{n}_1$ and $\mathbf{n}_2$ is

$$\mathbf{n}^1 = \frac{\mathbf{n}_2 \times (\mathbf{n}_1 \times \mathbf{n}_2)}{(\mathbf{n}_1 \times \mathbf{n}_2)^2}; \qquad \mathbf{n}^2 = \frac{(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_1}{(\mathbf{n}_1 \times \mathbf{n}_2)^2}; \qquad \mathbf{n}^3 = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{(\mathbf{n}_1 \times \mathbf{n}_2)^2}. \tag{9}$$

A convenient point P_0 on λ can be found by expressing $\overline{P_2P_1} = \mathbf{R}_1 - \mathbf{R}_2$ in terms of $\mathbf{n}^1$, $\mathbf{n}^2$, $\mathbf{n}^3$, i.e., $\mathbf{R}_1 - \mathbf{R}_2 = \alpha_1\mathbf{n}^1 + \alpha_2\mathbf{n}^2 + \alpha_3\mathbf{n}^3$, then

$$\alpha_2\mathbf{n}^2 = \overline{P_0P_1} = \mathbf{R}_1 - \mathbf{R}_0 = (\mathbf{R}_1 - \mathbf{R}_2)\cdot\mathbf{n}_2\frac{(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_1}{(\mathbf{n}_1 \times \mathbf{n}_2)^2},$$

and

$$\mathbf{R}_0 = \mathbf{R}_1 + \frac{(\mathbf{R}_2 - \mathbf{R}_1)\cdot\mathbf{n}_2}{(\mathbf{n}_1 \times \mathbf{n}_2)^2}(\mathbf{n}_1 \times \mathbf{n}_2) \times \mathbf{n}_1. \tag{10}$$

Reference

1. Coe and Rainich, Fundamental identities of vector algebra, this MONTHLY, 56 (1949) 175–179.

AN ELEMENTARY FIXED POINT THEOREM

R. H. COWEN, Yeshiva University

The following provides a simple and, I think, interesting example of the way an existence proof is sometimes effected by choosing a "suitable" function and showing that it has a fixed point, the existence of which turns out to be equivalent to the existence of the property we are interested in.

Suppose we have a two dimensional outline of the human form and we drop a smaller replica of the same form on the larger one in such a way that the little figure is completely within the big one. One might surmise after a few trials that there is always exactly one point with the property that it indicates the same location relative to both figures (clearly there is no more than one such point, for two would imply that both figures are of the same size); we shall call such a point, a *point of exact coincidence.* This conjecture is correct, as we shall show, but first we note that if there is in fact a point which indicates the same position on both figures, a point of exact coincidence, then it can be found as follows: assume that the smaller outline is a tattoo on the larger, and that this tattoo is inherited by all similar tattoos (in particular by itself) in exactly the position that the little figure occupies in the big one. In this way we get an infinite sequence f_n of ever smaller figures, each occurring in exactly the same place

relative to the next larger figure. We see that f_3 and f_2 must have a point of exact coincidence in the same anatomical location as that of f_2 and f_1. Therefore it must be the same point; hence the point of exact coincidence of f_1 and f_2 is contained in f_3. Continuing in this way we can show that the point must be in all the figures and hence is uniquely determined by them.

The existence of the point of exact coincidence is of course an immediate consequence of Brouwer's fixed point theorem (which states in two dimensions that any continuous transformation T of a set topologically equivalent to the unit disk (with its boundary) into itself has a fixed point, i.e., a point a, such that $T(a)=a$); for consider the transformation whose image for each point of the big figure is the corresponding point of the little figure. This transformation is surely continuous and into, and thus must have a fixed point which by definition of the transformation satisfies our condition.

The existence of the point of exact coincidence can, however, be readily established without Brouwer's theorem, and the remainder of this paper will be devoted to giving an "elementary" proof. We prove the theorem when the figures are rectangles; the same proof (with minor modifications) can be used for more "creative" figures.

THEOREM. *Let R_1, R_2 be two similar rectangles in the plane with R_2 properly contained in R_1. Let $(x, y)_1$ be a Euclidean coordinate system defined on R_1, and let $(x, y)_2$ be the coordinate system induced on R_2 by that of R_1 under the similarity transformation. Then there exists a point of the plane with the same coordinates in both systems.*

Proof. Define a transformation $f: R_1 \to R_2$ by $(a_1, a_2)_1 \to (a_1, a_2)_2$, for $(a_1, a_2)_1 \in R_1$. Thus the image under f of R_1 is R_2, and the image of R_2 is a rectangle, R_3, which has the same coordinates in (the system of) R_2, as R_2 has in (the system of) R_1. We represent this situation by f: $[R_1, R_2]_1 \to [R_2, R_3]_2$. Clearly $R_3 \subset R_2$.

Consider the rectangles R_1, R_2, R_3 in the system of R_1; their images under f are rectangles R_a, R_b, R_c in the system of R_2, and R_a, R_b, R_c have the same coordinates in R_2, as R_1, R_2, R_3 have in R_1; but as we saw above $R_a = R_2$, $R_b = R_3$, and so if we call R_c, "R_4," we have that R_2, R_3, R_4 have the same coordinates in R_2, as R_1, R_2, R_3 have in R_1, i.e., f: $[R_1, R_2, R_3]_1 \to [R_2, R_3, R_4]_2$. Also $R_4 \subset R_3$ (since $R_3 \subset R_2$ in the system of R_1, and f clearly preserves the inclusion relation), and so we have $R_4 \subset R_3 \subset R_2 \subset R_1$.

Continuing in the same way, we can show that for any k, the images of $R_1, R_2, \cdots, R_{k-1}, R_k$ in R_1 are, respectively, $R_2, R_3, \cdots, R_k, R_{k+1}$ in R_2, i.e., f: $[R_1, R_2, \cdots, R_{k-1}, R_k]_1 \to [R_2, R_3, \cdots, R_k, R_{k+1}]_2$, and also $R_{k+1} \subset R_k \subset \cdots \subset R_2 \subset R_1$.

Therefore for all n, f: $R_n \to R_{n+1}$, so certainly (since $R_{n+1} \subset R_n$)

$$(*) \qquad f: R_n \to R_n.$$

Let $d(R_n)$ be the length, measured in the system of R_1, of the diagonal of R_n. Then $d(R_2)/d(R_1) = s$, with $s < 1$. Therefore by the definition of $(x, y)_2$, and

since $f(R_i)=R_{i+1}$, $i\geqq 1$, we have $d(R_{i+1})/d(R_i)=s$. Thus

$$d(R_n)/d(R_1) = d(R_n)/d(R_{n-1})\cdot d(R_{n-1})/d(R_{n-2})\cdot \cdots \cdot d(R_2)/d(R_1) = s^{n-1},$$

i.e., $d(R_n)=s^{n-1}d(R_1)$.

Therefore, using (*), we see that all points in R_n move at most $s^{n-1}d(R_1)$, under f. But $R_1\supset R_2\supset R_3\supset\cdots$, and so by the nested interval property there is a point Q in all the R_n, and it is easily seen that Q must remain fixed under f, i.e., there exists real numbers a, b such that $Q=(a, b)_1=(a, b)_2$.

THE PERIODIC SOLUTION OF A NONLINEAR DIFFERENTIAL EQUATION

MARCIA PETERSON AND WILLIAM SWARTZ, Montana State College

1. Introduction. The purpose of this note is to point out that the existence of a periodic solution of a certain nonlinear differential equation may be proved by a method analogous to the familiar Cauchy-Picard procedure for initial value problems. We consider the differential equation

$$\frac{dx}{dt}+f(x) = p(t), \tag{1}$$

where $f'(x)$ is continuous, $0<a<f'(x)<b$, $p(t)$ is continuous, and $p(t+2\pi)=p(t)$. The application to an electric circuit with nonlinear resistance is apparent.

The approximating sequence of iterates $\{x_n(t)\}$ is obtained as follows. We first write (1) in the form

$$\frac{dx}{dt}+bx = p(t)+F(x), \tag{2}$$

where $F(x)=bx-f(x)$. Then $x_0(t)$ is the periodic solution of the linear equation resulting from the deletion of the nonlinear term $F(x)$. Now $x_0(t)$ is substituted into $F(x)$ in the right member of (2), and $x_1(t)$ determined as the periodic solution of this linear equation. In general, $x_n(t)$ is the periodic solution of the equation resulting from substituting $x_{n-1}(t)$ into the right member of (2). We use the readily verified fact that

$$x(t) = \int_{-\infty}^{t} e^{b(s-t)}g(s)ds$$

is the unique periodic solution of $x'+bx=g(t)$ where $b>0$, $g(t)$ is continuous, and $g(t+2\pi)=g(t)$. Hence, the iterate $x_n(t)$ is given by

$$x_n(t) = \int_{-\infty}^{t} e^{b(s-t)}[p(s)+F(x_{n-1}(s))]ds.$$

An inductive argument verifies the existence of all iterates.

We note the consequences of failure to add the term bx to both members of

(1). The iteration then consists of repeated quadratures and consequently the iterates are not unique.

2. Proof of convergence. We note that $F(x)$ satisfies a Lipschitz condition. The inequality $0<a<f'(x)<b$ is written $b>b-a>b-f'(x)>0$, or $0<F'(x)<K$, where $K=b-a$. If $y(t)$ and $z(t)$ are any two functions defined for all t, the Law of the Mean and the bound on $F'(x)$ give

$$|F(y(t)) - F(z(t))| \leqq K|y(t) - z(t)| \tag{3}$$

for all t.

Since F and x_0 are continuous and x_0 is periodic, $F(x_0(s))$ is continuous and periodic. Then a positive constant M exists such that $|F(x_0(s))| \leqq M$ for all s. This gives

$$|x_1(t) - x_0(t)| \leqq \int_{-\infty}^{t} e^{b(s-t)} M ds = \frac{M}{b}\cdot$$

Now using (3) we show inductively that for all t,

$$|x_n(t) - x_{n-1}(t)| \leqq \frac{K^{n-1}M}{b^n} \qquad (n = 1, 2, \cdots).$$

Thus the partial sums $x_n(t)$ of the series $x_0(t)+\sum_{k=1}^{\infty}[x_k(t)-x_{k-1}(t)]$ are dominated by the terms of a convergent series of constants. The sequence $\{x_n(t)\}$ converges uniformly on the infinite t interval to a limit $x(t)$. The iterates $x_n(t)$ are continuous so $x(t)$ is continuous. Since the sequences $\{x_n(t)\}$ and $\{x_n(t+2\pi)\}$ are identical and the convergence is pointwise, we have $x(t+2\pi)=x(t)$.

3. Verification of the solution. Here, and in the following uniqueness argument, we use the fact that $x(t)$ satisfies the integral equation

$$x(t) = \int_{-\infty}^{t} e^{b(s-t)}[p(s) + F(x(s))]ds, \tag{4}$$

if and only if $x(t)$ satisfies (2) and $x(t+2\pi)=x(t)$. Now one endeavors to show that $x(t)$ satisfies (4), by justifying the following interchanges of limiting processes.

$$\begin{aligned} x(t) &= \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} \int_{-\infty}^{t} e^{b(s-t)} F(x_{n-1}(s))ds \\ &= \int_{-\infty}^{t} e^{b(s-t)} \lim_{n\to\infty} F(x_{n-1}(s))ds = \int_{-\infty}^{t} e^{b(s-t)} F(x(s))ds. \end{aligned} \tag{5}$$

We give two lemmas.

LEMMA 1. *Let $f_n(s)$ be continuous and periodic for $n=0, 1, 2, \cdots$, and let $\{f_n(s)\}$ converge uniformly to $f(s)$ on $-\infty<s\leqq t_0$. Then, if $b>0$ and $t\leqq t_0$, $\int_{-\infty}^{t} e^{bs} f_n(s)ds$ converges uniformly with respect to n.*

Proof. The hypotheses insure that the $f_n(s)$ are uniformly bounded, i.e., $|f_n(s)| < M$ for $-\infty < s \leqq t_0$ and $n = 0, 1, 2, \cdots$. We then have

$$\left| \int_{\alpha_1}^{\alpha_2} e^{bs} f_n(s) ds \right| \leqq \left| \int_{\alpha_1}^{\alpha_2} e^{bs} M ds \right| = \frac{M}{b} \left| e^{b\alpha_2} - e^{b\alpha_1} \right| .$$

Clearly the right member tends to zero as $\alpha_1, \alpha_2 \rightarrow -\infty$, independently of n.

LEMMA 2. *Let $f_n(s)$ $(n=0, 1, 2, \cdots)$ be continuous and $\{f_n(s)\}$ converge uniformly to $f(s)$ on $-\infty < s \leqq t_0$. For $t \leqq t_0$, let $\int_{-\infty}^t f(s)ds$ converge, and $\int_{-\infty}^t f_n(s)ds$ converge uniformly with respect to n. Then*

$$\int_{-\infty}^{t} f(s)ds = \lim_{n\rightarrow\infty} \int_{-\infty}^{t} f_n(s)ds.$$

Proof. For $R < t$ we have

$$\left| \int_{-\infty}^{t} f(s)ds - \int_{-\infty}^{t} f_n(s)ds \right| \leqq \left| \int_{-\infty}^{R} f(s)ds \right| + \int_{R}^{t} \left| f(s) - f_n(s) \right| ds + \left| \int_{-\infty}^{R} f_n(s)ds \right| .$$

Let $\epsilon > 0$ be given. We may choose R so small that the first and third terms in the right member are each less than $\epsilon/3$ for all n. Also, there corresponds to the given ϵ an integer N such that $|f(s) - f_n(s)| < (\epsilon/3)(t_0 - R)$ for $n > N$ and $R \leqq s \leqq t_0$. Then, for $n > N$ we have

$$\left| \int_{-\infty}^{t} f(s)ds - \int_{-\infty}^{t} f_n(s)ds \right| < \epsilon/3 + \int_{R}^{t} \frac{\epsilon}{3(t_0 - R} ds + \epsilon/3 < \epsilon.$$

It follows immediately from (3) that $\{F(x_n(s))\}$ converges uniformly to $F(x(s))$ on $-\infty < s \leqq t_0$. On this interval e^{bs} is bounded so $\{e^{bs}F(x_n(s))\}$ converges uniformly to $e^{bs}F(x(s))$. Then, according to Lemma 1, the integral of $e^{bs}F(x_n(s))$ from $-\infty$ to t $(t \leqq t_0)$ converges uniformly with respect to n. The continuity and periodicity of $F(x(s))$ imply its boundedness, and it follows that the integral of $e^{bs}F(x(s))$ from $-\infty$ to t $(t \leqq t_0)$ is convergent. Now Lemma 2 justifies the third equality in (5). The last equality in (5) has been justified in the preceding remarks. Therefore, (4) is satisfied on any interval $t \leqq t_0$, where t_0 may be chosen to be any number.

Uniqueness. We need only show that (4) has a unique solution. Supposing $x(t)$ and $\bar{x}(t)$ to be solutions of (4) gives

$$\left| x(t) - \bar{x}(t) \right| \leqq \int_{-\infty}^{t} e^{b(s-t)} \left| F(x(s)) - F(\bar{x}(s)) \right| ds, \tag{6}$$

$$\left| x(t) - \bar{x}(t) \right| \leqq \int_{-\infty}^{t} e^{b(s-t)} K \left| x(s) - \bar{x}(s) \right| ds. \tag{7}$$

The functions $F(x(s))$ and $F(\bar{x}(s))$ are bounded and we may suppose this bound to be K (for $K=b-a$, and b may be replaced by any larger number without loss of generality). Then (6) gives $2K/b$ as a bound on $|x(t)-\bar{x}(t)|$. Using this result in (7) gives $2K^2/b^2$ as an improved bound. By induction, we find that for all n

$$|x(t) - \bar{x}(t)| \leqq 2(K/b)^n.$$

The right member tends to zero with increasing n. The left member is independent of n, hence is zero.

A COMPOSITE GENERATOR

R. C. ORR, Humboldt State College, Arcata, California

The prime generator problem has suggested the idea of a composite generator. The function $f(k, n, r, s)$ developed here generates all the composite positive integers and no others.

Let N be the set of all positive integers, and let all lower-case letters have only integral values.

Every positive integer except 1, 2, and 3 is of one of the forms, with $n\in N$: (1) $6n+1$, (2) $6n+2$, (3) $6n+3$, (4) $6n-2$, (5) $6n-1$, (6) $6n$.

Forms (2), (3), (4), and (6) obviously generate disjoint sets of composite numbers. Forms (1) and (5) generate disjoint sets containing the remainder of the composite numbers, and all the primes except 2 and 3.

Consider the function:

$$f(k, n, r, s) = (6n + k)\left(\frac{k(k+2)(k-2)(k-3)(6r+s)}{9k-3} + 1\right)$$

for $k\in\{0, \pm 1, \pm 2, 3\}$, $n, r\in N$, and $s\in\{0, -2\}$.
Then

$$f(k, n, r, s) = \begin{cases} 6n & \text{if } k = 0 \\ (6n+1)(6r+s+1) & \text{if } k = 1 \\ 6n+2 & \text{if } k = 2 \\ 6n+3 & \text{if } k = 3 \\ 6n-2 & \text{if } k = -2 \\ (6n-1)(6r+s+1) & \text{if } k = -1 \end{cases}$$

Hence $f(k, n, r, s)$ has only composite values; it cannot be a prime or 1.

Let P be the set of all primes.

Let $P_1=\{x \mid x\in P,\ x\equiv 1 \pmod 6\}$.

Let $P_5=\{x \mid x\in P,\ x\equiv -1 \pmod 6\}$.

Then $P=\{2, 3\}\cup P_1\cup P_5$.

$$p_1, p_2 \in P_1 \Rightarrow p_1p_2 \equiv 1 \pmod 6$$

$$q_1,\ q_2 \in P_5 \Rightarrow q_1q_2 \equiv 1 \pmod 6$$
$$p_1 \in P_1 \text{ and } q_1 \in P_5 \Rightarrow p_1q_1 \equiv -1 \pmod 6.$$

Let $m=\prod_{i=1}^{u} p_i^{\alpha_i}\prod_{i=1}^{v} q_i^{\beta_i}$, $\sum_{i=1}^{u}\alpha_i+\sum_{i=1}^{v}\beta_i>1$, where $p_i\in P_1$ and $q_i\in P_5$. Then

$$m \equiv \begin{cases} 1 \pmod 6) & \text{if } \sum_{i=1}^{v}\beta_i \equiv 0 \pmod 2) \\ -1 \pmod 6) & \text{if } \sum_{i=1}^{v}\beta_i \equiv 1 \pmod 2) \end{cases}$$

and every composite integer of either form (1) or form (5) is also of form m.

$$\prod_{i=1}^{u} p_i^{\alpha_i} = \prod_{i=1}^{u-1} p_i^{\alpha_i}p_u^{\alpha_u} = (6n+1)(6r+1)$$

$$\prod_{i=1}^{v} q_i^{\beta_i} = \prod_{i=1}^{v-1} q_i^{\beta_i}q_v^{\beta_v} = \begin{cases} (6n+1)(6r-1), & \sum_{i=1}^{v}\beta_i \equiv 1 \pmod 2) \\ (6n-1)(6r-1), & \sum_{i=1}^{v}\beta_i \equiv 0 \pmod 2) \end{cases}$$

$$\prod_{i=1}^{u} p_i^{\alpha_i}\prod_{i=1}^{v} q_i^{\beta_i} = \begin{cases} (6n_1+1)(6r_1+1)(6n_2+1)(6r_2-1) & \text{if } \sum_{i=1}^{v}\beta_i \equiv 1 \pmod 2) \\ (6n_1+1)(6r_1+1)(6n_2-1)(6r_2-1) & \text{if } \sum_{i=1}^{v}\beta_i \equiv 0 \pmod 2) \end{cases}$$

Hence

$$m = \prod_{i=1}^{u} p_i^{\alpha_i}\prod_{i=1}^{v} q_i^{\beta_i} = \begin{cases} (6n+1)(6r-1) \text{ or } (6n-1)(6r+1) & \text{if } m \equiv -1 \pmod 6) \\ (6n-1)(6r-1) \text{ or } (6n+1)(6r+1) & \text{if } m \equiv 1 \pmod 6). \end{cases}$$

It follows that $m=(6n\pm1)(6r+s+1)$ $s\in\{0,\ -2\}$ and $f(k,\ n,\ r,\ s)=x$ iff x is composite.

SETS OF GAUSSIAN RANDOM VARIABLES

J. E. POTTER, Massachusetts Institute of Technology

Firestone and Hanson pointed out in [1] the important distinction between sets of random variables with joint Gaussian distributions and sets of random variables whose individual distributions are Gaussian, but whose interrelation is not Gaussian. In this note we give a somewhat more intuitive example of two dependent uncorrelated random variables whose individual distributions are Gaussian. In addition, it will be proved that a set of random variables has a joint Gaussian distribution if, and only if, every linear combination of the random variables is normally distributed. This condition is of interest because it is a strengthening of the condition that each random variable be normally

distributed and because it does not involve multidimensional probability distributions.

To construct a pair of dependent uncorrelated random variables each of which is Gaussian when considered alone, let x be a Gaussian random variable with mean zero. Let y be an independent coin tossing random variable

$$y = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases}$$

and let

$$z = xy.$$

It will be shown that x and z have the desired properties. The property of the Gaussian distribution, which is fundamental to this example, is the symmetry of the distribution about its mean. Thus

$$\Pr(x < a) = \Pr(x > -a).$$

Since x and y are independent

$$\Pr(z < a) = \Pr(y = 1) \Pr(x < a) + \Pr(y = -1) \Pr(x > -a) = \Pr(x < a).$$

Hence x and z individually have Gaussian distributions. The variables x and z are dependent since $z = \pm x$. However, $E(xz) = E(x^2y) = E(x^2)E(y) = 0$.

It will now be shown that a set of random variables, $x_1, \cdots, x_n$, has a joint Gaussian distribution if, and only if, every linear combination of x_1 to x_n is normally distributed. The "only if" half of this statement is well known, so only the "if" half will be proved. Let x denote the column vector

$$x = (x_1, \cdots, x_n)^T$$

where the superscript T represents the transpose operation. Since, in particular, $x_1, \cdots, x_n$ are normally distributed, $E(x_1^2), \cdots, E(x_n^2)$ are finite and therefore, all first and second moments of the x_i's exist. Let the mean of x be denoted by

$$m = E(x) \tag{1}$$

and let C be the covariance matrix

$$C = E[(x - m)(x - m)^T]. \tag{2}$$

The proof will be carried out by showing that the characteristic function of the random variable x, $\phi(y) = E[\exp(iy^Tx)]$ is the characteristic function

$$\phi(y) = \exp\,(im^Ty - \tfrac{1}{2}y^TCy) \tag{3}$$

for a set of random variables having a joint Gaussian distribution with parameters m and C [2]. Since a probability distribution is uniquely determined by the associated characteristic function, x must have a joint Gaussian distribution.

Given the deterministic column vector y, it follows by hypothesis and equations (1) and (2) that the scalar random variable

$$u = y^T x \tag{4}$$

has a normal distribution with mean

$$a = y^T m \tag{5}$$

and variance

$$b = y^T C y. \tag{6}$$

Then

$$E[\exp(i\lambda u)] = \exp(ia\lambda - \tfrac{1}{2}b\lambda^2) \tag{7}$$

and substituting (4), (5), and (6) in (7), with $\lambda = 1$, yields (3).

References

1. C. D. Firestone and J. E. Hanson, Uncorrelated Gaussian dependent random variables, this MONTHLY 70 (1963) 659–660.

2. H. Cramer, Mathematical methods of statistics, Princeton, 1946.

SHORT PROOF OF A RESULT ON DETERMINANTS

GEORGE MARSAGLIA, Boeing Scientific Research Laboratories

In the reference below, Bush proves the following:

THEOREM. *If A is an $(n-1)\times n$ matrix of integers such that the row sums are all zero, then $|AA'| = nk^2$ where k is an integer.* (A' means A transpose, and $|A|$ means determinant of A.)

The proof uses an interesting application of the Cauchy-Binet theorem, prefaced by a remark on how the unwary may be led astray if they improperly use the rules for manipulating rows of a determinant. It is possible, by partitioning A, to get a simpler proof—one which suggests generalizations and also provides the value of k. Write $A = (B, B\beta')$, where B is $(n-1)\times(n-1)$ and $\beta = (-1, -1, \cdots, -1)$ is $1\times(n-1)$. Then

$$\begin{aligned} |AA'| &= \left|(B, B\beta')\begin{pmatrix} B' \\ \beta B' \end{pmatrix}\right| = |BB' + B\beta'\beta B'| = |B(I + \beta'\beta)B'| \\ &= |B|^2 |I + \beta'\beta|. \end{aligned}$$

Since $|I + \beta'\beta| = n$, we have $|AA'| = n|B|^2$.

Reference

1. K. A. Bush, An interesting result on almost square matrices and the Cauchy-Binet Theorem, this MONTHLY, 69 (1962) 648–649.

Comparisons between subject areas are difficult because of differences in the distribution of faculty by ranks. For example, the median teacher of law is a full professor, since the estimated 1400 law faculty, 960 are full professors! In the case of all faculty on academic year contracts, 49% are full and associate professors, while in the case of mathematics teachers, only 44% are at these ranks. Also when comparing academic year and calendar year salaries, it should be noted that 35% of all faculty are on calendar year contracts, while only 25% of mathematics teachers are employed on such contracts.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. Solutions to problems appearing in previous issues of the Monthly should not be sent to Professor Starke.

ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before June 30, 1965.

E 1757. *Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pa.*

Show that the equation $t''=2t'$ has infinitely many solutions in triangular numbers t' and t''.

E 1758. *Proposed by Michael Goldberg, Washington, D. C.*

Arrange four radial lines of lengths $a \geqq b \geqq c \geqq d$ so that their convex envelope is maximized.

E 1759. *Proposed by M. T. L. Bizley, London, England*

In how many ways can n rooks be placed on an $n \times n$ chessboard so that no rook can take another and so that the n occupied squares are symmetrically placed relative to a given diagonal of the board?

E 1760. *Proposed by I. I. Kolodner, University of New Mexico*

$\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that $\{b_n\}$ converges and $b_n = a_n - \alpha a_{n+1}$. Show that $\{a_n\}$ converges if $|\alpha| > 1$, or if $|\alpha| < 1$ and $\{a_n\}$ is bounded.

E 1761. *Proposed by John Burke, University of Vermont*

Let A be an $n \times n$ matrix with $a_{ij} = j^{i-1}$. Prove that

$$\det A = (n-1)(n-2)^2 \cdots 2^{n-2}.$$

E 1762. *Proposed by Underwood Dudley, University of Michigan*

Evaluate
$$\sum_{\substack{m=1 \\ (m,n)=1}}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2 n^2}.$$

E 1763. *Proposed by E. O. Thorp, New Mexico State University*

A computer programmer attempts to shuffle N cards by the following process. Let i be a random variable integer uniformly distributed over the integers $1 \leqq i \leqq N$. Generate an i. Interchange the first card and ith card. Then generate another i. Interchange the second card with the ith card. Continue until the Nth card has been interchanged with the ith card. For which $N \geqq 2$ is the shuffling random?

E 1764. *Proposed by Michael Gemignani, University of Notre Dame*

Let G be a group with identity 1, and A a subgroup of G such that $(G-A) \cup \{1\}$ is also a group. Prove that either $A = \{1\}$, or $A = G$.

E 1765. *Proposed by Andy Vince, Stanford University*

Given a set of $n+1$ integers chosen from the set $1, 2, 3, \cdots, 2n$, it is always possible to choose two elements from the $n+1$ integers such that one divides the other.

SOLUTIONS OF ELEMENTARY PROBLEMS

Ratio of volume of inscribed sphere to polyhedron

E 1671 [1964, 316]. *Proposed by D. C. Duncan, East Los Angeles College*

If a sphere is inscribed in a trirectangular tetrahedron, show that the ratio of the volumes of the sphere and the tetrahedron is equal to the ratio of their areas. (Ref. inscription on the tomb of Archimedes.)

Solution by Martin J. Cohen, 419 S. El Camino Dr., Beverly Hills, California. The problem posed is the special case $n = 3$ of the known formulas

$$V_n = 2\pi^{n/2} r^n / [n\Gamma(n/2)] \quad \text{and} \quad S_n = 2\pi^{n/2} r^{n-1} / \Gamma(n/2)$$

for the volume and surface area of an n-dimensional sphere of radius r, and the

THEOREM. *If an n-dimensional sphere of radius r is inscribed in a polyhedron P, then $V(P)/S(P)=r/n$, where $V(P)$ and $S(P)$ denote the volume and surface area respectively of P.*

Proof. Let λ be an n-dimensional pyramid with base area B and altitude h. If A_x is the area of a cross-section of λ at a distance x from the base, then $A_x=(h-x)^{n-1}B/h^{n-1}$ since each dimension of the cross-section is $(h-x)/h$ times the corresponding dimension of the base. Integrating from 0 to h gives

$$V(\lambda) = \int_0^h \frac{(h-x)^{n-1}B}{h^{n-1}}\,dx = hB/n.$$

If P has faces with areas $A_1, A_2, \cdots, A_k$, then

$$V(P) = \frac{r}{n}\sum_{i=1}^{k} A_i = rS(P)/n,$$

since the solid bounded by P is the union of the k pyramids with base areas $A_1, A_2, \cdots, A_k$ and altitudes r, the radius of the inscribed sphere.

Also solved by J. W. Baldwin, Merrill Barnebey, M. T. L. Bizley, F. L. Bookstein, F. P. Callahan, Michael Goldberg, Ned Harrell, J. O. Hassler, Stephen Hoffman, J. D. Konhauser, E. S. Langford, Tung-Po Lin, Andrzej Makowski, D. C. B. Marsh, Gus Mavrigian, Walter Meyer, David Perin, P. A. Scheinok, C. W. Trigg, Simon Vatriquant, J. J. Weinkan, and the proposer.

Cohen also deduces the formula

$$1/r = \sum_{k=1}^{n} 1/c_k + \left(\sum_{k=1}^{n} 1/c_k^2\right)^{1/2}$$

for the radius r of the sphere inscribed in the n-dimensional polyhedron bounded by the planes $x_1=x_2=\cdots=x_n=0$ and $\sum_{i=1}^{n}x_i/c_i=1$, $c_1, c_2, \cdots, c_n>0$. Meyer shows that the square of the area of the face opposite the right trihedral angle of a trirectangular tetrahedron is equal to the sum of the squares of the areas of the remaining faces.

Prime Divisors of 2^p-1

E 1672 [1964, 317]. *Proposed by Guy Torchinelli, State University College at Buffalo*

Prove that each prime divisor of 2^p-1, where p is a prime, is greater than p. (It is a corollary that the number of primes is infinite.)

Solution by Francis P. Callahan, 632 Deaver Drive, Blue Bell, Pennsylvania. There is nothing to prove if $p=2$, so we assume that p is odd. Suppose then that $2^p\equiv 1 \pmod q$, where q is a prime less than or equal to p. q is clearly odd. By Fermat's Theorem, $2^{q-1}\equiv 1 \pmod q$. If $p=q$, we have the contradiction $2\equiv 1 \pmod q$. If $q<p$, then $ap+b(q-1)=1$ for appropriate integers a and b, and then $2=2^{ap+b(q-1)}\equiv 1 \pmod q$, a contradiction. Therefore, $q>p$.

Also solved by J. C. Abad, Joseph Arkin, Joseph Bechely, M. T. L. Bizley, J. A. Burslem, Leonard Carlitz, M. J. Cohen, G. C. Dodds, R. B. Eggleton, Gregory Forster, Arthur Greenspoon, Hajna Janos, Erwin Just and Norman Schaumberger (jointly), Sidney Kravitz, E. S. Langford, Andrzej Makowski, D. C. B. Marsh, M. G. Murdeshwar, Stanton Philipp, G. B. Purdy, Camilo Schmidt, A. M. Vaidya, R. L. Wilson, K. L. Yocum, Charles Ziegenfus, and the proposer.

The problem is not new: Abad and Just and Schaumberger cite the

THEOREM (Fermat, 1640). *If p is an odd prime, then each prime divisor of 2^p-1 is of the form $2kp+1$ with k a positive integer.*

[L. E. Dickson, *History of the Theory of Numbers*, vol. 1, Chelsea (1952), p. 12 and Shanks, *Solved and Unsolved Problems in Number Theory*, vol. 1, p. 19.]

A permutation of three positive numbers

E 1673 [1964, 317]. *Proposed by Alexander Oppenheim, University of Malaya*

The three positive numbers x, y, z lie between the least and greatest of the three positive numbers a, b, c. If $x+y+z=a+b+c$ and $xyz=abc$, show that, in some order, x, y, z are equal to a, b, c.

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* It is easily verified that $bc(x-a)(y-a)(z-a) = ca(x-b)(y-b)(z-b) = ab(x-c)(y-c)(z-c)$, each expression being equal to $abc(ab+ac+bc-xy-xz-yz)$.

If no factor in any of the three expressions is zero, then one expression has all positive factors and another expression has exactly three negative factors, which is a contradiction. Thus each expression has one zero factor, and the required result follows.

II. *Solution by R. B. Eggleton, Avondale College, Cooranbong, N.S.W., Australia.* Let $s(u)$ and $t(u)$ be cubics in u with positive roots a, b, c and x, y, z respectively. Put $\alpha=a+b+c=x+y+z$, $\beta=ab+bc+ca$, $\beta'=xy+yz+zx$ and $\gamma=abc=xyz$. Then

$$s(u) = u^3 - \alpha u^2 + \beta u - \gamma \quad \text{and} \quad t(u) = u^3 - \alpha u^2 + \beta' u - \gamma.$$

Since x, y, z are to lie in the range determined by the least and greatest of the positive numbers a, b, c, the graphs of $s(u)$ and $t(u)$ must intersect at some point $u=u_0$ in the specified range. Therefore $s(u_0)-t(u_0)=(\beta-\beta')u_0=0$. But $u_0>0$, so $\beta=\beta'$. Thus $s(u)$ and $t(u)$ are identical, so their roots are identical; that is, x, y, z are equal in some order to a, b, c.

III. *Solution by Michael Goldberg, 5823 Potomac Ave., N.W. Washington, D. C.* The convex cubic surface represented by the equation $xyz=abc$ intersects the plane $x+y+z=a+b+c$ in a convex cubic curve. This curve passes through the six points (a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a). These are the vertices of a hexagon whose opposite sides are each parallel to the same coordinate plane. Each of the other points of the curve is closer to one of the coordinate planes than any of the six indicated points, or is further away than any of these

points. Its coordinates do not fall within the interval of the values a, b, and c. Hence, only the indicated six points satisfy the conditions.

IV. *Solution by M. G. Murdeshwar, University of Alberta, Edmonton.* Without loss of generality, assume $a \leqq b \leqq c$ and $x \leqq y \leqq z$. From $x+z=a+b+c-y$ and $xz=abc/y$, we get

$$2z = a + b + c - y + \sqrt{(a + b + c - y)^2 - \frac{4abc}{y}},$$

$$2x = a + b + c - y - \sqrt{(a + b + c - y)^2 - \frac{4abc}{y}}.$$

Using $z \leqq c$ in the first of these equations and eliminating the radical, we find that $(y-a)(y-b) \geqq 0$. Similarly, the inequality $x \geqq a$ and the second equation gives $(y-b)(y-c) \geqq 0$. Since $a \leqq y \leqq c$, it follows that y is one of a, b, and c and then that $x=a$, $y=b$, and $z=c$.

Also solved by Joseph Arkin, J. A. Burslem, F. P. Callahan, Leonard Carlitz, M. J. Cohen, D. M. Hancasky, Stephen Hoffman, R. F. Jackson, Leroy Junker, E. S. Langford, D. C. B. Marsh, Stanton Philipp, Horvath Sandor, M. S. R. K. Sastry, Simon Vatriquant, Andy Vince, and the proposer.

Another characterization of prime numbers

E 1674 [1964, 317]. *Proposed by C. A. Nicol, University of South Carolina*

Prove that a necessary and sufficient condition that n be a prime is that $\sigma(n)+\phi(n)=n\,d(n)$, where $\sigma(n)$ is the sum of the divisors of n, $\phi(n)$ is the Euler totient of n, and $d(n)$ is the number of divisors of n.

I. *Solution by Martin J. Cohen, 419 S. El Camino Dr., Beverly Hills, California.* We have

$$\begin{aligned}\sum_{d|n} \sigma(d)\phi(n/d) &= \sum_{d|n} \phi(n/d) \sum_{\delta|d} \delta = \sum_{\delta d|n} \delta\phi\left(\frac{n}{\delta d}\right) \\ &= \sum_{d|n} d \sum_{\delta|n/d} \phi\left(\frac{n}{\delta d}\right) = \sum_{d|n} n = nd(n),\end{aligned}$$

since $\sum_{d|k} \phi(d) = k$. Thus

$$nd(n) - [\sigma(n) + \phi(n)] = \sum_{\substack{d|n \\ d\neq 1, d\neq n}} \sigma(d)\phi(n/d)$$

is zero if and only if n is a prime.

II. *Solution by J. A. Fridy, University of North Carolina.* The necessity is obvious. On the other hand, for each positive integer n we have

$$\sigma(n) = 1 + \sum_{\substack{m|n \\ m>1}} m \leqq 1 + \sum_{\substack{m|n \\ m>1}} n = 1 + n\{d(n) - 1\};$$

therefore $nd(n) \geqq \sigma(n) + n - 1 \geqq \sigma(n) + \phi(n)$, since $\phi(n) \leqq n - 1$. Assuming that the left and right members are equal, we have $\phi(n) = n - 1$, in which case n is prime.

This same argument can be used to show that, for $x > 0$, $\sigma_x(n) + \phi_x(n) = n^x d(n)$ characterizes the primality of n, where $\sigma_x(n) = \sum_{m|n} m^x$ and $\phi_x(n) = n^x \prod_{p|n} (1 - 1/p^x)$.

Also solved by A. N. Aheart, W. J. Blundon, J. A. Burslem, Leonard Carlitz, S. Chowla, S. Chowla and R. A. Smith (jointly), S. Chowla and A. M. Vaidya (jointly), G. C. Dodds, R. B. Eggleton, Hajna Janos, E. S. Langford, C. C. Lindner, P. J. McCarthy, Andrzej Makowski, D. C. B. Marsh, R. A. Mullikin, M. G. Murdeshwar, Stanton Philipp, Joseph Raitberg, Charles Seguin, S. B. Seidman, Frank Servas, Jr., Lawrence Taylor, Simon Vatriquant, Andy Vince, and the proposer.

Makowski calls attention to Elem. Math., 15 (1960), pp. 39–40, Aufgabe 339.

An inequality involving the altitudes and exradii of a triangle

E 1675 [1964, 317]. *Proposed by Andrzej Makowski, Warsaw, Poland*

Prove that $h_1h_2 + h_2h_3 + h_3h_1 \leqq r_1r_2 + r_2r_3 + r_3r_1$, where the h_i are the altitudes and the r_i are the exradii of a triangle.

I. *Solution by Mannis Charosh, New Utrecht High School, Brooklyn, New York.* The following formulas may be found in most college level geometry text books: (1) $\sum 1/h_i = \sum 1/r_i$; (2) Area of triangle $= \Delta = a_1a_2a_3/4R = a_ih_i/2 = \sqrt{rr_1r_2r_3}$ (R = circumradius); and (3) Euler's Theorem: $(TP)^2 = R(R-2r)$, where TP is the length of the segment joining the incenter to the circumcenter, so that $TP = 0$ if and only if the triangle is equilateral. From (3), $2r \leqq R$; therefore, from (2), $8r\Delta \leqq a_1a_2a_3$. Thus $r \prod (2\Delta/a_i) \leqq \Delta^2 \leqq rr_1r_2r_3$. Hence $r \prod h_i \leqq r \prod r_i$. Therefore, $(\prod h_i) \sum 1/h_i \leqq (\prod r_i) \sum 1/r_i$. Whence, applying (1), we obtain the given result and we have equality if and only if the triangle is equilateral.

II. *Solution by A. Oppenheim, University of Malaya.* Since $a_1h_1 = 2\Delta = 2r_1(s - a_1)$ where $2s = a_1 + a_2 + a_3$, the given inequality is equivalent to the inequality

$$(1) \qquad (a_2 + a_3 - a_1)(a_3 + a_1 - a_2)(a_1 + a_2 - a_3) \leqq a_1a_2a_3.$$

Now if the positive numbers x, y, z lie between the least and greatest of the positive numbers p, q, r and if $x + y + z \geqq p + q + r$, then $xyz \geqq pqr$; equality occurs if and only if (in some order) the numbers x, y, z are equal to the numbers p, q, r. (See E 1673.) Take x, y, z to be a_1, a_2, a_3, and p, q, r to be $a_2 + a_3 - a_1$ etc. The inequality (1) follows at once.

III. *Solution by J. Schopp, Budapest, Hungary.* It is possible to prove more; namely, it will be shown that $w_1w_2 + w_2w_3 + w_3w_1 \leqq r_1r_2 + r_2r_3 + r_3r_1$ holds where

w_i ($i=1, 2, 3$) are the lengths of the inner angle bisectors.

Let a_i ($i=1, 2, 3$) be the sides of the triangle, $s=\frac{1}{2}(a_1+a_2+a_3)$ and put $s_i=s-a_i$ ($i=1, 2, 3$). If the area of the triangle is denoted by t, then $t=r_is_i$ holds for $i=1, 2, 3$t; herefore $r_1r_2s_1s_2=t^2$ and, using the Hero's formula, we obtain $r_1r_2=ss_3$. Summing this and the corresponding other equalities we get

$$r_1r_2 + r_2r_3 + r_3r_1 = s^2.$$

It was shown by Leuenberger (Elem. Math., 17 (1962), pp. 45–46) that $w_i \leqq (s\cdot s_i)^{\frac{1}{2}}$, therefore $w_i\cdot w_2 \leqq s(s_1\cdot s_2)^{\frac{1}{2}}$. The well-known inequality between geometrical and arithmetical means gives

$$w_1w_2 \leqq s\,\frac{s_1+s_2}{2}\,.$$

Summation of the corresponding other inequalities yields $w_1w_2+w_2w_3+w_3w_1 \leqq s^2$, what we have stated. The equality sign holds if and only if $s_1=s_2=s_3$, i.e., in the case of an equilateral triangle.

The original inequality is a simple consequence of ours, since $h_i \leqq w_i$ ($i=1, 2, 3$) implies $h_1h_2 \leqq w_1w_2$ and similar inequalities, which give $h_1h_2+h_2h_3+h_3h_1 \leqq w_1w_2 + w_2w_3+w_3w_1$, where the equality sign appears again only in the case of an equilateral triangle.

Also solved by A. N. Aheart, Leon Bankoff ,W. J. Blundon, Leonard Carlitz, Ragnar Dybvik, Michael Goldberg, D. C. B. Marsh, M. G. Murdeshwar and V. K. Rohatgi (jointly), J. I. Nasser, A. Oppenheim (another solution), Stanton Philipp, Horváth Sándor, M. S. R. K. Sastry, Simon Vatriquant, and the proposer.

An ellipse generated by an ellipse

E 1676 [1964, 317]. *Proposed by J. C. Van Rhijn, Vollenhove, Netherlands*

Given an ellipse E with foci F_1 and F_2, a point P outside E, and a positive number f. PR_1 and PR_2 are tangents to E. Find the locus of P if $(PR_1)(PR_2) = f(PF_1)(PF_2)$.

Solution by Stanton Philipp, Seal Beach, California. Choose a rectangular coordinate system so that the equation of E is $b^2x^2+a^2y^2=a^2b^2$ and P is the point (x, y). Let ϕ be the angle R_1PR_2 and let K be the area of the triangle R_1PR_2. The following three statements (which are not difficult to prove) occur as exercises 43, 54, and 56 on pages 180, 182 of C. Smith, *Conic Sections* (1914):

$$K = \frac{(b^2x^2+a^2y^2-a^2b^2)^{3/2}}{b^2x^2+a^2y^2}, \qquad \tan\phi = \frac{2\sqrt{b^2x^2+a^2y^2-a^2b^2}}{x^2+y^2-a^2-b^2},$$

$$\cos\phi = \frac{x^2+y^2-a^2-b^2}{(PF_1)(PF_2)}\,.$$

But $K=\frac{1}{2}(PR_1)(PR_2)\sin\phi$ and $\sin\phi=\cos\phi\tan\phi$. It follows that

$$(PR_1)(PR_2)/[(PF_1)(PF_2)] = (b^2x^2 + a^2y^2 - a^2b^2)/(b^2x^2 + a^2y^2).$$

Thus, the locus exists if and only if $0<f<1$, in which case it is the ellipse $(1-f)(b^2x^2+a^2y^2)=a^2b^2$.

Also solved by A. N. Aheart, D. C. B. Marsh, M. G. Murdeshwar and V. K. Rohatgi (jointly), Simon Vatriquant, and the proposer.

A nonsingular matrix

E 1677 [1964, 317]. *Proposed by J. E. Potter, Massachusetts Institute of Technology*

If A and B are positive semidefinite symmetric matrices, prove that $C=I+AB$ is nonsingular, if I represents the identity matrix.

I. *Solution by Paul J. Nikolai, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio.* We assume that A and B are Hermitian. If $Cx=0$ for some vector x, then the Generalized Schwartz Inequality [F, Reisz and B. Sz.-Nagy, *Leçons d'Analyse Fonctionnelle*, Budapest (1952), 260] gives

$$0 \leqq (x, x)^2 = (x, -x)^2 = (x, ABx)^2 \leqq (x, Ax)(Bx, ABx) = -(x, Ax)(x, Bx) \leqq 0,$$

whence $x=0$.

II. *Solution by Mathura D. Sawhney, New York University.* If $Cx=0$ for some vector x, and A and B are Hermitian, then $ABx=-x$, $BABx=-Bx$, and $\bar{x}^TBABx=-\bar{x}^TBx$. The left-hand side of the last-written equation is nonnegative and the right-hand side is nonpositive. Therefore, $\bar{x}^TBx=0$ and, then $Bx=0$ and $ABx=-x=0$. Hence, C is nonsingular.

III. *Solution by David Carlson, Oregon State University.* We prove more: If A and B are positive semi-definite Hermitian matrices, then all eigenvalues of AB are real and nonnegative.

Proof. (i) If A is nonsingular (i.e., definite), then there exists a nonsingular K for which $A=KK^*$. Hence AB is similar to $K^{-1}(KK^*B)K=K^*BK$. As B is positive semi-definite, by Sylvester's Law of Inertia, so is K^*BK, and AB and K^*BK both have all real nonnegative eigenvalues.

(ii) If A is singular, then there exists a unitary U so that

$$U^*AU = C = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

with C_{11} nonsingular. Let $U^*BU=D=[D_{ij}]$, $i, j=1, 2$, partitioned conformably. Clearly C and D are Hermitian and positive semi-definite, and if we can prove the result for CD we are done. We have

$$CD = \begin{bmatrix} C_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} C_{11}D_{11} & C_{11}D_{12} \\ 0 & 0 \end{bmatrix},$$

so that the nonzero eigenvalues of CD are those of $C_{11}D_{11}$. As C_{11} and D_{11} are positive semi-definite, and C_{11} is nonsingular, we may complete the proof by applying (i).

Also solved by Henry Cox, Carl Evans and J. T. Fleck (jointly), W. J. Hartman, J. C. Hickman, L. N. Howard, and the proposer.

A well-known eigenvalue problem

E 1678 [1964, 317]. *Proposed by Reuben Hersh, Stanford University*

What are the eigenvalues of the nth order matrix having $2a$'s along the main diagonal, b's along the two diagonals bordering the main diagonal, and 0's everywhere else?

Solution by M. G. Murdeshwar and V. K. Rohatgi, University of Alberta, Edmonton. Let A_n denote the det $(\lambda I-A)$, where A is the given matrix of order n. Expanding A_{n+1}, we have $A_{n+1}=(\lambda-2a)A_n-b^2A_{n-1}$. On writing $\cos\theta=(\lambda-2a)/2b$, and $y_n=A_n/b^n$, this becomes $y_{n+1}-2y_n\cos\theta+y_{n-1}=0$ of which the general solution is $y_n=C\cos(n\theta)+D\sin(n\theta)$. From the initial conditions, $C=1$ and $D=\cot\theta$, so that

$$y_n = \cos(n\theta) + (\cot\theta)\sin(n\theta) = \frac{\sin(n+1)\theta}{\sin\theta}\cdot$$

The zeros of this expression are $\theta=k\pi/(n+1)$, $k=1, 2, \cdots, n$. Hence, the required eigenvalues are $\lambda=2a+2b\cos(k\pi/n+1)$, $k=1, 2, \cdots, n$.

Also solved by F. P. Callahan, C. G. Cullen, Murray Geller, D. R. Hayes, Sidney Heller, R. H. Hines, R. F. Jackson, P. G. Kirmser, J. D. E. Konhauser, E. S. Langford, G. K. Liebschner, D. C. B. Marsh, Gus Mavrigian, Carl F. McLaren, R. V. Moody, F. D. Parker, Stanton Philipp, H. B. Rosenstock, M. D. Sawhney, P. A. Scheinok, Simon Vatriquant, and the proposer.

Denoting the given matrix by $M_n(a, b)$, most solvers observed that each eigenvalue of $M_n(a, b)$ is given by $2a-bx$, where x is an eigenvalue of $M_n(0, 1)$. They then obtained the recurrence relation $P_n(x)=xP_{n-1}(x)-P_{n-1}(x)$ with $P_0(x)=1$ and $P_1(x)=x$ for the characteristic equation of $M_n(0, 1)$, and observed that this recurrence relation defines the Chebyshev polynomials of the second kind.

The problem is apparently a standard one in quantum mechanics, solid state physics, and electrical engineering: M. Born and Th. v. Karman, Physikalische Zeitschrift, 13 (1912) p. 297; L. Brillouin, *Wave Propagation in Periodic Structures*, McGraw-Hill, 1946; W. Kauzmann, *Quantum Chemistry*, Academic Press, 1957; Rayleigh, *Theory of Sound*, vol. 1, Dover (1945), p. 174.

Kirmser found the problem as a special case of Problem 38 on page 215 of R. Bellman, *An Introduction to Matrix Analysis*, McGraw-Hill, 1960. He further calls attention to similar problems in F. B. Hildebrant, *Methods of Applied Mathematics*, Prentice-Hall (1952), pp. 335 and 366, and C. R. Wylie, Jr., *Advanced Engineering Mathematics*, 2nd. ed., McGraw-Hill (1960), p. 175. Other references: Mirsky, *Introduction to Linear Algebra*, chap. 1, and J. Todd, *The condition of a certain matrix*, Proc. Cambridge Phil. Soc., 46, Part 1, 116–118.

Probability that an integer-valued determinant is odd

E 1679 [1964, 317]. *Proposed by Harry Lass, California Institute of Technology*

Given an $n \times n$ matrix with randomly selected integer elements, what is the probability that the absolute value of the determinant of the matrix is an odd integer?

Solution by E. S. Langford, Autonetics Division, North American Aviation. We consider the following generalization. Let A be an $n \times n$ matrix with "random" integer elements. Let p be a fixed prime. Then what is $P(p, n)$, the probability that $\det A$ is not congruent to 0 (mod p)? (By symmetry, the probability that $\det A$ is congruent to k (mod p), for $k \neq 0$, is $P(p, n)/(p-1)$.) By "random," we mean that the numbers in A are equally likely to be congruent to anything (mod p).

Let $GF^n(p)$ be the vector space of n-tuples of elements of $GF(p)$ over $GF(p)$, with the usual operations. It is not difficult to see that $P(p, n)$ is the number of automorphisms of $GF^n(p)$ divided by the total number of endomorphisms; the latter number is evidently p^{n^2}.

Consider the basis $\{e_k\}$ for $GF^n(p)$, where e_k consists of 0's except for a 1 in the kth place. Let A be an automorphism of $GF^n(p)$. Let E_k be the subspace spanned by $Ae_1, Ae_2, \cdots, Ae_k$. Since $\{Ae_k\}$ must also be a basis for $GF^n(p)$,

$$\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_n$$

(strict inclusion). Since the order of E_k (for any k) must divide p^n, evidently the order of E_k is p^k.

Thus one has $(p^n - 1)$ choices for Ae_1, $(p^n - p)$ choices for $Ae_2, \cdots, (p^n - p^{n-1})$ choices for Ae_n. Since the automorphism is completely defined by its effect on the basis $\{e_k\}$, this gives the total number of automorphisms of $GF^n(p)$. Thus:

$$P(p, n) = p^{-n^2} \prod_{k=0}^{n-1} (p^n - p^k) = \prod_{k=1}^{n} (1 - p^{-k}).$$

Also solved by A. H. Booker and E. A. Flinn (jointly), Joel Brawley, Jr., Carl Evans and J. T. Fleck (jointly), D. C. B. Marsh, and the proposer.

An anomaly concerning Picard successive approximations

E 1680 [1964, 317]. *Proposed by R. A. Struble, University of North Carolina*

Show that the limit of a uniformly convergent subsequence of Picard successive approximations need not satisfy the associated differential equation.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* For $y' = f(x, y)$, where

$$f(x, y) = \begin{Bmatrix} x^{1/2}/y & \text{if } 0 \leqq x \leqq 1,\ 0 < y < 1, \\ 1 & \text{if } 0 \leqq x \leqq 1,\ 0 = y, \end{Bmatrix}$$

under the initial condition $y(0)=0$, the Picard approximations are $y_{2n}=2^n x^{1/2}$, $y_{2n+1}=2^{-n}x$. The subsequence with odd subscripts converges uniformly to 0, which does not satisfy the differential equation.

II. *Solution by the proposer.* The sequence of successive approximations for the equation (i) $dx/dt=-|x|^\alpha$, $0<\alpha<1$ beginning with $x_0=t$ contains subsequences uniformly convergent on $[0,\beta]$ for any $\beta>0$ which do not converge to solutions of (i).

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before August 31, 1965.

5240. [1964, 1047] *Correction. Proposed by D. G. Bourgin, University of Illinois*

Prove the following identity:

$$\frac{4\Gamma(\frac{3}{4})\Gamma(n+\frac{5}{4})}{\Gamma(2(n+1))}=\sum_{j=0}^{2n-1}(-1)^{[j/2]}\frac{\Gamma(\frac{j}{2}+\frac{3}{4})\Gamma(n-\frac{j}{2}+\frac{1}{4})}{(j+2)\Gamma(j+1)\Gamma(2n-j)}.$$

5261. *Proposed by M. Rajagopalan, University of Illinois*

Let R^2 be the complex plane with the usual addition operation. Let S be the set $\{x+i\sin x \mid 0\leqq x\leqq\pi\}$. What is the additive group generated by S?

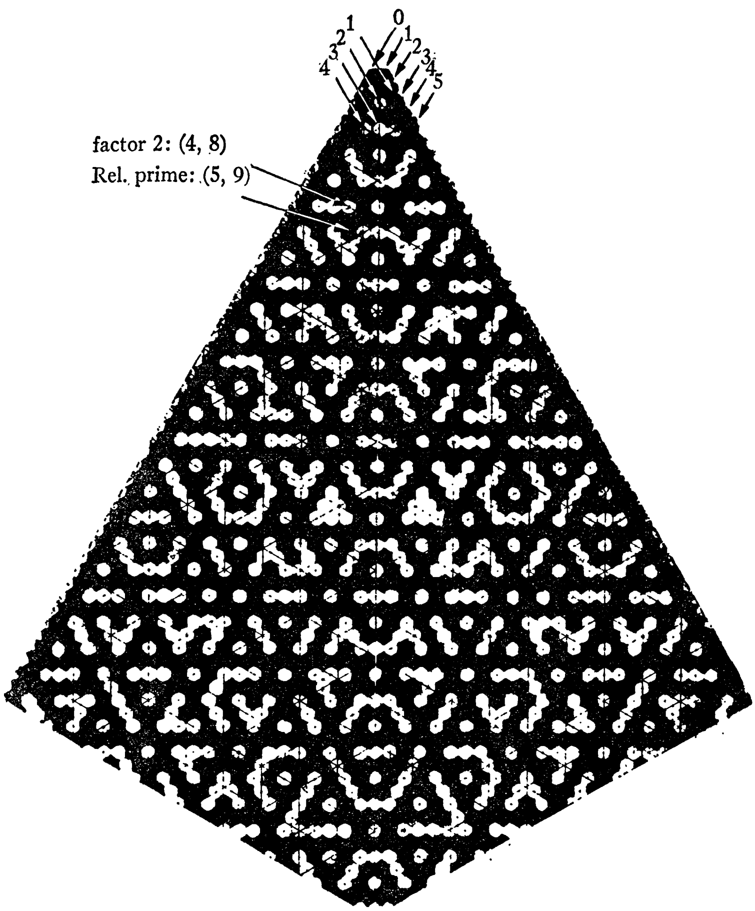
5262. *Proposed by George Bergman, Harvard University*

The Dirichlet Unit Theorem says that, if R is an integrally closed integral domain with finite additive basis, over the integers, then the multiplicative group of units in R is finitely generated, with rank $m-1$ (and torsion subgroup, of course, equal to the group of roots of unity in R); where m is the number of embeddings of R in the complexes, modulo complex conjugacy. (E.g., $m=2$ for $Z[\sqrt{2}]$, $m=1$ for $Z[\sqrt{-2}]$.)

Prove that the condition that R be integrally closed can be omitted without affecting the conclusion.

5263. *Proposed by Art Winfree, Cornell University*

Consider an angular region, of angle 60°, filled with a honeycomb of regular hexagons (that is, one sixth of the complete regular tessellation $\{6, 3\}$). Number the rows of hexagons 1, 2, 3, $\cdots$ from one side of the angle and again 1, 2, 3, $\cdots$ from the other side, thus assigning a pair of coordinate integers to each hexagon.



Prove or disprove that the hexagons with relatively prime coordinates (constituting $6/\pi^2$ of the whole) form a connected set, whereas those whose coordinates have a common factor greater than 1 occur in isolated clusters. (See the figure where the construction is carried out to about 50 by 50.)

5264. *Proposed by D. E. Knuth, California Institute of Technology*

Let V be a vector space over the real numbers, and let the operator $\|x\|$ be defined for all $x \in V$, satisfying the following conditions:

(1) $\|x\|$ is a nonnegative real number.

(2) $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$ (parallelogram law).

Prove that the triangle inequality holds:

$$\|x + y\| \leqq \|x\| + \|y\|.$$

5265. *Proposed by E. Ehrhart, Strasbourg, France*

For which integral values of a, b, c is the following sum defined and equal to an integer:

$$\sum_{n=1}^{a-1} \frac{\sin (bn\pi/a)}{\sin (cn\pi/a)}.$$

5266. *Proposed by J. D. Pryce, University of Newcastle, England*

Let F be a positive, nondecreasing, convex function on $[0, \infty)$ with $F(0)=0$ and with $y=x-\alpha$ as an asymptote, $[\alpha>0]$. It is easily seen that the function $H(x)=\frac{1}{2}F(F(2x))$ again has $y=x-\alpha$ as asymptote. Prove $H(x)\leqq F(x)$ for all x.

Hence prove that if we define a sequence $\{F_n\}$ by

$$F_1 = F, \qquad F_n(x) = \tfrac{1}{2}F_{n-1}(F_{n-1}(2x)),$$

then every F_n is a positive nondecreasing convex function and $F_n(x) \to \max(0, x-\alpha)$ uniformly on $[0, \infty)$.

5267. *Proposed by Paul Erdös, Israel Institute of Technology*

Let there be given a system of circles of radius 1 in the plane such that every line meets at least one of them. Prove that, for every k, there is a line which meets at least k of them.

5268. *Proposed by V. H. Keiser, University of Colorado*

(a) A transitive nilpotent permutation group which is not of prime degree is imprimitive.

(b) A primitive permutation group which is not of prime degree has no center.

5269. *Proposed by Michael Gemignani, University of Notre Dame*

Prove that any first countable topological space in which every compact subset is closed is T_2.

SOLUTIONS OF ADVANCED PROBLEMS

Unitary modules over division rings

5145 [1963, 1014]. *Proposed by Seth Warner, Duke University*

Prove that if E is a nonzero unitary module over a semi-simple ring K with identity, then every set of generators of E contains a basis of E if and only if K is a division ring.

Solution by Bertram Walsh, University of California, Los Angeles. It is well known that every unitary module (left vector space) over a division ring has the property that every set of generators contains a basis. Suppose that the

unitary left-K-module E has the property in question; then so does every direct summand of E, for if $E=E_1\oplus E_2$ and $\{x_i\}_{i\in I}$ generates E_1, then $\{x_i\}\cup E_2$ generates E. E consequently contains a basis $\{x_i\}_{i\in J}\cup\{y_k\}_{k\in L}$ where $J\subseteq I$ and each $y_k\in E_2$, and it follows easily that $\{x_i\}_{i\in J}$ is a basis for E_1. Since E is certainly free and thus has direct summands left-K-isomorphic to K itself, K must have the property that every set of left-K-generators of K contains a left-K-basis of K.

But let $\{e_i\}_{i\in A}$ be a left-K-basis of K. Then $1=\sum_{i\in A} a_ie_i$ for uniquely determined a_i's, almost all of which are zero. Then for any $j\in A$ we have

$$e_j = e_j\cdot 1 = \sum_{i\in A} (e_ja_i)e_i$$

and so (1) $e_ja_i=0$ if $i\neq j$, and (2) $e_ja_j=1$. From (2) it is clear that a_j generates K and is thus a basis of K. Hence a_j cannot be a right zero divisor; interchanging the roles of i and j in (1) we see that $e_i=0$ for $i\neq j$. Thus $\{e_i\}_{i\in A}$ has exactly one element e, and there exists a with $ea=1=ae$.

Now, given any nonzero $b\in K$, by semisimplicity there exists a maximal left ideal $M\subset K$ with $b\notin M$. Then $\{b\}\cup M$ generates K; thus K contains a basis. But we have just seen that this basis must contain exactly one element, and it must be invertible, hence not in M. But then this element is b, and b is invertible as desired.

Also solved by Earl Willard and James Murtha, and by the proposer.

Integral of functions in $L_1(-\infty,+\infty)$

5171 [1964, 215]. *Proposed by P. T. Bateman and L. A. Rubel, University of Illinois.*

Suppose that f is a real-valued function belonging to $L_1(-\infty,+\infty)$. Put $H(t)=\sum' tf(kt)$, where $\sum'$ indicates that the summation is over all nonzero integers k.

(1) Prove that $\sum'|f(kt)|<\infty$ for almost all positive t.
(2) Prove that

$$\operatorname{ess}\liminf_{t\to 0+} H(t) \leqq \int_{-\infty}^{\infty} f(x)dx \leqq \operatorname{ess}\limsup_{t\to 0+} H(t).$$

(3) Construct an f for which

$$\operatorname{ess}\liminf_{t\to 0+} H(t) \neq \operatorname{ess}\limsup_{t\to 0+} H(t).$$

Solution by the proposers. Let D denote the characteristic function of the open interval $(-1,+1)$. Since $\sum' f(kt)$ is a function of t, we have formally for any positive λ

$$(*)\qquad \begin{aligned}\frac{1}{\lambda}\int_0^\lambda H(t)dt &= \frac{1}{\lambda}\int_0^\lambda t\{\textstyle\sum' f(kt)\}dt\\ &= \frac{1}{2\lambda}\int_{-\infty}^{\infty}|t|\,D\left(\frac{t}{\lambda}\right)\{\textstyle\sum' f(kt)\}dt\\ &= \frac{1}{2\lambda}\sum'\frac{1}{k^2}\int_{-\infty}^{\infty}|x|\,D\left(\frac{x}{k\lambda}\right)f(x)dx\\ &= \int_{-\infty}^{\infty}K\left(\frac{x}{\lambda}\right)f(x)dx,\end{aligned}$$

where

$$K(u) = \frac{|u|}{2}\sum'\frac{1}{k^2}\,D\left(\frac{u}{k}\right) = |u|\sum_{k>|u|}^{\infty}\frac{1}{k^2}\cdot$$

Since

$$K(u) < |u|\sum_{k>|u|}^{\infty}\left(\frac{1}{k-\frac{1}{2}} - \frac{1}{k+\frac{1}{2}}\right) = \frac{|u|}{[|u|]+\frac{1}{2}} < 2,$$

the calculation in (*) is justified by the absolute convergence of the end-result.

Since the same calculation as in (*) can also be made with $|f(x)|$ in place of $f(x)$, we conclude that

$$\int_0^\lambda t\,\{\textstyle\sum'|f(kt)|\}\,dt$$

is finite. Thus $\sum'|f(kt)| < +\infty$ for almost all t in $(0, \lambda)$. Since λ is arbitrary, (1) is established.

Since

$$K(u) > |u|\sum_{k>|u|}^{\infty}\frac{1}{k(k+1)} = \frac{|u|}{[|u|]+1},$$

we see that $\lim_{u\to+\infty}K(u)=1$. Thus, by the Lebesgue dominated convergence theorem

$$\lim_{\lambda\to 0+}\int_{-\infty}^{\infty}K\left(\frac{x}{\lambda}\right)f(x)dx = \int_{-\infty}^{\infty}f(x)dx.$$

Hence (*) gives

$$\lim_{\lambda\to 0+}\frac{1}{\lambda}\int_0^\lambda H(t)dt = \int_{-\infty}^{\infty}f(x)dx,$$

and (2) follows at once.

To obtain (3) consider the particular f defined as follows: Let $f(x)=n^2$ if $n^{-1}\leqq x\leqq n^{-1}+n^{-4}$ for some positive integer n, and let $f(x)=0$ otherwise. Then if $n^{-1}\leqq t\leqq n^{-1}+n^{-4}$ for some positive integer n, we have $H(t)\geqq tf(t)=tn^2\geqq n$. Thus

$$\operatorname{ess}\limsup_{t\to 0+} H(t) = +\infty.$$

On the other hand

$$\int_{-\infty}^{\infty} f(x)dx = \sum_{n=1}^{\infty}\frac{n^2}{n^4} < +\infty,$$

so that, by (2)

$$\operatorname{ess}\liminf_{t\to 0+} H(t) < +\infty.$$

We note finally that assertions (1) and (2) may also be established in a somewhat different way by combining Lemmas 2 and 3 of Bateman's paper, *The Minkowski-Hlawka theorem in the geometry of numbers*, Archiv der Mathematik, 13 (1962) 357–362.

Integral domains embedded in primitive rings

5172 [1964, 216]. *Proposed by Kwangil Koh, University of North Carolina*

An associative ring R, not necessarily commutative, is said to be an integral domain if it has no zero divisor. It is known that any commutative integral domain can be embedded in a field [1], and there is an integral domain which cannot be embedded in a skew field [2]. Prove that an integral domain can be embedded in a primitive ring (a ring with faithful irreducible right module [3]).

Solution by George Bergman, Harvard University. We shall say that a ring R "has exact characteristic p" if p is a nonnegative integer such that for every integer n and nonzero $r\in R$, $nr=0$ implies $p|n$. It is immediate that an exact characteristic must be a prime or 0, and that any integral domain has an exact characteristic.

Any ring R with an exact characteristic can be embedded in an algebra R' over a field F. If the characteristic is $p\neq 0$, R is indeed already an algebra over the field of integers mod p. If R has exact characteristic 0, then $R\otimes_Z Q$ is an algebra over the field Q of rationals, and R is embedded therein.

Let $\overline{R}$ be the ring of all linear maps (written as acting on the right) of the algebra R' as an F-vector-space into itself. Since R' is an irreducible right $\overline{R}$-module, $\overline{R}$ is primitive. Finally, R' and R are embedded in $\overline{R}$.

It is not hard to show that having an exact characteristic is a necessary as well as sufficient condition for R to be embeddable in a primitive ring.

It seems, then, that this embeddability property is not strictly a generalization of embeddability in a field, since there are many commutative rings that satisfy it, and yet have zero divisors.

Also solved by E. R. Gentile (Argentina), and by the proposer.

References

1. Birkhoff and MacLane, *A Survey of Modern Algebra*, New York, 1947.
2. A. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann., 113 (1936) 689–691.
3. Nathan Jacobson, *Structure of rings*, Amer. Math. Soc., Colloq. Publ., 37, 1956.

An integral inequality

5173 [1964, 216]. *Proposed by James E. Potter and Robert Fitzgerald, Massachusetts Institute of Technology*

Given that $v(t)$ is an n-dimensional vector valued function, that $P(t)$ is a one parameter family of $n\times n$ positive definite matrices, that μ is a positive measure on the real line, and that $v(t)$ and $P^{-1}(t)$ are in $L^1(\mu)$, show that

$$\int v^*(t)P(t)v(t)\mu(dt) \geqq \int v^*(t)\mu(dt)\left(\int P^{-1}(t)\mu(dt)\right)^{-1}\int v(t)\mu(dt),$$

where * denotes conjugate transpose.

Solution by L. J. Wallen, Stevens Institute of Technology. We may as well consider the problem in any n-dimensional unitary space V with an inner product (,). For $w_1, w_2\in V$ and any positive definite Q, Schwartz's inequality gives

$$\begin{aligned}|(w_1, w_2)|^2 &= |(Q^{-1}Qw_1, w_2)|^2 \leqq (Q^{-1}Qw_1, Qw_1)\cdot(Q^{-1}w_2, w_2)\\ &= (Qw_1, w_1)\cdot(Q^{-1}w_2, w_2).\end{aligned}$$

Then for any constant vector $w\in V$,

$$\begin{aligned}\left|\left(\int v(t)\mu, w\right)\right|^2 &\leqq \left\{\int |(v(t), w)|\,\mu\right\}^2\\ &\leqq \left\{\int (P(t)\,v(t), v(t))^{1/2}\cdot(P^{-1}(t)w, w)^{1/2}\mu\right\}^2\\ &\leqq \left\{\int (P(t)\,v(t), v(t))\mu\right\}\cdot\left\{\left(\left[\int P^{-1}(t)\mu\right]w, w\right)\right\}.\end{aligned}$$

The result follows on setting $w=[\int P^{-1}(t)\mu]^{-1}\int v(t)\mu$.

Also solved by the proposers.

Derivates of slowly varying functions

5174 [1964, 216]. *Proposed by Ranko Bojanic, University of Notre Dame*

In A. Zygmund's Trigonometric Series, v. 1, Ch. V, a positive and continuous function $l(x)$ defined for $x\geqq x_0$ is called a slowly varying function if for every $\delta>0$ there exists $x_\delta\geqq x_0$ such that

$$x^\delta l(x) \text{ increases and } x^{-\delta} l(x) \text{ decreases if } x \geqq x_\delta. \tag{1}$$

Prove that a positive and continuous function $l(x)$ is slowly varying in the above sense if and only if

$$\begin{aligned} -\infty < D_+ l(x) \leqq D^+ l(x) < +\infty \qquad & (x \geqq x_1) \\ x D_+ l(x) = o(l(x)) \quad \text{and} \quad x D^+ l(x) = o(l(x)) \qquad & (x \to \infty). \end{aligned} \tag{2}$$

Here $D_+ l(x)$ and $D^+ l(x)$ are the lower and the upper right hand derivatives of $l(x)$ respectively.

Solution by Martin J. Cohen, Beverly Hills, California. Suppose that $l(x)$ is slowly varying. Then, with $x \geqq x_1$, for any $\delta > 0$, $(x+h)^\delta l(x+h) - x^\delta l(x) > 0$ and $(x+h)^{-\delta} l(x+h) - x^{-\delta} l(x) < 0$ or $(1+h/x)^\delta l(x+h) - l(x) > 0$ and $(1+h/x)^{-\delta} l(x+h) - l(x) < 0$. We now assume $\delta < 1$ and $|h| < x$ and use the easily proved inequalities $(1+\gamma)^\delta < 1 + 2\delta\gamma$, $(1+\gamma)^{-\delta} > 1 - \delta\gamma$, valid for $0 < \delta < 1$ and $0 < \gamma < 1$.

We then have

$$(1 + 2\delta h/x) l(x + h) - l(x) > 0 \quad \text{and} \quad (1 - \delta h/x) l(x + h) - l(x) < 0$$

or

$$\frac{l(x + h) - l(x)}{h} > \frac{-2\delta l(x + h)}{x}, \qquad \frac{l(x + h) - l(x)}{h} < \frac{\delta l(x + h)}{x}$$

or, upon letting $h \to 0$,

$$D_+ l(x) \geqq -2\delta l(x)/x, \qquad D^+ l(x) \leqq \delta l(x)/x$$

since l is continuous. Therefore

$$-2\delta l(x) \leqq x D_+ l(x) \leqq x D^+ l(x) \leqq \delta l(x).$$

δ may be chosen arbitrarily small, hence (1) implies (2).

On the other hand, suppose that $-\infty < D_+ l(x) \leqq D^+ l(x) < \infty$ and $x D_+ l(x) = o(l(x))$, $x D^+ l(x) = o(l(x))$. This means, for any $\epsilon > 0$, for large enough x, and for all h, $0 < h < h_\epsilon$, we have

$$-\epsilon l(x) < x \cdot \frac{l(x + h) - l(x)}{h} < \epsilon l(x). \tag{*}$$

Suppose that, for some $\delta > 0$ and some x_0, $x^\delta l(x)$ decreases at x_0. Then for $0 < h_1 < h_0$, $(x_0 + h_1)^\delta l(x_0 + h_1) - x_0^\delta l(x_0) \leqq 0$, or $(1 + h_1/x_0)^\delta l(x_0 + h_1) - l(x_0) \leqq 0$, or

$$(1 + \delta h_1/x_0) l(x_0 + h_1) - l(x_0) \leqq 0$$

or

$$\frac{l(x_0 + h_1) - l(x_0)}{h_1} \leqq \frac{-\delta l(x_0 + h_1)}{x_0}.$$

But if this is true for x_0 large enough so that (*) holds, then

$$\frac{-\delta l(x_0+h_1)}{x_0} > -\frac{\epsilon l(x_0)}{x_0}$$

or $\epsilon l(x_0) > \delta l(x_0+h_1)$, which is a contradiction for small enough ϵ since l is continuous and positive. A similar contradiction may be obtained from the assumption that $x^{-\delta}l(x)$ increases at a point.

That (2) implies (1) now follows from the continuity of $l(x)$.

Also solved by E. S. Langford, and by the proposer.

Oscillations of sections in a Riemann integral

5175 [1964, 216]. *Proposed by Ranko Bojanić, University of Notre Dame*

Let f be continuous for all $x \geqq 0$ and

$$\eta(x) = \sup_{a\geqq 0}\left(\int_x^{x+a} f(t)dt\right).$$

Then $\eta(x)\to 0 (x\to\infty)$ if and only if there exists a Riemann integrable function g such that $f(x) \leqq g(x)$ for all x and $\int_0^\infty g(t)dt$ exists.

I. *Solution by Leroy Junker, Stanford University.* First, suppose we are given such a function g. Then

$$0 \leqq \eta(x) \leqq \sup_{\alpha\geqq 0}\left(\int_x^{x+\alpha} g(t)dt\right) \to 0 \qquad (x\to\infty)$$

and hence $\eta(x)\to 0$ as $x\to\infty$.

Conversely, suppose that $\eta(x)\to 0$ as $x\to\infty$. Then we have several possibilities:

(a) If $\int_0^\infty f(t)dt$ exists and is finite, let $g(x) \equiv f(x)$.

(b) If $\int_0^\infty f(t)dt = +\infty$, this would contradict the assumption that $\eta(x)\to 0$.

(c) If $\int_0^\infty f(t)dt$ were undefined, the assumption would be contradicted also, because this would mean that there were numbers $p<q$ such that, for some sequence $\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots$ approaching infinity, $\int_0^{\alpha_n} f(t)dt \leqq p$ and $\int_0^{\beta_n} f(t)dt \geqq q$, so that

$$\eta(\alpha_n) \geqq \int_{\alpha_n}^{\beta_n} f(t)dt \geqq q-p > 0,$$

contradicting the assumption that $\eta(x)\to 0$.

(d) The only remaining possibility is $\int_0^\infty f(t)dt = -\infty$.

If this is true, let $f^+(t) = \max(f(t), 0)$ and $f^-(t) = \min(f(t), 0)$. We define the sequence $\{x_n\}_{n=0}^\infty$ inductively as follows: $x_0=0$, and $x_{n+1}=x_n+k_n$, where k_n is the smallest positive integer such that

$$\int_{x_n}^{x_n+k_n} f(t)dt \leqq 0.$$

(This always exists since $\int_0^\infty f(t)dt = -\infty$.) In the interval $[x_n, x_{n+1})$, define $g(t) = f^+(t) + \lambda_n f^-(t)$, with $\lambda_n \in [0, 1]$ chosen so that

$$\int_{x_n}^{x_{n+1}} g(t)dt = 0.$$

To complete the proof we note that $g(t) \geqq f(t)$, $\int_0^\infty g(t)dt = 0$.

II. *Solution by D. Ž. Djoković, University of Belgrade, Yugoslavia.* To prove the existence of a $g(x)$, given $\eta(x) \to 0$, we note that, given $\epsilon > 0$ there exists $\alpha \geqq 0$ such that $0 \leqq \eta(x) < \epsilon (x \geqq \alpha)$. For $x > \alpha$ we have $\int_0^x f(t)dt < \int_0^x f(t)dt + \epsilon$, implying that $\int_0^x f(t)dt$ is bounded from above. We set

$$h(x) = \sup_{y \geqq x} \int_0^y f(t)dt.$$

This function is monotone decreasing and continuous. Moreover, the derivative $h'(x) (\leqq 0)$ exists except for a finite or countable number of points. We can take $g(x) = f(x) - h'(x) \geqq f(x)$. Then we have

$$\begin{aligned} \int_0^x g(t)dt &= \int_0^x f(t)dt - h(x) + h(0) \\ &= \int_0^x f(t)dt - \sup_{y \geqq x} \int_0^y f(t)dt + h(0) \\ &= h(0) - \sup_{y \geqq x} \int_x^y f(t)dt = h(0) - \eta(x) \to h(0) \quad (x \to \infty). \end{aligned}$$

Hence the Riemann integral $\int_0^\infty g(t)dt$ exists.

Also solved by N. G. de Bruijn (Netherlands), P. G. Engstrom, and D. J. Schaefer.

Dependence of rows and columns in a matrix

5176 [1964, 217]. *Proposed by Orrin Frink, Pennsylvania State University*

Let there be given a square array of elements of a division ring (e.g. quaternions). Prove that if the rows of the array are left linearly dependent, then the columns are right linearly dependent.

Solution by D. Ž. Djoković, University of Belgrade, Yugoslavia. Let the left linear relation between the rows be

$$\sum_{i=1}^{n} \lambda_i r_i = 0 \quad (\text{with some } \lambda_i \neq 0), \tag{1}$$

where n is the number of rows in the given array, and r_i is the ith row vector. Suppose that the columns of the array are right linearly independent. We consider the following two types of elementary transformations:

(a) addition of $c_i\mu$ to c_j where c_k is the kth column vector,

(b) transposition of c_i and c_j.

It is obvious that the new row vectors (obtained by the elementary transformations) satisfy the same linear relation (1) and the new column vectors are still right linearly independent. By applying the elementary transformations we can reduce the given array to the lower triangular form with nonzero entries on the main diagonal. But such an array cannot satisfy (1). This contradiction insures that right linear dependence of the columns must obtain.

Also solved by E. F. Assmus, Jr., Hwa S. Hahn, J. van Ijzeren (Netherlands), J. G. Mauldon (England), M. H. Murdeshwar, Veselin Peric (Yugoslavia), R. F. Rinehart, W. C. Waterhouse, G. P. Weller, Oswald Wyler, and the proposer.

Several contributors pointed out that the problem was in essence that column rank is equal to row rank in a square matrix with elements from a skew field. References cited include Emil Artin, *Geometric Algebra*, Ch. I; Nathan Jacobson, *Lectures in Abstract Algebra*, vol. II.

A potential integral independent of a parameter

5177 [1964, 217]. *Proposed by P. R. Vein, Leatherhead, Surrey, England*

Find $q(x)$ such that the integral

$$u = -\int_{-1}^{1} q(x) \log \left\{x^2 - 2x \cosh b(r^2 - \sinh^2 b)^{1/2} + r^2\right\} dx \tag{1}$$

is independent of r in the interval $\sinh b \leqq r \leqq \cosh b$.

Solution by the proposer. u can be interpreted physically as the potential at a point r units distant from the origin on the elliptic cylinder $(x/\cosh b)^2 + (y/\sinh b)^2 = 1$, due to a nonuniform distribution of electric charge of density $q(x)$ per unit length perpendicular to the xy-plane between the foci ± 1. The potential is

$$u = -2\int_{-1}^{1} q(x) \log (x^2 - 2xr\cos\theta + r^2)^{1/2} dx$$

and this is the same as (1) when the equation of the cylinder is written in polar form: $r\cos\theta = \cosh b(r^2 - \sinh^2 b)^{1/2}$.

Now, if u is to be independent of r, the elliptic cylinder must be an equipotential surface. But it is known from conformal transformation theory that when a family of confocal ellipses represents a family of equipotential surfaces, the charge density on the flat strip lying between ± 1, i.e., the degenerate ellipse, is

$$q(x) = q/\pi(1 - x^2)^{1/2},$$

q being the total charge per unit length perpendicular to the xy-plane between

the foci. This must therefore be the solution of (1).

The solution may also be checked by direct calculation of the potential u for the given $q(x)$.

Combinatorial sums

5178 [1964, 217]. *Proposed by Robert Breusch, Amherst College*

Find A_i ($i=0, 1, 2, \cdots$) if for every nonnegative integer n,

$$\sum_{i=0}^{n}\binom{2n-2i}{n-i}A_i=\binom{2n+1}{n}\quad\left[\text{with}\binom{0}{0}=1\right].$$

I. *Solution by M. T. L. Bizley, London, England.* Let

$$f(x) = A_0 + A_1x + A_2x^2 + \cdots.$$

Then, since

$$\binom{2n-2i}{n-i}$$

is the coefficient of x^{n-i} in $(1-4x)^{-1/2}$,

$$\sum_{i=0}^{n}\binom{2n-2i}{n-i}A_i$$

is the coefficient of x^n in $(1-4x)^{-1/2}f(x)$. But by hypothesis this coefficient is equal to

$$\binom{2n+1}{n},$$

which is the coefficient of x^n in $\{(1-4x)^{-1/2}-1\}/2x$.

Since this is true for all nonnegative integral values of n, we have

$$(1-4x)^{-1/2}f(x) = \{(1-4x)^{-1/2}-1\}/2x.$$

Therefore $f(x)=\{1-(1-4x)^{1/2}\}/2x$ and $A_i=\binom{2i}{i}/(i+1)$.

II. *Solution by M. T. Bird, San Jose State College.* The values of A_i are given by the equation $A_i=\binom{2i}{i}/(i+1)$. This fact is established by using the identity

$$\sum_{i=0}^{n}\binom{2n-2i}{n-i}\binom{2i}{i}\frac{1}{i+1}=\frac{1}{2}\binom{2n+2}{n+1}=\binom{2n+1}{n}.$$

The foregoing identity is a consequence of placing $j=0$ in the more general form

$$\sum_{i=j}^{n}\binom{2n-2i}{n-i}\binom{2i}{i}\frac{1}{i+1}=\frac{n-j+1}{2(n+1)}\binom{2j}{j}\binom{2n+2-2j}{n+1-j}$$

which is valid for all nonnegative integers n and for all integers j in the range

$j=0, 1, 2, \cdots, n$. This more general identity is seen to be valid for $j=n$ and the validity for $j=k$ may be deduced as a consequence of its validity for $j=k+1$.

III. *Solution by S. G. Mohanty, State University of New York, Buffalo.* A generalized Vandermonde convolution formula [H. W. Gould, this MONTHLY, 1956, 84–91, (11)] is

$$\sum_{i=0}^{n} \binom{\alpha+\beta(n-i)}{n-i} A_i = \binom{\alpha+\nu+\beta n}{n}$$

where

$$A_i = \nu\binom{\nu+\beta i}{i}/(\nu+\beta i).$$

The present problem is a special case of this formula with $\alpha=0$, $\nu=1$, $\beta=2$.

An interpretation of the generalized convolution in terms of lattice paths is as follows: The set of minimal lattice paths from (0, 0) to $(\alpha+\nu+(\beta-1)n, n)$, not touching the line $x=\alpha+(\beta-1)y$ beyond $(\alpha+(\beta-1)(n-i), n-i)$, $i=0, \cdots, n$ is the same set of lattice paths from (0, 0) to $(\alpha+\nu+(\beta-1)n, n)$.

Also solved by George Bergman, W. J. Blundon, J. Boersma (Netherlands), R. G. Buschman, L. Carlitz, M. J. Cohen, P. G. Engstrom, N. J. Fine, W. D. Fryer, H. W. Gould, R. E. Greenwood, Cornelius Groenewoud, R. P. Kelisky, John B. Kelly, J. Koekoek (Netherlands), E. S. Langford and T. L. Austin, J. H. Lieske, A. E. Livingston, Imanuel Marx, J. W. Moon (England), B. J. M. Morselt (Netherlands), R. C. Mullin, M. G. Murdeshwar and V. K. Rohatgi, Stanton Philipp, John Riordan, Hermann Schulte (Germany), Arnold Singer, James Singer, W. F. Trench, and the proposer.

Several solvers directed attention to the literature for other studies and interpretations of the numbers A_i, in particular the Cayley formula, $A_{n+1}=A_0A_n+A_1A_{n-1}+\cdots+A_nA_0$ [*Collected Mathematical Papers*, pp. 112–115]. See also H. G. Forder, *Some problems in combinatorics*, Math. Gazette, vol. 45, pp. 199–201 for several problems which are solved by the numbers A_i. A generalization of the present problem by Gould appears as no. 5231 [1964, 923].

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, University of California, Berkeley and
E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. Programmed Materials: K. O. May, Univ. of California, Berkeley, California 94704. Films: E. P. Vance, Oberlin College, Oberlin, Ohio 44074.

A Course in Mathematical Analysis, Vol. II, Intermediate Analysis. By N. B. Haaser, J. P. LaSalle, and J. A. Sullivan. Blaisdell, New York, 1964. xii+677 pp. $11.75.

Although this is the second part of a two-volume work, the authors have

Concepts of Real Analysis. By Charles A. Hayes, Jr., Wiley, New York, 1964. 190 pp. $6.50.

This book was originally designed for use in a summer institute for college mathematics teachers. Half the book is devoted to set theory (including a discussion of finite and infinite sets), the real number system (not constructed), and an exposition of the validity of "definition by induction." The other half deals with the limiting processes that occur in elementary analysis. Sequential limits are dealt with first and then the methods and results are applied to the more general theory of real functions. No applications to the calculus are given. The analysis is phrased in terms of the extended real number system.

Certainly every college teacher of mathematics should be familiar with everything in this book. In fact, any student who is about to take a reasonably good course in advanced calculus should be familiar with the bulk of the book.

Many theorems are proved that are ordinarily, even in rigorous courses of analysis, taken for granted or left as exercises. Thus the book often presents a landscape that is flat and that bristles with notation. There is some danger that the reader may become impatient with the apparent lack of distinction between the trivial and the significant.

But the book appears to be teachable. There are motivational passages and many exercises of varying difficulty. It can usefully be recommended to anyone who has gone through two years of college mathematics and still feels insecure about sets, numbers, and limits.

J. B. ROBERTS, Reed College

Elementary Theory of Analytic Functions of One or Several Complex Variables. By Henri Cartan. Addison-Wesley, Reading, Mass., 1963. 228 pp. $10.75.

This is a translation, by John Standring and H. B. Shutrick, of Cartan's monograph *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Collection Enseignement des Sciences, Hermann, Paris, 1960. Comments are therefore in order as well on the content as on the translation.

Analyticity is viewed, with Weierstrass, as the property of having, at each point, a convergent power series expansion. This enables Cartan to treat real- as well as complex-analytic functions in the same breath. Holomorphy is taken to be the property (of a function of complex variables) of having a complex derivative with respect to each variable. The equivalence of these two notions is proved after the presentation of useful algebraic results on formal power series and a (somewhat jazzed-up) Ahlfors-like version of winding numbers.

The usual elementary results on analytic functions (Cauchy integral theorem, Laurent expansion, residues, maximum modulus, d'Alembert's (fundamental) theorem (of algebra), relations with harmonic functions, Dirichlet problem, etc.) are followed by chapters on: the topology of the spaces of functions continuous, holomorphic, or meromorphic, respectively, on a given domain; holomorphic

transformations, conformal mapping, complex manifolds, and Riemann surfaces; and analytic systems of differential equations.

Each of the seven chapters is followed by numerous exercises prepared by M. Reiji Takahashi. The clear and economical presentation of the text is neatly complemented by this opportunity to try one's own hand.

As for the translation itself, one has small bones to pick. Two of the eighteen errata listed in the original edition recur here. Among noteworthy examples of mistranslation or poor mathematical English we mention only the beginning of the last paragraph of the preface, the statement of Liouville's theorem (p. 80), the use of "revision" for "review" (pp. 17, 66), and the introductory remarks in the section on Weierstrass's $\wp$-functions (p. 153). Fortunately, the mathematical thought is nowhere more than briefly obscured, if at all.

Some errata that reached the reviewer's notice are: p. 8, line 8 from below: replace "who are from" by "who, fortified by" or "who, fresh with"; p. 8, line 6 from below: replace "exersices" by "exercises"; p. 50, last line: the summation should extend from 1 to $n+1$; p. 97, 4 lines after (5.2): cancel the last "m" on "parallelogramm"; p. 140, second display: replace $f_n(z)$ by $f_\nu(z)$; p. 222: replace $3a_3b_1b_2$ by $3a_3b_1^2b_2$ in the expression P_4.

The last two corrections are listed on the errata sheet bound into the French edition.

F. E. J. LINTON, Wesleyan University

Elements of Point Set Topology. By John D. Baum. Prentice-Hall, Englewood Cliffs, N. J., 1964. x+150 pp. $5.95.

Over the past ten or fifteen years several books have been published which aim at providing undergraduate students with a background in linear or abstract algebra. Only in the past two or three years have texts at a comparable level in topology become available. In the reviewer's opinion, Mr. Baum's text is a good one for a one quarter or one semester course. It presents the basic topology which is used in analysis with an axiomatic-geometric approach. The proofs are rigorous and (in the first part of the book) are presented in exhaustive detail so that the student can learn to recognize a proof and to recognize when steps in a proof have been omitted or lightly touched upon. The language and motivation of many of the proofs are geometric so the student's geometric background will enable him to follow and remember the main line of a proof.

A few random comments: Topological spaces are based on neighborhood systems. Examples and exercises appear together, with each example being partly an exercise and many exercises providing examples. The Axiom of Choice and Zorn's Lemma are both stated and used, but neither is proved from the other. The book appears to be singularly free from misprints; in a fairly careful reading, the reviewer found only one point at which to cavil: "thus" in line 21, p. 111 might better have been "on the other hand." If that is the only objectionable feature of the book, it must be a very good one indeed.

B. H. ARNOLD, Oregon State University

of use in a differential equations-advanced calculus sequence. The author is to be commended on the clarity of exposition in which the student is eased into conceptual material with interesting examples in such a manner that a reasonableness prevails. By linearly grading the material over the four chapters from a quite elementary level of reasoning to a mature approach at the very end, a hurdling effect is eliminated. An ample supply of exercises having an even distribution of the computational drill type, a testing of the student's grasp of concepts, and material not included in the text proper follows each section. Answers and hints to solutions of many problems are given at the end of the book. In general, examples are plentiful and wisely chosen. The first chapter would make excellent outside reading for students during the latter part of the introductory calculus sequence.

Some may object to the non-postulational and less algebraic approach that is used and to the superscript notation, $X^1, X^2, \cdots, X^n$. It is felt that the material could have been arranged with more sectional headings; e.g., the first chapter of 108 pages is divided into only six sections. The over-all use of Euclidean two-space and 2×2 matrices for the introduction of a concept from which generalizations are immediately made, at times masks evolution by the usual inductive processes; in particular this is evident in the sections on the adjoint of an operator and on determinants. Any other objections would depend more on individual tastes than on the soundness of development.

This text should be given serious consideration for a first course in linear algebra and is very suitable for a sophomore level program.

HOMER BECHTELL, Bucknell University

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Associate Secretary, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Wade Ellis of Oberlin College has been awarded the rank of Commander in the Order of Magisterial Palms of Peru for services rendered to national education in that country.

Professor Emeritus Solomon Lefschetz, Princeton University, and Professor Emeritus Marston Morse, Institute for Advanced Study, have been awarded the 1964 National Medal of Science.

Professor M. Gweneth Humphreys, Randolph-Macon Woman's College, represented the Association at the inauguration of Marion C. Brewer as President of Lynchburg College on October 17, 1964.

Professor W. J. Klimczak, Trinity College, represented the Association at the installation of Rev. William C. McInnes, S.J., as President of Fairfield University on October 24, 1964.

Professor E. H. Matthews, Southwest Missouri State College, represented the Association at the inauguration of E. S. Brandenburg as President of Drury College on November 6, 1964.

Professor D. W. Robinson, Brigham Young University, represented the Association at the inauguration of James C. Fletcher as President of the University of Utah on November 5 and 6, 1964.

University of Akron: Professor L. D. Rodabaugh, Ohio Northern University, has been appointed Associate Professor; Assistant Professor W. H. Beyer has been promoted to Associate Professor.

University of Alberta: Professor A. L. Dulmage, University of Manitoba, has been appointed Professor and Head of the Department of Mathematics; Drs. T. A. Burton, Washington State University, and J. S. W. Wong, California Institute of Technology, have been appointed Assistant Professors; Dean Max Wyman has been appointed Vice-President of the University.

University of Arizona: Visiting Assistant Professor M. S. Cheema and Assistant Professor D. E. Myers have been promoted to Associate Professors.

Stephen F. Austin State College: Dr. M. R. Hagan, Oklahoma State University, has been appointed Assistant Professor; Mr. H. E. Bunch has been promoted to Assistant Professor.

Bowling Green State University: Dr. Richard Eakin, Washington State University, has been appointed Assistant Professor; Associate Professor Louis Graue has been promoted to Professor.

University of California, Santa Barbara: Associate Professor Hans Schneider, University of Wisconsin, has been appointed Visiting Professor; Assistant Professor R. C. Thompson, University of British Columbia, has been appointed Assistant Professor; Dr. David Outcalt, Claremont Men's College, and Dr. Melvin Rosenfeld, University of California, Los Angeles, have been appointed Lecturers; Dr. E. C. Johnsen has been promoted to Assistant Professor; Assistant Professor A. M. Bruckner has been promoted to Associate Professor.

California State Polytechnic College: Professor Simon Green, Arizona State University, and Dr. Sidney Spital, Hughes Aerospace, El Segundo, California, have been appointed Associate Professors; Assistant Professor Samuel Gendelman, Los Angeles State College, has been appointed Assistant Professor; Mr. T. J. Flynn has been promoted to Assistant Professor; Associate Professor H. F. Simmons has been appointed Head of the Department of Mathematics.

Clarkson College of Technology: Dr. A. R. Amir-Moéz, Monterey Park, California, has been appointed Professor; Mr. G. H. Ryder, Carnegie Institute of Technology, has been appointed Assistant Professor.

Denison University: Associate Professor Andrew Sterrett has been promoted to Professor; Assistant Professor W. N. Prentice has been promoted to Associate Professor.

East Los Angeles College: Mr. Arthur Hadley, State University of New York at Albany, has been appointed Assistant Professor; Assistant Professor Edward Rosenberg has been promoted to Associate Professor; Mr. Robert V. Kester has been promoted to Associate Professor.

Fordham University: Dr. Salvatore Anastasio, New York University, and Dr. Donald Goldsmith, University of Pennsylvania, have been appointed Assistant Professors; Assistant Professor Michael Aissen has been promoted to Professor.

Georgetown University: Associate Professors A. K. Aziz and M. W. Oliphant have been promoted to Professors.

Harpur College: Dr. C. J. Houghton, Ohio State University, and Dr. H. V. Kronk, Michigan State University, have been appointed Assistant Professors.

University of Illinois, Chicago: Professor Joseph Landin, University of Illinois, Urbana, has been appointed Head of the Department of Mathematics; Associate Professors Flora Dinkines and L. L. Pennisi have been promoted to Professors.

Illinois Institute of Technology: Dr. Arthur Grad, Stanford University, has been appointed Professor and Dean of the Graduate School; Mr. Adam Czarnecki and Mr. John Synowiec have been promoted to Assistant Professors.

State University of Iowa: Professor G. P. Weeg, Michigan State University, has been appointed Professor and Director of the Computer Center; Dr. R. H. Oehmke, Institute for Defense Analysis, has been appointed Professor; Dr. M. A. Geraghty, University of Alabama, and Dr. T. M. Price, University of Wisconsin, have been appointed Assistant Professors; Professor H. T. Muhly has been appointed Chairman of the Department of Mathematics; Associate Professors Sterling Berberian and D. W. Wall have been promoted to Professors; Assistant Professor J. F. Jakobsen has been promoted to Associate Professor.

Kent State University: Associate Professor K. B. Cummins has been promoted to Professor and appointed Temporary Chairman of the Department of Mathematics; Assistant Professor T. N. Bhargava has been promoted to Associate Professor; Dr. J. G. Maxwell has been promoted to Assistant Professor.

University of Maryland: Drs. R. S. Bucy, R. I. A. S., Towson, Maryland; J. H Henkelman, Saugus High School, Saugus, Massachusetts; W. E. Kirwan, Rutgers, The State University; A. S. Strauss, University of Wisconsin; and R. J. Whitley, New Mexico State University, have been appointed Assistant Professors; Mr. T. H. Dyer has been promoted to Assistant Professor.

Massachusetts Institute of Technology: Assistant Professor R. G. Heyneman, on leave from Cornell University, and Dr. R. W. Goodman, Harvard University, have been appointed Lecturers; Associate Professors L. N. Howard and Hartley Rogers, Jr., have been promoted to Professors; Assistant Professor W. G. Strang has been promoted to Associate Professor; Professor Philip Franklin retired on June 30, 1964 with the title of Professor Emeritus.

Michigan State University: Assistant Professor M. R. Parameswaran, Washington State University, has been appointed Visiting Associate Professor; Assistant Professor P. K. Wong, Lehigh University, and Dr. D. A. Moran, University of Chicago, have been appointed Assistant Professors; Associate Professor J. G. Hocking has been promoted to Professor; Assistant Professor D. W. Hall has been promoted to Associate Professor; Miss Dorothy Bollman has been promoted to Assistant Professor.

Mississippi State University: Professor W. E. Koss, Louisiana Polytechnic Institute, has been appointed Professor; Drs. J. L. Tilley, Clemson College, and Robert Heller, University of Houston, have been appointed Associate Professors; Mrs. Helen McPherson Boyd and Mr. R. P. Finley have been promoted to Assistant Professors.

University of Missouri: Assistant Professors M. D. George and J. N. Younglove have been promoted to Associate Professors.

University of Nebraska: Dr. J. A. Eidswick, Purdue University, has been appointed Assistant Professor; Professor Edwin Halfar has been appointed Chairman of the Department of Mathematics.

University of New Mexico: Dr. Reuben Hersh, Stanford University, has been appointed Assistant Professor; Professor J. R. Blum has been appointed Chairman of the Department of Mathematics.

New York University: Dr. A. B. Novikoff, Stanford Research Institute, has been appointed Visiting Associate Professor; Assistant Professor Donald Ludwig, University of California, Berkeley, has been appointed Associate Professor; Dr. J. J. Benedetto, University of Toronto, has been appointed Assistant Professor; Mr. C. W. Langley has been promoted to Assistant Professor.

State University of New York at Albany: Associate Professor Paul Schaefer has been promoted to Professor; Assistant Professor John Therrien has been promoted to Associate Professor.

North Texas State University: Associate Professors D. F. Dawson and J. T. Mohat have been promoted to Professors; Assistant Professor R. G. Bilyeu has been promoted to Associate Professor.

Northeast Louisiana State College: Dr. D. R. Bedgood, Oklahoma State University, and Mr. C. D. Tabor, Texas Christian University, have been appointed Assistant Professors.

Old Dominion College: Professor J. J. Barron, Marshall University, has been appointed Professor and Chairman of the Department of Mathematics; Assistant Professor C. A. Schulz, Frederick College, has been appointed Assistant Professor.

Pratt Institute: Dr. A. B. Finkelstein, Professor of Engineering Science, has been appointed Professor of Mathematics; Associate Professor G. C. Helme has been promoted to Professor.

University of Rhode Island: Assistant Professor Arnold Seiken, Michigan State University, and Dr. George Martin, University of Michigan, have been appointed Assistant Professors; Professor H. A. Bender retired October 1964.

Ripon College: Assistant Professor J. A. Berton, Indiana State College, has been appointed Associate Professor; Assistant Professor Wayne Larson has been promoted to Associate Professor.

Rockefeller Institute: Professor Emeritus Hans Rademacher, University of Pennsylvania, has been appointed Visiting Professor; Assistant Professor Louis Solomon, Haverford College, has been appointed Assistant Professor.

San Diego State College: Mr. Euel Kennedy, University of Utah, has been appointed Lecturer; Mr. James Carter, University of California, Los Angeles, has been appointed Assistant Professor; Associate Professor C. V. Holmes has been promoted to Professor; Assistant Professors Leonard Fountain and Joseph Moser have been promoted to Associate Professors.

San Fernando Valley State College: Associate Professor J. C. Smith has been promoted to Professor; Assistant Professor Tung-Po Lin has been promoted to Associate Professor.

San Jose State College: Dr. R. H. DeVore, University of Arizona, has been appointed Assistant Professor; Mr. A. E. Halteman, Sylvania Electronic Defense Laboratories, Mountain View, California, and Mr. G. K. McLeod, University of San Francisco, have been appointed Lecturers; Associate Professors Richard Post and J. R. Smart have been promoted to Professors.

Southeastern Louisiana College: Assistant Professor N. S. Andrews has been promoted to Associate Professor: Mr. L. B. Wheeler has been promoted to Assistant Professor.

Southern Illinois University: Professor Andrew Sobczyk, University of Miami, has been appointed Professor; Associate Professor Dilla Hall retired September 1, 1964 with the title of Associate Professor Emeritus.

University of Tennessee: Assistant Professor R. M. McConnel, University of Arizona, has been appointed Assistant Professor; Professor J. H. Barrett has been appointed Acting Head of the Department of Mathematics.

Temple University: Assistant Professor Hwa Tsang, San Fernando Valley State College, has been appointed Assistant Professor; Assistant Professor Seymour Lipschutz, Polytechnic Institute of Brooklyn, has been appointed Associate Professor.

Texas Christian University: Drs. D. O. Cutler, New Mexico State University, and D. S. Nymann, University of Kansas, have been appointed Assistant Professors.

University of Toledo: Associate Professor Nand Kishore, Idaho State University, has been appointed Professor; Professor S. S. Blakney, Grambling College, has been appointed Associate Professor; Assistant Professor Edward Ebert has been promoted to Associate Professor.

Trenton State College: Mr. Jack Irwin, Western Illinois University, has been appointed Assistant Professor; Miss Jane McLaughlin has been promoted from Assistant Professor II to Assistant Professor I.

Tulane University: Associate Professor F. D. Quigley has been promoted to Professor; Assistant Professor F. T. Birtel has been promoted to Associate Professor.

Union College: Assistant Professor S. M. Robinson, Smith College, has been appointed Assistant Professor; Professor A. H. Fox has been appointed Chairman of the Department of Mathematics; Professor O. J. Farrell retired June 1964 with the title of Professor Emeritus.

University of Virginia: Assistant Professor L. H. Lanier, Jr., Ohio State University, and Dr. L. E. Snyder, Purdue University, have been appointed Assistant Professors; Assistant Professor N. F. G. Martin has been promoted to Associate Professor.

Washington University: Assistant Professor R. H. McDowell has been promoted to Associate Professor and is on leave of absence as Associate Director of the MAA Committee on the Undergraduate Program in Mathematics; Assistant Professor C. D. Gorman has been promoted to Associate Professor; Professor I. I. Hirschman, Jr., is on leave of absence and has been appointed Visiting Professor at Stanford University.

College of William and Mary: Associate Professor H. Margaret Elliott, Washington University, has been appointed Acting Professor; Professor L. T. Conner, Bluefield College, has been appointed Assistant Professor.

Wisconsin State University, Whitewater: Mrs. Karamaneh Oschwald, Carroll College, has been appointed Assistant Professor; Mr. J. S. Dombek has been promoted to Assistant Professor; Professor C. E. Flanagan will be on leave of absence for two years to work on a teacher education project in Northern Nigeria.

University of Wisconsin, Madison: Drs. W. C. Holland, University of Chicago, F. L. Sandomierski, Pennsylvania State University, and Michael Voichick, Dartmouth College, have been appointed Assistant Professors; Associate Professor John Nohel has been promoted to Professor; Professor S. C. Kleene has been named Cyrus C. MacDuffee Professor of Mathematics; Professor R. H. Bing has been named Research Professor of Mathematics.

University of Wyoming: Assistant Professor J. E. Shockley, College of William and Mary, has been appointed Associate Professor; Dr. D. R. Anderson, Massachusetts Institute of Technology, and Dr. A. D. Porter, University of Colorado, have been appointed Assistant Professors; Associate Professor P. O. Steen has been promoted to Professor.

Assistant Professor R. O. Atkinson, North Georgia State College, has been appointed Assistant Professor at Tennessee Polytechnic Institute.

Assistant Professor Felice Bateman, University of Illinois, has been appointed a Visiting Member of the faculty of Science and Mathematics at Sarah Lawrence College.

Assistant Professor J. O. Brooks, Haverford College, has been appointed Assistant Professor at Villanova University.

Mr. T. R. Caplinger, Florida State University, has been appointed Assistant Professor at Memphis State University.

Dr. Gary Chartrand, Michigan State University, has been appointed Assistant Professor at Western Michigan University.

Assistant Professor Donato DeFelice, Duquesne University, has been promoted to Associate Professor.

Rev. M. S. Delaney, Immaculate Heart College, has been promoted to Assistant Professor.

Associate Professor J. W. Dettman, Case Institute of Technology, has been appointed Professor at Oakland University.

Associate Professor R. A. Dobyns, McNeese State College, has been promoted to Professor.

Associate Professor R. E. Dowds, Butler University, has been promoted to Professor.

Mr. A. H. Garness, Concordia College, has been promoted to Assistant Professor.

Mr. G. E. Haborak, Wayne State University, has been appointed Assistant Professor at the U. S. Naval Academy.

Dr. C. B. Hanneken, Marquette University, has been appointed Chairman of the Department of Mathematics.

Dr. H. H. Hartzler, Professor of Physics, Mankato State College, has been appointed Professor of Mathematics.

Dr. Robert Heller, University of Houston, has been appointed Associate Professor at Mississippi State University.

Dr. B. G. Hodges, Winthrop College, has been appointed Professor at Illinois State University.

Dr. J. M. Horner, University of Alabama, has accepted a position as an Associate Senior Research Mathematician at General Motors Research Laboratories, Warren, Michigan.

Dr. F. M. Hudson, University of Texas, has been appointed Associate Professor and Chairman of the Department of Mathematics at McMurry College.

Dr. R. E. Hughs, Sandia Corporation, Albuquerque, New Mexico, has been appointed Assistant Professor at Carleton College.

Associate Professor G. P. Johnson, Wesleyan University, has been appointed Associate Professor at the University of the South.

Assistant Professor J. R. King, Providence College, has been promoted to Associate Professor.

Dr. G. B. Lampton, Jr., University of South Carolina, has been appointed Assistant Professor at Winthrop College.

Mr. Gerald Leibowitz, Northwestern University, has been promoted to Assistant Professor.

Assistant Professor F. M. Lister, Western Washington State College, has been promoted to Associate Professor.

Dr. N. E. Losey, University of Wisconsin, Milwaukee, has been appointed Assistant Professor at the University of Manitoba.

Assistant Professor R. A. McHaffey, on leave from the University of Massachusetts, has been appointed Assistant Professor at Adelphi University.

Dr. B. H. McLemore, Jackson State College, has been appointed Associate Professor at Fenn College.

Assistant Professor W. S. Mahavier, University of Tennessee, has been appointed Assistant Professor at Emory University.

Assistant Professor J. W. Meux, Kansas State University, has been appointed Assistant Professor and Chairman of the Department of Mathematics at Midwestern University.

Professor Emeritus W. L. Miser, Vanderbilt University, has been appointed Visiting Lecturer in Mathematics at Sweet Briar College.

Assistant Professor J. G. Moser, Rose Polytechnic Institute, has accepted a position as an Associate with Daniel H. Wagner, Associates, Paoli, Pennsylvania.

Professor F. W. Perkins, Dartmouth College, retired June 1964 with the title of Professor Emeritus.

Assistant Professor Wallace Raab, California State Polytechnic Institute, has been appointed Professor at the University of South Dakota.

Dr. D. S. Ray, University of Tennessee, has been appointed Associate Professor at Bucknell University.

Dr. Zalman Rubinstein, Harvard University, has been appointed Associate Professor at Clark University.

Dr. J. L. Sieber, Shippensburg State College, has been appointed Acting Chairman of the Department of Mathematics.

Visiting Professor Louis Silverman, University of Houston, retired on May 31, 1964 with the title of Professor Emeritus.

Sister Mary Job Ternes, Rosary College, has been promoted to Assistant Professor.

Professor M. B. Smith, University of Wisconsin, Madison, has been named Vice Provost of the University Center system.

Associate Professor W. E. Timon, Jr., Northwestern State College, has been promoted to Professor.

Associate Professor Evelyn K. Wantland, University of Mississippi, has been appointed Professor and Chairman of the Department of Mathematics at Illinois Wesleyan University.

Assistant Professor J. W. Warner, College of Wooster, has been promoted to Associate Professor.

Associate Professor A. B. Willcox, Amherst College, has been promoted to Professor.

Associate Professor C. W. Williams, Washington and Lee University, has been promoted to Professor.

Assistant Professor Israel Zuckerman, Queens College, has been appointed Assistant Professor at Vassar College.

THE CITY UNIVERSITY OF NEW YORK—DOCTORAL PROGRAM IN MATHEMATICS

The doctoral program in mathematics of The City University of New York is happy to announce that Professor Marston Morse will give a year's course on Differential Geometry in the Large.

Our regular staff will include Professors Louis Auslander, Gilbert Baumslag, Alex Heller, Stanley Kaplan, Richard Sacksteder and Leo Zippin. Professors Elliott Mendelson, Moses Richardson, and Robert Schatten of the CUNY doctoral faculty and Dr. Henry Landau and Dr. Henry O. Pollak of the Bell Telephone Laboratories will also each give a year's course.

The topics covered will include Lie Algebras, Category Theory, Group Theory, the geometric theory of differential equations, Game Theory and Programming, Advanced Topics in Analysis, real and complex, and partial differential equations.

A number of substantial Fellowships are available and also teaching opportunities in the many colleges of CUNY (usually only reserved for students with the M.A.). Address inquiries to Professor Leo Zippin, The City University of New York, 33 West 42 Street, New York, N. Y. 10036.

Editorial Note. Professional mathematicians as well as laymen will be interested in the article: "Mathematics and the Layman," by F. A. Ficken, which appears in *American Scientist* for December 1964, pp. 419–430.

H.M.G.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 2

PART II

Computers and Computing

Number 10
of the
HERBERT ELLSWORTH SLAUGHT
MEMORIAL PAPERS

FEBRUARY 1965

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamilar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $8.00 ($10.00 effective August–September 1965).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

CONTENTS

Editorial Notes: Collaborating Editors for this issue were Jack Heller, P. D. Lax, D. H. Lehmer, and R. D. Richtmyer to whom the Editor-in-chief expresses his sincere thanks.

Readers may be interested in the *Recommendations on the Undergraduate Mathematics Program for Work in Computing*, copies of which may be obtained gratis by writing to: CUPM, P.O. Box 1024, Berkeley, California 94701.

IMPACT OF COMPUTERS

R. W. HAMMING, Bell Telephone Laboratories, Murray Hill, N. J.

1. Introduction. In discussing the impact of computers it is easy to cite the number of computers installed, the projected number of installations in the next decade, the number of persons involved, the size of the computer industry in billions of dollars spent annually, and the presence of computers throughout our society such as in accounting, process control, research, etc., but in writing for mathematicians it seems better to ignore these material aspects and to concentrate on the intellectual aspects of the impact.

In order to approach the effects of computers on our intellectual outlook it seems necessary to dispose first of a number of misconceptions that are widely held. These misconceptions have their roots, it seems to me, not in a rational, considered approach, but rather in a mixture of 1) emotional fear, and 2) the conservative approach that wishes to believe that nothing essentially new is occurring.

2. Misconceptions. The simplest error is to suppose that *computing machines merely do arithmetic.* The fact is that they are symbol manipulators. It is often convenient to identify the symbols with numbers; it is not the form of the representation, however, but rather the interrelationships between the symbols that gives them meaning. Since by suitable programming we can give the symbols any interrelationships we please, we are free to give them a wide variety of meanings. Every mathematician knows the power that comes from the ability to manipulate symbols according to a given set of rules.

Another argument that continually arises is that *machines can do nothing that we cannot do ourselves*, though it is admitted that they can do many things faster and more accurately. The statement is true, but also false. It is like the statement that, regarded solely as a form of transportation, modern automobiles and aeroplanes are no different than walking. One can walk from coast to coast of the U. S. so that statement is true, but is it not also quite false? Many of us fly across the U.S. one or more times each year, once in a while we may drive, but how few of us ever seriously consider walking more than 3000 miles? The reason the statement is false is that it ignores the order of magnitude changes between the three modes of transportation: we can walk at speeds of around 4 miles per hour, automobiles travel typically around 40 miles per hour, while modern jet planes travel at around 400 miles per hour. Thus a jet plane is around two orders of magnitude faster than unaided human transportation, while modern computers are around six orders of magnitude faster than hand computation. It is common knowledge that a change by a single order of magnitude may produce fundamentally new effects in most fields of technology; thus the change by six orders of magnitude in computing have produced many fundamentally new effects that are being simply ignored when the statement is

made that computers can only do what we could do for ourselves if we wished to take the time.

One often hears the remark that *computers can do only what they are told to do.* True, but that is like saying that, insofar as mathematics is deductive, once the postulates are given all the rest is trivial. An outsider not well acquainted with mathematics is apt to admit readily the truth of this last statement, but the mathematician well knows how much is being concealed when one says that all of deductive mathematics is trivial. Similarly, any person who has had reasonably extensive experience with modern computers well knows how much is being concealed when one admits that machines can do only what they are told to do. The truth is that in moderately complex situations, such as the postulates of geometry or a complicated program for a computer, it is not possible on a practical level to foresee all of the consequences. Indeed, there is a known theorem that there can be no program which will analyze a general program to tell how long it will run on a machine without actually running the program (or a direct simulation of the program).

One often hears, especially in military circles, remarks to the effect that machines may make mistakes (due to poor programming, not machine failures) that a skilled human would not. The error that is being made is *to compare ideal human behavior with carelessly programmed machine behavior.* In practice it is the human, of course, who is unable to be careful all the time, who has the lapses of judgment, etc., especially on jobs which are monotonous and involve day in and day out activities of the same sort. Similarly, the mathematician is apt to compare the behavior of the best mathematicians at their inspired moments with that of a computing machine doing routine work; if we were to pick the average mathematician when plodding along then the results would often be reversed and the machine would look a good superior.

Finally, another argument that is frequently met is the attempted *comparison of the human mind and the computer.* Mysteriously arrived at numbers are cited for the number of neurons in the human nervous system and these are arbitrarily compared with the number of components in a machine. Since the machine has a repair man to replace defective parts promptly, while we poor humans have to put up with failing neurons at the rate of one every few seconds (so they claim), we evidently have to use much of our capacity in the form of protective redundancy. A realistic comparison would also point out that human signaling speeds are at best 125 meters per second, while computers signal at close to the speed of light, that human pulse repetition rates are rarely close to 1000 per second while computers use rates of millions to almost billions per second, that humans are incredibly prone to lapses such as forgetting their best friend's name, careless slips while playing chess, etc., while machines have failure rates that are far, far, lower. But fundamentally it seems to me that one cannot compare computers and human minds generally but only on specific tasks, and then only when the tasks are described in some detail so that we can have a good idea of what is being done in both cases.

3. The impact on experimental science. Having, I hope, cleared the air of a number of misconceptions, let us turn to our main goal of examining the impact of computers on our intellectual world. One observation I have often used is that not too long ago 90% of the experiments done at the Bell Telephone Laboratories were done in the lab and only 10% were done on computers, but in the not too distant future 90% will be done on computers and only 10% in the lab. This remark illustrates the fundamental change in our basic approach to the physical world that is occurring due to the availability of computers.

Let us examine the remark more closely. Many of the early uses of computers were to reduce the data *after* an experiment was done (and I shall not make the distinction between using machines to gather the data while the experiment is actually running and using it after the experiment is done). A few early applications of computers, and increasingly so these days, put the computer *before* the experiment and simulate what would happen if we did the experiment. For example, at Los Alamos during the Second World War the behavior of an atomic bomb was computed *before* any were actually tried out in the field. Currently we are "flying" thousands and thousands of trajectories on computers for every one we actually fly in space. We are gradually using computers *during* the experiment in a control loop, reducing the data so rapidly that the results are available to control the next steps in the experiment.

Besides these three uses we are also using computers *instead of* experiments. In many areas we have almost completely abandoned the laboratory equipment and use only simulations. Of course there must always be some checking of our models against reality, but think of how seldom we bother to check classical mechanics in practical matters; we simply assume that if we let go of a plate it will drop to the floor and the higher it is held the harder will be the impact. Rarely do we do experiments in this well-tested area.

The net effect of these applications of computers on experimental science is yet to be widely felt, but many individuals have reorganized their approaches to their work and are working in a fundamentally new framework of ideas of how to plan and carry on their research.

Two examples recently done at Bell Telephone Laboratories come to my mind. (I apologize for using Bell Telephone Laboratories as if it were the only place using machines, which of course it is not, but excuse myself on the grounds that naturally I am more familiar with the work there than work done elsewhere.) One was a study of binocular vision. In this field one of the main experimental problems is to separate the effects of the recognition of previously learned patterns from the direct recognition by the central nervous system. Bela Julesz used a computer [1] to produce square patterns of some 10,000 spots of randomly selected shades of light and dark. By suitably arranging a pair for stereo vision he could demonstrate and study some problems of depth perception with reasonable assurance that he had eliminated the learned pattern effects.

The second example of a new approach due to the availability of a computer

is that of the work of Roger Shepard [2] who used a digital computer and programs already produced by others to generate tones without the presence of any musical instruments. This remark needs a small qualification. The computer produces a digitized version of the sound track and this is run through a digital to analog converter plus a simple smoothing circuit to produce the actual sound track. One of the sound tracks so produced had a series of "notes" that seemed to rise as each note was played, but in fact was simply circular and might be called an "auditory illusion" in analogy to "optical illusions."

In both of these cases it was the presence of a computer which stimulated and made possible the basic approach to the research; the awareness of the computer and its possibilities had to be there before the idea to try the experiment could arise.

4. The impact on theoretical science. It might be supposed that if computers have had so large an effect on experimental science then their impact on theoretical science would be even greater, but the facts seem not to be so. The methods of pure theory are still the same as they were before computers were available; a man speculates and puzzles over a field of knowledge until a new insight arises. It is true that the theoreticians do make wide use of computing when the analytical tools of classical mathematics fail them. Computers have also affected the starting points of theories; the concept of a suitable end point of a theory may perhaps now be a much more complex array of formulas than before; the subject matter of the theory has often been affected; but so far as I can see the *way* the theoreticians work is still much the same. Thus while it is necessary for the experimentalist to reorganize his approach, the theoretician need only add a few additional tools to his armory.

"Experiments" have been carried out on computers to study genetic problems, traffic patterns, etc., but I interpret these as the use of computers *instead of* experiments. They have not affected the approach to the theory except to encourage a wider range of speculation since now we can check the speculation against observation fairly easily.

5. The impact on some other areas of intellectual activity. Computers have had a large effect, relatively speaking, on music. Not only have experiments been performed in the area of musical composition according to accepted standards of composition, but many other experiments have also been tried.

One of the most interesting is that of having the computer calculate the sound track directly without the presence of any musical instruments, as mentioned before. This is interesting for the composer because it opens the door to the entire world of possible sounds, not merely those that can be made with real instruments. All kinds of effects may be produced which are beyond human production, such as simultaneous vibrato in the various instruments which are being simulated.

In the entertainment world increasing amounts of background music are

being produced by artificial means, and computers will undoubtedly carry an increasing share of this type of "music."

This kind of work is stimulating some musicians to try new kinds of approaches to their tasks, in composing, production, even in the evaluation of music.

I have already mentioned two experiments in the area of psychology as examples of the impact of computers; there are many more examples.

Computers have also had a fair sized effect on language studies. Attempts to produce programs to translate from one natural language, such as Russian, to another, such as English, have stimulated a great deal of research. Furthermore, the attempts of computer experts to devise synthetic languages for communicating with computers, and the building of translaters from these languages to the languages of the machines themselves have given us a great many new insights into language. We are now much more aware of language as an engineering device to which engineering criteria such as efficiency, reliability of communication, need for redundancy, etc., can be applied.

Computers have also stimulated the growth of an almost completely new field called "artificial intelligence." The goals of this field are varied, as is natural in a new field, and the various workers disagree, sometimes violently, over what should be, or has been, done. Some are trying to explore how humans operate by simulating various proposed models. Others are simply trying to get machines to do tasks which were traditionally supposed to require thinking, without trying to do it in ways that are thought to be similar to those used by humans. There are all gradations of activity in between these extremes, methods which use analog as well as digital simulations, those which try to be precise and those which attempt to use random methods (sometimes merely to cover ignorance!), those which try to produce detailed models and those which merely simulate gross effects, etc.

The field is interesting because for the first time computers give us the possibility of exploring models of "creativity," whatever that means. We may not be able to define "creativity," but in many, many phases of our life, from mathematics to fashion styles, we put a high premium on it.

Another way of examining the impact of computers is to note that computers use algorithms, and this very fact has caused many of us to examine the world with eyes searching for *processes* to describe what we see rather than *words*. (The displacing of the dominance of words is not unique to the computing field but seems to be a somewhat broad effect in the whole of our society.)

These are only a few of the ways computers have affected some fields of intellectual activity, but space and time forbid going on to fields such as medicine, law, cosmology, and archeology.

6. The impact on mathematics. Finally we come to the topic of the impact of computers on mathematics. It depends on what you think mathematics is

as to how much you think computers have had, or will have, an effect, and it also depends on what you mean by "have an effect"; hence there is a wide range of opinions on the matter. If you think of mathematics as being the more abstract parts, such as are so often published in the purer mathematical journals, then it is apparent that computers have had very little effect and it is difficult to see how it will soon be much different. Thus I cannot imagine how computers will ever affect the field of Lebesgue integration, though I do not deny the possibility. On the other hand if, like the author, you believe that much of mathematics has its ultimate roots in attempts to explore and understand the universe, then it is apparent that there have been great effects, and the future will show even greater effects.

Let me illustrate in a general way what I mean. Much of mathematics has arisen from the observation of special cases. Computers now enable us to compute many more special cases than we could by hand, to see much more detail in those we do examine, and consequently have led to many more insights.

Mathematics depends not only on numerical examples, counting of subgroups, enumeration of branches of a tree, etc., but also occasionally on doing a large mass of algebra. A number of efforts to program a computer to perform the typical operations of high school algebra have been made and some have met with modest successes. Thus at BTL the so-called ALPAK (*Al*gebra *Pack*age) has been used [3] to help prove theorems in a number of fields such as queueing theory, celestial mechanics, crystals, and wave propagation, simply by letting the computer perform a large mass of well-organized algebraic manipulation. When the final answer condenses down to a few terms then the result can often be used, but when there is simply a sea of thousands of terms left, then the result is almost useless except in a negative sense. In the successful cases there is, of course, the suspicion and hope that, knowing the answer, by some ingenuity one can find a simple derivation which by-passes the mass of algebra done by the machine.

D. H. Lehmer has used computers [4] to follow out all the various branches of particular complex proofs in Number Theory. Others have tried to write programs for computers so that the computer could "find" proofs in broad fields, such as High School geometry or the early parts of Whitehead and Russell's *Principia.* The results have been very interesting but at least for this author somewhat discouraging—at present we seem to be very far from the day when we can give a machine a very difficult theorem such as Riemann's hypothesis and hope to have the machine produce a mathematical proof.

To those who use mathematics to explore the universe, computers are of tremendous help. For us a numerical solution to a 10 body problem is often far more useful than an analytic solution to a 3 body problem. We use computers in a rather uninhibited fashion, to make a preliminary numerical exploration of a situation, to simulate systems of various degrees of idealization, and to compute solutions to situations for which there are no known existence theorems. In practice we care surprisingly little for a rigorous mathematical proof

of convergence so long as we get an answer we like; on the other hand, mathematical rigor based on an inappropriate model does not impress us much. In a way, we have gone back to the classical mathematicians who created the mathematics they felt was appropriate to the physical situation at hand, who ignored the mathematics they did not like, and in the absence of rigorous proof went ahead anyway.

I happen to believe that this uninhibited approach to mathematics will eventually enrich mathematics by giving it many new fruitful and useful models to explore—but I appear to be in the minority! Most of my mathematically trained friends still feel that computing, especially Numerical Analysis, should stay within the currently accepted bounds of mathematical theory and rigor; if it does, then I believe that the impact of computing on pure mathematics will be much less than it could be.

It is hard to be sure when some mathematics is inspired by computers. Thus the whole area of "computability" in abstract logic might be thought to have been started from considering current computing machines, but the fact is that Turing's basic paper appeared in 1937 long before the modern era of computers! How much the later elaboration of the field is due to computers is a matter for conjecture.

One final effect of computers on mathematics should be mentioned; namely, that many of the most able people in computing were attracted there from mathematics and probably represent a loss to mathematics. On the other hand, the glamor and money in computing probably have attracted some people to mathematics. The net balance would be hard to measure.

References

1. Bela Julesz, (i) Binocular depth perception of computer generated patterns, Bell System Tech. J., 39 (1960) 1125–1162; (ii) Binocular depth perception without familiarity cues, SCIENCE, July 24, 145 (1964) 356–362.

2. Roger Shepard, Circularity in judgments of relative pitch, (BTL internal memo).

3. W. S. Brown, J. P. Hyde, B. A. Tague, The ALPAK system for nonnumerical algebra on a digital computer—II: Rational functions of several variables and truncated power series with rational-function coefficients, Bell System Tech. J., (1964) 785–804.

4. Walter Freiberger and William Prager (editors), Applications of digital computers, Ginn and Co., Boston, 1963.

THE POST-WAR COMPUTER DEVELOPMENT

R. D. RICHTMYER, New York University

Of the automatic computers that existed at the end of 1945, only the Eniac was really electronic, and none qualified as a computer by today's definition. None had a central control unit, a vocabulary or list of instructions, or a memory in the modern sense. (In Harvard's Mark I relay calculator, however, the sequence of operations was controlled by a pre-punched paper tape.)

Electronic computer technology really began with the Eniac, which was designed and built by J. P. Eckert and J. W. Mauchly at the Moore School of Electrical Engineering at the University of Pennsylvania and later moved to the Army's Aberdeen Proving Ground. Eckert and Mauchly pioneered in the systematic use of gates and flip-flops and in the general design of digital circuitry, and the Eniac was quite fast, even in comparison with most computers of the next several years. Many large computations in pure and applied science were done on it, but its utility for the largest problems was hampered by the necessity of making hundreds of cable connections and of setting hundreds of switches on counting devices, etc., in order to organize the Eniac for solving a given problem. Later, this difficulty was eliminated by the development of the so-called converter code, which will be described briefly below.

Meanwhile, several people were giving serious thought to the theory of automatic computing, and there emerged a number of basic principles so fundamental that computer design became completely revolutionized. These principles seem almost obvious in retrospect, and people tend to forget that many of the notions that are now commonplace in the computing world, and which still dominate the design of 1964's computers, had not been thought of before 1945, although their origin can perhaps be traced to Babbage. The most influential and clear thinking of the people in the field in the middle and late 40's was John von Neumann; he and his co-workers turned out a series of reports at the Institute for Advanced Study's Computer Project, at Princeton, in 1945–47, which established the pattern that the subsequent evolution of computers has followed.

One basic principle was that the presentation of a calculation to the computer and the functioning of the computer should both be in terms of a *vocabulary* or *machine language*. That is, the computer should be wired, once and for all, so that it can perform any one of a list of a few dozen basic operations, or *instructions*, on receipt of a suitable initiating impulse; to solve a particular problem, the machine should then be supplied (from a device not yet described) with a long sequence of initiating impulses, planned in advance by writing down a sequence of instructions, or a *code* for the problem. The code is a description of the calculational procedure, of more-or-less the same kind that one might give to a hand computer; it tells which arithmetical operations are to be performed, on which numbers, and in which order, and it tells when portions of the sequence are to be repeated until, say, some test shows convergence; it tells

when new data are to be brought into the computer from an outside source, like a deck of punched cards, when to print out results, and when to stop. Designing a computer so that it can, in effect, read, interpret, and execute such a sequence of instructions was the goal of the Princeton project.

It was evident that any finite computational program, whether numerical or otherwise, can be expressed in terms of the instruction list chosen, and symbol manipulation and formal operations were often as important as numerical processes in many of the large calculations then being planned.

Another basic principle was the separation of various functions and of the units in the computer that performed them. (The separation was to some extent more conceptual than physical, in the actual construction.) A computer was thereby recognized to consist of an arithmetic or operation unit, a control unit, a memory or storage unit, and input and output units.

The *arithmetic unit* consisted of the circuits for adding, subtracting, multiplying, dividing, possibly taking square roots, shifting numbers relative to the decimal or binary point, extracting certain digits from a number, and so on. This unit also contained several registers for holding one or possibly both of the operands before an operation, and the result after the operation. In the Princeton arrangement, which persists in most of today's computers, there were two of these: an *accumulator register* A and a *multiplier-quotient register* MQ.

The *storage unit* or *memory* consisted of devices for storing a very large number of bits, or elementary units of information; these could represent the binary or decimal digits of numbers, letters or other characters, or the designations of instructions. The unit was divided into a thousand or so memory *locations*, each of about forty bits, and each designated by a number, its *address*. It was decided that the code should be stored in the same memory as the numerical data; just which locations were allotted to which, for a given problem, was left to the programmer to decide. The string of bits stored in a location was called a *word*; the number of bits in a word was fixed so that it could represent a number of ten to twelve decimal digits (or the equivalent number of binary digits); it then turned out that a word of this length could contain either one or two instructions, depending on the arrangement.

Associated with the memory, there was a *memory address register* (MAR) and circuits for bringing out or *fetching* from the memory that word whose address was then sitting in the MAR and sending it to the arithmetic unit (in case it happened to be a number) or to the control unit (in case it happened to be an instruction word). For purely technical reasons, this was usually done via a *storage register*, where the word was held temporarily. A word could similarly be stored in the memory, at any desired address, the word previously stored at that address thereby being wiped out.

There was much debate as to the best number of addresses per instruction. In a *three address code*, a typical instruction was "multiply X, Y, Z," meaning "multiply together the numbers stored at X and Y, and store the product at location Z." (A three address machine needs no registers.) In a *one address code*,

which was favored by the Princeton group and has been almost universally adopted since, the corresponding instruction, "multiply X," meant "multiply the number stored at X by the number in the MQ register and leave the product in the accumulator"; it was followed either by a *store* instruction or an instruction one of whose operands is the number in the accumulator.

The principle of storing the code in the same memory as the numerical data, i.e., the principle of the *stored-program computer*, made it possible for the code to be modified (i.e. to modify itself) during the running of a problem, by use of any of the same instructions that operate on numbers. For example, addresses of operands and of results could be modified in successive passes through a subroutine; numbers could be taken from the right place in a stored table by suitably modifying the address of a fetch instruction; the exit from a subroutine could be modified so as to return to different places in the main code on different occasions. Furthermore, codes could even be generated by other codes especially prepared for this purpose. This made possible *assemblers* and *compilers*, which first appeared in about 1949 and are basic to most coding systems today. In these systems, the code is written, usually in somewhat symbolic form, in pieces, each of which ignores the other pieces; the assembler or compiler then puts the pieces together, assigns storage locations, and computes and inserts the relative addresses and the addresses of cross-references between the pieces.

The *control unit* contained a control counter (or "instruction location counter"), an instruction register, and a decoding matrix. The *control counter* held the address of the instruction to be performed next; this address was automatically increased by 1 while an instruction was being performed, so long as one wanted to have the instructions performed in the order in which they appeared in the memory, but could be set to a new value by a transfer instruction if one wanted to change the sequence. The instruction to be performed was held in the *instruction register*, from which its operation part was communicated, for as long as needed, to the decoding matrix, and whose address part was communicated to the memory address register. The *decoding matrix* was a network of resistors and diodes with as many inputs as there were bits in an instruction and as many outputs as there were separate instructions; it had the remarkable property that if the bit combination that described any one of the operations was supplied to its inputs, exactly one of the output lines was energized; this line provided the initiating impulse for the operation to the appropriate arithmetic unit. This manner of control, which was von Neumann's idea, persists today, with variations.

Input and output were straightforward. Primary input was from a manually punched paper tape or a deck of punchcards. When a *load-button* on the console was pushed, consecutive words were automatically read from the tape or cards and stored in consecutively numbered storage locations until the end of the paper tape or card deck was reached. Primary output was equally simple: a *print* instruction caused the word whose address appeared in the instruction to be printed on a typewriter or punched on a tape.

In addition, the computers were planned to have (and they all eventually acquired) an auxiliary memory, often in the form of one or more magnetic tapes, of more-or-less unlimited length, onto which words could be written in sequence, automatically under control of the stored program, and from which they could be read back later into the machine. The primary memory was limited in size both by its cost (it usually contained electrostatic tubes or a magnetic drum) and by the number of bits allotted to an address; for example, if just 10 bits were available for an address, in each instruction, only 2^{10} or 1024 words could have separate addresses. The auxiliary memory did not have separate addresses for individual words and was much slower, but these disadvantages could be overcome by a little care, during the coding of a problem, in planning the transfer of information back and forth between the two memories. Today, both the main and the auxiliary memories have increased greatly in size and speed, but the distinction between them has remained.

In 1947, before the foregoing ideas had been fully incorporated in any of the computers then under construction, von Neumann proposed that the mode of operation of the Eniac should be modified to put some of these ideas into effect. A list of sixty instructions was drawn up, and each was designated by two, four, or sometimes six decimal digits. After a code had been written, the list of instructions could be set up on huge banks of ten-position switches. (These banks were originally intended for storage of tables, but were not in fact much used for this purpose, because it is generally easier, in an automatic computation, to compute function values as needed, by the same methods used for preparation of the tables, than to store the tables in advance.) Each row of switches was assigned an address, a number from 1 to 300. Of the Eniac's twenty accumulators, devices for adding or storing a number, one was used as a control counter, to keep track of the address of the row of instructions being executed, one served as a central clearing house for numbers, similar to the accumulator register of the Princeton design; two others were reserved for special purposes, and the rest were available for general storage. Wiring up the Eniac so that it could then read, interpret, and execute the instruction sequence on the switch banks was a huge task accomplished by the Aberdeen Computing Laboratory in collaboration with people from Princeton and Los Alamos. When this was done, not only did one have the advantage that the Eniac was easier to use, because the wiring never had to be changed again, but also the much greater advantage that the planning of a calculation was enormously simplified: the scientist had only to *program* the calculation and then set up the resulting code on the switches, rather than arrange a set of circuit connections about as complicated as a moderate-sized telephone exchange. The Eniac became thus the first machine to have a vocabulary.

The Princeton group also studied the problem of the mathematical formulation of large computations. One powerful idea that emerged was that of the *flow diagram*, introduced by von Neumann. This idea has fallen somewhat into disuse, lately, partly because of the emphasis on rather smaller and simpler prob-

lems than those that occupied von Neumann. For difficult and especially novel problems in the physical sciences a flow diagram, as originally understood, that is, a diagram providing a complete precise specification of the entire computational process, not merely a vague general plan, is still almost essential.

Although the fundamental principles have changed very little since the late nineteen-forties, a number of additional ideas have been introduced. The first is that of *floating point operation*, which was considered in 1945, but was discarded then, mainly because the technology of that time did not seem adequate for designing the needed circuits in a reliable way. Whenever numbers are operated on, their exponents are automatically adjusted; one usually takes as *normal form* that representation $y \cdot 10^x$ of a number in which the left-most nonzero digit of y lies immediately to the right of the decimal point, that is, $0.1 \leqq y < 1.0$, and in a binary machine one writes $y \cdot 2^x$ where $1/2 \leqq y < 1$; y and x are normally stored in one word of memory.

A more recent idea, that of *significance arithmetic*, also goes back to the practice of hand computers, who usually designate, for example by an underline, the last digit that is considered significant, thus writing either $3.7\underline{6}0$ or $3.76\underline{0}2$ or $3.7602\underline{9}3$, depending on the estimated accuracy. During a calculation, the significance of each intermediate result is estimated by a simple set of rules. (Usually one more digit is carried along than the last significant one, to decrease the accumulation of rounding errors.) Experienced computers know that unless this practice is carefully followed it is very easy in a long calculation to end up, without knowing it, with results that have no significance at all. This is especially likely to happen in calculations where there can be cancellation due to subtraction of nearly equal quantities. Scientific computations made on today's computers are generally much longer and more difficult to analyze than those formerly made by hand, and it is perhaps surprising that only three or four computers, even now, have automatic provision for the monitoring of significance. Several methods of doing this are under study, however, and probably one of them will soon be adopted generally.

A basic idea, introduced around 1950, is that of *index registers*, to provide for automatic modification of addresses in those instructions containing a *tag* in one or more bits reserved for this purpose in the instruction word, by increasing the address by the amount in the index register; this enables one in effect to deal with subscripted variables. Some machines have up to 100 index registers, any one of which (or sometimes several) can be specified by a corresponding tag in any instruction.

Index registers and other innovations, such as indirect addressing, instructions for logical operations, push-down storage banks, etc. have made it often possible to do in one instruction things that previously took several.

A number of ideas for speeding up the execution of the instructions have been introduced into computer circuit designs, whereby operations are made concurrent whenever possible, for example, by overlapping of input and output, or by having the control unit look ahead, while it is performing one instruction,

and make as much preparation as possible for the next few instructions to come, or by the use of several multipliers or other arithmetic devices in parallel, and so on.

The so-called associative memories, now under consideration, in which a word can be given an arbitrary symbolic address, located in part of the word itself, will probably have great value in some applications, though probably not in basic mathematical science.

The main advance in computers since 1950 is the very great progress in the technology of computer manufacture. This shows up primarily in connection with *reliability*. A typical calculation may contain 10^9 basic operations, each depending on, say, 100 electrical components; in order to be 99.9% sure that the result is not invalidated by a machine error, each component has to be so reliable that it only makes one mistake in 10^{14} actuations. This is like permitting a typist only one error in a million years of continuous typing, and is a degree of reliability far beyond anything that was attainable in electrical or mechanical parts until recently. Simultaneously, speeds have been increased almost a thousandfold since 1950, and memory capacities almost a hundredfold. Ferrite cores and disc-files have revolutionized the storage problem. Input and output devices have been greatly improved with respect to both speed and the variety of characters that can be read and printed.

The advances in computer technology have unfortunately not been paralleled by a corresponding advance in the art of scientific computing. In many areas of physical science, the calculation procedures now used are similar to those formerly used in hand calculations, although they have been considerably improved in detail and are applied to much bigger and more complex problems. It is clear on combinatorial grounds alone that today's computers could handle procedures based on vastly more sophisticated mathematical principles, if we only knew how to find those principles. Contrary to popular belief, we are more often limited in what we can compute by the lack of sufficiently powerful mathematical methods than by the lack of sufficiently powerful computers.

The chief advance in the computing art has been the development of simplified coding schemes, such as formula translators, that help people who have not-too-large problems of a somewhat standardized nature. In a formula-translator system, the programmer writes a description of the problem as a series of statements and simple formulas, according to a certain set of rules, and the *formula translator*, or *compiler*, which is a lengthy code in itself, turns the statements and formulas into a running code. The compiler is usually embedded in an elaborate *system* for using the computer, which sequences the problems through the computer, one after another, and supervises them in various ways. This system, together with the library of subroutines, routines for control of input and output formats, and so on, has come to be called *software* in the industry (as opposed to hardware); a manufacturer is expected to supply a considerable amount of software with the machines he sells.

A curious phenomenon that has accompanied the development of software

is a tendency for the hardware to become dependent on it. For example, machines of a few years ago had a "print" instruction, which caused one or more words to be printed (either on-line or off-line) directly in recognizable form. This is no longer true of many present machines, which require a complicated input-output "package" of subroutines, also format statements and the like, before anything can be printed.

The amount and complexity of the software have increased enormously in the last few years, and its preparation has become a rather large industry. It is the writer's conviction (shared by at least some of his colleagues) that, although the systems now in use are successful in industrial and commercial applications, they are a serious obstacle to basic research in mathematical science, especially to research whose aim is to find fundamentally new and different methods of computation. To change existing methods requires nonconformity, which is virtually impossible in the face of the vast array of rules and regulations in the present systems. The overall system is so complicated that it is almost impossible to find out what has happened whenever one of the rules (it may be an unwritten one) has been violated. The testing of novel methods is an almost continual process of debugging, not of the code but of the mathematical processes taking place. For this, one has to work directly with the machine language code, even if the program was written in some super language, but the codes turned out by the compilers are almost impossible to follow, and the monitor systems prevent any other than very standard and conventional manipulation of the code. The manoeuvres that one has to go through not only waste machine time but also retard progress enormously. Also, present machine languages are designed for the systems people, who write compilers, etc., and are not well planned for scientific users.

The fundamental spirit of von Neumann's ideas was an emphasis on directness and simplicity. It is the writer's belief that these notions are *not* incompatible with the advances in computer technology and that a return to them is necessary before the inherent promise of modern computers in basic science can be realized.

References

John von Neumann, Collected works, A. H. Taub (editor), vol. 5: Design of Computers, Theory of Automata, and Numerical Analysis, Macmillan, New York, 1963; see also various Ballistic Research Laboratory reports on the Eniac Converter Code, Aberdeen Proving Ground, Aberdeen, Md.

THE PRIMALITY OF RAMANUJAN'S TAU-FUNCTION

D. H. LEHMER, University of California, Berkeley

Introduction. The function $\tau(n)$ introduced by Ramanujan in 1916 [1] as a natural outgrowth of the functions $\sigma_k(n)$, the sum of the kth powers of the divisors of n, has been the subject of numerous investigations ever since. It is defined most simply as the coefficient of X^n in the expansion of the product

$$X\prod_{m=1}^{\infty}(1-X^m)^{24}=\sum_{n=1}^{\infty}\tau(n)X^n = X - 24X^2 + 252X^3 + \cdots.$$

Although a number of remarkable properties of $\tau(n)$ have been established, some of which are cited below, there remains a number of unsolved questions about $\tau(n)$; for example: What is the exact order of magnitude of $\tau(n)$ (see [2])? Is $\tau(n)=0$ for some $n>0$ (see [3])? In this note we address ourselves to the question: Is $\tau(n)$ ever a prime? We answer this question by

THEOREM A. *The integer $\tau(n)$ is composite for $2\leqq n\leqq 63000$, but*

$$\tau(63001) = 80561663527802406257321747$$

is a prime number.

Since published tables of $\tau(n)$, [4], extend to $n=1000$ and unpublished tables to $n=10000$, [5], it is clear that to prove Theorem A requires the use of some of the known properties of $\tau(n)$, namely the formulas and congruence properties listed below. Numbers in square brackets give references to papers where these results are established. In what follows p always designates a prime.

Required Properties.

(1) If $(a, b) = 1$, then $\tau(ab) = \tau(a)\tau(b)$. [6]

(2) $\tau(p^{\alpha+1}) = \tau(p)\tau(p^{\alpha}) - p^{11}\tau(p^{\alpha-1}), \quad (\alpha > 0)$. [6]

As an immediate consequence of (1) and (2)

(3) If $p \mid \tau(p)$ then $p \mid \tau(np), \quad (n > 0)$.

(4) If n is odd $\tau(n) \equiv \sigma(n) \pmod 8$. [7]

Setting $p=2$ in (3) and using (4) we easily derive

(5) $\tau(n)$ is odd if and only if n is an odd square.

(6) If n is odd, $\tau(n) \equiv \sigma_3(n) \pmod{32}$. [8]

(7) If $(n, 3) = 1$, $\tau(n) \equiv \sigma(n) \pmod 3$. [9]

(8) If $3p = u^2 + 23v^2$, $\tau(p) \equiv -1 \pmod{23}$. [10], [3]

(9) $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$. [2], [11]

Proof of Theorem A. We begin by assuming there exists a least integer $n_0 \leqq 63000$ for which $\tau(n_0)$ is a prime. If n_0 is not a power of a prime then $n_0=ab$

with $1<a<b<n_0$ and $(a, b)=1$. By (1), either $\tau(a)$ or $\tau(b)$ is a prime, contrary to the minimal property of n_0; hence

$$n_0 = p^\alpha \qquad (\alpha \geqq 1). \tag{10}$$

We now separate two cases:

Case I. $|\tau(n_0)|=2$.

In this case it is clear that $p\neq 2$ because $\tau(2)=-24$ is a multiple of 8 and so, by (2), is $\tau(2^\alpha)$. Hence n_0 is odd but not an odd square, by (5). Therefore $n_0=p^{2\beta+1}$, $(p>2, \beta\geqq 0)$. Writing (2) in the form

$$\tau(p^{2k+1}) - p^{11}\tau(p^{2k-1}) = \tau(p)\tau(p^{2k}), \tag{11}$$

we see, by induction on k, that $\tau(p)$ divides $\tau(p^{2\beta+1})=\pm 2$. Now $|\tau(p)|=1$ would contradict (5). Hence $|\tau(p)|=2$. Because n_0 is minimal this implies $n_0=p$. Now the case $\tau(p)=-2$ is impossible by (7). In fact, $p\neq 3$ because $\tau(3)=252 \neq -2$. Hence (7) would give

$$-2 = \tau(p) \equiv \sigma(p) = 1 + p \pmod 3.$$

This implies $p=3$, a contradiction. Hence we are left with

$$\tau(p) = 2. \tag{12}$$

To complete Case I we have to show the impossibility of (12) with $p<63000$. This is easily done by using the congruences (6) and (9). In fact, (6) gives

$$2 = \tau(p) \equiv 1 + p^3 \pmod{32}$$

which implies

$$p \equiv p^{1+2\cdot 16} \equiv p^{33} \equiv (p^3)^{11} \equiv 1^{11} \equiv 1 \pmod{32}. \tag{13}$$

Also (9) gives $2=\tau(p)\equiv 1+p^{11} \pmod{691}$ which implies

$$p \equiv p^{1+4\cdot 690} \equiv p^{2761} \equiv (p^{11})^{251} \equiv 1 \pmod{691}.$$

Combining this with (13) gives $p=22112x+1$. Since 22113 and 44225 are not primes, $p>63000$. This disposes of Case I.

Case II. $|\tau(n_0)|>2$.

Since $|\tau(n_0)|$ is now an odd prime, (5) and (10) give $n_0=p^{2\beta}(p>2, \beta\geqq 1)$.

We show next that p does not divide $\tau(p)$, for otherwise by (2) p^2 would divide $\tau(p^2), \tau(p^3), \cdots$, so that $\tau(n_0)=\tau(p^{2\beta})$ could not be a prime. Since $p|\tau(p)$ for $p=3, 5, 7$ it follows that $p\geqq 11$. Since

$$63000 < 83521 = 17^4 < 1771561 = 11^6,$$

the case of $\beta>1$ reduces to the consideration of

$$\tau(11^4) = -8154467755666712757895$$

and

$$\tau(13^4) = 1528680442488998435984621.$$

Neither one of these numbers is a prime, the latter being divisible by 25741.

There remains the case $n_0 = p^2$, $11 \leqq p < 251$. We see from (7) that if $p = 6x+1$ then

$$\tau(p^2) \equiv \sigma(p^2) \equiv 1 + p + p^2 \equiv 0 \pmod 3.$$

This would imply $\tau(p^2) = \pm 3$. We would infer from (9) that

$$\pm 3 \equiv \sigma_{11}(p^2) \equiv 1 + p^{11} + p^{22} \pmod{691}.$$

Solving these two congruences gives the solutions $p \equiv 1, 21, 33, 348 \pmod{691}$. This disagrees with the assumption that p is a prime less than 251. Hence $p \neq 6x+1$.

Next suppose that

$$3p = u^2 + 23v^2. \tag{14}$$

Then

$$p^{11} \equiv \left(\frac{p}{23}\right) \equiv \left(\frac{3p}{23}\right) = \left(\frac{u^2}{23}\right) \equiv 1 \pmod{23}$$

so that (2) and (8) give $\tau(p^2) = (\tau(p))^2 - p^{11} \equiv (-1)^2 - 1 \equiv 0 \pmod{23}$. This would require $\tau(p^2) = \pm 23$. But then (9) would give

$$\pm 23 \equiv \sigma_{11}(p^2) \equiv 1 + p^{11} + p^{22} \pmod{691},$$

which has solutions $p \equiv 92, 340, 410, 432 \pmod{691}$ contrary to $p < 251$. Hence the prime p is of the form $6x-1$ but not of the form (14). The fifteen such primes <251 are the arguments of Table 1. For each argument, we give one or more small factors q of $\tau(p^2)$ to show that $\tau(p^2)$ is not a prime. In those cases in which only one q is given $|\tau(p^2)| \neq q$ (see [12]).

TABLE I

p	q	p	q
17	842087·15936629	113	31
23	11·13	137	11
53	17	149	49139
59	137	167	137
83	61·71	173	89·1567·38833
89	33107	191	2357·308117
101	8731	227	51869
107	43·211		

Test for Primality of $\tau(251^2)$. To complete the proof of Theorem A we must establish the primality of $\tau(63001)$. The standard procedure for testing numbers N as large as this is to apply some valid converse of Fermat's theorem [**13**]. These require the knowledge of some large prime factor of $N-1$. In our case

$$N - 1 = 2 \cdot 397 \cdot 1014630523020181439009 09,$$

where the large factor was found to be composite but rather difficult to decompose into its prime factors, but the factorization of $N+1$ is relatively easy, namely

$$N + 1 = 2^2 \cdot 3^2 \cdot 7 \cdot 23 \cdot 29 \cdot 1249 \cdot 1767401 \cdot 217122342553.$$

Hence the following Fibonacci type test was used [**14**].

THEOREM. *Let $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, be the sequence of Fibonacci. If F_n is divisible by N for $n = N+1$ but not for $n = (N+1)/p$, where p ranges over the prime factors of $N+1$, then N is a prime.*

Although this theorem speaks of Fibonacci numbers that are incredibly large for $N = \tau(63001)$, one does not deal with such large numbers themselves but only their remainders on division by N. Furthermore, one does not use the defining recurrence to compute F_n modulo N but instead skips over all but $O(\log N)$ values of n by a duplication formula. All told, the application of the theorem represents an effort proportional to only log N. For $N = \tau(63001)$ the hypothesis of the theorem was verified in about twenty seconds. Hence $\tau(63001)$ is indeed the first prime value of $\tau(n)$.

References

1. S. Ramanujan, On certain arithmetical functions, Trans. Camb. Phil. Soc., 22 (1916) 159–184.
2. G. H. Hardy, Ramanujan, Cambridge, Ch. X (1940) 161–185.
3. D. H. Lehmer, The vanishing of Ramanujan's function $\tau(n)$, Duke Math. J., 14 (1947) 429–433.
4. G. N. Watson, A table of Ramanujan's function $\tau(n)$, Proc. London Math Soc., 51 (1949) 1–13.
5. D. H. Lehmer, Manuscript Table of $\tau(n)$; $n = 1$ (1) 10000.
6. L. J. Mordell, On Mr. Ramanujan's empirical expansions of modular functions, Proc. Camb. Phil. Soc., 19 (1920) 117–124.
7. H. Gupta, Congruence properties of $\tau(n)$, Proc. Benares Math. Soc., 5 (1943) 17–22.
8. R. P. Bambah, Two congruence properties of Ramanujan's function $\tau(n)$, J. London Math. Soc., 21 (1946) 91–93.
9. H. Gupta, A congruence relation between $\tau(n)$ and $\sigma(n)$, J. Indian Math. Soc., 9 (1945) 59–60.
10. J. R. Wilton, Congruence properties of Ramanujan's function $\tau(n)$, Proc. London Math. Soc., 31 (1930) 1–10.
11. D. H. Lehmer, Properties of the coefficients of the modular invariant $J(\tau)$, Amer. J. Math., 64 (1942) 488–502.
12. These factors were discovered by John Brillhart using the IBM 7090 at Stanford University's Department of Computer Sciences under grant No. NSF-GP948.
13. D. H. Lehmer, Tests for primality by the converse of Fermat's theorem, Bull. Amer. Math. Soc., 33 (1927) 327–340.
14. A brief discussion of such tests will appear elsewhere.

THE PRIME FACTORS OF CONSECUTIVE INTEGERS

D. H. LEHMER, University of California, Berkeley

In 1892, J. J. Sylvester [1] proved that every product of k consecutive integers greater than k is divisible by a prime greater than k. This beautiful result is by no means the strongest possible. In fact, a theorem of Størmer [2] implies that the product of only two sufficiently large consecutive integers must be divisible by a prime greater than k. That is, for each integer k there exists a number M_k such that $n(n+1)$ is divisible by a prime greater than k for every $n > M_k$. Incidentally, it follows from recent results [3] that $\log M_k < Ck^2e^{k/2}$ for some constant C.

Thus, for each k, the statement: "$n(n+1)$ is divisible by a prime greater than k" is true except for only a finite number of n's. The same is true, of course, for the statement: "$n(n+1) \cdots (n+h-1)$ is divisible by a prime greater than k" no matter what $h \geqq 2$ is chosen.

Let $p_1 = 2, p_2 = 3, \cdots$ be the primes in ascending order. Then from the above considerations we can define the function $U_h(t)$ to be the greatest integer $n > p_t$, if any, such that $n(n+1) \cdots (n+h-1)$ is divisible by no prime greater than p_t. It is clear that $U_h(t)$ will cease to exist if h becomes too large. In fact Sylvester's result shows that this has already happened when $h = p_t$. The least value of h for which there is no $U_h(t)$ we denote by $f(p_t)$, following Erdös [4]. In proving a conjecture of Erdös, Utz [5] has recently shown that

$$f(2) = 2, \quad f(3) = 3, \quad f(5) = f(7) = 4, \quad \text{and} \quad f(13) \geqq 6.$$

It is the purpose of this note to show that

$$\begin{aligned} &f(11) = 4, \quad f(13) = f(17) = f(19) = f(23) = f(29) = f(31) = f(37) = 6, \text{ and,} \\ &f(41) = 7. \end{aligned} \tag{1}$$

These results follow from the evaluation of $U_h(t)$ for all possible h and all $t \leqq 13$.

The determination of $U_h(t)$ for a particular h and t is a small theorem in its own right whose proof, for $t > 6$, becomes rapidly too laborious for humans to carry out. For $t \leqq 14$ such detailed proofs are well within the capabilities of today's computers.

The general method is easy enough to understand. For each t we find all the numbers S such that $S(S+1)$ is divisible by no prime greater than p_t. A practical method of discovering all such S is explained in [3] and complete lists of such S are given for each $t \leqq 13$. We have only to arrange each set of S's in order and search the list of runs for $h-1$ consecutive integers. The largest such S is clearly $U_h(t)$. These results are as tabulated in Table 1.

For example, the entry $U_4(9) = 322$ means that

$$322 = 2 \cdot 7 \cdot 23, \quad 323 = 17 \cdot 19, \quad 324 = 2^2 \cdot 3^4, \quad 325 = 5^2 \cdot 13$$

is the last instance of four consecutive numbers whose prime factors do not exceed $23=p_9$. A glance at each row of Table 1 establishes the results (1).

TABLE 1

t	p_t	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$
2	3	8				
3	5	80	8			
4	7	4374	48			
5	11	9800	54			
6	13	123200	350	63	24	
7	17	336140	440	63	48	
8	19	11859210	2430	168	48	
9	23	11859210	2430	322	48	
10	29	177182720	13310	322	54	
11	31	1611308699	13454	1518	152	
12	37	3463199999	17575	1518	152	
13	41	63927525375	212380	1680	286	285

References

1. J. J. Sylvester, On arithmetical series, Messenger Math., 21 (1892) 1–19, and 87–120; Collected Mathematical Papers, 4 (1912) 687–731.

2. G. Størmer, Quelques théorèmes sur l'équation de Pell $x^2-Dy^2=\pm 1$ et leurs applications, Skr. Norske Vid. Akad. Oslo, I no. 2 (1897).

3. D. H. Lehmer, On a problem of Størmer, Illinois J. Math., 8 (1964) 57–79.

4. P. Erdös, A theorem of Sylvester and Schur, Nieuw Arch. Wisk., 3 (1955) 124–128.

5. W. R. Utz, A conjecture of Erdös concerning consecutive integers, this MONTHLY, 68 (1961) 896–897.

COMBINATORIAL ANALYSIS AND COMPUTERS

MARSHALL HALL, JR. AND D. E. KNUTH, California Institute of Technology

1. Introduction. The "logical decision-making" characteristics of computers enable them to attack many problems which are not basically numerical. Combinatorial constructions and searches are applications of this kind in which computers have been very successful. The two principal ways in which computers have been used in these problems are: 1) generation of a sequence of combinatorial patterns as part of a larger problem, and 2) constructions and searches for which hand calculations are unfeasible.

Since the size of combinatorial problems grows very rapidly, we can usually expect the computer to do only about one case larger than can be done by hand. For example, the problem of constructing a finite plane of order n is equivalent to choosing, in a restricted way, $(n-1)^2$ of the $n!$ permutations on n letters; thus, changing n to $n+1$ will make the problem many orders of magnitude larger.

2. Combinatorial sequences. Many computer applications call for the generation of combinatorial sequences such as the set of all permutations on n elements, the set of all combinations of m things taken n at a time, and so on.

The simplest combinatorial sequence is the set of all ordered n-tuples $(x_1, x_2, \cdots, x_n)$ for which $0 \leqq x_i < m$, $1 \leqq i \leqq n$. This set can, of course, be generated by repeatedly adding 1 in the m-ary number system; in applications, however, it is usually more useful if the n-tuples are generated in such a way that only one x_i changes at each step [5, 7]. The simplest method for doing this is the following: At the kth step, $1 \leqq k < m^n$, add 1 to x_{n-i}(modulo m) if m^i is the highest power of m dividing k. Whenever k is a multiple of m, it is not difficult to prove that this will be the digit immediately left of the rightmost nonzero digit, assuming $(0, 0, \cdots, 0)$ was the starting n-tuple. The case $m=2$ gives rise to the so-called Gray binary numbers which have found many engineering applications, and the sequence is the well-known procedure for solving the classical Chinese Ring Puzzle [2].

The problem of generating all permutations has stimulated much interest [1, 27, 40]. A method which is at least 150 years old [6], and which is still being rediscovered fairly often, is the following: Permutations can be generated in lexicographic order if we (1) obtain the successor of $(x_1, x_2, \cdots, x_n)$ by finding r, s such that

$$x_r < x_{r+1} \geqq x_{r+2} \geqq \cdots \geqq x_n, \qquad x_{r+s} > x_r \geqq x_{r+s+1},$$

then (2) interchange x_r and x_{r+s}; and (3) replace $(x_{r+1}, x_{r+2}, \cdots, x_n)$ by $(x_n, \cdots, x_{r+2}, x_{r+1})$. This method is valid even when the elements permuted are not distinct.

As above, however, it is usually advantageous to generate the permutations in such a way that successive permutations are "near" each other, i.e., only two

elements have been interchanged. The first method of this kind was due to Wells [**45**], but an improved algorithm was later discovered almost simultaneously by Trotter [**42**], and by Johnson [**22**]. In the latter method, all permutations of n distinct objects are obtained by the successive interchange of two adjacent elements, by using essentially the following recursive rule: Let $(y_1, y_2, \cdots, y_{n-1})$ be the kth permutation of the sequence for all permutations on $(1, 2, \cdots, n-1)$; then the $(kn+1)$-st, $(kn+2)$-nd, $\cdots$, $(k+1)n$-th permutations on $(1, 2, \cdots, n)$ are

$$(n, y_1, y_2, \cdots, y_{n-1}), (y_1, n, y_2, \cdots, y_{n-1}),$$
$$(y_1, y_2, n, \cdots, y_{n-1}), \cdots, (y_1, y_2, \cdots, y_{n-1}, n)$$

if k is even, and the same sequence in reverse order if k is odd. For example, here is the set of all 24 permutations on 4 elements obtained by repeated interchanges of adjacent elements:

1234	3124	2314
1243	3142	2341
1423	3412	2431
4123	4312	4231
4132	4321	4213
1432	3421	2413
1342	3241	2143
1324	3214	2134

This technique is the fastest method for generating the complete set of permutations on a computer.

Similarly, an algorithm is known for successively choosing all combinations of m things, n at a time, by changing the choice of only one object each time. Algorithms for obtaining all partitions of an integer n are given in [**38**]; the partitions of a set are produced by the algorithm in [**19**]; and many similar techniques are known.

3. Backtrack. The basic method [**9, 43**] used for most combinatorial searches has been christened "backtrack" technique by D. H. Lehmer. A great many combinatorial problems can be stated in the form, "Find all vectors $(x_1, x_2, \cdots, x_n)$ which satisfy p_n," where $x_1, x_2, \cdots, x_n$ are to be chosen from some finite set of m distinct objects, and p_n is some property. The combinatorial sequences of the preceding section all fit into this framework; for example, if we are to generate all *permutations* of distinct objects, the property p_n states simply that $x_i \neq x_j$ if $i \neq j$. If we wish to generate all *combinations* of the numbers $(1, 2, \cdots, m)$ taken n at a time, p_n is the property

$$1 \leqq x_1 < x_2 < \cdots < x_n \leqq m.$$

The backtrack method consists of defining properties p_k for $1 \leqq k \leqq n$ in such a way that whenever $(x_1, x_2, \cdots, x_n)$ satisfies p_n, then $(x_1, \cdots, x_k)$ necessarily satisfies p_k. The computer is programmed in a straightforward manner to consider only those partial solutions $(x_1, \cdots, x_k)$ which satisfy p_k; if p_k is not satisfied, the m^{n-k} vectors $(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)$ are not examined by the program. If the properties p_k can be chosen in an efficient way, comparatively few cases are considered.

Although the backtrack approach frequently gives satisfactory results, there are equally many cases on record for which it has failed; in fact, it is not uncommon (see [**31**]) to find situations in which thousands of centuries would be required for the algorithm to be completed! For this reason, it is desirable to have some convenient means for estimating the computational time before putting the algorithm onto a computer. One can proceed by playing the following little game: Let a_1 be the number of (x_1) which satisfy p_1; then let y_1 be one such solution, chosen at random with probability $1/a_1$. Similarly let a_k be the number of x_k for which $(y_1, \cdots, y_{k-1}, x_k)$ satisfies p_k; randomly choose y_k to be one such x_k, with probability $1/a_k$. If $a_k = 0$ or $k = n$, the game terminates. It is not difficult to verify that the expected value of

$$a_1 + a_1a_2 + a_1a_2a_3 + \cdots + a_1a_2 \cdots a_n$$

will be precisely the number of cases which are examined by the backtrack algorithm, and therefore by playing this game several times one obtains a good idea of the size of this number. When the number is greater than, say, 10 million, the backtrack method is probably unfeasible with contemporary equipment.

This random process can be combined with backtracking to give "random" solutions $(x_1, x_2, \cdots, x_n)$ to the combinatorial problem (see [**30**]). The solutions, however, are not truly random in general, i.e., they are not necessarily obtained with equal probability. For example, of the 10×10 latin squares shown in Fig. 1, square B [**31, 32**] would be obtained with probability 9.0×10^{-27}, assuming the first row and first column are fixed; square A, which certainly appears to be "less random," is actually produced with the much higher probability $9^{-1}8^{-3}7^{-5}6^{-7}5^{-6}4^{-4}3^{-2} = 1.3 \times 10^{-21}$.

A

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	0	1
3	4	5	6	7	8	9	0	1	2
4	5	6	7	8	9	0	1	2	3
5	6	7	8	9	0	1	2	3	4
6	7	8	9	0	1	2	3	4	5
7	8	9	0	1	2	3	4	5	6
8	9	0	1	2	3	4	5	6	7
9	0	1	2	3	4	5	6	7	8

B

0	1	2	3	4	5	6	7	8	9
1	8	3	2	5	4	7	6	9	0
2	9	5	6	3	0	8	4	7	1
3	7	0	9	8	6	1	5	2	4
4	6	7	5	2	9	0	8	1	3
5	0	9	4	7	8	3	1	6	2
6	5	4	7	1	3	2	9	0	8
7	4	1	8	0	2	9	3	5	6
8	3	6	0	9	1	5	2	4	7
9	2	8	1	6	7	4	0	3	5

C

0	1	2	3	4	5	6	7	8	9
9	6	5	4	7	8	3	0	1	2
1	3	8	5	2	6	0	9	4	7
7	8	4	0	5	3	2	1	9	6
5	2	9	6	3	7	8	4	0	1
3	9	6	2	0	1	4	5	7	8
8	4	0	1	6	9	7	2	5	3
6	7	3	9	1	0	5	8	2	4
2	0	1	7	8	4	9	6	3	5
4	5	7	8	9	2	1	3	6	0

FIG. 1. Three latin squares of order 10. Squares B and C are orthogonal.

The backtrack algorithm usually produces its results in lexicographic order, and this fact can often be utilized to shorten the computational time greatly by eliminating isomorphic solutions. If it can be established that any completion of the partial solution $(x_1, x_2, \cdots, x_k)$ must be isomorphic to a solution $(y_1, y_2, \cdots, y_n)$ with $(y_1, \cdots, y_k) < (x_1, \cdots, x_k)$ in the lexicographic order, we may omit consideration of $(x_1, x_2, \cdots, x_k)$. (This principle is to be incorporated into the property p_k.) This and other methods for rejecting isomorphic solutions are discussed by Swift [**39**]. A technique for possibly speeding up backtrack solutions by preliminary identification of objects has been suggested by Tompkins [**41**].

Backtrack has been used in most of the applications discussed later in this paper; it has also been used to determine all 80 Steiner triple systems of order fifteen [**13**], to construct certain codes [**20, 46**], to discover an Hadamard matrix of order 92 [**8**], to study the four-color problem [**47**], and to do problems with a more "recreational" flavor such as the fitting together of geometric shapes [**36**]. The techniques of dynamic programming [**3**] have proved useful as an alternative to backtrack in some combinatorial applications.

4. Latin Squares. Parker [**32**] has discovered an ingenious way to apply the backtrack method for determining latin squares orthogonal to a given one. A straightforward method would be to successively fill each cell of the potential orthogonal mate. Parker's method, however, proceeds in two steps: He first applies backtrack to obtain the set of all *transversals* of the given square; a transversal is a set of cells with exactly one in each row, one in each column, and one containing each symbol. The problem of finding an orthogonal mate to the square is equivalent to finding a set of disjoint transversals to cover the square, so the second stage is to use backtrack again in order to find such coverings by transversals. With 10×10 latin squares, this reduces the number of cases to be considered from approximately 10^{17} to approximately 5×10^5, a factor of some 200 billion! The total time required, when the problem is split into two parts in this way, may be thought of as the sum of the time for two separate steps rather than the product.

Parker used his program to show that square C of Fig. 1 is the only latin square orthogonal to square B. On the other hand, square A is known to have no transversals, hence no orthogonal mate, by theorems proved in [**12**]. Parker's program has been applied to nearly one hundred 10×10 latin squares and the majority of these have possessed orthogonal mates. This is quite remarkable, since Euler's conjecture that no such squares exist had been believed for so many years. Yet no triple of mutually orthogonal squares has ever been found after much computer searching, and more theoretical advances will be necessary before this problem can be reduced to a size computers can handle.

A set of five mutually orthogonal 12×12 latin squares has been found by computer [**4, 21**]. Starting with the multiplication table of a group which has the elements $x_1, x_2, \cdots, x_n$, the square formed by permuting the rows so that the first column has the form $y_1, y_2, \cdots, y_n$ will be orthogonal if and only if

$x_1y_1^{-1}, x_2y_2^{-1}, \cdots, x_ny_n^{-1}$ is a permutation of the group elements. We may assume y_1 is the identity. If we start with the Abelian group having elements $x_i(0 \leqq x < 6,\ 0 \leqq i < 2)$, with product $x_iy_j = (x+y) \bmod 6_{(i+j) \bmod 2}$, the following first columns will yield five mutually orthogonal latin squares:

$$\begin{array}{cccccccccccc}
0_0 & 1_0 & 2_0 & 3_0 & 4_0 & 5_0 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 5_1 \\
0_0 & 2_0 & 1_0 & 4_1 & 3_1 & 5_1 & 3_0 & 5_0 & 4_0 & 1_1 & 0_1 & 2_1 \\
0_0 & 4_1 & 4_0 & 2_1 & 2_0 & 0_1 & 3_1 & 1_0 & 1_1 & 5_0 & 5_1 & 3_0 \\
0_0 & 5_0 & 3_1 & 2_0 & 4_1 & 1_1 & 2_1 & 4_0 & 5_1 & 1_0 & 3_0 & 0_1 \\
0_0 & 0_1 & 5_0 & 1_1 & 5_1 & 4_0 & 2_0 & 2_1 & 4_1 & 3_0 & 1_0 & 3_1
\end{array}$$

It has been shown that no set of six mutually orthogonal latin squares of this kind exist. Similarly, if we start with the non-Abelian group having elements x_i (x a permutation on 3 letters, $0 \leqq i < 2$), and multiplication defined by multiplying the permutations and adding the subscripts (mod 2), we can get three mutually orthogonal squares:

$$\begin{array}{cccccccccccc}
(1)_0 & (12)_0 & (13)_0 & (23)_0 & (123)_0 & (132)_0 & (1)_1 & (12)_1 & (13)_1 & (23)_1 & (123)_1 & (132)_1 \\
(1)_0 & (123)_1 & (12)_0 & (12)_1 & (23)_1 & (13)_0 & (13)_1 & (23)_0 & (123)_0 & (132)_1 & (1)_1 & (132)_0 \\
(1)_0 & (132)_1 & (123)_0 & (13)_1 & (23)_0 & (23)_1 & (132)_0 & (123)_1 & (13)_0 & (12)_1 & (12)_0 & (1)_1
\end{array}$$

Starting with the alternating group on 4 elements, there are 3840 pairs of orthogonal squares but no three mutually orthogonal. From this data it appears that as the group gets less Abelian, less orthogonal squares are present, although no theoretical grounds for such behavior are known at this time.

5. Projective planes and symmetric block designs. More computer studies have been made relating to finite projective geometries than to any other branch of combinatorial analysis. A survey of this work appears in [16] and we mention only the principal results here.

It is easy to show by hand computation that the projective planes of orders 2, 3, 4, and 5 are unique. The Bruck-Ryser theorem shows that there is no plane of order 6. There is a unique plane of order 7; this result was first established by Norton [29], being incidental to his listing of the 147 nonisomorphic latin squares of order 7. Norton found only 146; an omission was found by Sade [34] who confirms the completeness of the list of 147. A more direct proof of the uniqueness of the plane of order 7 is given by Hall [10] and Pierce [33].

In a plane of order 8, the multiplication table of nonzero elements forms a 7×7 latin square. Using the list of 147 such squares and a few further simplifications, a computer was programmed to help establish the uniqueness of the plane of order 8 [11]; this proof is a good example of the effective interplay of machine and hand methods. There are four known projective planes of order 9, and work is under way to determine whether these are in fact the complete set. This problem appears to lie at the borderline of computational feasibility with respect to an exhaustive search for planes of a small order. Since planes exist for every order which is a prime or prime power, the question as to the existence of any plane of nonprime-power order is of considerable interest, and the smallest order

of this kind permitted by the Bruck-Ryser theorem is order 10. A search is under way for a plane of order 10 with a nontrivial collineation, but even this is a large undertaking. A plane of order 10 is equivalent to a set of nine mutually orthogonal 10×10 latin squares, but as stated above not even three such squares are known.

If we can find residues $b_1, b_2, \cdots, b_k$(modulo v) where every difference $d \not\equiv 0 \pmod v$ has exactly λ representations $d \equiv b_i - b_j$ (modulo v) we call the b's a difference set modulo v. The relation $k(k-1) = \lambda(v-1)$ must necessarily hold. If $b_1, \cdots, b_k$ are a difference set modulo v, then the "blocks" B_j: b_1+j, b_2+j, $\cdots$, $b_k+j \pmod v$ $j=0, \cdots, v-1$ form a symmetric block design. In the special case that $\lambda=1$, $k=n+1$, $v=n^2+n+1$, the block design is a finite projective plane of order n, and the blocks are the lines of the plane. For example, the following difference sets may be used to construct the planes of small orders:

$$n = 2: \quad 1, 2, 4 \pmod 7$$

$$n = 3: \quad 0, 1, 3, 9 \pmod{13}$$

$$n = 4: \quad 3, 6, 7, 12, 14 \pmod{21}.$$

By a theorem of one of the authors [**14**], under certain conditions we may assume that the difference set $b_1, \cdots, b_k$ is fixed by "multipliers," where a multiplier is a prime p dividing $k-\lambda$. If p is such a multiplier then $pb_1, \cdots, pb_k \pmod v$ are the same as $b_1, \cdots, b_k$ in some order. This fact considerably simplifies a computer search for difference sets, and those of small order are discussed in [**14, 17**]. In particular, the SWAC computer was used to show that all difference sets with $\lambda=13$, $k=40$, $v=121$ are isomorphic to one of the following four:

1, 3, 4, 7, 9, 11, 12, 13, 21, 25, 27, 33, 34, 36, 39, 44, 55, 63, 64, 67, 68, 70, 71, 75, 80, 81, 82, 83, 85, 89, 92, 99, 102, 103, 104, 108, 109, 115, 117, 119.

1, 3, 4, 5, 9, 12, 13, 14, 15, 16, 17, 22, 23, 27, 32, 34, 36, 39, 42, 45, 46, 48, 51, 64, 66, 69, 71, 77, 81, 82, 85, 86, 88, 92, 96, 102, 108, 109, 110, 117.

1, 3, 4, 7, 8, 9, 12, 21, 24, 25, 26, 27, 34, 36, 40, 43, 49, 63, 64, 68, 70, 71, 72, 75, 78, 81, 82, 83, 89, 92, 94, 95, 97, 102, 104, 108, 112, 113, 118, 120.

1, 3, 4, 5, 7, 9, 12, 14, 15, 17, 21, 27, 32, 36, 38, 42, 45, 46, 51, 53, 58, 63, 67, 68, 76, 79, 80, 81, 82, 83, 96, 100, 103, 106, 107, 108, 114, 115, 116, 119.

Parker has recently shown that these four lead to nonisomorphic designs, by having a computer determine the intersection patterns of three blocks at a time.

When $\lambda=1$, the difference set will always give a projective plane, and all primes p dividing $n=k-\lambda$ must be multipliers. Singer [**37**] proved that all finite Desarguesian planes can be constructed by using difference sets; Mann and Evans [**28**], by hand computation, have shown that for $n \leqq 1600$ no projective planes can be constructed from a difference set unless n is the power of a prime.

Other searches have been bases on the algebraic properties of the planes. Killgrove [**23**] showed that no plane of order 9 has a cyclic additive group, and

it was determined in [15] that the only planes with an elementary Abelian additive group are the four known ones. Therefore addition in a further plane of order 9, if there is one, cannot be a group.

Kleinfeld [24] found all Veblen-Wedderburn systems of order 16 which have a left nucleus of order 4. These systems lead to four planes: the Desarguesian plane, the Hall plane, and two new planes coordinatized by semifields (nonassociative division rings). The two latter planes inspired constructions of several new types of projective planes [18, 25, 35].

Non-Desarguesian planes of all orders p^n, where p is prime, $n>1$, were known except for the cases 2^p for prime p. As mentioned above, there are no non-Desarguesian planes of order 2^2 or 2^3, and so there was interest in the next smallest case 2^5. A search was undertaken for all semifields of order 32, independently by Walker [44] and one of the authors, and the complete set of 2502 nonisomorphic systems was found. These yield five non-Desarguesian planes whose structure was also determined by computer [25]. Fortunately, it was possible to observe a pattern in one of the semifields, and this led to a construction [26] of semifields of all orders 2^{mn} where $mn>3$ and $n>1$ is odd. In this way the solution for all remaining cases 2^p was obtained as a byproduct of a computer examination of the smallest case; such results are the main goals of combinatorial computer explorations.

The preparation of this paper was supported, in part, by NSF grant number GP-212.

References

1. Algorithm index, Commun. ACM, 7 (1964) 147.

2. W. W. Rouse Ball, Mathematical Recreations and Essays, rev. by H. S. M. Coxeter, Macmillan, New York, 1947, 305–310.

3. R. Bellman, Dynamic programming treatment of the traveling salesman problem, J. ACM, 9 (1962) 61–63.

4. R. C. Bose, I. M. Chakravarti, and D. E. Knuth, On methods of constructing sets of mutually orthogonal latin squares using a computer, Technometrics, 2 (1960) 507–516; *ibid.*, 3 (1961) 111–117.

5. Martin Cohen, Affine m-ary Gray codes, Information and Control, 6 (1963) 70–78.

6. L. J. Fischer and K. Chr. Krause, Lehrbuch der Combinationslehre und der Arithmetik, Dresden, 1812.

7. Ivan Flores, Reflected number systems, IRE Trans., EC-5 (1956) 79–82.

8. S. W. Golomb and L. D. Baumert, The search for Hadamard matrices, this MONTHLY, 70 (1963) 12–17.

9. ———, Backtrack programming, submitted for publication.

10. Marshall Hall, Jr., Uniqueness of the projective plane with 57 points, Proc. Amer. Math. Soc., 4 (1953) 912–916; correction *ibid.*, 5 (1954) 994–997.

11. Marshall Hall, Jr., J. D. Swift, and R. J. Walker, Uniqueness of the projective plane of order eight, Math. Tables Aids Comput., 10 (1956) 186–194.

12. Marshall Hall, Jr. and L. J. Paige, Complete mappings of finite groups, Pacific J. Math., 5 (1955) 541–549.

13. Marshall Hall, Jr. and J. D. Swift, Determination of Steiner triple systems of order 15, Math. Tables Aids Comput., 9 (1955) 146–152.

14. Marshall Hall, Jr., A survey of difference sets, Proc. Amer. Math. Soc., 7 (1956) 975–986.

15. Marshall Hall, Jr., J. D. Swift, and R. Killgrove, On projective planes of order nine, Math. Tables Aids Comput., 13 (1959) 233–246.

16. Marshall Hall, Jr., Numerical analysis of finite geometries, IBM J. Res. Dev. (to appear).

17. Harry S. Hayashi, Computer investigation of difference sets, submitted for publication.

18. D. R. Hughes and Erwin Kleinfeld, Semi-nuclear extensions of Galois fields, Amer. J. Math., 82 (1960) 389–392.

19. G. Hutchinson, Partitioning algorithms for finite sets, Commun. ACM, 6 (1963) 613–614.

20. B. H. Jiggs, Recent results in comma-free codes, Canad. J. Math., 15 (1963) 178–187.

21. Diane M. Johnson, A. L. Dulmage, and N. S. Mendelsohn, Orthomorphisms of groups and orthogonal latin squares, I, Canad. J. Math., 13 (1961) 356–372.

22. Selmer M. Johnson, Generation of permutations by adjacent transposition, Math. Comp., 17 (1963) 282–285.

23. R. Killgrove, A note on the nonexistence of certain projective planes of order nine, Math. Comput., 14 (1960) 70–71.

24. E. Kleinfeld, Techniques for enumerating Veblen-Wedderburn systems, Jour. ACM, 7 (1960) 330–337.

25. Donald E. Knuth, Finite semifields and projective planes, to appear in J. of Algrebra.

26. ———, A class of projective planes, to appear in Trans. Amer. Math. Soc.

27. D. H. Lehmer, Teaching combinatorial tricks to a computer, AMS Proc. Symp. Appl. Math., 10 (1960) 179–193.

28. H. B. Mann and T. A. Evans, On simple difference sets, Sankhya, 11 (1951) 357–364.

29. H. W. Norton, The 7×7 squares, Ann. Eugenics, 9 (1939) 269–307.

30. R. T. Ostrowski and K. D. Van Duren, On a theorem of Mann on latin squares, Math. Comp., 15 (1961) 293–295.

31. L. J. Paige and C. B. Tompkins, The size of the 10×10 orthogonal latin square problem, AMS Proc. Symp. Appl. Math., 10 (1960) 71–84.

32. E. T. Parker, Computer investigation of orthogonal latin squares of order ten, AMS Proc. Symp. Appl. Math., 15 (1962) 73–82.

33. W. A. Pierce, The impossibility of Fano's configuration in a projective plane with eight points per line, Proc. Amer. Math. Soc., 4 (1953) 908–912.

34. A. Sade, An omission in Norton's list of 7×7 squares, Ann. Math. Statist., 22 (1951) 306–307.

35. Reuben Sandler, Autotopism groups of some finite nonassociative algebras, Amer. J. Math., 84 (1962) 239–264.

36. Scott, Programming a combinatorial puzzle, Dept. Elec. Eng. Princeton U., 10 June 1958.

37. J. Singer, A theorem in finite projective geometry and some applications to number theory, Trans. Amer. Math. Soc., 43 (1938) 377–385.

38. F. Stockmal, Algorithms 95 and 114, Commun. ACM, 5 (1962) 344, 434.

39. J. D. Swift, Isomorph rejection in exhaustive search techniques, AMS Proc. Symp. Math., 10 (1960) 195–200.

40. C. B. Tompkins, Machine attacks on problems whose variables are permutations, AMS Proc. Symp. Appl. Math., 6 (1956) 195–212.

41. ———, Methods of successive restrictions in computational problems involving discrete variables, AMS Proc. Symp. Appl. Math., 15 (1962) 95–106.

42. H. F. Trotter, Algorithm 115, Commun. ACM, 5 (1962) 434–435.

43. R. J. Walker, An enumerative technique for a class of combinatorial problems, AMS Proc. Symp. Appl. Math., 10 (1960) 91–94.

44. ———, Determination of division algebras with 32 elements, AMS Proc. Symp. Appl. Math., 15 (1962) 83–85.

45. Mark B. Wells, Generation of permutations by transposition, Math. Comp., 15 (1961) 192–195.

46. Wozencraft and Reissen, Sequential Decoding, MIT Press, 1961.

47. H. Yamabe and D. Pope, A computational approach to the four-color problem, Math. Comp., 15 (1961) 250–253.

THE QUEST OF THE PERFECT SQUARE

W. T. TUTTE, University of Waterloo, Ontario, Canada

1. Introduction. A *squared rectangle* is a rectangle dissected into a finite number, two or more, of squares. If no two of these squares have the same size the squared rectangle is *perfect*. The *order* of a squared rectangle is the number of constituent squares. It has been found that there are just two perfect rectangles of order 9, and none of order less than 9 ([1], [2]).

The case in which the rectangle is itself a square is of special interest. It was long conjectured that no "perfect square" could exist. However an example was published by R. Sprague [5] in 1939, and many others have been found since. The outstanding problem at present is that of finding the lowest possible order for a perfect square. The record is held by T. H. Willcocks with a perfect square of order 24 ([11], [12]), and C. J. Bouwkamp, A. J. W. Duijvestijn and P. Medema have found, using computers, that there is no perfect square of order less than 20 [3].

It is convenient to describe a squared rectangle as *compound* or *simple* according as it does or does not contain a smaller squared rectangle. Willcocks' perfect square, shown in Figure 2, is compound. For some years the simple perfect square of lowest order was one of order 37, also due to Willcocks. But this record has now been lowered to 25 by J. C. Wilson. His square is shown in Figure 3.

The object of the present paper is to give some account of the methods which have been used to find low-order perfect squares. Following P. J. Federico we use the term "low-order" for squared squares of order 28 or less. A list of 24 perfect squares of this kind has been published by Federico [4].

2. Squared rectangles and graphs. Let R be any squared rectangle. We distinguish one pair of parallel sides of R as *horizontal*, and the other pair as *vertical*. The union of the horizontal sides of the constituent squares of R consists of disjoint horizontal segments, among them the upper and lower horizontal sides of R. We call these segments the *horizontal dissectors* of R. There is of course an analogous family of *vertical dissectors*.

We can represent R by a graph P as follows. The vertices of P correspond to the horizontal dissectors of R, and the edges of P to the constituent squares of R. Each edge joins the vertices representing the two horizontal dissectors containing sides of the corresponding square. The relation between R and P is illustrated in Figure 1, in which each vertex of P is shown as a point in line with the corresponding horizontal dissector.

We call P the *horizontal polar net* of R, and say that the vertices corresponding to the upper and lower horizontal sides of R are its *positive* and *negative poles* respectively. Interchanging these poles corresponds to the trivial operation of turning R upside down. On joining the poles of P by a new edge we obtain the *horizontal completed net*, or *c-net*, of R. There is likewise a *vertical polar net*, and a *vertical c-net* of R.

With each edge of P we associate a "current" whose magnitude is measured by the side of the corresponding square. It is directed from the vertex representing the upper side of the square to that representing the lower side. It is easy to see that the currents satisfy Kirchhoff's Laws for a network of unit resistances, provided that the total current is supposed to enter P at the positive pole and to leave at the negative. The currents are indicated by numbers and arrows in Figure 1.

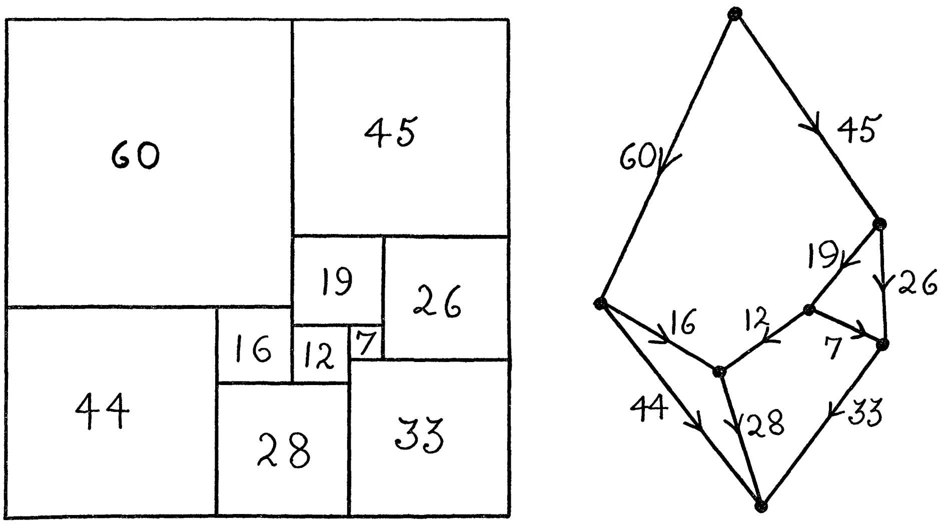


FIG. 1.

For the theory of squared rectangles, their polar nets and c-nets, the reader may turn to [2]. In what follows we quote many of the theoretical results of that paper without proof.

It can be shown that a c-net is a nonseparable planar graph, and that a polar net can be drawn in the sphere or closed plane so as not to separate its poles. Moreover a c-net G of a simple squared rectangle R is *3-connected.* This means that G, in addition to being nonseparable, cannot be represented as the union of two subgraphs, each with at least two edges, such that their intersection has no edges and only two vertices.

3. Catalogues of squared rectangles. Methods for constructing polar nets of perfect squares by modifying graphs with suitable kinds of symmetry have been described in the literature ([2], [6], [7]). Unfortunately no such method has been made to yield a perfect square of low order, and so we do not discuss them here.

One way of searching for low-order perfect squares is simply to construct a large number of perfect rectangles in the hope that eventually a square will appear among them. This simple-minded approach has not yet been made to

work, but a number of workers who have constructed such lists of rectangles have found that two or more of their specimens could be fitted together, with a few extra constituent squares, so as to make a perfect square. Sprague's square was obtained in this way. It is not of low order, however, having 55 constituent squares.

A complete catalogue of the simple perfect rectangles up to the 13th order was constructed by Brooks, Smith, Stone and Tutte, though only part of it was published. The method used was as follows.

Let G be a 3-connected planar graph. It is known that such a graph can be drawn in the 2-sphere, or closed plane, in essentially only one way ([9], [10]). Let A be an edge of G, with ends x and y, and let P be the graph obtained from G by deleting the edge A. We use Kirchhoff's equations to obtain the currents in P on the assumption that the total current enters at x and leaves at y, and that the edges of G are unit resistances. It is shown in [2] that the resulting distribution of currents in P determines a squared rectangle R. Usually P is the horizontal polar net of R, though this rule breaks down when P has a zero current, or when two vertices of P, not separated by G in the 2-sphere, have equal potentials. Every simple squared rectangle can be derived, however, from its horizontal c-net by applying the above construction to the appropriate edge A.

It follows from the above considerations that all simple squared rectangles of the nth order can be determined provided that we can first list the 3-connected planar graphs of $n+1$ edges.

In the determination of the currents in P it is convenient to take the total current equal to the *complexity* $C(P)$ of P. This is the number of spanning trees of P, that is the number of subgraphs of P which are trees and which include all the vertices of P. With this convention it is found that the currents in P are all integers. We describe these currents and the corresponding potential differences in P as *full*. The *reduced* currents are obtained from the full ones by dividing by their highest common factor. It is the reduced currents that are the side-lengths of the constituent squares of R "in its lowest terms."

The full potential drop from vertex a to vertex b when the total current enters P at x and leaves at y is conveniently denoted by $(ab\cdot xy)$. It can be regarded as a function either of P or of G, for it takes the same value in each network.

Brooks, Smith, Stone and Tutte found two distinct simple perfect rectangles of sides 422 and 593. Both are of the 13th order, and the 26 constituent squares are all of different sizes. These two rectangles can be combined with two squares of sides 422 and 593 to form a perfect square of order 28 and side 1015 [4]. There was also a 12th-order rectangle of sides 231 and 377, and a 13th order one of sides 377 and 608. These can be combined with a square of side 231 to form a 26th order perfect square of side 608 ([2], [4]).

The square of side 608 held the low-order record for some years, until it was replaced by Willcocks' square, whose side is only 175. (See Figure 2.)

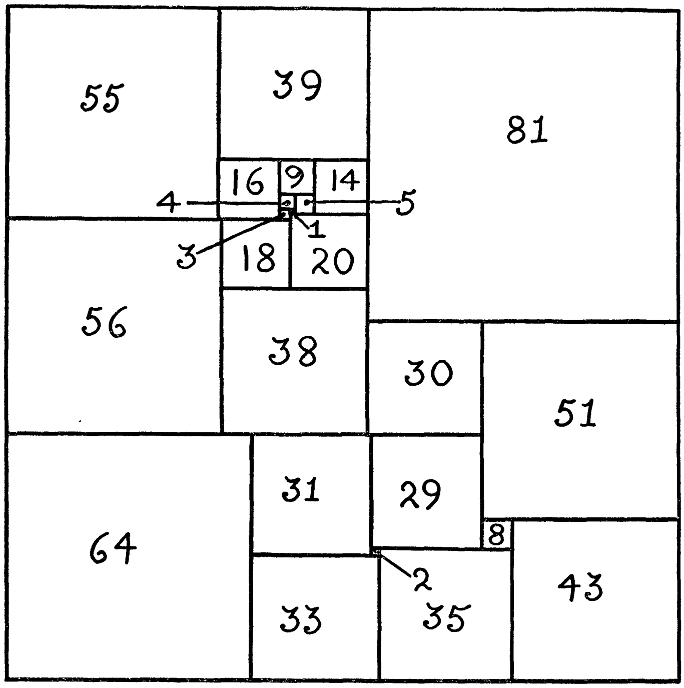


FIG. 2.

Willcocks was able to find this square because he did not confine his catalogue to simple and perfect squared rectangles. He included also rectangles with just two equal squares, one of these being in a corner and the other being adjacent to it. In his square he uses one such rectangle of sides 94 and 111, the repeated constituent square being of side 30, one perfect rectangle of the same size, and two extra squares. The two squared rectangles are made to overlap in the corner square of side 30 which accordingly does not appear in the final dissection.

4. The use of computers. Another catalogue of simple squared rectangles of orders 9–13 was made by C. J. Bouwkamp, working independently, and later the two catalogues were checked by comparison. At this stage the problem of finding all the 3-connected planar graphs of a given order was becoming acute. However it was solved by a theory later published in [8].

The essential theorem is as follows. Let L_n be a complete list of nonisomorphic 3-connected planar graphs of n edges, where $n \geqq 6$. Then the members of L_{n+1} are (i) the graphs derivable from some $G \in L_n$ by joining two nonadjacent vertices not separated by G in the 2-sphere, (ii) the duals of the graphs so derivable, and (iii) when n is an odd number $2t-1$ the graph of the pyramid with a t-sided base.

It is easily verified that L_6 contains only the graph of the tetrahedron. The lists L_7, L_8, L_9 and so on can then be constructed by repeated application of the above theorem. L_7 is null, while L_8 contains only a pyramidal graph. The main difficulty in applying this method is that of recognizing duplicates.

C. J. Bouwkamp, A. J. W. Duijvestijn and P. Medema succeeded in programming a computer to carry the lists of 3-connected planar graphs and simple squared rectangles up to the 15th order. The resulting catalogue [1] has become essential for workers in this field. The researcher's main object was to demonstrate the applicability of computers to problems in pure graph theory, but of course they were also looking for perfect squares. But no such square appeared in the catalogue.

A theoretical reason can be given why perfect squares of such low orders are not to be expected. Let G be any network of unit resistances, let C be its complexity, and let a, b, x and y be vertices of G. Then it is shown in [2] that

$$(ab\cdot ab)(xy\cdot xy) \equiv (ab\cdot xy)^2 \bmod C.$$

If G is the polar net, with poles x and y, of a squared square S we have $C=(xy\cdot xy)$. Writing $C=mk^2$, where m is square-free, we find that mk divides $(ab\cdot xy)$. Thus to find the reduced currents corresponding to the squares of S we must divide the associated full currents by the large reduction factor mk. But for squared rectangles in general the reduction factors encountered are usually small, commonly unity. It would seem therefore that the probability of a squared square of a given order being perfect must be much less than that of a squared rectangle of the same order being perfect.

Another conclusion to be drawn from the above theorem is that in searching for perfect squares we need consider only c-nets whose complexity is of the form mk^2, where k is reasonably large. (Complexities can be computed recursively or as determinants.) Making use of this observation Bouwkamp and his colleagues extended their search to the 19-th order and concluded that there was no perfect square within this range. Details of the investigation have been given by Duijvestijn [3].

5. Recent results. In 1962 a new method was introduced by P. J. Federico [4]. He makes a catalogue of squares which are dissected into a number of unequal squares and one rectangle. These figures are obtained by a process of distortion applied to known squared rectangles. In each such figure the shape of the rectangle is uniquely determined. Reference to the catalogue of squared rectangles may show that the rectangle can be squared perfectly. If so a squared square results and this may or may not be perfect.

Using this method Federico has found many new low-order perfect squares, including the first known perfect square of order 25. But he has found no perfect square of lower order than this.

A new application of an electronic computer to the problem has been made by J. C. Wilson. He has examined a large number of graphs of about 25 edges, each graph being the union of two polar nets with only their poles in common. The polar nets are taken from Bouwkamp's tables, and the complexities of their combinations are readily computable. Many of the graphs were found to have square complexities, corresponding to the case $m = 1$ of the theory given in Section 4. These gave rise to several simple perfect squares. One of these squares, of order 25, is shown in Figure 3. Its discovery was announced at the IBM symposium held at White Plains, New York, in March 1964.

The writer would prefer to find perfect squares by a theoretical method not demanding exhaustive computational searches, but must confess that so far he has seen no hint of such a process. Instead we have a pioneering example of the construction by computers of graphs satisfying specified nontrivial conditions.

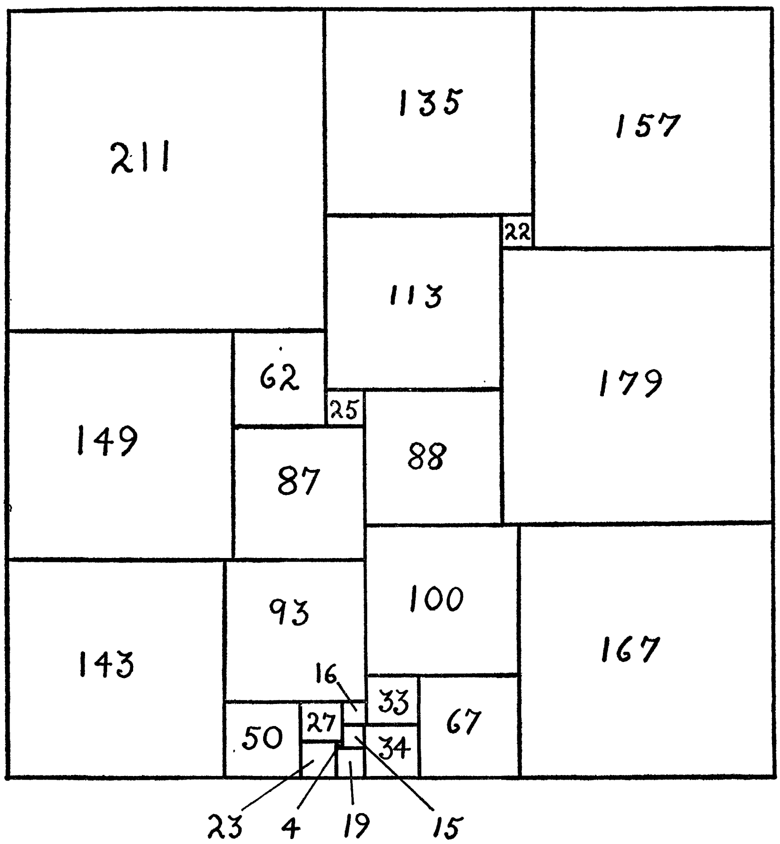


FIG. 3

References

1. C. J. Bouwkamp, A. J. W. Duijvestijn and P. Medema, Catalogue of simple squared rectangles of orders nine through fifteen, Eindhoven, 1960.

2. R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, The dissection of rectangles into squares, Duke Math. J., 7 (1940) 312–340.

3. A. J. W. Duijvestijn, Electronic computation of squared rectangles. Printed thesis, 1962. (Address of author: Philips Computing Centre, Eindhoven.)

4. P. J. Federico, Note on some low-order perfect squared squares, Canad. J. Math., 15 (1963) 350–362.

5. R. Sprague, Beispiel einer Zerlegung des Quadrats in lauter verschiedene Quadrate, Math. Z., 45 (1939).

6. W. T. Tutte, Squaring the square, Canad. J. Math., 2 (1960) 197–209.

7. ———, Squaring the square, Scientific American (Nov. 1958) 136–142. Repr. with Addendum and Bibliography in The 2nd Sci. Amer. Book of Math. Puzzles and Diversions, Martin Gardner, New York, 1961.

8. ———, A theory of 3-connected graphs, Indag. Math., 23 (1961) 441–455.

9. H. Whitney, Nonseparable and planar graphs, Trans. Amer. Math. Soc., 34 (1932) 339–362.

10. ———, 2-isomorphic graphs, Amer. J. Math., 55 (1933) 245–254.

11. T. H. Willcocks, Fairy Chess Review, 7 (Aug./Oct. 1948).

12. ———, A note on some perfect squared squares, Canad. J. Math., 3 (1951) 304–308.

PERMUTATION BY ADJACENT INTERCHANGES

D. H. LEHMER, University of California, Berkeley

1. Introduction. Many problems of optimization involve the examination of permutations of distinct or possibly nondistinct objects. Often the number of these permutations becomes too great to deal with effectively even at high speeds. In other cases the number is modest enough to contemplate an exhaustive examination. In such cases one can consider optimizing the procedures used. The following problem is one of these. It could be considered as a special case of the *Travelling Salesman Problem* [1] but it turns out to be sufficiently special to warrant a solution peculiar to its own type. In any case, solutions have been found to problems which if stated in the general context would have involved more than a thousand cities for the Salesman to visit. We call the problem the *Motel Problem* and we show its connection with another more general problem which we call the *Travelling Burglar Problem.* For this we present a procedure and give solutions for the *Motel Problem* for motels having up to seven units.

2. The Motel Problem. Mr. Smith is the manager of the Magnolia, a standard type, thin walled motel, consisting of n units arranged in a long straight row, with adjacent units supplied with an interconnecting door. Mr. Smith is a profound student of the behavioral sciences and a good combinatorial analyst. There is no vacancy; in fact the present guests are all scheduled to stay at the Magnolia for m days. Mr. Smith wants to study the effect of rearranging the guests in all possible ways so that every morning a new arrangement is to be made. Unfortunately it rains every day and the guests don't like to move. So Mr. Smith plans to minimize the discomfort of his guests by interchanging the occupants of only two adjacent units each morning. This interchange can be made via the interconnecting door without anyone's luggage getting wet. The problem is complicated, however, by the fact that among the guests there are several large families. Whenever members of the same family occupy adjacent units they naturally unlock the connecting door and circulate to such an extent that scheduling an interchange of these two units is a waste of time. Mr. Smith's problem is to work out a schedule of interchanges that will most nearly meet the above conditions.

Of course, if m is too small, for example, less than the total number of really different permutations of the guests, the problem becomes clearly hopeless. More precisely, if we suppose that the units are occupied singly and that there are t families with a_i guests belonging to the ith family, so that

$$a_1 + a_2 + \cdots + a_t = n$$

is one of the partitions of n into t parts, then the number of permutations in Mr. Smith's project is

$$P_t(n) = n!/(a_1!a_2! \cdots a_t!).$$

Even if $m=P_t(n)$ the problem may be impossible. For example if $n=4$, $t=2$, $a_1=a_2=2$, then $P_2(4)=6$. Denoting the 4 guests by 0, 0, 1, 1, the six permutations are, in dictionary order,

(1) 0011, 0101, 0110, 1001, 1010, 1100.

Try as he will, Mr. Smith cannot arrange these 6 permutations in a sequence such that each is derived from its predecessor by the interchange of adjacent elements. Thus 6 days will not suffice. However, he soon solves the problem for $m=7$ days as follows

(2) 0011, 0101, 0110, 1010, 1001, 1010, 1100.

This problem from "real life" can occur in a much more realistic way in mathematics. For example, let $f(x)$ be a function of a real variable whose values are unpredictable and expensive to calculate. Let

$$\mathbf{b} = (b_1, b_2, \cdots, b_n)$$

be a given vector some of whose components may be equal and let S be the set of all distinct vectors obtained by permuting the components of $\mathbf{b}$. Let $\mathbf{a}$ be a given vector. We wish to find with least expense the maximum

$$M = \max_{\mathbf{x}\in S} f[(\mathbf{a}, \mathbf{x})],$$

where $(\mathbf{a}, \mathbf{x})$ is the inner product and $\mathbf{x}$ ranges over all the members of S. Here the given components of $\mathbf{b}$ play the roles of the Magnolia guests. The successive inner-products $(\mathbf{a}, \mathbf{x})$ are most cheaply calculated recursively by leaving the components of $\mathbf{x}$ unaltered except for the interchange of x_i and x_{i+1} (unless, of course, $x_i=x_{i+1}$), where i runs through an appropriate sequence.

3. The Travelling Burglar Problem. One of the purposes of this paper is to point out that the Motel Problem is a special case of the following problem.

A competent burglar plans to rob a bank in each of 50 large cities. He travels only by air. Although each large city has an airport, some have rather limited service to the other cities, especially in the way of nonstop flights which our friend insists upon. To avoid being recognized, he resolves, after his first robbery, not to land twice at the same airport, at least, not if he can help it. As he will have plenty of money, the cost of the flights is immaterial. Fortunately our burglar has as a trusted friend a travel agent who is also interested in combinatorial problems, and who can therefore solve his transportation problem.

4. Graph formulation. The connection between these two problems is the graph that represents both of them. The graph of the Travelling Burglar Problem is pretty obvious. Its vertices, or nodes, are the 50 cities and the edges, or lines, joining each node to certain others are the available nonstop flights. The problem consists of starting from a certain node and choosing a continuous path

through 48 other nodes to a final node without revisiting any node, if possible. Such a path, if possible, is called a perfect Hamiltonian path since W. R. Hamilton proposed a similar problem for the graph of the dodecahedron in 1862 [2]. If, for a given graph, such a perfect path is impossible we may become reconciled to an imperfect path in which one or more nodes are visited twice. For reasons which will appear later, we allow ourselves to revisit a node N only by returning immediately to N from the node N' to which we first set out from N, thus making a "spur" in the path, the path from N to N' being double. The simplest example of such an imperfect graph is shown in Figure 1.

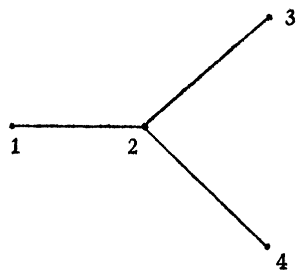


FIG. 1

For example, the path through the points 1, 2, 3, 2, 4 has the spur on the segment (2, 3).

A graph need not be drawn to be appreciated. It may be presented in tabular form so that even a digital computor can understand it. One convenient and concise tabular representation consists in placing in the argument column the names of the nodes, which we may as well take to be the numbers $k=1(1)n$. The kth row of the table consists of a list of those nodes that are directly connected to the node k. Thus the graph of Figure 1 can be tabulated as follows

1	2
2	1, 3, 4
3	2
4	2

If the graph is connected, it is possible to number its nodes in such a way that every node, except the first, is connected to at least one node of smaller serial number. The number of entries corresponding to a given argument, that is, the number of lines leading into and out of a given node is called the valency or multiplicity of the node.

The way in which the Motel Problem leads to a graph is only a little less obvious. Let us take again the example in which the motel has 4 units occupied by 4 guests two of which belong to one family and two to another. The 6 permu-

tations are given in (1). Each of these is transformed into one or more of the others by interchanging adjacent elements. Numbering these permutations from 1 to 6 we form the following table:

1	2
2	1, 3, 4
3	2, 5
4	2, 5
5	3, 4, 6
6	5

in which opposite each argument entry we have listed the serial numbers of those permutations that result from interchanging adjacent elements of the given permutation. Drawing the graph corresponding to this table gives Figure 2.

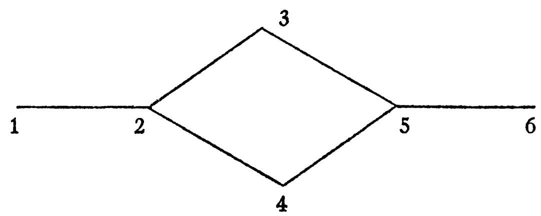


FIG. 2

It is at once evident from Figure 2 that a perfect solution of the Motel Problem is impossible. The path 1, 2, 3, 5, 4, 5, 6 gives a compromise solution with a spur on the segment (4, 5). This is one of the 14 imperfect Hamilton paths with a single spur traversing this graph.

Although the characterization of graphs arising from Motel Problems appears to be a difficult problem, they have a few special features that are easily discussed.

All such graphs are colorable, that is, their nodes can be colored in red and blue so that the two nodes at the ends of each edge of the graph are of opposite color. This follows from the fact that the parity of a permutation is altered by an interchange. The nodes corresponding to the even permutations can be colored red. As a consequence, any loop in the graph must have an even number of sides. Most loops have 4 sides but some have 6. For instance, the graph corresponding to the permutations of the marks 0012 consists of two hexagons and two quadrilaterals, and that of the marks 00012 consists of three hexagons and six quadrilaterals.

Let n_r and n_b denote the respective numbers of red and blue nodes of a colorable graph. We define $d = |n_r - n_b|$ as the defect of the graph of $n = n_r + n_b$ nodes. Since on any Hamilton path the nodes must alternate in color, a perfect

path is possible only when $d=0$ or 1 according as n is even or odd. It is impossible to find a Hamilton path with fewer than $d-1$ spurs. In fact, if a path has s spurs, these could be erased to produce a residual graph of $n-s$ nodes with a perfect Hamilton path and hence of defect 0 to 1. But removing s nodes can reduce the defect by at most s, so that $d-s$ does not exceed 1. That is, s is at least $d-1$.

Another important feature of the Motel Problem graph is that each node has a modest multiplicity, in fact less than n since at most $n-1$ adjacent interchanges are possible. Figure 2 is a rare exception to the usual fact that there are no nodes of multiplicity 1. This can occur only when there are two families and only for the extreme permutation $000 \cdots 011 \cdots 11$ and its reverse, for in these two cases only is there but a single possible interchange. As a more typical example, we may cite the case of $n=7, t=3, a_1=3, a_2=a_3=2$ so the marks being permuted are 0001122. Here $P_t(n)=7!/(3!2!2!)=210$. The 210 nodes are distributed as to multiplicity as follows:

There are 6 nodes of multiplicity 2,
27 nodes of multiplicity 3,
63 nodes of multiplicity 4,
79 nodes of multiplicity 5,
35 nodes of multiplicity 6.

The graph contains 475 lines. There are 108 red nodes and 102 blue ones, so the defect is 6. There is a path with 5 spurs for this graph.

We may describe the Motel Problem by using the notations of partitions. Thus in the above example the family distribution of 0001122 can be written 32^2. In general, the families may be ordered by size and the number α_i of families of size p_i can be reported in this as $p_i^{\alpha_i}$. This gives in general the partition $p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ of n, where

$$\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s = n(p_1 > p_2 > \cdots > p_s, \alpha_i > 0, i \geqq 1).$$

The number of Motel Problems for a motel of n units is thus $p(n)$, the number of unrestricted partitions of n. Incidentally

$$p(n) \sim \frac{1}{n\sqrt{48}} \exp\left(\pi\sqrt{2n/3}\right)$$

and even $p(10)=42$. These $p(n)$ problems range from the completely trivial partition n itself in which the guests all belong to the same family to the most frequently encountered partition 1^n in which the guests are entirely unrelated. This last case has been solved by Selmer M. Johnson [3] who found a perfect Hamilton path for every n. In between this case and the trivial case no complete solution is known, but a great many cases are solved by the following theorem.

THEOREM. *If a perfect solution of the Motel Problem is known for a certain set of n guests then there is a constructive method of solving the problem for the set of $n+1$ guests obtained by adjoining a new guest unrelated to the others.*

Proof. We can think of the given perfect solution as a sequence of $P_t(n)$ permutations on n marks each of which is obtained from its predecessor by a single interchange of adjacent marks. We now expand this list of permutations by replacing each permutation by $n+1$ copies of itself. Let M be the new mark representing the new guest so that M is distinct from the other marks. We begin by annexing the mark M to the first permutation at the extreme right. In the succeeding permutations we interchange M with its left neighbor until M arrives at the extreme left. Here M remains for one step while the old marks undergo an interchange. M now infiltrates from left to right during the next $n+1$ steps. This process continues back and forth to the end of the list. We now have the $P_{t+1}(n+1)=(n+1)P_t(n)$ permutations on the $n+1$ marks giving a perfect solution to the new Motel Problem.

In particular if all the marks are distinct this gives Johnson's method for generating permutations by adjacent interchange.

Cayley [4] called the partition $p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ *unitary* in case $p_s=1$ and gave a table of the nonunitary partitions of $n \leqq 100$ into parts less than 7. The above theorem says that if the Motel Problem for a unitary partition fails to have a perfect solution it must also fail for the nonunitary partition obtained by suppressing the unit parts of the unitary partition. However, the converse proposition is not generally true. We have seen that for the four marks 0011 there is no perfect solution. Likewise, there is none for the five marks 00112 corresponding to the unitary partition 2^21. However, a perfect solution does exist for the 6 marks 001123 and so for all further partitions of the form 2^21^α with $\alpha>1$. Again, the problem has a perfect solution for the partitions 321^α for $\alpha>0$, but not for 32.

5. Interchange digits. In Section 8 we append solutions, perfect or best possible, for $n \leqq 7$ of all cases not covered by our theorem. Instead of listing the succession of permutations it suffices (and it is much more compact) to give a list of successive interchanges to be applied to an initial permutation of the given marks. By convention this initial permutation is taken to be the first in dictionary order. Thus, instead of the 7 permutations (2), we give the 6 "interchange digits" 213112. Here the first digit 2 signifies that, beginning with 0011 we interchange the second and third marks, counting from the right, obtaining 0101. The next digit 1 tells us to interchange the first and second marks obtaining 0110 and so forth. The two consecutive equal marks 11 indicate a spur, the one running from 1010 to 1001 and back to 1010.

The interchange digits are intended to be stored in packed form in the high-speed memory of a computer. In processing the interchange digits recursively the computer saves the last digit used and compares it with the incoming new digit. In case of equality the resulting "new" permutation is not used, since it is in reality an old one. Thus in those problems in which a considerable amount of time is spent processing a permutation the occurrence of one or more spurs is of little consequence.

It goes without saying that the interchange digits are used only to organize

"address arithmetic." The actual objects being permuted can be any kind of information each item occupying part of a machine word, one machine word, or indeed any number of machine words.

6. Machine method. We give now a very brief account of the methods used to obtain the results given in the next section. Some of these methods are direct and straightforward such as those used to produce the $P_t(n)$ permutations of the given set of marks and the generation of the graph in tabular form. Although there is abstractly a single graph associated with a set of marks there is the possibility of assigning different serial numbers to the $P_t(n)$ nodes and thus producing different looking incidence tables of the same graph. The importance of this possibility will appear presently. Different serial numbering of the $P_t(n)$ permutations can be accomplished simply by using different methods of generating the $n!$ permutations of the n marks with regard to the equalities existing among the n marks. The machine saves only the $P_t(n)$ distinct permutations numbering them in order of their appearance. In practice three different methods of generation were made available to the machine. Once the graph is constructed the machine proceeds to color it and compute its defect.

Now the machine is ready to begin the search for a Hamiltonian path through the graph. This is done by carrying out the formal heuristic procedure given below. In this procedure the machine goes from step n to step $n+1$ unless directed to some other step. We can picture the machine in the middle of the process standing at a particular node trying to decide which next path to take. We designate this node by B in what follows. The next path selected leads to the node called N. There are two convenient auxiliary tallies being kept: the node tally and the spur tally. The former keeps track of the number of nodes visited and the latter the number of spurs produced.

Hamiltonian Path Procedure

Step 1 Set node tally at 1.
Step 2 Set spur tally at 0.
Step 3 The first node becomes B.
Step 4 If there is no path leaving B go to Step 16.
Step 5 Among the nodes connected to B of least multiplicity select the node N of least serial number.
Step 6 If the multiplicity of N is 1 go to Step 12.
Step 7 Store interchange digit from B to N in next storage place.
Step 8 Disconnect B from all connecting nodes thus reducing by 1 the multiplicity of each such node.
Step 9 N becomes B.
Step 10 Node tally plus 1 replaces node tally.
Step 11 Go to Step 4.
Step 12 Store interchange digit from B to N in the next two storage places.
Step 13 Spur tally plus 1 replaces spur tally.
Step 14 Disconnect B and N.
Step 15 Go to Step 10.
Step 16 Output initial marks, spur and node tallies, and list of interchange digits.
Step 17 Halt.

7. Application. To illustrate the application of this algorithm to a more general graph we consider the Travelling Burglar Problem where the cities are the largest in each of the 50 states. An examination of the Official Airline Guide (March 1964) reveals at once that a perfect solution is impossible. In fact, six of the fifty cities Billings, Burlington, Cheyenne, Fargo, Manchester, and Portland, Me. have nonstop flights to only one of the 44 other cities. These flights are Manchester-Boston, Portland-Boston, Burlington-New York, Cheyenne-Denver, Billings-Minneapolis, Fargo-Minneapolis. Therefore, if we consider the graph of 44 nodes, ignoring these 6 cities, and find a perfect Hamilton path beginning with Boston and ending with Minneapolis, or vice versa, we can insert the 6 cities and obtain an imperfect but optimal solution of the Travelling Burglar Problem with only 4 spurs.

The graph in question has 44 nodes and 195 edges. It is undirected but not colorable (that is, nonstop flights have nonstop return flights). Four nodes (Anchorage, Columbia, Little Rock, and Wilmington) have multiplicity 2, the maximum being Chicago with multiplicity 26. The average is, of course, $8.86 = 195/22$.

The fixing of the two end points of the proposed Hamilton path complicates the search somewhat. The algorithm, once started, is completely determined by the order in which serial numbers have been attached to the nodes. By renumbering the 44 nodes and starting at one end, one can try to reach the other by trial and error. Alternatively, starting anywhere, one may look for Hamilton paths that are circuits and in which the desired end points are made adjacent. Breaking the loop at this point gives the desired path. One can also try a combination of these strategies. For this graph the IBM 7090 search for a path takes less than a second. It found first the solution in Table A.

TABLE A

1. Manchester	18. Memphis	35. Wichita
2. Boston	19. Houston	36. Chicago
3. Portland, Me. [Boston]	20. New Orleans	37. Anchorage
4. Providence	21. Jackson	38. Seattle
5. Hartford	22. Birmingham	39. Honolulu
6. Philadelphia	23. Atlanta	40. Portland, Ore.
7. Wilmington	24. Columbia	41. Salt Lake City
8. Newark	25. Charlotte	42. Boise
9. Milwaukee	26. Baltimore	43. Denver
10. Detroit	27. Norfolk	44. Cheyenne [Denver]
11. Miami	28. New York	45. Omaha
12. Cleveland	29. Burlington [New York]	46. Des Moines
13. Charleston	30. Phoenix	47. Sioux Falls
14. Louisville	31. Las Vegas	48. Minneapolis
15. Indianapolis	32. Albuquerque	49. Billings [Minneapolis]
16. St. Louis	33. Los Angeles	50. Fargo
17. Little Rock	34. Oklahoma City	

8. Table of interchange digits. Table B gives the interchange digits for all nonunitary partitions of n and those unitary partitions that are not simple consequences of Theorem A for $n \leqq 7$. The trivial partition $n = n$ is omitted.

TABLE B

n	t	$P_t(n)$	partition	marks	defect	Interchange digits
4	2	6	2^2	0011	2	213112
5	2	10	32	00011	2	2131412443
5	3	30	2^21	00112	2	123421411312121421212312321412
6	2	15	42	000011	3	211312433215I234
6	2	20	3^2	000111	0	3452541453252154123
6	3	90	2^3	001122	6	453521543245352154324535215431 322435452543534525351535454525 431514322435152543543224352115 4552
6	3	60	321	000112	0	123452154321212343545154252413 21512313521513515412345354353
6	4	180	2^21^2	001123	0	123451254345354545254545154545 325215454512354345354245351541 514515354545254315414345354121 231345354353451543534535253525 431323432545452545415254252453 52535235315253545154545351535
7	2	21	52	0000011	3	21322141235443216I2345
7	2	35	43	0000111	3	321412562616316512524165152446 365224
7	3	210	32^2	0001122	6	213141244321341424312321341424 536162636465215646164636126251 412443564534562346451256436326 463645616525636563245141516123 364626153461626162361636141216 512515612331564142142162142412 422651563651565316515351524216 1414
7	3	105	421	0000112	3	123456216154143141245626164654 354212124212151612162161316512 524165152534564654633456361651 6541626334626526

TABLE B—(*Continued*)

n	t	$P_t(n)$	partition	marks	defect	Interchange digits
7	4	630	$2^3 1$	0011223	6	12345621654145354342414212421З 123242526146546456162654426236 546543215646362616261626362636 264626352515454564515243262326 321212621212534561451536562656 562646515456565451651525634321 234645123456463653635432423464 342436234513456565431365631413 121251231321421212315432123132 343121612313261636512354345634 652313534143634143465656156254 246423452515342141512535125234 245146563656563626565636546261 623132621324261516136515646265 262562646535626512523515613216 525636525632652515253563212321 213126163616241421652313631232 325326216261636162612363236162 612326244163614261626162613161 235231212131451254633426614156 163612146142521525122561436316 36654

9. Further questions. In conclusion we raise a few questions about the above procedure and about Motel Problem graphs with some partial answers.

Are there connected graphs without Hamilton paths even if spurs are allowed? Of course there are. One has only to put more nodes on the arms of Figure 1, but any such graph with a "pigtail" of length 2 or more cannot be the graph of a Motel Problem. In fact the end node, with its multiplicity of 1, would have to correspond to a permutation of the form

$$aaa\dot{a} \cdots abbb \cdots b$$

and the node adjacent to this one would have multiplicity 3, not 1.

We have observed that the multiplicity m of a node of a Motel Problem graph cannot exceed $n-1$. Can anything stronger be said about m? If the partition notation for the graph is

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \qquad (p_1 > p_2 > \cdots > p_s)$$

then

$$\alpha_1 + \alpha_2 + \cdots + \alpha_s - 1 \leqq m \leqq \min(n-1, 2(n-p_1))$$

and the first inequality is best possible since there always is a node of the graph

for which equality holds. To see this we need only observe that

$$S = \alpha_1 + \alpha_2 + \cdots + \alpha_s$$

is the number of distinct marks in each permutation. There are therefore at least $S-1$ adjacent interchanges possible in each permutation and exactly this many when the marks are arranged in ascending order, for example. In fact there are $S!$ such nodes of minimum multiplicity. As for nodes of maximal multiplicity, these correspond to permutations with the maximum number of pairs of adjacent but distinct marks. This number cannot exceed $2(n-p_1)$. In fact, $n-p_1$ is the number of marks not equal to 0, the most popular mark. If $p_1 > (n+1)/2$ we consider permutations of the type

$$0\ m_1\ 0\ m_2\ 0 \cdots 0\ m_{n-p_1} 0\ 0 \cdots 0,$$

where the m's are the nonzero marks. The number of pairs of adjacent distinct marks is clearly $2(n-p_1)$ and this is the maximum possible. If $p_1 \leqq (n+1)/2$ then $n-1 \leqq 2(n-p_1)$ and in any case $m < n-1$ as we have seen.

Whether the heuristic algorithm given above always leads to a solution is an unsolved problem. So far no failure was encountered. Another unsolved problem is whether by adjoining a sufficiently large number of unrelated guests to a given motel registration one always can obtain a perfect solution of the Motel Problem. Another question is whether a graph of motel type always has a Hamilton path of $d-1$ spurs where d is the defect of the graph. Finally one can ask for a procedure for obtaining all solutions of the Hamilton path problem for a given graph. The "backtrack" method of exhaustive search [5], though expensive, will solve this problem.

References

1. G. Dantzig, R. Fulkerson and S. Johnson, Solution of a large-scale travelling salesman problem, Operations Res., 2 (1954) 393–410.

2. A. Herschel, Sir Wm. Hamilton's Icosian Game, Quart. J. Math., 5 (1862) 305. Perhaps the first example of a Hamilton path problem is the known chess problem of the knight's tour of the early 18th century. See W. W. R. Ball, Mathematical Recreations and Essays, Macmillan, London, 1940, 174–185; especially the account of Warnsdorff's method. For Hamilton paths in general see O. Ore, Theory of Graphs, Amer. Math. Soc. Colloq. Publ., 38 (1962) 52–57.

3. S. M. Johnson, Generation of permutations by adjacent transposition, Math. Comp., 17 (1963) 282–285. See also H. F. Trotter, Algorithm 115, Comm. ACM, 5 (1962) 434–435.

4. A. Cayley, Non-unitary partition tables, Amer. J. Math., 7 (1885) 57–58.

5. R. J. Walker, An enumerative technique for a class of combinatorial problems, Amer. Math. Soc. Symposium on Applied Math., Proc., 10 (1960) 91–94.

ERROR ESTIMATION IN COMPUTER CALCULATION

R. L. ASHENHURST AND N. METROPOLIS
Institute for Computer Research, University of Chicago

Introduction. The problem of error analysis has several facets, and confusion often results from the failure to distinguish clearly between them. To illustrate, suppose that a number

$$y = \sum_{k=0}^{n} \frac{2}{2k+1} x^{2k+1}$$

is calculated, by some means or other; the nature of the "error in y" which results depends on the conceptual framework in which the calculation is conceived. If the calculation is regarded as the evaluation of a polynomial $P(x)$, where the argument x is exact as represented, the only source of error affecting the result is that arising from rounding or otherwise truncating the results of the arithmetic operations. If x is taken to be only an approximation of some "true" argument, however, this must also be taken into account in attributing an error to the value $y=P(x)$. Finally, one may not wish to think of this as the evaluation of a polynomial at all, but rather as the evaluation of the function $\log [(1+x)/(1-x)]$ for x near 0; the polynomial is then only an approximation to an infinite series, and the error in y due to neglecting terms of order greater than $2n+1$ must also be taken into account. Here one would still have to distinguish between cases according as x is assumed exact or is taken only as an approximation. One sees that care must be taken in making statements about the "error in y," to provide a context for interpretation.

It is convenient, and by now more or less traditional, to distinguish three sources of error, designated (here) *generated*, *inherent* and *analytic*. Generated error reflects inaccuracies due to the necessity of rounding or otherwise truncating the numeric results of arithmetic operations, inherent error reflects inaccuracies in initially given arguments and parameters, and analytic error reflects inaccuracies due to the use of a computing procedure which calculates only an approximation to the theoretical result desired.

In practice, error from any of these sources is introduced at successive points in a calculation, and affects the subsequent results. The study of the *propagation* of error in calculations of various types is an important part of error analysis. Error propagation may in general be studied without reference to the classification of error by source; it is usually possible, however, to break down the final error in a calculation and ascribe part of it to each of the several error sources [1, ch. 1]. On the other hand, it is often convenient to adopt a relative point of view concerning sources of error; thus at any stage of a calculation all previously accumulated error, from whatever source, might be regarded as inherent error for subsequent stages. Another expedient which has been successfully used is to

redefine generated error as inherent error, by attributing the result of inexact arithmetic on exact arguments to exact arithmetic calculation with inexact arguments [**2**, ch. 1].

The difficulties and subtleties involved in a thoroughgoing analysis of even the seemingly simplest computation are not at once evident; these will be illustrated as the investigation progresses. What is desired is a straightforward way of expressing the outcome of a computational process which depends on many parameters (e.g., all the coefficients as well as the argument of a polynomial, or all the elements of a matrix) subjected to a complicated and interwoven sequence of arithmetic operations. The critical dependence on the particular combination of parameter values and the particular way in which the computation is structured makes it imperative to have some sort of analysis technique; the efficiency with which such an analysis can be obtained, however, turns out to be dependent on certain features of the way arithmetic is done. Thus one must be concerned with basic questions of computer design as well as with issues of computational technique.

Error analysis. To facilitate the discussion of computer arithmetic, it is convenient to postulate a mode of number representation sufficiently general to comprehend the specific conventions found in practice. Let e be an integer and f a real number satisfying $|f| \leqq 1$; then the pair (e, f) may be taken to represent the number $2^e f$. The *exponent* e specifies a "scale factor" (here taken as a power of 2 in anticipation of the application to binary arithmetic), while the *coefficient* f carries the detailed information specifying the value of the number. The exponent is sometimes called the "characteristic" and the coefficient (inappropriately) the "mantissa" or (more appropriately) the "fixed-point part" of the number.

In practice, the possible values of f are restricted to the set in the interval $[-1, 1]$ representable by a standard length sequence of digits; that is, assuming a binary representation, the possible values of f are multiples of some value 2^{-p}, p being the number of binary places allocated to the representation of f. (It is also true that usually one and possibly both of the values $f = \pm 1$ are excluded, but this becomes important only as a detail.) The value p here may be said to characterize the *precision* of the representation.

If e is to be represented explicitly, it is also restricted in a similar manner; since e is an integer, however, the effect here is merely to restrict the *range* of e values.

Now it is possible, for purposes of computer processing, to have only f represented explicitly, e being kept track of by the programmer; alternatively, both e and f may be carried explicitly and used in the implementation of arithmetic rules. In the former case, one speaks of *fixed-point* representation and arithmetic, and in the latter of *floating-point*. In either case, arithmetic rules are not uniquely

determined, since a result (e, f) could be equivalently represented as $(e+d, 2^{-d}f)$, with only a discrepancy of low order being incurred due to the digital nature of the representation of f. The manner in which the magnitude of a number is shared between e and f is called the *adjustment* of the number. The adjustment of a given number is uniquely specified by giving either its exponent e, or alternatively the number m of *leading digits* in the representation of f; the latter is formally defined, for $f \neq 0$, as the number which gives $1/2 \leqq 2^m|f| < 1$. A number for which $m = 0$, that is, for which $1/2 \leqq |f| < 1$, is said to be *normalized*; the conventional manner of resolving the adjustment ambiguity in floating-point work is to impose the requirement that all nonzero numbers (e, f) be manipulated in normalized form. Although this seems to have the advantage of retaining as many "significant digits" as possible, it also makes it difficult to estimate the number of digits which are truly significant; this is discussed in more detail subsequently.

Suppose that (e, f) is derived as a computational approximation to a number x; that is, there is some "true value" f^T (not necessarily representable in a finite number of digits) such that $x = 2^e f^T$, with the discrepancy between f and f^T being attributed to one or more of the error sources already described. Let $\delta = f - f^T$; then $2^e\delta = 2^e f - x$ may be defined as the *absolute error* of the approximation, and $r = \delta/f^T$ as the *relative error*. The latter is usually more interesting; in fact, the number of "significant digits" in a result is essentially a measure of the relative error, being approximately $-\log_2 |r|$ (assuming binary representation as before). In the present context, however, it is convenient to take the value δ itself as the fundamental error measure, and call it the *coefficient error*. Since if (e, f) is also given the absolute error is immediately expressible as $2^e\delta$, and the relative error is approximately expressible as $r \approx \delta/f$, nothing much is lost by thinking in terms of δ. It should be noted that if (e, f) is a normalized representation, then $|\delta| \leqq |\delta/f| \leqq 2|\delta|$, so that δ itself is a good order-of-magnitude measure of relative error; in general, where (e, f) is not restricted to normalized form, however, this is not the case.

Error propagation. Consider the determination of a value $y = F(x)$ of a function F of one argument. Let (e_0, f_0) represent y and (e_1, f_1) represent x, and suppose the coefficient errors in function value and argument to be δ_0 and δ_1, respectively; this supposition implies that the relationship

$$2^{e_0}(f_0 - \delta_0) = F[2^{e_1}(f_1 - \delta_1)]$$

is satisfied exactly, not approximately. Define α, the *amplification factor* for the function evaluation, by the relation

$$|\delta_0| = \alpha|\delta_1|.$$

It is then possible to study the nature of α (as it depends on x) under various

assumptions concerning the computational method and sources of error introduced. To begin with, if one assumes that $F[2^{e_1}f_1]$ can be derived exactly (i.e., without the introduction of either analytic or generated error), then δ_0 is solely the propagated effect of the inherent error δ_1, and as such is determined by the nature of the function F and the adjustment of (e_0, f_0) in relation to that of (e_1, f_1). In practice, it may be possible to minimize the effect of analytic and generated error (i.e., produce a value such that $2^{e_0}f_0 = F[2^{e_1}f_1]$ to within the accuracy permitted by the p-digit representation); the propagation of inherent error is then the main error effect which must be taken into account.

To illustrate, consider a very simple functional relationship $y = x - 1$. If (e_0, f_0) and (e_1, f_1) are normalized floating-point approximations to y and x, respectively, then $\alpha \gg 1$ when x is very near 1, and $\alpha \ll 1$ when x is very near 0. Another way of expressing this is to say that the function is "significance-losing" near 1 (i.e., y has fewer significant digits than x) and "significance-gaining" near 0 (i.e., y has more significant digits than x). This very simple example shows that there is no law tending to "preserve" significance, a fact that is obvious on the face of it but often lost sight of in arguments concerning error.

The behavior of α in normalized calculation reflects the behavior of relative error, because of the circumstance, mentioned earlier, that $|f|$ is constrained to be of the order of unity. The fact that α is subject to essentially unbounded fluctuation in normalized work makes its estimation very difficult in a calculation of any complexity. The above point of view suggests that this might be remedied by not requiring a normalized representation (e, f), but instead adjusting so that α is at worst of the order of unity. A criterion which accomplishes this may be called a *significance* criterion. If such a criterion is used, the evaluation of $y = x - 1$ for unnormalized (e_1, f_1) produces an (e_0, f_0) such that $|f_0| \ll |f_1|$ when x is near 1, and $|f_0| \gg |f_1|$ when x is near 0; this is effectively equivalent to the condition that (e_0, f_0) have fewer significant digits than (e_1, f_1) in the former case, and more in the latter.

A simple adjustment rule applicable to this case is expressible as $e_0 = e_1$, with the provision that if this leads to violation of the inequality $|f_0| \leqq 1$, then (e_0, f_0) is developed in normalized form. When this rule is used for the function at hand, it can be stated definitively that $\alpha = 1$ except in the cases where $e_0 > e_1$ in order to preserve $|f_0| \leqq 1$, in which case $\alpha < 1$. Thus coefficient error is never amplified, and is deamplified only as required by formal properties of the floating-point representation. This criterion is automatically implemented if $y = x - 1$ is evaluated in unnormalized "significant-digit" floating-point arithmetic, as described in the next section. It is important to realize, however, that in the case of function evaluation an adjustment criterion which makes α of the order of unity is defined independently of the manner of computation of the function value, and in general also independently of the actual value of δ_1 (which is ordinarily unknown); furthermore, for many basic functions an adjustment criterion can

be expressed in terms of the exponent or some other simple formal property of the argument representation (e_1, f_1), thereby making it not unduly onerous to apply (see [3]).

It is desirable to extend the concept of amplification factor to the case in which the result depends on several arguments, all of which may be in error. A reasonable way to do this is to define α with respect to the largest of the argument errors; that is, if $y = F(x_1, x_2, \cdots, x_n)$ and (e_0, f_0) with error δ_0 represents the value of y derived from $(e_1, f_1), (e_2, f_2), \cdots, (e_n, f_n)$ with errors $\delta_1, \delta_2, \cdots, \delta_n$ representing $x_1, x_2, \cdots, x_n$, then α is defined by the relation

$$|\delta_0| = \alpha \cdot \max(|\delta_1|, |\delta_2|, \cdots, |\delta_n|).$$

Besides applying to the evaluation of actual multi-argument functions, this definition can also be applied to the case where a function of one argument also depends on one or more "parameters" (e.g., a polynomial), and to the analysis of arithmetic rules (viewed as specifying the evaluation of functions of two arguments).

Computer arithmetic. The concepts of the previous section can be applied to the analysis of propagation of error through arithmetic operations; this in turn gives insight into the way in which inherent and generated error affect arithmetically computed results, and suggests methods for monitoring and controlling such error. It is even possible occasionally to say something concerning analytic error in this context, although in general, of course, the arithmetic operations defining an approximate evaluation of (say) a function of one argument carry no information as to what the function "really" is.

The adjustment of numbers in conventional fixed-point arithmetic is determined by the assumed requirement $e = 0$ for operands and result, and the normalization convention $m = 0$ plays a similar role for the usual floating point arithmetic. An error-amplification analysis can be carried out for arithmetic operations under both these rules, using the extended definition of α given in the preceding section. It turns out that in fixed-point $(e = 0)$ arithmetic, $\alpha \approx 1$ necessarily for addition or subtraction, but multiplication may be sharply error-deamplifying and division may be sharply error-amplifying. By contrast, in normalized floating-point arithmetic the situation is reversed; $\alpha \approx 1$ for multiplication or division, but it may be that $\alpha \gg 1$ in addition or subtraction. Specifically, the latter obtains where the sum or difference is considerably smaller in magnitude than the operands, so that major left-shifting is required to bring it to normalized form. This particular situation, usually unpredictable in advance because it involves the comparative magnitudes of intermediate results, makes it possible for catastrophic loss of significance in floating-point calculation to occur without any explicit indication of the fact in the final results (since coefficients are all expressed to p-digit precision).

If the normalization requirement is relaxed, it becomes possible to specify

floating-point rules which are more reasonable with regard to significance propagation, and which may be said to share essentially the advantages of both the standard fixed- and floating-point arithmetics without the disadvantages. Specifically, this is achieved by the adjustment rules

$$e_0 = \max(e_1, e_2)$$

for addition and subtraction and

$$m_0 = \max(m_1, m_2),$$

for multiplication and division; here m is as before the number of leading digits (so $p-m$ is the number of "significant" digits) associated with a coefficient f. The result of addition or subtraction may be described as obtained by adjusting the operand with the smaller exponent to the larger (which decreases δ), then adding or subtracting with no post-normalization operation such as would occur in standard floating-point arithmetic. The result of multiplication or division may be described as obtained by multiplying or dividing the associated normalized coefficients, and then adjusting the result so as to have the same number of apparently significant digits as that operand with the fewer such.

A detailed analysis of these arithmetic rules has been carried out in [4]; it shows that the "worst case" amplification factor for all operations is a number of the order of 2 or 3, and that "on the average" $\alpha \approx 1$. This is the basis for the informal name "significant-digit arithmetic" for this particular variety of unnormalized floating-point arithmetic. The arithmetic unit of the Maniac III computer at the University of Chicago has been designed to incorporate, among others, these arithmetic rules (see [5]).

It should be kept in mind clearly that despite the name there is no guarantee that computing in "significant-digit arithmetic" will produce only results for which all digits are meaningful, in terms of any error criterion in which one may happen to be interested at the moment. Naive trust on this point can lead to early and unjustified disillusionment with the potentialities of unnormalized arithmetic. In fact it would be possible to specify adjustment rules so that only "truly" significant digits of results were generated, assuming all operand digits to be "truly" significant; these would require $\alpha < 1$ by enough margin to take into account the additional error made in rounding, and would thus tend to discard digits which often should be retained. In any case, operands are not in general error-free, so the general propagative effects of arithmetic rules must be taken into account. From this point of view, the stability achieved by the $\alpha \approx 1$ criterion, and the accompanying formal consistency of the rules given above, seems to recommend them over more conservative rules which guarantee that $\alpha < 1$. In succeeding sections some of the possibilities afforded by the use of these arithmetic rules, in conjunction with additional adjustments made on the basis of analysis are discussed.

Error monitoring and control. As indicated in the previous section, the use of unnormalized arithmetic permits the establishment of estimates for and bounds on the amplification of coefficient error through arithmetic operations, such estimates and bounds being essentially independent of the particular operands involved. This permits the "monitoring" of coefficient error as it propagates through a series of calculational steps; the possibility of exponent adjustment even permits a certain degree of "control" to be exercised, since adjusting (e, f) to $(e+d,\ 2^{-d}f)$ in general adjusts the coefficient error δ to $2^{-d}\delta$, regardless of what δ may be (this statement must of course be modified to take into account the use of p-place representation, but its force is clear).

It may be seen from the foregoing that the estimation of computational error in the context of unnormalized arithmetic depends on the assessment of a variety of interacting factors. Furthermore, arithmetic rules by themselves cannot be expected to solve the problem of obtaining sufficiently accurate estimates; this is because their application is strictly "local" in a computational scheme, and they cannot reflect certain "global" or long-run aspects of its structure. In this connection, two cases must be considered; that where the error builds up more rapidly than predicted, thus leading to results which appear more significant than they actually are, and that where there is unexpected deamplification of error, so that digits which are actually significant are discarded.

The unnormalized arithmetic rules described insure against sharp amplification of error by virtue of the bounding inequalites on α; thus large increases in coefficient error can be associated only with long calculational sequences. In fact, striking examples of error buildup have been noticed in connection with summation of series (see [6]); in general, however, the phenomenon is marked mainly when the summation is carried out to many more terms than practical computation would ordinarily involve.

The contrary phenomenon, error deamplification, is troublesome because it need not occur gradually, it being impossible to design arithmetic operations so as to insure that errors do not accidentally cancel each other. Of course, when it occurs such cancellation is usually not really accidental, but depends on the fact that the arithmetic operands are mathematically related in some way. To illustrate, consider the calculation, in unnormalized arithmetic, of $y=x(1-x)$. Let it be assumed that the particular x involved is such that $1-x$ can be represented at the same exponent as x. The product adjustment for unnormalized multiplication is based on the assumption of independent errors in the factors; here, however, the coefficient errors in x and $1-x$ are equal in magnitude and opposite in sign. For $x \approx 1/2$, these errors are multiplied by approximately the same coefficients and added to get the error in the product; this error is therefore much less than the adjustment allows for, and in fact may be less than the rounding error which occurs. In the latter case, the result obtained using normalized arithmetic is more accurate; one can take advantage of this addi-

tional accuracy, however, only if analysis has shown it to be there, and the same knowledge could be used to compensate for the effect in the unnormalized case.

This latter point emphasizes the necessity for having available adjustment operations over and above those built into the unnormalized arithmetic operations. In planning Maniac III, consideration was given to the various ways of specifying adjustment that might be useful in practice: adjustment by a given amount d, adjustment to a given exponent e, and so forth. A flexible capability of this nature is needed if full advantage is to be taken of the error monitoring and control potentialities of unnormalized arithmetic.

The present section has indicated how the judicious use of unnormalized arithmetic and supplementary adjustment operations can lead to error estimates which are "computed" along with the associated numerical results. Although some of the considerations discussed can be applied to the case of normalized arithmetic, the constraint on the representation of f makes difficult the analysis of α, either beforehand or afterwards. In effect, the value m given by the number of leading digits of an unnormalized result is an evaluator for the error which is not available in the normalized case. Mention should be made of an "index" scheme which has been proposed, in which computation is normalized but an index value equivalent to m is kept with each number as a significance indicator (see [**7**]). This requires that extra storage be assigned to the digits which represent the index, which could otherwise serve to give more precision in the representation of f; aside from this, however, the system combines the accuracy of normalized arithmetic with a capability of monitoring significance. It must be realized, of course, that in cases such as those discussed above, in which the significance indication is misleading, one must still use analytic criteria to determine what extra accuracy can be claimed.

Computational case studies. Besides illustrating some of the general notions introduced earlier, the following examples indicate the manner in which techniques applicable to various specific classes of computational tasks may be developed.

a) *Function Evaluation.* As already stated, adjustment rules for values of a function can be specified independently of the manner of their computation. In several elementary cases, however, the rules appear in a certain sense "natural" from the computational point of view also. Consider the case of the square-root function, where amplification analysis (see [**3**]) shows the rule $m_0 = m_1 - 1$ to be appropriate. The conventional method for computing a square-root value $y = \sqrt{x}$ uses the Newton-Raphson iteration formula

$$y^{(k+1)} = \tfrac{1}{2}(y^{(k)} + x/y^{(k)}).$$

Assume that x is represented in unnormalized form and $y^{(k)}$ is a normalized approximation to $\sqrt{x}$. Then $x/y^{(k)}$ is computed at the same adjustment as x (in terms of number of leading digits) by the unnormalized division rule. If the

normalized $y^{(k)}$ is added to this, it is adjusted to the same exponent as $x/y^{(k)}$, which it approximately equals; the sum, being approximately double each of the terms, is therefore formed with the same exponent and one less leading digit than the terms. Performing the multiplication by normalized 1/2 according to the unnormalized multiplication rule does not alter this situation, so the new approximation $y^{(k+1)}$ will satisfy $m_0=m_1-1$. Repetition of the process starting with normalized $y^{(k+1)}$ again produces a value of so adjusted; hence the adjustment of the final value y also satisfies the rule. Note that it is specified that each intermediate approximation be normalized before being used in further iteration. This is a general rule, to be applied in cases where an approximation is to be improved according to some analytic criterion, since then it may be regarded as "arbitrary" and hence accurate; the naive supposition that approximations should be introduced with low significance simply because they are approximations is inappropriate. Actually, what is required is only that each successive approximation have enough digits so that its apparent significance does not condition the result; this shows that the intermediate normalization in the square-root case is not strictly necessary, as long as $y^{(k)}$ has fewer leading digits than x.

b) *Determination of Polynomial Zeros.* Consider a general polynomial of degree n, defined by

$$P(x) = a_0 + a_1x + \cdots + a_nx^n.$$

Iteration to an isolated zero $\bar{x}$ may be accomplished by the Newton process

$$x^{(k+1)} = x^{(k)} - P(x^{(k)})/P'(x^{(k)}),$$

where P' is the derivative of P. The repeated evaluation of $P(x)$ and $P'(x)$ necessary here can be accomplished by the combined synthetic division $p_0=a_n$, $q_0=0$;

$$p_i = a_{n-i} + xp_{i-1}, \qquad q_i = p_i + xq_{i-1}, \qquad i = 1, 2, \cdots, n.$$

This yields $P(x)=p_n$ and $P'(x)=q_n$.

Now if the polynomial coefficients a_i are known exactly, it is theoretically possible to get an arbitrarily accurate determination of $\bar{x}$. But practically, the attainable accuracy is limited by the accuracy with which $P(x)/P'(x)$ can be evaluated near $\bar{x}$, using the available numerical precision. Even if the a_i are representable exactly (as in the case when they are integers), the occurrence of rounding errors in the evaluation process renders the computed value of $P(x)/P'(x)$ inaccurate. The degree to which this rounding error is amplified in the evaluation of $P(x)$ near $\bar{x}$ is said to reflect the "condition" of the zero $\bar{x}$; if a zero is "ill-conditioned" it is in general impossible to get a determination of its value which is relatively accurate in terms of the computational precision (see [2] ch. 2).

When normalized arithmetic is used, the attempt to compute a value $P(x)/P'(x)$ near $\bar{x}$ must in any case give a normalized number, most of the digits of which are essentially meaningless. The crucial question then becomes whether the exponent of this meaningless number is small enough so that it does not affect x appreciably when added to it. Ill-conditioning is signified by failure of this "zero value" to become sufficiently small, giving rise to appreciable fluctuations in the successive iterates which never "damp out." Since in automatic computation one does not have the opportunity to see that this is happening, the procedure for detecting that "convergence" has been achieved must be complicated accordingly.

By contrast, when unnormalized arithmetic is used, the representation of a value $P(x)/P'(x)$ near an ill-conditioned zero $\bar{x}$ consists mainly of leading digits, and the ill-conditioning is essentially reflected only in the size of the exponent. This results in the value $x-P(x)/P'(x)$ being formed with a greater exponent than that of normalized x, and hence the sum has "lost significance." If each successive iterate is computed by first normalizing x (see the remarks above pertaining to the iterative evaluation of the square-root function), then the series of computed iterates will "settle down" to a sequence of values which fluctuate only in the last one or two places, if at all; the number of digits achieved then reflects the accuracy attainable in the determination of the zero $\bar{x}$. Iteration can in this case be terminated by a coefficient tolerance criterion, which is appropriate to both the well- and ill-conditioned cases.

(*Example*) The polynomial of degree 8 defined by

$$P(x) = x^8 - 36x^7 + 546x^6 - 4536x^5 + 22449x^4 - 67284x^3 + 118124x^2 - 109584x + 40320$$

has zeros at the integer points $\bar{x}=1, 2, \cdots, 8$. It is a member of a class of polynomials whose zeros tend to be ill-conditioned (see [2] ch. 2). Starting from the normalized approximation $x^{(0)}=4.75$ (so convergence to the zero at $\bar{x}=5$ is expected), the Newton process was applied unnormalized and normalized on the Maniac III to generate the iterative sequences of Table 1.

Caution: The reader should not conclude from this single example that (a) the unnormalized iteration always settles down to a single value (it may cycle through a sequence of slightly differing values); (b) that the normalized iteration always arrives at a cycle of such short period so early. It should also be noted that care must be used when drawing conclusions from results displayed in decimal; the binary number represented by the convergent value in the unnormalized case was

$$101.0000000000000000000001,$$

so the value was actually in error by only one bit in the last place, and the greater apparent discrepancy in the decimal value was a product of the binary-decimal conversion.

TABLE 1. Sequences of unnormalized and normalized iterates.

Unnormalized	Normalized
5.0814474	5.08144719801
5.0001462	5.00014621355
4.9999998	4.99999985429
5.0000006	4.99999979387
5.0000002	5.00000000331
5.0000002	4.99999971107
.	4.99999998841
.	4.99999986837
.	4.99999982947
(unchanged	5.00000013163
thereafter)	5.00000017053
	4.99999986837
	.
	.
	.
	(repeats cycle of four iterates thereafter)

c) *Matrix Pivoting.* Many matrix procedures for calculating inverses and determinants, and solving equations, are based on a succession of stages involving the selection of a pivot element and the application of a transformation derived from the classical method of elimination. In numerical work considerations of accuracy dictate the adoption of an appropriate pivot selection strategy, and this is customarily handled by specifying that at each stage the pivot of largest magnitude in some set of eligible elements be selected. An elegant analysis justifying this strategy for normalized floating-point arithmetic has been given (see [2] ch. 3); it pertains mainly to the buildup of rounding error, however, and shows, essentially, that if the largest-magnitude strategy is adopted then the loss of accuracy from rounding error is comparable to that arising from inherent error (which can be substantial in ill-conditioned situations).

An analysis from the point of view of the present paper indicates that this strategy is not the appropriate one if unnormalized arithmetic is used. The rule suggested for pivot selection is based, not merely on largest magnitude, but on a combination of magnitude and accuracy of representation. Let $x_{ij} = (e_{ij}, f_{ij})$ be an element of a square matrix. A measure of magnitude is the quantity $\mu_{ij} = e_{ij} - m_{ij}$, where m_{ij} is the number of leading digits in x_{ij}. It may be noted that the number of significant digits in x_{ij} is given by $\sigma_{ij} = p - m_{ij}$, where p is the precision of representation, as before. The operation that transforms the majority of the elements x_{ij} at any one stage of pivoting may be written

$$x_{ij} - x_{ic}x_{rj}/x_{rc} \longrightarrow x_{ij},$$

where x_{rc} is the pivot element. In the execution of the subtractive operation the term with the smaller exponent is first adjusted until exponent match is achieved, then the subtraction is performed. It is evidently desirable to select as pivot an

element that effects a minimization in the exponent of the subtractive term, so that as few digits of x_{ij} are lost as possible. The pivot selection criterion suggested is to maximize $\lambda_{ij}=\mu_{ij}+\sigma_{ij}$ (or in practice $\lambda_{ij}=\mu_{ij}-m_{ij}$, since p is a constant); this tends to select elements of large magnitude, but not when they are relatively inaccurate. If all elements are normalized, this reduces to the usual largest-magnitude criterion; if however elements are unnormalized in varying degrees, reflecting varying significance, the use of the present rule can make a substantial difference. A detailed justification of the rule will be given elsewhere; the present discussion is intended to give a heuristic basis for its use in unnormalized computation.

Research on which this paper is based supported by the Atomic Energy Commission under Contract No. AT(11-1)-614.

References

1. A. S. Householder, Principles of Numerical Analysis, McGraw-Hill, New York, 1953.

2. J. H. Wilkinson, Rounding Errors in Algebraic Processes, Prentice-Hall, Englewood Cliffs, N. J., 1963.

3. R. L. Ashenhurst, Function evaluation in unnormalized arithmetic, J. Assoc. Comput. Mach. no. 2, 11 (April 1964) 168–87.

4. R. L. Ashenhurst, and N. Metropolis, Unnormalized floating-point arithmetic, J. Assoc. Comput. Mach. no. 3, 6 (July 1959) 415–28.

5. R. L. Ashenhurst, The Maniac III arithmetic system, Proc. Joint Comput. Conf., 21 (1962) 195–202.

6. R. H. Miller, An example in "significant-digit" arithmetic, Comm. ACM, no. 1, 7 (January 1964) 21.

7. H. L. Gray, and C. Harrison Jr., Normalized floating-point arithmetic with an index of significance, Proc. Joint Comp. Conf., 16 (1959) 244–8.

MATRIX EIGENVALUE PROBLEMS

BERESFORD PARLETT, Stevens Institute of Technology, Hoboken

1. Matrix theory is easy, so why should matrix computations be difficult? Mathematics is the art of avoiding calculations. Should the mathematician therefore shun the computation of matrix eigenvalues? The fact that these numbers are often of importance to physicists, engineers and other scientists does not guarantee that methods for finding eigenvalues have in themselves much mathematical interest. The structure of a matrix in terms of its eigenvalues and eigenvectors is well understood and so it is rather surprising that for only a few of the current computational problems are there satisfactory computer programs.

In the hope of illustrating the difficulties we have chosen to follow two examples in some detail. We will make no attempt to survey the development of the many topics covered by the title of this essay. For example we shall omit the development of the sophisticated and highly satisfactory methods now employed for symmetric (or Hermitian) matrices. Indeed programs exist which not only give the eigenvalues (and eigenvectors too, if required) but derive from the calculation rigorous estimates of the possible error in each root.

2. Eigenvalue calculations: *the computer's speed makes them possible, the computer's inexact arithmetic makes them interesting.* A frequent request is for those numbers λ which make the matrix $A\lambda^2+B\lambda+C$ singular. Here A, B, and C are conformable square matrices, usually all singular. There are many other forms which the eigenvalue problem assumes but we shall confine ourselves here to the fundamental, classical one of the location of the zeros of the characteristic polynomial, $p(\lambda)=\det(A-\lambda I)$, associated with the $n\times n$ matrix A and the $n\times n$ identity matrix I. Note that we are being asked to find an algorithm which will work *automatically* on *any* digital matrix. Here is one reason for the difficulty in obtaining good programs.

Whatever method we employ the arithmetic labor required to find the eigenvalues increases sharply with n. Using pencil and paper and calculating each number to 5 significant decimals a man might take most of a day to resolve a 5×5 real matrix. On a high speed digital computer all the eigenvalues of a 100×100 real matrix can be found in a few minutes. Thus it is automatic computing devices which give us the hope of finding the eigenvalues of a wide variety of matrices.

The price paid for the high speed of a digital computer is that the arithmetic operations are not performed exactly. It is this limitation of the tool we are obliged to use which has given the problem its mathematical interest.

A computer is given a matrix and an algorithm. The algorithm is a sequence of instructions which tell the machine how to use the data. In our case the oper-

ations on the data should produce the eigenvalues.

We must study an algorithm from two different points of view. Firstly we must understand the method in the context of exact arithmetic operations on the full set of real numbers. Here is the place to show for what class of matrices the method works in theory. Yet conventional analysis of the *theoretical* algorithm is but a preliminary to the newer, less orthodox, task of assessing the *practical* algorithm; i.e. the effect of inexact arithmetic operations on the finite, discrete set of digital numbers (the only ones which the machine can handle). Will the work of this hasty but imperfect servant be an acceptable imitation of the theoretical process?

Interest in a large number of techniques has resulted from the variety of forms in which the practical problem is posed (e.g. find a few eigenvalues, all eigenvalues, no eigenvectors, all eigenvectors) and from the inadequacy of earlier methods.

3. Error versus uncertainty. When asked to multiply together two digital numbers x and y the computer produces another digital number z which contains a small error relative to the true product xy. In a machine which uses 3 decimals for each number the product of 12.3 and .234 is given as 2.88 (or, in some machines, 2.87) instead of 2.8782, the sum of these numbers is given as 12.5 instead of 12.534.

It seems possible that the concatenation of very many of these small round-off errors in the course of a long computation could lead to large errors in the results. But this is not the only, nor even the main danger in computing. We must examine more closely the meaning of the term error. The error in a result is the difference between that result and the true solution. What is the true solution?

In by far the majority of cases which occur in practice the matrix elements in the computer are not exact real (or complex) numbers. They may come from physical observation, from previous computation, or they may be genuine real numbers forced into the strait-jacket of machine representation. Thus $1/3$ and $\sqrt{2}$ would appear in our 3 decimal machine as .333 and 1.41 respectively. Thus for each element we know only the first few digits of an infinite expansion. Our digital matrix stands for an infinite set of real matrices each of which has a set of eigenvalues with a valid claim to be called correct. In fact the eigenvalues of a digital matrix should be defined as the totality of these eigenvalue sets. Little progress has been made so far in describing these sets in practice. It has been maintained that the unrepresented digits of our matrix should be conventionally taken as zero in order to give us an exact matrix and a true set of eigenvalues. This seems a rather arbitrary decision.

We see that the uncertainty in the data implies an uncertainty in the eigenvalues and in general this lack of precision can vary greatly from one eigenvalue

to another. This uncertainty has nothing to do with any particular method for computing the eigenvalues. It is a property of the data. There are matrices with elements given to 8 significant decimals which determine some of their eigenvalues to 8 and others to 2 significant decimals.

Here is a limitation on the results to be obtained from any numerical eigenvalue program. The uncertainty in an answer is seen to arise from two causes. Amplifying the inherent lack of precision attributable to the data is the uncertainty arising from the approximate execution of the algorithm. Only the latter depends on the method which is being used and an ideal technique would be one which gives rise to uncertainties which are no greater than those stemming from the data. When these sources are confounded then the imperfections of a particular matrix may be visited on the method.

It is not easy to assess these uncertainties. Error analysis, as the subject is called, suffers the reputation of being both difficult and tedious. Recently, however, the situation has improved with the fruitful idea of "backward" analysis. We ask not for the error in our answer but to what problem is our result the exact solution. When this can be answered we are often in a position to say that the computer algorithm has actually solved a problem sufficiently close to the given one as to be entirely acceptable. The solutions of the two problems can only differ substantially if the given problem is sensitive to small changes in the data.

4. A simple error analysis. One of the major difficulties in a practical analysis is that of description. An ounce of analysis follows a pound of preparation.

In order to avoid inessential difficulties we have chosen a problem and an algorithm which are easy to follow theoretically. The analysis of the practical algorithm retains, however, many important features of full scale analyses. The result is surprising in the sense that a method which could with justice be called bad is shown to be quite satisfactory for the purpose in hand. Dividing by small numbers is not always fatal. Indeed careful analysis has resuscitated more than one technique which lay on the mound of forsaken methods.

A common approach to the eigenvalue problem is to use some "safe" similarity transformations to obtain a matrix of simpler form. Yet the simpler the form the less safe the methods seem to be. Frequently, then, a method uses some polynomial root finder and obtains the required trial values of the characteristic polynomial (and possibly some of its derivatives) by determinant evaluation. Our example is of determinant evaluation of Hessenberg matrices, i.e. matrices which are upper triangular together with nonzero elements immediately below the main diagonal. The matrix is 3×3 for ease of analysis; the conclusions do not depend on the order.

In what follows Roman letters denote digital numbers and Greek letters denote real numbers.

THE PROBLEM. *Calculate the determinant* δ *of the matrix* A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{23} & a_{33} \end{bmatrix},$$

where a_{ij} *is a real digital number.*

Theoretical Algorithm. We use the method of elimination. Subtract μ_1 times row 1 from row 2 so that the new element in position (2, 1) is zero. Subtract μ_2 times the new row 2 from row 3 to eliminate the (3, 2) element. In matrix terminology we pre-multiply A by the matrix μ to obtain a triangular matrix β,

$$\begin{bmatrix} 1 & 0 & 0 \\ -\mu_1 & 1 & 0 \\ 0 & -\mu_2 & 1 \end{bmatrix} A = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ 0 & \beta_{22} & \beta_{23} \\ 0 & 0 & \beta_{33} \end{bmatrix}.$$

We assume that neither β_{11} nor β_{22} vanish. By direct calculation,

$$\text{(T)} \quad \begin{cases} \beta_{1j} = a_{1j}, & (j = 1, 2, 3) \\ \mu_1 = a_{21}/\beta_{11}, & \\ \beta_{2j} = a_{2j} - \mu_1 a_{1j}, & (j = 2, 3) \\ \mu_2 = a_{32}/\beta_{22}, \quad \beta_{33} = a_{33} - \mu_2\beta_{23}, \quad \delta = \beta_{11}\beta_{22}\beta_{33}. & \end{cases}$$

Practical Algorithm. This depends on the particular computer being used. All we need to know is that our computer, when given x and y, produces the digital numbers $(x+y)(1+\epsilon_a)$ and $xy(1+\epsilon_m)$ instead of $x+y$ and xy which may not be digital numbers. Here ϵ_a and ϵ_m depend on x, y, and the computer. However, $|\epsilon_a| < \epsilon$, $|\epsilon_m| < \epsilon$, where ϵ is a small number depending only on the computer. Any subscripted ϵ is to be taken as a rounding error bounded by ϵ. When instructed to form $xy+z$ our computer first produces $xy(1+\epsilon_1)$ and then tries to add z, finally obtaining the digital answer $(xy(1+\epsilon_1)+z)(1+\epsilon_2)$.

When we try to implement the theoretical algorithm (T) on our computer we obtain

$$\text{(P)} \quad \begin{cases} b_{1j} = a_{1j}, & (j = 1, 2, 3) \\ m_1 = a_{21}(1 + \epsilon_{21})/b_{11}, & \\ b_{2j} = [a_{2j} - m_1 a_{1j}(1 + \epsilon_{1j})](1 + \epsilon_{2j}), & (j = 2, 3) \\ m_2 = a_{32}(1 + \epsilon_{32})/b_{22}, & (\text{we assume } b_{22} \neq 0) \\ b_{33} = [a_{33} - m_2 b_{23}(1 + \epsilon'_{23})](1 + \epsilon_{33}), & \\ d = (b_{11}b_{22}(1 + \epsilon_1))b_{33}(1 + \epsilon_2). & \end{cases}$$

We have assumed that $b_{11} \neq 0$, $b_{22} \neq 0$ but one or both of them may be very small and thus one or both of m_1 and m_2 may be very large compared with the elements

of A. We shall be concerned with the consequences of this.

Forward Analysis. We could pursue the type of analysis in which we express each quantity in (P) as a true value plus an error, e.g. $b_{22}=\beta_{22}+\eta_{22}$. Eventually we would obtain a complicated expression for $\eta_\delta=d-\delta$. If we could obtain a satisfactorily small bound for $|\eta_\delta|$ in terms of a's and ϵ then we would be sure that the practical algorithm is useful. In practice η_δ may be so complicated that a realistic estimate is hard to make.

Suppose that a realistic error estimate seems very large. Is it the fault of the method or of the data? The determinants of some matrices A may be so sensitive that the best of methods, working to the given accuracy, would produce such an error.

Backward Analysis. A more illuminating approach is to seek the matrix whose triangular decomposition we actually found with (P). We see that

$$\begin{bmatrix} 1 & 0 & 0 \\ m_1 & 1 & 0 \\ 0 & m_2 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{23} \end{bmatrix} = A + \mathcal{E},$$

$$\mathcal{E} = \begin{bmatrix} 0 & 0 & 0 \\ \epsilon_{21}a_{21}, & \epsilon_{22}a_{22}-(\epsilon_{12}+\epsilon_{22}+\epsilon_{12}\epsilon_{22})m_1a_{12}, & \epsilon_{23}a_{23}-(\epsilon_{13}+\epsilon_{23}+\epsilon_{13}\epsilon_{23})m_1a_{13} \\ 0 & \epsilon_{32}a_{32} & \epsilon_{33}a_{33}-(\epsilon_{23}'+\epsilon_{33}+\epsilon_{23}'\epsilon_{33})m_2b_{23} \end{bmatrix}.$$

We have observed that m_1 or m_2 (or both) may be very large. Thus $A+\mathcal{E}$ can be substantially different from A. We conclude that the practical algorithm does not produce a satisfactory triangular decomposition of A and hence $d=\det(A+\mathcal{E})(1+\epsilon_1)(1+\epsilon_2)$ must be suspect as an approximation to δ. Notice that even if each b_{ij} is correct to working accuracy, i.e. b_{ij} is the machine number closest to β_{ij}, we will still have a poor decomposition of A when the m_i are large. The fact that the b_{ij} have very small relative errors is irrelevant.

It follows that this is a bad method to use as part of a program to solve $Ax=y$ since the result will actually satisfy $(A+\mathcal{E})x=y'$, a very different equation when the m_i are large.

In fact, however, d is quite satisfactory. However large $\mathcal{E}$ may be, d is a "good" approximation to δ because it happens that $A+E$ has the same determinant as another matrix A' which *is* very close to A. This surprising conclusion would be hard to reach by analyzing $\det(A+\mathcal{E})$. Yet it becomes obvious if we write out the b_{ii} in terms of the original matrix in the following way:

$$\text{(P}'\text{)}\quad \begin{aligned} b_{11} &= [a_{11}], \quad m_1 = [(1+\epsilon_{21})a_{21}]/b_{11}, \\ b_{22} &= [(1+\epsilon_{22})a_{22}] - m_1[(1+\epsilon_{12})(1+\epsilon_{22})a_{12}], \quad m_2 = [(1+\epsilon_{32})a_{32}]/b_{22}, \\ b_{33} &= [(1+\epsilon_{33})a_{33}] - m_2[(1+\epsilon_{23}')(1+\epsilon_{33})(1+\epsilon_{23})a_{23}] \\ &\quad + m_2m_1[(1+\epsilon_{23}')(1+\epsilon_{33})(1+\epsilon_{23})(1+\epsilon_{13})a_{13}]. \end{aligned}$$

Comparison with the theoretical algorithm (T) shows that b_{11}, b_{22}, b_{33} are identical to the diagonal elements which the exact algorithm (T) would produce from the matrix

$$A' = \begin{bmatrix} a_{11} & (1+\epsilon_{12})(1+\epsilon_{22})a_{12}, & (1+\epsilon_{13})(1+\epsilon_{23})(1+\epsilon'_{23})(1+\epsilon_{33})a_{13} \\ (1+\epsilon_{21})a_{21} & (1+\epsilon_{22})a_{22} & (1+\epsilon_{23})(1+\epsilon'_{23})(1+\epsilon_{33})a_{23} \\ 0 & (1+\epsilon_{32})a_{32} & (1+\epsilon_{33})a_{33} \end{bmatrix}.$$

We have not yet accounted for the rounding errors incurred in multiplying together the b_{ii}. If we multiply the last column of A' by $(1+\epsilon_1)(1+\epsilon_2)$ we obtain a matrix A'' with $\det(A'')=d$. The greatest perturbation involves changing a_{13} to $a_{13}(1+\epsilon_{13})(1+\epsilon_{23})(1+\epsilon_{33})(1+\epsilon'_{23})(1+\epsilon_1)(1+\epsilon_2)$. Normally $\epsilon<10^{-8}$ so that this is a small perturbation.

Note that the matrix (b_{ij}) is not the upper triangular factor of A' because our choice of a'_{13} and a'_{23} made b_{33} exact at the expense of b_{13} and b_{23}. This is in accordance with our observation that the elements of $\mathcal{E}$ may be large.

The success of this backward analysis depended crucially on the fact that $a_{31}=0$. When $a_{31}\neq 0$ the elimination algorithm allows of no small perturbations of the a_{ij} for which the b_{ii} would be exact because elements a_{12} and a_{13} would appear in more than one equation in (P′). Thus, in general, a'_{12} and a'_{13} would have to take on at least two different values simultaneously to make (P′) exact. In this case the algorithm is good for nothing.

5. Similarity transformations. As the limitations of known methods were revealed numerical analysts were driven to invent more sophisticated techniques. Most eigenvalue procedures fall into two categories; those which utilize only a finite number of similarity transformations and those whose theoretical algorithm demands an infinite number of them. The former approach was discussed in Section 4.

The latter approach is to derive from the given matrix a sequence of similarity transformations which, in the limit, will yield a triangular matrix. In the computer each matrix is written over its predecessor and the diagonal elements will ultimately yield the eigenvalues. In general the sequence must be infinite because the similarity transformations we use will produce only rational functions of the matrix elements and the eigenvalues are not in general rational in these elements. This sort of consideration is irrelevant for the practical algorithm but is germane to the convergence proofs of the theoretical procedure.

In order to illustrate the analysis of theoretical algorithms we will now describe an ingenious method from the second class, the LR Transformation of Rutishauser.

The LR Transformation. Most matrices can be factorized into the product of a left triangular matrix L and a right triangular matrix R. One way to effect this is by Gaussian elimination, the familiar technique used in solving systems

of simultaneous linear equations $A_1x=y$. The multipliers are the elements to the left of the diagonal of the matrix L_1 and they are chosen so that $L_1^{-1}A_1=R_1$.

The diagonal elements of L_1 are all unity. This transformation of A_1 is not a similarity, however, and therefore R_1 and A_1 do not have the same eigenvalues. When we complete the similarity transformation on A_1 we obtain $L_1^{-1}A_1L_1 = R_1L_1$ and we lose our triangular matrix R_1. The remarkable idea of Rutishauser is that we should now repeat the same process on the new matrix $R_1L_1(=A_2)$. This will give $L_2^{-1}A_2=R_2$ and a new matrix $R_2L_2(=A_3)$. If we continue in this way we will produce a sequence $\{A_k\}$ of matrices all similar to A_1. Under quite mild conditions on A_1 this sequence converges to a triangular matrix. The sequence is defined by

$$\text{(D)} \qquad A_k = L_kR_k, \qquad A_{k+1} = R_kL_k, \qquad k = 1, 2, \cdots,$$

where L_k and R_k are derived from A_k by Gaussian elimination (assuming no breakdown).

From the definition (D) there follow two basic formulas. Now $A_{k+1}=L_k^{-1}A_kL_k$ and, using this formula on A_k, we obtain $A_{k+1}=L_k^{-1}L_{k-1}^{-1}A_{k-1}L_{k-1}L_k$ and, eventually,

$$\text{(1)} \qquad A_{k+1} = L_k^{-1} \cdots L_1^{-1}A_1L_1 \cdots L_k = M_k^{-1}A_1M_k,$$

where $M_k=L_1 \cdots L_k$ and is also left triangular with 1's on the diagonal. If we define S_k by $S_k=R_k \cdots R_1$ (note the order of the factors) and use (1) in the form $M_kA_{k+1}=A_1M_k$ we obtain

$$\text{(2)} \qquad M_kS_k = M_{k-1}A_kS_{k-1} = A_1M_{k-1}S_{k-1} = A_1^2M_{k-2}S_{k-2} = \cdots = A_1^k.$$

So (2) shows that M_k and S_k are the triangular factors obtained by using Gaussian elimination on A_1^k.

Now $L_k=M_kM_{k-1}^{-1}$ and therefore, if M_k converges as $k\to\infty$, its limit M_∞ will have 1's on the diagonal and thus be nonsingular. In this case then $L_k\to M_\infty M_\infty^{-1} =I$, the identity matrix. If $L_k\to I$ then $R_k=L_k^{-1}A_k$ shows that A_k must tend to right triangular form. Thus the proof depends on obtaining the triangular factorization of A_1^k.

We will sketch here an elementary and illuminating proof of convergence which was given recently by Wilkinson.

THEOREM. *Let* $A_1=XDX^{-1}$ *where* $D=\text{diag}\{\lambda_1, \cdots, \lambda_n\}$. *For convenience write* $Y=X^{-1}$. *If* (i) $|\lambda_1|>|\lambda_2|> \cdots >|\lambda_n|>0$, (ii) *X and Y can be factorized into* $X=L_xR_x$, $Y=L_yR_y$, *then, as* $k\to\infty$, A_k *converges to right triangular form.*

Proof. The key is to express A_1^k, in a simple but ingenious way, as a product of a left and a right triangular matrix.

$$A_1^k = XD^kY = XD^kL_yR_y = X(D^kL_yD^{-k})(D^kR_y).$$

Let l_{ij} be the (i, j) element of L_y, then the (i, j) element of $D^kL_yD^{-k}$ is $l_{ij}(\lambda_i/\lambda_j)^k$ and, by (i), this tends to zero, as $k \to \infty$, for $i > j$. Also $l_{ii} = 1$. Thus

$$D^kL_yD^{-k} = I + E_k, \qquad E_k \to 0,$$

where the nonzero elements of E_k are below the diagonal. Thus

$$A_1^k = L_xR_x(I + E_k)D^kR_y = L_x(I + R_xE_xR_x^{-1})R_xD^kR_y = L_x(I + F_k)R_xD^kR_y,$$

where $F_k \to 0$. Now $R_xD^kR_y$ is right triangular. The triangular factors of $I + F_k$ must both tend to I and hence the left triangular factor of A_1^k, namely M_k, must converge to L_x. This ensures, as desired, the convergence of A_k to right triangular form.

The practical LR algorithm is not satisfactory when applied to general matrices for several reasons. The main defect is closely related to the limitations, described in Section 4, of Gaussian elimination. Very large numbers can occur in the L and R matrices given by $A = LR$ and such numbers wreck the practical algorithm. Moreover considerable calculation is required to obtain L, R, and RL.

The appreciation of the numerical dangers involved in the LR method has stimulated research into alternative factorizations of a matrix which will also generate, by multiplication of the factors in reverse order, a sequence of matrices converging to triangular form. One such development, the QR transformation of Francis, uses the decomposition of a matrix into a unitary matrix multiplied by a triangular matrix. The practical QR algorithm is numerically "safe" and, with the aid of ingenious tricks, has overcome almost all the limitations of the basic LR transformation to become one of the most successful general eigenvalue algorithms at the present time.

References

1. J. H. Wilkinson, The numerical eigenvalue problem, Oxford University Press, 1965.

2. B. N. Parlett, The development and use of methods of LR type, SIAM Review, 6 (1964) 275–295.

ITERATIVE METHODS FOR SOLVING MATRIX EQUATIONS

R. S. VARGA, Case Institute of Technology

1. Introduction. Iterative methods are, in concept, well known to us all since it is likely that each of us has at one time or another used Newton's method to find square roots of numbers. It is probably not as well known that iterative methods, utilizing the great speeds of modern computing machines, are extensively used today in practical computations for solving matrix equations which arise from finite difference approximations to elliptic partial differential equations of reactor technology and petroleum technology.

The object of this paper is to illustrate the use of finite difference techniques, and to illustrate the nature of iterative methods for solving the associated matrix equations. We shall do this by means of a very simple one-dimensional problem. It is hoped that this simple example will serve as an elementary introduction to the theory of such iterative methods which is covered in much more detail in [1, 2, 3, 5, 6].

2. A simple two-point boundary value problem. Consider the solution of the two-point boundary value problem:

$$(1) \qquad -y^{(2)}(x) + \sigma y(x) = f(x), \quad 0 < x < 1, \qquad y^{(2)} = d^2y/dx^2,$$

where

$$(2) \qquad y(0) = \alpha, \qquad y(1) = \beta.$$

We assume that α, β, and σ are given constants with $\sigma \geqq 0$, and $f(x)$ is a given function such that $y^{(4)}(x)$ exists in $0 \leqq x \leqq 1$, and

$$(3) \qquad |y^{(4)}(x)| \leqq M, \qquad 0 \leqq x \leqq 1.$$

By means of Taylor's Theorem, we now express $-y^{(2)}$ in terms of a three-point central difference approximation plus an error:

$$(4) \quad -y^{(2)}(x_i) = [2y(x_i) - (y(x_i + h) + y(x_i - h))]/h^2 + h^2 y^{(4)}(x_i + \theta_i h)/12,$$

where $|\theta_i| < 1$, $x_i \equiv ih$, $1 \leqq i \leqq N$, and $h \equiv 1/(N+1)$. With $y(x_i) \equiv y_i$, the differential equation (1) can be written for the particular values x_i, $1 \leqq i \leqq N$, in matrix form, as

$$(5) \qquad A\mathbf{y} = \mathbf{k} + \boldsymbol{\tau}(y),$$

where A is an $N \times N$ real matrix, and $\mathbf{y}$, $\mathbf{k}$, and $\boldsymbol{\tau}(y)$ are vectors, all given explicitly by

$$(6)\qquad A = \frac{1}{h^2}\begin{bmatrix} 2+\sigma h^2 & -1 & & \\ -1 & 2+\sigma h^2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2+\sigma h^2 \end{bmatrix}; \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_N \end{bmatrix}; \quad \mathbf{k} = \begin{bmatrix} f_1 + \alpha/h^2 \\ f_2 \\ \cdot \\ \cdot \\ \cdot \\ f_N + \beta/h^2 \end{bmatrix},$$

$$\boldsymbol{\tau} = \frac{-h^2}{12}\begin{bmatrix} y^{(4)}(x_1 + \theta_1 h) \\ \cdot \\ \cdot \\ \cdot \\ y^{(4)}(x_N + \theta_N h) \end{bmatrix}.$$

Neglecting the vector $\boldsymbol{\tau}(y)$ in (5) gives us the matrix equation

$$(7)\qquad A\mathbf{z} = \mathbf{k},$$

whose solution $\mathbf{z}$ is defined to be our discrete approximation to the solution $y(x)$ of (1)–(2), i.e., z_j, the jth component of $\mathbf{z}$, is to approximate $y_j = y(jh)$.

It is obvious that A is real and symmetric. Noting that all the diagonal entries of A are equal, we can express the matrix A as

$$(8)\qquad A = \frac{(2+\sigma h^2)}{h^2}[I - B],$$

where B is an $N \times N$ real symmetric matrix given explicitly by

$$B = \frac{1}{(2+\sigma h^2)}\begin{bmatrix} 0 & 1 & & 0 \\ 1 & 0 & 1 & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 0 \end{bmatrix}.$$

Because of the tridiagonal form of B, it is easy to verify that the vector $\mathbf{x}^{(j)}$, with k-th component $x_k^{(j)}$ defined by $x_k^{(j)} = \sin(jk\pi h)$, $1 \leqq k \leqq N$, is an eigenvector of B for *each* j, with the corresponding eigenvalue μ_j given by

$$(10)\qquad \mu_j = 2\cos(j\pi h)/(2+\sigma h^2), \qquad 1 \leqq j \leqq N.$$

Then, it follows that the spectral radius $\rho(B) \equiv \max_{1 \leq j \leq N} |\mu_j|$ is less than unity, and thus, from (8), A is evidently positive definite. Moreover, since $\rho(B) < 1$, we can express A^{-1} by means of the convergent matrix infinite series

$$(11)\qquad A^{-1} = \gamma(I - B)^{-1} = \gamma\{I + B + B^2 + \cdots\}, \quad \gamma \equiv h^2/(2+\sigma h^2).$$

Note that all entries of B are nonnegative from (9), so that all powers of B and hence A^{-1} have only nonnegative real entries. This last fact will be useful in proving in the next section that the solution $\mathbf{z}$ of (7) is "close" to the solution $y(x)$ of (1)–(2), evaluated at the points $x_i = ih$, $1 \leqq i \leqq N$.

3. Convergence of the discrete approximation. To measure the discrepancy between the components z_i of the vector $\mathbf{z}$ of (7) and the numbers $y_i = y(ih)$, we subtract (7) from (5) giving $A(\mathbf{y}-\mathbf{z}) = \boldsymbol{\tau}(y)$, or, since A is nonsingular,

$$\mathbf{y} - \mathbf{z} = A^{-1}\boldsymbol{\tau}(y). \tag{12}$$

Now, from (3), each component of $\boldsymbol{\tau}(y)$ is bounded above by $Mh^2/12$. Hence, since A^{-1} has only nonnegative components, $|y_i - z_i| \leqq (Mh^2/12)(A^{-1}\boldsymbol{\xi})_i$ for all $1 \leqq i \leqq N$, where $\boldsymbol{\xi}$ is the vector with all components unity. Thus, if $\mathbf{w}$ is any vector with $A\mathbf{w} \geqq \boldsymbol{\xi}$, we obtain the bound $|y_i - z_i| \leqq Mh^2 w_i/12$. But it is easy to verify that the choice $w(x) \equiv x(1-x)/2$ with $w_i = w(ih)$ satisfies $A\mathbf{w} \geqq \boldsymbol{\xi}$, so that

$$|y_i - z_i| \leqq Mh^2[ih(1 - ih)]/24 \leqq Mh^2/96, \tag{13}$$

and last inequality following since $w(x) \leqq 1/8$ for $0 \leqq x \leqq 1$. This means that $\mathbf{z}$ is close to $\mathbf{y}$ if the mesh spacing h is sufficiently small.

4. The Jacobi iterative method. For h small, the result of the previous section shows that solving $A\mathbf{z} = \mathbf{k}$ will give us a reasonable approximation to the solution of the original differential equation (1)–(2). To solve $A\mathbf{z} = \mathbf{k}$ for $\mathbf{z}$, let us combine (7) with (8) to form

$$\mathbf{z} = B\mathbf{z} + \mathbf{g}, \tag{14}$$

where $\mathbf{g} \equiv (h^2\mathbf{k})/(2+\sigma h^2)$. Familiarity with the method of successive substitutions suggests that we consider the following iterative method, called the Jacobi iterative method,

$$\mathbf{z}^{(m+1)} = B\mathbf{z}^{(m)} + \mathbf{g}, \qquad m \geqq 0, \tag{15}$$

where $\mathbf{z}^{(0)}$ is some initial estimate of the unique solution of (14). Writing $\mathbf{z}^{(m)} = \mathbf{z} + \boldsymbol{\varepsilon}^{(m)}$ to define the error $\boldsymbol{\varepsilon}^{(m)}$ at each iteration, we see from (14) and (15) that $\boldsymbol{\varepsilon}^{(m+1)} = B\boldsymbol{\varepsilon}^{(m)}$, from which we deduce that

$$\boldsymbol{\varepsilon}^{(m)} = B^m\boldsymbol{\varepsilon}^{(0)}, \qquad m \geqq 0. \tag{16}$$

Since B is a real symmetric $N \times N$ matrix, the eigenvectors $\{\mathbf{x}^{(j)}\}_{j+1}^{N}$ of B can be normalized to form an orthonormal basis for the associated N dimensional vector space. Thus, there exist constants c_j such that $\boldsymbol{\varepsilon}^{(0)} = \sum_{j=1}^{N} c_j\mathbf{x}^{(j)}$, where $B\mathbf{x}^{(j)} = \mu_j\mathbf{x}^{(j)}$, and it follows from (16) that

$$\boldsymbol{\epsilon}^{(m)} = \sum_{j=1}^{N} (\mu_j)^m c_j \mathbf{x}^{(j)}, \qquad m \geqq 0. \tag{17}$$

Because all μ_j are less than unity in modulus, $\boldsymbol{\varepsilon}^{(m)}$ evidently tends to the zero vector for *any* initial $\boldsymbol{\varepsilon}^{(0)}$. Equivalently, the Jacobi iterative method converges for *any* initial $\mathbf{z}^{(0)}$. What is more, from (10) each component of the error $\boldsymbol{\varepsilon}^{(m)}$ with respect to the orthonormal basis $\{\mathbf{x}^{(j)}\}_{j=1}^{N}$ is reduced per iteration by a factor not exceeding in modulus

$$\rho(B) = 2\cos \pi h/(2 + \sigma h^2) = 1 - (\pi^2 + \sigma)h^2/2 + O(h^4), \qquad h \downarrow 0. \tag{18}$$

Having just shown that the iterative method of (15) is convergent, it is our sad duty to report that this iterative method can be *extremely* slowly convergent for small h. To illustrate this, let $h=10^{-2}$ and $\sigma=1$. Then, to reduce each component of $\boldsymbol{\varepsilon}^{(0)}$ by 10 would require approximately m iterations, where $[\rho(B)]^m \approx 0.1$, and m turns out to be about 4200. Even for fast computing machines, this would be a slow process, and obviously there is a need for faster iterative methods.

5. The successive overrelaxation iterative method. To introduce just one such faster iterative method, we first express the matrix B of (9) as the sum $L+L^T$, where L is a strictly lower triangular matrix. Multiplying both sides of (14) by the real parameter ω, and then adding $\mathbf{z}$ to both sides of the result yields, after rearrangement,

$$(I - \omega L)\mathbf{z} = \{(1 - \omega)I + \omega L^T\}\mathbf{z} + \omega\mathbf{g}. \tag{19}$$

Since L is strictly lower triangular, then $(I-\omega L)$ is nonsingular for any choice of ω. Thus, we can multiply both sides of (19) on the left by $(I-\omega L)^{-1}$, which gives

$$\mathbf{z} = (I - \omega L)^{-1}[(1 - \omega)I + \omega L^T]\mathbf{z} + \omega(I - \omega L)^{-1}\mathbf{g}. \tag{19'}$$

Simply inserting iteration superscripts then serves to define the successive overrelaxation (SOR) iterative method:

$$\mathbf{z}^{(m+1)} = (I - \omega L)^{-1}[(1 - \omega)I + \omega L^T]\mathbf{z}^{(m)} + \omega(I - \omega L)^{-1}\mathbf{g}, \qquad m \geqq 0. \tag{20}$$

At first glance, this iterative method of (20) looks rather formidable, and would appear to be implicit. From the definition of B in (9), however, this iterative method can also be written equivalently as

$$z_i^{(m+1)} = z_i^{(m)} - \omega[(2 + \sigma h^2)z_i^{(m)} - z_{i-1}^{(m+1)} - z_{i+1}^{(m)} - h^2 f_i]/(2 + \sigma h^2), \qquad 1 \leqq i \leqq N. \tag{21}$$

Starting with $i=1$, $z_0=\alpha$ is fixed by the boundary condition (2), and we can solve for the new component $z_1^{(m+1)}$ in (21) in terms of the old components $z_1^{(m)}$ and $z_2^{(m)}$. Continuing from left to right, we see that this is an *explicit* iterative method.

In analogy with the Jacobi iterative method, we similarly seek the eigenvalues λ of the matrix

$$\mathcal{L}_\omega \equiv (I - \omega L)^{-1}\{(1 - \omega)I + \omega L^T\}. \tag{22}$$

Thus, we seek the roots of det $\{\lambda I-\mathcal{L}_\omega\}=0$. Since $\det(I-\omega L)=1$, it follows that

$$0 = \det(I - \omega L)\det(\lambda I - \mathcal{L}_\omega) = \det[(\lambda + \omega - 1)I - \omega(\lambda L + L^T)]. \tag{23}$$

Previously, we explicitly determined the eigenvectors $\mathbf{x}^{(j)}$ and eigenvalues μ_j of the matrix B of (9). In the same manner, one can verify that the vector $\mathbf{w}^{(j)}$, with kth component $w_k^{(j)}$ defined by $\tau^{k/2} \sin (jk\pi h)$, is an eigenvector of $\tau L + L^T$ for any τ, with corresponding eigenvalue $\tau^{1/2}\mu_j$. Consequently, with $\tau = \lambda$ there is a natural pairing from (23) of the eigenvalues λ of $\mathcal{L}_\omega$ with the eigenvalues μ of B through

$$(24) \qquad (\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2,$$

which is the fundamental result of Young [8]. The real significance of this expression (24) is that it can be used to explicitly determine the real parameter ω which *minimizes* the spectral radius of $\mathcal{L}_\omega$. Indeed, it is now well known [8] that

$$(25) \qquad \min_{\omega} \rho(\mathcal{L}_\omega) = \rho(\mathcal{L}_{\omega_b}) = \omega_b - 1, \quad \text{where} \quad \omega_b = 2/(1 + \sqrt{1 - \rho^2(B)}).$$

Using the result of (18), we see that

$$(26) \qquad \rho(\mathcal{L}_{\omega_b}) = 1 - 2(\sqrt{\pi^2 + \sigma})h + O(h^2), \qquad h \downarrow 0.$$

Comparison of the exponents of h in the leading terms of (18) and (26) shows that an *order of magnitude* improvement in convergence rate has been made, simply by the adroit choice of the parameter ω. To illustrate this numerically, again let $h = 10^{-2}$ and $\sigma = 1$. Then, the least positive integer m such that $[\rho(\mathcal{L}_{\omega_b})]^m \leqq 0.1$ now turns out to be approximately 35, as compared with 4200. For the slight increase in arithmetic requirements per iteration in passing from (15) to (21), a great over-all improvement in computational efficiency has been achieved.

6. Alternating direction implicit methods. After all is said and done, the solution of the discrete one-dimensional problem $A\mathbf{z} = \mathbf{k}$ in (7) would *not* be determined by means of iterative methods, except, of course, in expository papers on the subject! Rather, straight-forward Gaussian elimination applied to the matrix problem $A\mathbf{z} = \mathbf{k}$ is not only efficient, but it is very stable relative to the growth of rounding errors, thanks to the positive definite tridiagonal nature of A. The Gaussian elimination algorithm for solving this simple problem can be expressed as

$$(27) \qquad \begin{cases} w_1 = -1/b; & w_i = (-1/(b + w_{i-1}), \quad 2 \leqq i \leqq N-1, \quad b \equiv 2 + \sigma h^2, \\ g_1 = h^2 k_1/b; & g_i = (h^2 k_i + g_{i-1})/(b + w_{i-1}), \quad 2 \leqq i \leqq N, \\ z_N = g_N; & z_i = g_i - w_i z_{i+1}, \quad 1 \leqq i \leqq N-1. \end{cases}$$

Our simple one-dimensional problem can be useful once more to us in quickly introducing a particular variant of the alternating direction implicit (ADI) methods, for the discussion above shows that discrete approximations to the one-dimensional problem $-u_{xx} + \sigma u = f$ can be directly solved. Consider then

the second order elliptic partial differential equation

$$(28) \qquad -u_{xx}(x, y) - u_{yy}(x, y) + 2\sigma u(x, y) = f(x, y), \qquad 0 < x, y < 1,$$

in the unit square R with Dirichlet boundary conditions

$$(29) \qquad u(x, y) = g(x, y), \qquad (x, y) \in \partial R,$$

where g is specified on ∂R, the boundary of R. Writing (28) as

$$(30) \qquad [-u_{xx} + \sigma u] + [-u_{yy} + \sigma u] = f,$$

each term in brackets represents a differential operator in one of the space variables. Thus, with

$$(31) \qquad \begin{cases} Hu(x_0, y_0) \equiv [(2 + \sigma h^2)u(x_0, y_0) - (u(x_0 + h, y_0) + u(x_0 - h, y_0))]/h^2, \\ Vu(x_0, y_0) \equiv [(2 + \sigma h^2)u(x_0, y_0) - (u(x_0, y_0 + h) + u(x_0, y_0 - h))]/h^2, \end{cases}$$

representing discrete approximations to these differential operators on a uniform mesh $h = 1/(N+1)$, the discrete matrix problem corresponding to (28)–(29) is

$$(32) \qquad H\mathbf{u} + V\mathbf{u} = \mathbf{k},$$

where H and V are real $N^2 \times N^2$ matrices. It is not difficult to see that, with suitable numbering of the mesh points of the square, H and V are direct sums of the particular matrix A of (6). An important application of the discussion above concerning Gaussian elimination is that matrix problems of the form $H\mathbf{x} = \mathbf{s}$ and $V\mathbf{x} = \mathbf{t}$ can be solved *directly* for $\mathbf{x}$, which is to be a basic part of the iterative method to be described. Writing (32) as the pair of equations

$$(33) \qquad (H + rI)\mathbf{u} = (rI - V)\mathbf{u} + \mathbf{k}, \qquad (V + rI)\mathbf{u} = (rI - H)\mathbf{u} + \mathbf{k},$$

we insert iteration superscripts to define the Peaseman-Rachford alternating direction iterative method [4]

$$(34) \qquad \begin{aligned} (H + r_m I)\mathbf{u}^{(m+1/2)} &= (r_m I - V)\mathbf{u}^{(m)} + \mathbf{k}, \\ (V + r_m I)\mathbf{u}^{(m+1)} &= (r_m I - H)\mathbf{u}^{(m+1/2)} + \mathbf{k}, \qquad m \geqq 0, \end{aligned}$$

where $\{r_m\}$ is any sequence of positive acceleration parameters. Note that carrying out a single complete iteration of (34) requires the direct solution of matrix equations on horizontal mesh lines, then on vertical mesh lines, hence the name *alternating directions.*

For the particular problem (28)–(29), we shall now show that the iterative method (34) converges for *any* fixed positive acceleration parameter $r > 0$. In analogy with section 4, we first exhibit explicitly the eigenvectors of the $N^2 \times N^2$ matrices H and V. Defining the column vector $\boldsymbol{\alpha}^{(k,l)}$ with N^2 components $\alpha_{i,j}^{(k,l)}$ by

$$(35) \qquad \alpha_{i,j}^{(k,l)} = \sin(k\pi ih)\sin(l\pi jh), \qquad 1 \leqq i, j \leqq N,\ 1 \leqq k, l \leqq N,$$

where $\alpha_{i,j}^{(k,l)}$ refers to the component of $\boldsymbol{\alpha}^{(k,l)}$ at the ith column and jth row of the uniform mesh on the unit square, we see from (31) that

$$\begin{aligned} H\boldsymbol{\alpha}^{(k,l)} &= [4\sin^2(k\pi h/2) + \sigma h^2]\boldsymbol{\alpha}^{(k,l)}; \\ V\boldsymbol{\alpha}^{(k,l)} &= [4\sin^2(l\pi h/2) + \sigma h^2]\boldsymbol{\alpha}^{(k,l)} \qquad 1 \leqq k, l \leqq N. \end{aligned} \tag{36}$$

For any fixed $r>0$, the error vectors $\boldsymbol{\varepsilon}^{(m)}$ associated with (34) evidently satisfy, in analogy with (16),

$$\boldsymbol{\varepsilon}^{(m+1)} = T_r\boldsymbol{\varepsilon}^{(m)},\ m \geqq 0, \text{ where } T_r \equiv (V+rI)^{-1}(rI-H)(rI+H)^{-1}(rI-V). \tag{37}$$

Thus, from (36), we have that $T_r\boldsymbol{\alpha}^{(k,l)} = \lambda_{k,l}\boldsymbol{\alpha}^{(k,l)}$ where

$$\lambda_{k,l} = \frac{[r-(4\sin^2(k\pi h/2)+\sigma h^2)][r-(4\sin^2(l\pi h/2)+\sigma h^2)]}{[r+(4\sin^2(k\pi h/2)+\sigma h^2)][r+(4\sin^2(l\pi h/2)+\sigma h^2)]}, \qquad 1 \leqq k, l \leqq N,$$

and clearly, $|\lambda_{k,l}|<1$ for any $1\leqq k,\ l\leqq N$ since $r>0$. Because the $\boldsymbol{\alpha}^{(k,l)}$ form, after normalization, an orthonormal basis for the associated N^2 dimensional vector space, it follows from $|\lambda_{k,l}|<1$ that $\boldsymbol{\varepsilon}^{(m)}$ tends to the zero vector as $m\to\infty$ for *any* initial vector $\boldsymbol{\varepsilon}^{(0)}$, which proves that the iterative method of (34) converges for any fixed $r>0$. If we now choose the parameter r to be

$$\hat{r} = \{(4\sin^2(\pi h/2)+\sigma h^2)(4\cos^2(\pi h/2)+\sigma h^2)\}^{1/2},$$

it turns out [6, p. 216] that the spectral radius $\rho(T_{\hat{r}})$ is *exactly equal* to that of the successive overrelaxation iterative method applied to (28)–(29), so that from (26),

$$\rho(T_{\hat{r}}) = 1 - 2(\sqrt{\pi^2+\sigma})\,h + O(h^2), \qquad h \downarrow 0. \tag{38}$$

The really interesting results for such ADI methods are obtained by using a sequence of optimally selected positive parameters [7; 6, p. 223], since the use of such parameters can lead to a further order of magnitude improvement in convergence rates. Specifically, with such optimally selected parameters, we merely state that the average error reduction in norm per iteration turns out to be asymptotically

$$1 - \frac{(\pi^2+\sigma)}{\ln(1/h)}, \qquad h \downarrow 0, \tag{39}$$

which should be contrasted with (38). It must be mentioned, however, that the result of (39) depends heavily on the assumption that the matrices H and V have the common set of eigenvectors $\boldsymbol{\alpha}^{(k,l)}$, which is equivalent to the assumption that H and V commute, i.e., $HV=VH$. For cases in which $HV\neq VH$, the results concerning convergence rates of ADI methods are much less complete.

There is a rich and growing literature on the subject of iterative methods; our aim here was to introduce quickly some of these methods. More complete results, beyond the scope of this paper, can be found in [2, 3, 5, 6].

References

1. D. K. Faddeev and V. N. Faddeeva, Computational methods of linear algebra, translated by R. C. Williams, Freeman, San Francisco, 1963.
2. G. E. Forsythe and W. R. Wasow, Finite difference methods for partial differential equations, Wiley, New York, 1960.
3. A. S. Householder, The theory of matrices in numerical analysis, Blaisdell, New York, 1964.
4. D. W. Peaceman and H. H. Rachford, Jr., The numerical solution of parabolic and elliptic differential equations, J. Soc. Indust. Appl. Math., 3 (1955) 28–41.
5. John Todd, (editor) Survey of numerical analysis, McGraw-Hill, New York, 1962.
6. R. S. Varga, Matrix iterative analysis, Prentice-Hall, Englewood Cliffs, N. J., 1962.
7. E. L. Wachspress, Optimum alternating-direction-implicit iteration parameters for a model problem, J. Soc. Indust. Appl. Math., 10 (1962) 339–350.
8. D. M. Young, Iterative methods for solving partial difference equations of elliptic type, Trans. Amer. Math. Soc., 76 (1954) 92–111.

NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

P. D. LAX, New York University

No one will quarrel with the statement that computing is bound to revolutionize applied mathematics by providing new and powerful tools for getting numerical answers, the life and blood of the subject. The versatility of the new computers has already stimulated the systematic study of numerical methods; this has attracted a number of pure mathematicians who, before, had been repelled by the *ad hoc* nature of the methods used by the hapless applied mathematician.

This brief article is about the numerical solution of partial differential equations. A very general (although by no means the only) method for solving these is to convert them into difference equations through the replacement of derivatives by difference quotients of some sort. The effectiveness of this method is attested to by the vast literature on it which has sprung up during the last 15 years—a literature so extensive that it is hopeless to try to summarize it. Instead of trying to present some of the highlights of the new developments I shall try to convey its flavor. My aim is to convince a skeptical reader who may regard using finite differences as the last resort of a scoundrel that the theory of difference equations is a rather sophisticated affair, more sophisticated than the corresponding theory of partial differential equations. My argument will be based on two contentions:

1) In order to prove that solutions of a sequence of difference equations converge one needs estimates for difference operators which are analogous to the estimates needed in the existence and uniqueness theory for solutions of differential equations.

2) Estimates for difference operators are much harder to derive than the corresponding estimates for differential operators.

These contentions will be illustrated on a very simple linear equation; the reader interested in the general theory should consult R. D. Richtmyer's book (and the research papers listed at the end, most of which have been incorporated into the second edition of his book).

I regret that there is no room to describe difference approximations to nonlinear equations; this part of the theory has many surprises and many unsolved problems. An intriguing approach to a certain class of such problems is discussed by Frank Harlow in an article appearing in this issue.

1. In problems involving partial differential equations we want to determine a function or set of functions u which satisfies a partial differential equation

$$Lu = f \tag{1}$$

in some domain D and which satisfies certain conditions on a subset of D. These additional conditions, usually called boundary (or initial) values of u, together with the given function f constitute the so-called *data* of u.

According to the modern theory of linear partial differential equations, the data determine a unique solution u if and only if u can be estimated in terms of its data, i.e. an inequality of the form

$$\|u\| \leqq \text{constant } \|Lu\| \tag{2}$$

holds, and a similar inequality holds for the so-called adjoint problem. (For simplicity we have assumed that the boundary data of u are zero; the symbol $\|\ \|$ denotes some norm which measures magnitude.) For linear operators L it follows from (2) that

$$\|u_1 - u_2\| \leqq \text{constant } \|Lu_1 - Lu_2\|.$$

This clearly implies that u is uniquely determined by $Lu=f$; in fact it says substantially more; it says that u is a *continuous* function of f in the sense of the norm employed. For problems of mathematical physics nothing short of this continuous dependence of solutions on the data will suffice since, as Hadamard pointed out, the data here come from observations which are never exact; thus observations (i.e. approximate data) must suffice to determine approximate solutions. For this reason an operator which satisfies inequality (2) is sometimes called *stable*.

We turn now to difference approximations. The differential operator L is replaced by a difference operator, denoted by L_Δ, where Δ stands for the width of the mesh to be employed in the difference scheme. Denote by u_Δ the solution of

$$L_\Delta u_\Delta = f. \tag{1_Δ}$$

Since L_Δ approximates L, $L_\Delta u = f_\Delta$ approximates $Lu=f$, that is $\|f - f_\Delta\|$ is small

for small Δ. Therefore, if we knew that L_Δ satisfies an inequality analogous to (2):

$$(2_\Delta) \qquad \|v\| \leqq \text{constant } \|L_\Delta v\|$$

with a constant that does not depend on the mesh width Δ, then by applying (2_Δ) to $v=u-u_\Delta$, we would obtain

$$\|u - u_\Delta\| \leqq \text{constant } \|f - f_\Delta\|$$

or, in other words, that u_Δ is a good approximation of u for small Δ.

The converse proposition also holds: if for each f with finite norm the solution u_Δ of (1_Δ) tends to the solution u of (1) then according to the principle of uniform boundedness (2_Δ) holds with a constant independent of Δ.

A difference scheme for which inequalities (2_Δ) are satisfied uniformly is called *stable*; we can summarize the result obtained above by saying that a difference scheme is *convergent* for all data f if and only if it is *stable.* This bears out the first contention made in the introduction—that in order to prove the convergence of a difference scheme one has to derive the same kind of inequalities for the difference operators as those needed for the differential operator in order to prove the existence and uniqueness of solutions of the differential equation.

The second contention made in the introduction is that the inequalities for the difference operators are harder to prove than the corresponding inequality for the differential operator. We shall illustrate this for the operator

$$(3) \qquad L = \partial_t + a\partial_x$$

acting on a scalar function u of two variables x and t; the coefficient a is a given function of x and t. The domain in question is some slab $0 \leqq t \leqq T$ and the prescribed initial data are the values of u at $t=0$,

$$(4) \qquad u(x, 0) = g(x).$$

The differential equation

$$(5) \qquad Lu = u_t + au_x = f$$

states that the directional derivative of u in the direction

$$(6) \qquad \frac{dx}{dt} = a$$

is equal to f. Given the initial values (4) and the function f, our task is to determine u uniquely. To find u, we have to solve the ordinary differential equation (6) and then integrate f along the trajectories of (6). But even without performing any integrations, we can give *a priori* estimates of u in terms of its data, f and g. For simplicity we take the case when f is zero; then the differential equation (5) asserts that u is constant along the trajectories of (6). Since any point in the upper half plane can be connected to some point of the initial line

by a trajectory, it follows that the range of values of u for $t \geqq 0$ is the same as the range of values of u at $t=0$. So

$$|u|_{\max} = |g|_{\max}. \tag{7}$$

We shall also derive an *a priori* estimate for u in the L_2 norm. We multiply (5) by u and integrate, getting

$$\int uu_t dx + \int auu_x dx = 0. \tag{8}$$

Now introduce the quantity E, sometimes called *energy*, as

$$E(t) = \frac{1}{2} \int u^2 dx;$$

then the first term in (8) can be written as E_t. In the second term we write uu_x as $\frac{1}{2}(u^2)_x$ and integrate by parts; assuming that u tends to zero as $|x|$ becomes large we can rewrite (8) as

$$E_t - \frac{1}{2} \int a_x u^2 dx = 0. \tag{9}$$

Denote by M the supremum of a_x:

$$a_x \leqq M. \tag{10}$$

It follows from (9) that $E_t - ME \leqq 0$; multiplying this by e^{-Mt} we obtain $(e^{-Mt}E)_t \leqq 0$. This means that $e^{-Mt}E$ is a nonincreasing function of t; so

$$E(t) \leqq e^{Mt}E(0); \tag{11}$$

this is called an energy inequality.

We shall set up and study three different difference analogues of the differential equation (5). In the first scheme we replace u_t by a forward difference quotient and u_x by a centered difference quotient:

$$\begin{aligned} u_t &\cong [u(x, t + \Delta t) - u(x, t)]/\Delta t, \\ u_x &\cong [u(x + \Delta x, t) - u(x - \Delta x, t)]/2\Delta x. \end{aligned} \tag{12}$$

We introduce the abbreviations $u(k\Delta x, t) = u_k$, $u(k\Delta x, t+\Delta t) = v_k$ and

$$a \frac{\Delta t}{\Delta x} = b. \tag{13}$$

Using the approximations (12) in (5) where we assume that $f=0$, we obtain the difference equation

$$v_k = \frac{b}{2} u_{k-1} + u_k - \frac{b}{2} u_{k+1}. \tag{I}$$

The second method is very much like the first one except that in the forward difference formula for u_t we replace $u(x, t)$ by the average quantity $[u(x+\Delta x, t)+u(x-\Delta x, t)]/2$. The resulting difference equation can be written as

$$v_k = \frac{1+b}{2} u_{k-1} + \frac{1-b}{2} u_{k+1}. \tag{II}$$

The third method is a little more complicated; we start with the truncated Taylor series

$$u(x, t+\Delta t) \cong u + \Delta t u_t + \frac{(\Delta t)^2}{2} u_{tt}. \tag{14}$$

Differentiating (5) with respect to x and t gives

$$u_{tt} = -au_{xt} - a_t u_x, \qquad u_{xt} = -au_{xx} - a_x u_x.$$

By substituting the expression for u_{xt} into that for u_{tt}, we deduce that for all solutions of (5) $u_{tt}=a^2u_{xx}-\dot{a}u_x$, where $\dot{a}=a_t-aa_x$. Using this and (5) we can express the t derivatives in (14) by x derivatives; we replace these by difference quotients, u_x as in (12) and u_{xx} by

$$u_{xx} \cong \frac{u_{k+1} - 2u_k + u_{k-1}}{(\Delta x)^2}.$$

The resulting difference equation is

$$v_k = \frac{b^2+b}{2} u_{k-1} + (1-b^2)u_k + \frac{b^2-b}{2} u_{k+1}. \tag{III}$$

All three difference equations can be written in the form

$$v = u(t+\Delta t) = S_\Delta u(t), \tag{15}$$

where S_Δ is a spatial difference operator

$$S_\Delta u = v_k = \sum c_j u_{k-j}. \tag{16}$$

If we assume for simplicity that the coefficients of the operator S_Δ depend on x but not on t then the solution of the initial value problem for (15) can be written as

$$u_\Delta(t) = S_\Delta^n u(0), \qquad n = t/\Delta t. \tag{17}$$

According to estimates (7) and (11) the solutions $u(t)$ of (5) at time t depend boundedly on the initial values in the sense of the maximum norm and the L_2 norm. We called a difference scheme stable in one of these norms if $u_\Delta(t)$ is uniformly bounded in its dependence on the initial data. In view of the operator expression (17) for $u_\Delta(t)$ we can express this as follows:

The difference scheme (15) *is stable if and only if the norms of the operators*

$$S_\Delta^n \tag{18}$$

are uniformly bounded in the range $n\Delta t \leqq T$.

We shall regard the solutions of the difference equation (15) as being defined on the discrete set of points $x = k\Delta x$, with k an integer, and shall use the discrete maximum and L_2 norms

$$|u|_{\max} = \operatorname{Sup} |u_k|, \qquad \|u\|^2 = \sum |u_k|^2. \tag{19}$$

THEOREM: (A) *The difference scheme* (I) *is stable if and only if*

$$\frac{\Delta t}{(\Delta x)^2} \leqq \text{const.} \tag{20}$$

is satisfied.

(B) *The difference scheme* (II) *is stable with respect to both the maximum and* L_2 *norms if*

$$\left(a\,\frac{\Delta t}{\Delta x}\right) \leqq 1 \tag{21}$$

and is unstable if $(a(\Delta t/\Delta x)) > 1+\delta$, $\delta > 0$.

(C) *The difference scheme* (III) *is stable or unstable in the* L_2 *norm under the same conditions as* (II).

In order to evaluate the efficiency of a given numerical method we have to know the number of operations which have to be carried out. For a scheme of type (15), (16) the operations are the calculation of the coefficients c_j and the indicated multiplications in (16); these have to be carried out N times, N being the number of lattice points used in the calculation. N is proportional to $(\Delta x \Delta t)^{-1}$.

Condition (20) forces us to choose a much smaller value for Δt than condition (21), and therefore the evaluation of the approximation (17) by method (I) needs many more iterations of the operator S_Δ than by methods (II) or (III). This makes (I) quite impractical and in fact (I) is never used; of the other two, (III) is much more accurate than (II) and only moderately more cumbersome. Therefore (III) would be used in preference to (II) except when only a very crude answer is required.

We turn now to part A of the theorem; we shall show using a method due to von Neumann that if condition (20) is violated then scheme (I) is unstable in the maximum norm by exhibiting a sequence of initial data g_Δ such that $|g_\Delta|_{\max}$ remains bounded but $|S_\Delta^n g_\Delta|_{\max}$, $\Delta t n = 1$, tends to infinity. Clearly it pays to

choose g_Δ as an element for which $S_\Delta^n g_\Delta$ is easy to evaluate; such an element is an eigenvector of S_Δ. We consider now the special case when a is constant; then the operator S_Δ commutes with translation and so its eigenvectors are exponentials. Taking $g_\Delta(k) = e^{i\xi k}$ we have by (I)

$$(S_\Delta g_\Delta)(k) = \frac{b}{2} e^{i\xi(k-1)} + e^{i\xi k} - \frac{b}{2} e^{i\xi(k+1)} = \lambda g_\Delta(k),$$

where $\lambda = 1 - ib \sin \xi$. Then

$$S_\Delta^n g_\Delta = \lambda^n g_\Delta;$$

clearly $|g_\Delta|_{\max} = 1$ and $|S_\Delta^n g_\Delta|_{\max} = |\lambda|^n$. The largest value of $|\lambda|^2 = 1 + b^2 \sin^2 \xi$ occurs when $\xi = \pi/2$. For this value of ξ

$$|\lambda|^n = (1 + b^2)^{n/2} \cong e^{(b^2 n)/2}.$$

Using the definition (13) of b and $n = 1/\Delta t$ we can write

$$\frac{b^2 n}{2} = \frac{a^2 \Delta t}{2(\Delta x)^2};$$

clearly if (20) is violated, $b^2 n$ tends to infinity and so does $|S_\Delta^n g_\Delta|_{\max}$. This proves the negative portion of part A of the theorem; since this discredits scheme (I) we don't bother to prove the positive side but go on to part (B). We shall use the following stability criterion:

If there exists a constant M such that for all Δ

$$\|S_\Delta\| \leqq 1 + M\Delta t \tag{22}$$

then the scheme is stable.

The validity of this criterion follows from the chain of inequalities

$$\|S_\Delta^n\| \leqq \|S_\Delta\|^n \leqq (1 + M\Delta t)^n \leqq e^{Mn\Delta t} = e^{Mt}.$$

We turn to scheme (II); here v_k is a linear combination of u_{k-1} and u_{k+1} with weights $(1 \pm b)/2$; the sum of the weights is one and if condition (21) is satisfied *the weights are nonnegative.* So it follows that

$$\operatorname{Max} |v_k| \leqq \operatorname{Max} |u_k|, \tag{23}$$

which is inequality (22) in the maximum norm with $M = 1$.

Next we prove the stability of (II) in the L_2 norm; multiplying (II) by v_k we get

$$v_k^2 = \frac{1+b}{2} u_{k-1} v_k + \frac{1-b}{2} u_{k+1} v_k.$$

Using the inequality

$$uv \leqq \tfrac{1}{2}(u^2 + v^2) \tag{24}$$

twice on the right we obtain

$$v_k^2 \leqq \frac{1+b}{4}(u_{k-1}^2 + v_k^2) + \frac{1-b}{4}(u_{k+1}^2 + v_k^2),$$

which is equivalent to

$$v_k^2 \leqq \frac{1+b_k}{2}u_{k-1}^2 + \frac{1-b_k}{2}u_{k+1}^2,$$

where b_k stands for $a(k\Delta x)(\Delta t/\Delta x)$. Sum with respect to k; re-indexing on the right we get

$$\sum v_k^2 \leqq \sum\left(1 + \frac{b_{k+1} - b_{k-1}}{2}\right)u_k^2. \tag{25}$$

Recalling the definition of M as sup a_x we see that $(b_{k+1} - b_{k-1}) \leqq 2M\Delta t$; using this in (25) we get

$$\|v\|^2 = \sum v_k^2 \leqq (1 + M\Delta t)\sum u_k^2 = (1 + M\Delta t)\|u\|^2. \tag{26}$$

This is inequality (22) in the L_2 norm.

The estimates (23) and (26) derived here are clear-cut analogues of the estimates (7) and (11) for solutions of the differential equation. Similar estimates can be derived by the same arguments for all difference operators of the form (16) where the weights c_j are nonnegative, add up to one and vary Lipschitz continuously with x. Friedrichs has observed that the L_2 estimates can be derived also in the much more interesting case when u is a vector-valued function and the coefficients c_j in (16) are matrices which are positive in the sense of quadratic forms. The only change necessary in the reasoning is to replace inequality (24) by the Schwarz inequality $u \cdot cv \leqq \frac{1}{2}(u \cdot cu + v \cdot cv)$.

The necessity of condition (21) for convergence has been demonstrated by Courant, Friedrichs and Lewy in their classical paper in the following elegant fashion: The difference scheme (II) expresses the value of u at $(x, t+\Delta t)$ in terms of its values at $(x \pm \Delta x, t)$. When we express $u_\Delta(x, t)$ in terms of the initial values, therefore, we need the initial values only at points inside the interval $(x - (\Delta x/\Delta t)t,\ x + (\Delta x/\Delta t)t)$. On the other hand the value $u(x, t)$ of the exact solution equals the value of the initial function at that point y which can be connected to (x, t) by a trajectory of (6). If $\Delta x/\Delta t < a - \delta$ for arbitrarily small Δt then that point y will fall outside the above interval at least for some x and t. In this case the value of $u_\Delta(x, t)$, and therefore also its limit, does not depend on the value of the initial function at y and therefore cannot in general be equal

to $u(x, t)$. This shows the divergence of scheme (II) and also of (III) if (21) is violated.

Lastly we prove the stability of III in the L_2 norm; we treat first the case when a is constant. We introduce the Fourier transform $\tilde{u}$ of $\{u_k\}$ as

$$\tilde{u}(\xi) = \sum u_k e^{ik\xi}.$$

For constant b the operator S_Δ defined by III is a convolution; a brief calculation shows that $\tilde{v}(\xi) = \sum v_k e^{ik\xi}$ is given by

$$\tilde{v}(\xi) = \lambda(\xi)\tilde{u}(\xi), \tag{27}$$

where $\lambda(\xi) = 1 - b^2 + b^2 \cos \xi - ib \sin \xi$. Another brief ($\sim$15 minute) calculation gives

$$|\lambda(\xi)|^2 = 1 - (\cos \xi - 1)^2(b^2 - b^4); \tag{28}$$

this formula shows that if condition (21) is satisfied then $|\lambda(\xi)| \leqq 1$; therefore it follows from (27) that then $|\tilde{u}(\xi)| \leqq |\tilde{v}(\xi)|$ and *a fortiori*

$$\int |\tilde{v}(\xi)|^2 d\xi \leqq \int |u(\xi)|^2 d\xi. \tag{29}$$

According to the Parseval identity

$$||u||^2 = \sum |u_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{u}(\xi)|^2 d\xi,$$

$$||v||^2 = \sum |v_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\tilde{v}(\xi)|^2 d\xi.$$

From these and (29) it follows that $||v||^2 \leqq ||u||^2$, which proves the stability of scheme (III) when b is constant.

For variable coefficients we prove stability as follows: squaring (III) gives

$$v_k^2 = \left(\frac{b^2 + b}{2} u_{k-1} + (1 - b^2)u_k + \frac{b^2 - b}{2} u_{k+1}\right)^2.$$

We add to the right side the quantity

$$\left(\frac{b^2 - b^4}{4}\right)(u_{k-1} - 2u_k + u_{k+1})^2. \tag{30}$$

If condition (21) is satisfied (30) is positive, so we get an inequality which, after some algebraic manipulation, can be written in the form

$$v_k^2 \leqq \frac{b^3 + b^2}{2} u_{k-1}^2 + (1 - b^3)u_k^2 + \frac{b^2 - b^3}{2} u_{k+1}^2 + (b^3 - b)(u_{k+1}u_k - u_k u_{k-1}). \tag{31}$$

The right side is a quadratic form in u_{k-1}, u_k, u_{k+1}; the matrix of this quadratic form is

$$\begin{bmatrix} \dfrac{b^3+b^2}{2} & \dfrac{b-b^3}{2} & 0 \\ \dfrac{b-b^3}{2} & 1-b^2 & \dfrac{b^3-b}{2} \\ 0 & \dfrac{b^3-b}{2} & \dfrac{b^2-b^3}{2} \end{bmatrix}.$$

This matrix has the following properties which are crucial for deriving the L_2 inequality:

The sum of the elements along the diagonal equals one.

The sum of the elements along a subdiagonal is zero.

We sum (31) over all k and re-index the right-side, obtaining

$$\begin{aligned} \sum v_k^2 &\leqq \sum \tfrac{1}{2}[b_{k+1}^3 + b_{k+1}^2 + 2(1-b_k^2) + b_{k-1}^2 - b_{k-1}^3]u_k^2 \\ &\quad + \sum [b_k^3 - b_k - b_{k+1}^3 + b_{k+1}]u_{k+1}u_k. \end{aligned} \tag{32}$$

It follows from the properties of the matrix listed above that if b were independent of k then the right side of (32) would be just $\sum u_k^2$. If b is variable then, denoting $\sup |a_x|$ by M_1 we have, just as before, the inequality $|b_{k+1}-b_k| \leqq M_1\Delta t$. Using this we deduce that the right side of (31) is less than

$$(1 + \text{const. } M_1\Delta t) \sum u_k^2;$$

this completes the proof of the stability of (III).

The reader may justly ask how the quantity (30) was picked; we give away the trade secret by pointing out that the crucial property of the resulting quadratic form follows by Fourier transformation of the trigonometric identity (28). Identity (28) itself may be regarded as a special instance of the Fejér-Riesz representation for nonnegative trigonometric polynomials; all this is explained fully in [6].

What about the stability of scheme (III) in the maximum norm? Recently Thomée has shown, by a subtle analysis, that (III) is unstable in the maximum norm. The instability is very mild and doesn't interfere with the effectiveness of (III) except possibly for very wild initial data.

We hope that the foregoing bears out our second contention.

References

1. R. Courant, K. O. Friedrichs and H. Lewy, Über die partiellen Differenzialgleichungen der mathematischen Physik, Math. Ann., 100 (1928) 32–74.

2. F. John, On the integration of parabolic equations by difference methods, Comm. Pure Appl. Math., 5 (1952) 155–211.

3. H. O. Kreiss, Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differential-gleichungen approximieren, BIT, 2 (1962) 153–181.

4. ———, On difference approximations of the dissipative type for hyperbolic differential equations, Comm. Pure Appl. Math., 17 (1964) 335–353.

5. P. D. Lax, Differential equations, difference equations and matrix theory, Comm. Pure Appl. Math., 11 (1958) 175–195.

6. ———, The scope of the energy method, Bull. Amer. Math. Soc., 66 (1960) 32–35.

7. ———, On the stability of difference schemes, Comm. Pure Appl. Math., 15 (1962) 363–371.

8. P. D. Lax and B. Wendroff, Difference schemes for hyperbolic equations with high order of accuracy, Comm. Pure Appl. Math., 17 (1964) 381–398.

9. M. Lees, Von Neumann difference approximation to hyperbolic equations, Pacific J. Math., 10 (1960) 213–222.

10. R. D. Richtmyer, Difference methods for initial value problems, Interscience Tracts in Pure and Appl. Math., Vol. 4, Interscience, New York, 1957.

11. W. G. Strang, Trigonometric polynomials and difference methods of maximum accuracy, J. Math. and Phys., 16 (1962) 147–154.

12. ———, Polynomial approximations of Bernstein type, Trans. Amer. Math. Soc., 105 (1962) 525–535.

NUMERICAL FLUID DYNAMICS

F. H. HARLOW, University of California, Los Alamos

1. Computer representation of fluid dynamics. It has been suggested that the numerical solution of a fluid dynamics problem is a good example of "experimental arithmetic." We shall see that this is true in several respects, but first it is necessary to consider how the computer can represent a fluid, and to see how both the instantaneous representation and the calculation of changes involve approximations.

For each instant in the running of a calculation, the computer must contain a complete account of everything required to describe the fluid configuration, together with all data necessary for calculating the changes which currently are occurring. Certainly the computer cannot remember the necessary information for *every* point within the fluid; infinite storage capability would be required. This, then, introduces the first approximation: The computer memory can have stored in it the values of the field variables (such as velocity, temperature, and density) at only a *finite* number of points. The greater the number, the more detailed will be the resolution and the more accurate the results.

It is convenient to imagine each of these points within the fluid to be the center of a tiny cell. Then the value of each field variable at the center position

represents an average of that quantity over the volume of the cell. The cell concept is also useful for calculating the changes with time of the field variables. The changes are described by a set of coupled nonlinear partial differential equations, and the spatial derivatives in these must be represented for computational purposes by finite difference approximations. As an example, the pressure gradient in the x direction, $\partial p/\partial x$, is approximated by $(p_{j+1}-p_j)/\delta x$, where the index j labels cell number, and δx is the length of a cell side. With this and similar prescriptions for the other space derivatives, it is possible to calculate the instantaneous rates of change of the dependent variables by simple algebraic substitution of values of the appropriate cell-wise quantities.

To proceed with the calculation, another approximation is required: The advancement of time in the problem must proceed through a set of finite time increments. The configuration at the end of a time cycle is determined from that at the beginning in a fashion illustrated by the velocity:

$$u_j(t + \delta t) = u_j(t) + A_j(t)\delta t.$$

δt is the time increment per cycle and $A_j(t)$ is the acceleration calculated using finite-difference gradients of quantities available from the previous cycle, or from the initial conditions of the problem. The results resemble the frames of a motion picture; if the time cycles are close enough together, the output will appear nearly continuous.

2. Some properties of fluid flows. It is convenient to classify fluid flows into the loosely defined categories, incompressible and compressible. More precisely, these terms actually refer to the *speed* of the fluid; in the first category the material speed is small compared with that of sound, while the second includes all faster processes. The properties of the two flow types are so different that they may be considered as separate subjects for investigation. We shall mention a few of the peculiar features of each which are closely related to the numerical techniques used for computer solutions.

Under many circumstances, the changes in a compressible fluid take place by one of two processes, the compression and the rarefaction. Since, by definition, the fluid speeds are at least comparable to that of sound, much can happen in one part of a fluid with no effect being felt elsewhere. Only through a rarefaction or compression wave will other regions receive communication of the disturbance.

The extreme manifestation of a rarefying process is the production of a cavity. Thus, for example, if a piston is withdrawn from a cylinder of gas more rapidly than the "escape speed," the gas is unable to follow. The front surface of the gas moves toward the piston at escape speed, while an ever widening rarefaction wave eats its way into the gas down the cylinder. For air under ordinary conditions, the escape speed is five times greater than the speed of a

sound signal, so that only under the most unusual circumstances would cavitation occur.

The extreme manifestation of a compressing process is the production of a shock. If the piston of the previous example were pushed rapidly *into* the gas, the gas would be compressed. A continued acceleration of the piston would produce a succession of compressions, each catching up with the front of the first one. It is this "piling up" of compressions that forms the shock.

Shocks are common phenomena, being produced by explosions, bullets, and various other rapidly moving sources of compression. Under ordinary circumstances, the entire transition from undisturbed gas ahead of the shock to completely compressed and heated gas behind takes place within a tiny fraction of a millimeter. This remarkable thinness is kept from being vanishingly small only by the molecular properties of the gas. For many theoretical purposes, the shock can be treated as a true discontinuity; in numerical calculations, however, special techniques are required. The most common uses an artificial viscosity, *increasing* the shock thickness to the width of several computational cells.

While true viscous effects are often negligible in compressible flows, they usually are of considerable importance when the flow is incompressible. Viscosity is responsible for generating stress in regions of shear, meaning, for example, that whenever a wall moves in its own plane, the nearby fluid will be pulled along, too. Viscosity thus serves as a somewhat different type of mechanism for propagating disturbances away from a region of motion.

There is another mechanism for the propagation of disturbances in incompressible fluids which may be called "action at a distance." If, for example, the piston is moved slowly into a cylinder of water, motion commences "instantaneously" at all points within the fluid because of the incompressibility. (Actually, of course, motion at any point is initiated by the arrival of a compression wave. Since this travels at least with sound speed, the effect is essentially instantaneous when the piston speed is very small. In many cases, it is a very accurate approximation to consider the sound speed to be *actually* infinite.)

3. Instability, real and computational. One property shared by both incompressible and compressible flows is the tendency to *instability*. An example of common occurrence is the instability of a rising filament of smoke. The turbulent wake behind a boat or a bullet is another manifestation of fluid dynamic instability. Unfortunately, *computational* instability is a common fault that must be overcome in the development of computer methods for solving problems in fluid dynamics. As knowledge gradually increases on how to cure the numerical instabilities, so also are meaningful calculations being applied to a wider and wider scope of interesting studies.

Analysis can be brought to bear upon the question of computational instability, and many useful results have been derived as guidance for avoiding the difficulty. In most cases, the stability criterion is, in effect, a restriction on the size of δt, the time increment per cycle: δt must be so small that even the fastest signal does not travel more than a cell width in any one cycle. For com-

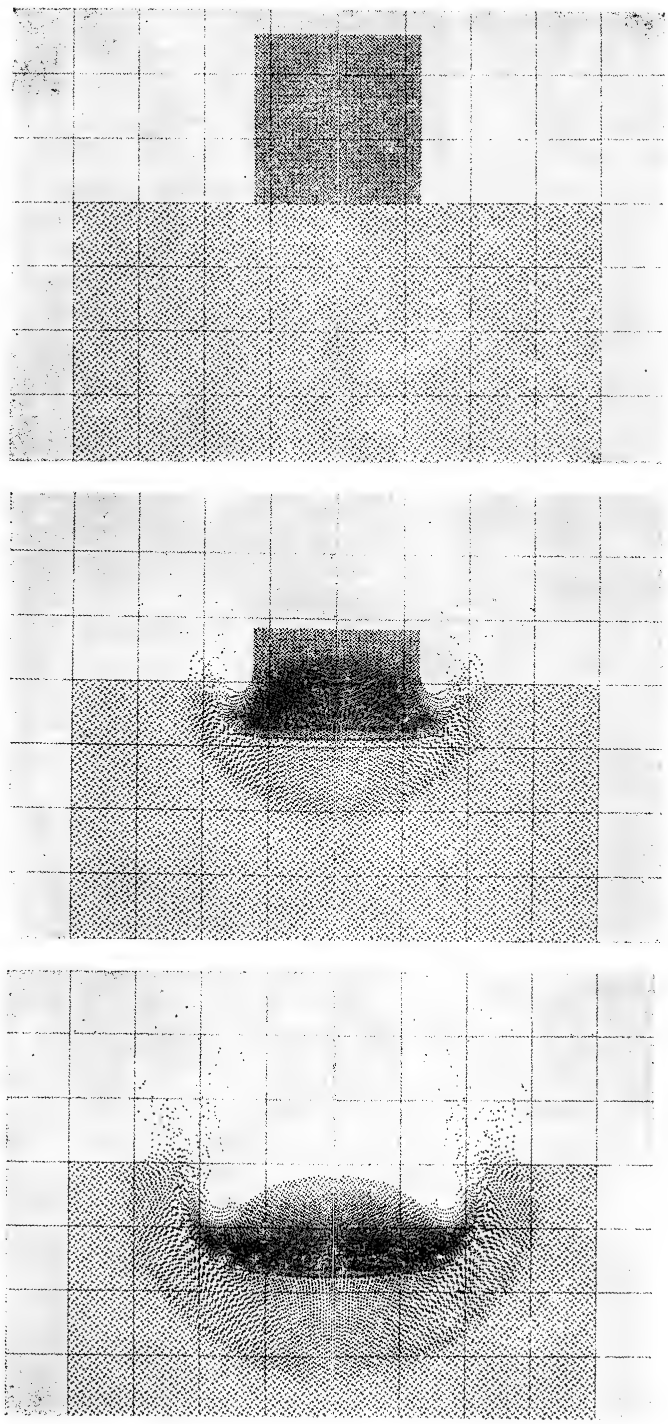


FIG. 1

pressible flows, the fastest restrictive signal usually is the sound speed. For incompressible flows, however, some way must be found to eliminate the sound signal restriction, since sound speeds are essentially infinite. It turns out that there are iterative techniques whereby the solution for each cycle can be made to converge sufficiently close to the desired result, even though sound signals have moved over many cells. For such incompressible flow calculations the crucial restriction then prohibits the propagation of *shear* disturbances (through viscous drag) by more than a cell per cycle.

Such stability requirements are *necessary* for accurate computer solutions, but are by no means *sufficient.* Once a new technique can calculate without instability, there generally are many variations which must be tried before sufficient accuracy can be achieved.

4. The analogy with experiments. Now it can be seen how the computer studies of fluid dynamics are experimental arithmetic in at least two respects. First, as described in the previous section, the process by which a new computing technique is developed requires a large amount of true experimentation. Numerous variations must be tried until the answers are sufficiently accurate for test examples with known solutions.

It also can be seen that in many respects, the computer calculation resembles a laboratory experiment. Just as in the wind tunnel laboratory, the computer operator prepares his equipment (the code of instructions by which the stepwise solution proceeds) and sets up his initial and boundary conditions. He pushes the "start button" and the results come forth. With both the computer and the wind tunnel there are uncertainties; the wind tunnel technician has limits to the precision of his measurements, and these are closely analogous to the computer accuracy limitations imposed by the coarseness of the mesh of computing cells. Carrying the analogy even further, we may note the resemblance between a computational instability in which the calculation "blows up," and a fan motor instability in which the laboratory equipment may *literally* blow up. In both cases the outflow of useful results promptly ends.

5. Some examples. The most picturesque applications of computers to fluid dynamics have been for the solution of problems in several space dimensions involving large distortions of the fluids. Three examples are presented here, illustrating both compressible and incompressible flows. In every case, the figures have been obtained directly from the computer through the Stromberg-Carlson SC-4020 Microfilm Recorder, and have not been retouched in any way.

In Fig. 1 there are three selected frames from the calculation of a hypervelocity impact problem. ("Hypervelocity" implies a projectile speed much greater than that of sound.) The initial configuration shows a cylindrical projectile (the darker material) just at the instant of striking the target.

Note that the fluid elements are represented by point particles. The actual

mesh of finite difference cells is not shown; sixty-four were present in each grid square. The computer technique is called the "Particle-in-Cell" method [1].

This hypervelocity impact calculation is a good example of a compressible fluid flow problem. The second frame of Fig. 1 demonstrates the basic elements, the compression and the rarefaction, quite well. Thus, there are two shocks, one traveling downward into the target ahead of the interface and the other traveling upwards into the projectile. Likewise there are two rarefactions. One is shown in the lateral expansion of the shocked projectile while the other is seen in the splash of target material.

The last frame of Fig. 1 is at twice the time, after impact, of the middle frame. By this time, the shock has reached the top of the projectile and a rarefaction is eating its way back down again. Projectile material is being plastered against the sides during this stage of crater formation, and both projectile and target are contributing material to the splash.

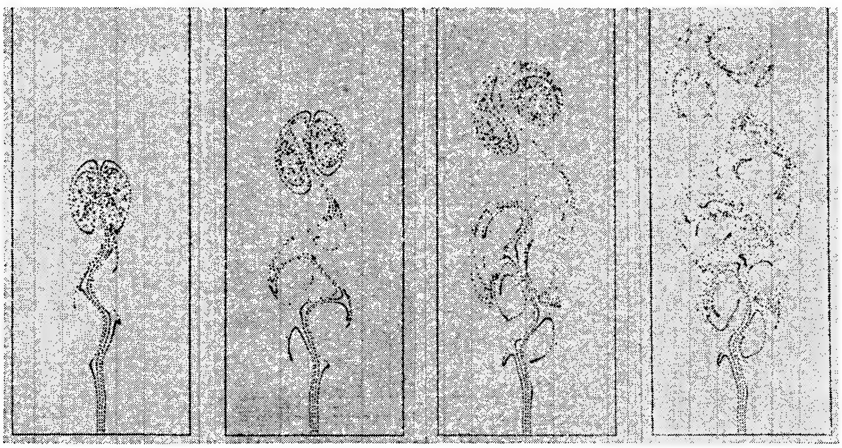


FIG. 2

The second example is illustrated in Fig. 2, showing an incompressible flow calculation using the method of Fromm [2]. The appearance is familiar to all who have watched a rising filament of smoke. Here, the particles follow only the incoming material of the jet, the surrounding material of the same density is left unmarked. Likewise, not shown is the mesh of finite difference computational cells, which is twenty-three cells wide by fifty-five high. Four of the numerous frames are illustrated, showing the progress of the instability from the earlier simple contortions to the stages in which a transition to turbulence is beginning to occur.

The third example shows an application of Fromm's method to calculate the incompressible flow of cold water past a heated obstruction in a channel. Again, hydrodynamic instability contributes the most interesting features to the

flow pattern. In Fig. 3, we have illustrated but one instant in the flow history, shown in four different ways. All of the different modes of illustration show one striking feature at a glance. Instead of a steady laminar flow, the pattern is formed of a sequence of vortices which are alternately shed from the back corners of the obstruction. The top part shows contours of equal "vorticity," a function which measures the strength of rotation. Just below this is a plot of lines of

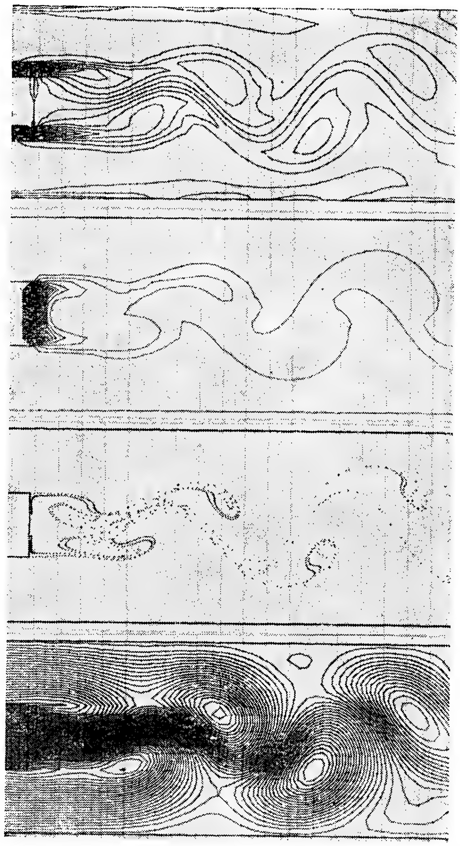


FIG. 3

equal temperature, and it is apparent that concentrations of vorticity and heat are closely associated. Next below is the configuration of massless test particles which have been injected into the flow (like filaments of dye into water) at four places along the back edge of the obstruction. Again, there is a close correlation in spots of concentration with those of the heat and vorticity, showing that the latter two very nearly move with the fluid. Finally, at the bottom is a con-

figuration of streamlines. Even these indicate a correlation among centers of concentration. Experience has shown, however, that this last correlation will not always be as close as in the present example.

6. Conclusion. The development of very large and fast electronic computers has opened new paths of theoretical investigation in almost every branch of science and technology, and the subject of fluid dynamics has proved to be particularly rich in useful applications. Numerous examples, such as those illustrated in the present discussion, have been compared often with the results of experiments, showing excellent agreement.

While analytical studies usually require idealizations for tractability, it should be emphasized that one of the most powerful properties of the computer approach is that many of these idealizations are not at all necessary. In many cases, it is almost as easy to include all the pertinent features as it is to ignore one or several of them. Thus, for example, in the viscous flow of a fluid, the coefficient of viscosity may vary with temperature and density in a complicated fashion with negligible addition of computer solution difficulty.

There are many uses of computer solutions in fluid dynamics problems. Such solutions give a detailed description of processes which otherwise would defy laboratory measurement or analytical solution. Thus they are useful for the design and interpretation of experiments, for the prediction of events, but most especially they are useful as sources of understanding from which analytical generalizations can be made. No wonder, then, that one of the principal applications of computers today is to a host of fascinating and useful fluid dynamics studies.

Acknowledgments. I am grateful to A. A. Amsden and J. E. Fromm, who performed the calculations illustrating this discussion. This work was performed under the auspices of the United States Atomic Energy Commission.

References

1. Francis H. Harlow, The Particle-in-Cell Method for Numerical Solution of Problems in Fluid Dynamics, Proc. Symposia Appl. Math., XV (1963) 269–288.

2. Jacob E. Fromm, A Method for Computing Non-Steady, Incompressible Viscous Fluid Flows, Los Alamos Sci. Lab. Report No. LA-2910 (1963).

COMPUTERS IN SOLID MECHANICS:—A CASE HISTORY

H. B. KELLER AND E. L. REISS
Courant Institute of Mathematical Sciences, New York University

1. Introduction. The real power and significance of modern digital computers in applied mathematics stems from the fact that with appropriate numerical methods they furnish a fairly universal method for obtaining approximate solutions to very general classes of problems. In particular computers have had a profound influence on recent work in elasticity theory and solid mechanics in general by "solving" some of the difficult differential equation problems which arise. Such problems are rarely, if ever, rigorously solved by numerical computations but it is in the very nature of most theories in continuum mechanics that exact solutions are no more closely related to the physical phenomena in question than are good approximate solutions.

Solid mechanics is concerned with the deformations and stresses in solid bodies which result from the application of external force and/or displacement fields. The solids here are not the rigid bodies of ordinary mechanics but "deformable" bodies of various types. The most familiar are elastic bodies which return to their original state when the external fields have been removed. However many other kinds of solids have been defined (e.g. plastic, viscoelastic, elastic-plastic, etc.) to better describe various "real" materials. The theories proposed to describe the motion of these solids are usually in the form of nonlinear partial differential equations (say of hyperbolic type for most elastic bodies) derived from the application of Newton's laws or else from the application of variational principles. When equilibrium deformations are of interest the inertial terms and other time variations are eliminated and the resulting equations are usually of a different type (frequently elliptic for elastic bodies). If the deformations and stresses are "sufficiently small" then the equations can be simplified to get the linear theories which are best known and more fully developed.

The vast majority of significant problems in solid mechanics cannot be solved explicitly. In fact present approximation procedures, including numerical methods, are still inadequate for most of these problems. But as the speed and capacity of computers is increased and better numerical methods are developed this situation is steadily improved. As a specific example we shall consider an important area of solid mechanics—the buckling of shells—in which numerical computations play a crucial role.

2. Buckling of a spherical cap. A typical and technologically significant shell buckling problem is that of a thin, shallow, elastic spherical shell segment clamped around its edge and subject to uniform external pressure on its convex surface. (For example, slice off a polar cap of a ping-pong ball, clamp its edge and blow down on it.) Experimentally it is well known that as the pressure, say P, increases slowly from zero the cap deviates only slightly from the spherical

shape until some critical pressure, $P=P_B$, is reached. Then the cap suddenly jumps or snaps into a non-spherical shape with relatively large deformations; we call this the buckled state. The fundamental problem of cap buckling and in fact of all shell buckling is to explain the mechanism which initiates or "triggers" the jump and to obtain theoretical estimates of the buckling load, P_B.

It is clear that buckling occurs only when the state of the cap is in some sense "unstable." To be precise we say that an equilibrium state of the cap is stable at a given load if it depends continuously on the load. Then if $D(P)$ is an appropriate norm of the deformation at load P the cap will not buckle if $D(P)$ is a continuous function of P. Thus the buckling load must be at a point of discontinuity of $D(P)$. It has been conjectured by von Karman and Tsien [7] that the curve of $D(P)$ versus P, for caps which buckle, is similar to that in Fig. 1. Clearly $D(P)$ is not single valued for $P_L \leqq P \leqq P_U$ and hence not continuous in this interval. Thus the buckling load must lie in this interval and we call P_L and P_U the lower and upper buckling loads, respectively (since the cap cannot buckle for $P<P_L$ and it must have buckled when $P>P_U$). We refer to the equilibrium states that correspond to points on branch $\overline{OU}$ as "unbuckled," on $\overline{LN}$ as "buckled" and on $\overline{UL}$ as "unstable."

A further conjecture by Friedrichs [3] assumes that for some intermediate load, P_M, the buckled states have more potential energy than the unbuckled states when $P_L \leqq P < P_M$ and conversely when $P_M < P \leqq P_U$. If in addition a shell will not jump into a state of higher energy then the buckling load must lie in $P_M \leqq P \leqq P_U$. Before any of the above concepts can be shown to be relevant the assumed form of $D(P)$ and the existence of P_M must be verified.

Although the actual process of buckling is dynamic it is assumed that a static or equilibrium theory can explain the mechanism which initiates this process (because the initial unbuckled and final buckled states are essentially equilibrium states). The differential equations which determine these states have been derived by several authors. If we assume rotational symmetry they can be written in a dimensionless form as:

$$(1) \qquad \text{a) } L\alpha = \rho[\alpha\gamma + Px^2], \qquad \text{b) } L\gamma = \rho[x^2 - \alpha^2].$$

Here

$$L\cdot \equiv x \frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\, x\cdot\right),$$

P is the loading parameter, ρ is a geometric parameter, $\alpha(x)$ is the slope of the deformed midsurface and $\gamma(x)$ is a stress function. From regularity and symmetry at the center of the cap, $x=0$, and rigid clamping at the edge, $x=1$, it follows that the boundary conditions are:

$$(2) \qquad \text{a) } \alpha(0) = \gamma(0) = 0, \qquad \text{b) } \alpha(1) = 1,\ \gamma'(1) = \nu\gamma(1).$$

It is the solution of this rather harmless looking boundary value problem that

can be expected to shed much light on the nature of shell buckling. Analytical attempts to solve this problem have been essentially fruitless.

3. Numerical cap snapping. One of the first attempts to solve the cap problem numerically employed power series on the AEC UNIVAC at New York University [9]. It follows from (1) and (2) that only odd powers can be present and so

$$\alpha(x) = \sum_{n=1}^{\infty} \alpha_n x^{2n-1}, \qquad \gamma(x) = \sum_{n=1}^{\infty} \gamma_n x^{2n-1}.$$

Then, the coefficients can be determined recursively, in terms of P, ρ, α_1 and γ_1, by requiring like degree terms in (1) to vanish. It only remains to satisfy the conditions (2b) which can now be written as

$$\text{(3)} \qquad F(\alpha_1, \gamma_1) \equiv \sum_{n=1}^{\infty} \alpha_n - 1 = 0, \qquad G(\alpha_1, \gamma_1) \equiv \sum_{n=1}^{\infty} (2n - 1 - \nu)\gamma_n = 0.$$

Thus the problem is reduced, aside from convergence questions, to solving the transcendental equations (3) for α_1 and γ_1 (given any values of P and ρ). Newton's method was used for this purpose and it was terminated when, say $|F| < \epsilon$ and $|G| < \epsilon$. Of course the infinite series also had to be terminated at a finite value of n, say N, such that

$$|\alpha_N| + |\alpha_{N-1}| + |\alpha_{N-2}| < \delta, \quad |\gamma_N| + |\gamma_{N-1}| + |\gamma_{N-2}| < \delta.$$

These are only necessary conditions for convergence but unfortunately in numerical computations we can almost never be assured of satisfying sufficient conditions for convergence of power series or of sequences of iterates. This is one of the reasons why computing is at times an art requiring great care and insight.

Starting with small values of P, for a given ρ, there is no difficulty in finding accurate roots (α_1, γ_1) since the shell is nearly spherical (i.e. $\alpha \equiv \gamma \equiv 0$). Then with ρ fixed P is increased in small increments, an initial estimate of each new root is obtained by extrapolation of previous roots, and in this way a sequence of roots of (3) is generated. The procedure is terminated for either of two reasons: a) lack of "convergence" of the Newton scheme after many (say 10) iterations; or b) lack of "convergence" of the power series after many terms (say $N = 100$). The failure when it occurs is usually spectacular in that a small change in P completely destroys the convergence. In fact this is analogous to a lack of continuous dependence on P and so the pressure at which this first occurs was called the "initial buckling load." This computational procedure is in a sense a numerical experiment. As in the physical experiments the buckling load is attained when for a small increment in load no neighboring solution can be found. Of course the series for $\alpha(x)$ and $\gamma(x)$ may not converge in $0 \leqq x \leqq 1$ for all P and ρ. If this is the case the "initial buckling loads" may be spurious. An

obvious way to eliminate such effects would be to employ analytic continuation. Calculations based on this idea have been performed on an IBM 704 computer [11].

A completely different and more fruitful approach to the numerical solution of the boundary value problem (1)–(2) is based on finite difference calculations [4]. For this purpose we place the $J+1$ net points $x_j = j\Delta x$, $j=0, 1, \cdots, J$ with $\Delta x = 1/J$ on $[0, 1]$. On this net we approximate $\{\alpha(x_j), \gamma(x_j)\}$ by net functions $\{\bar{\alpha}_j, \bar{\gamma}_j\}$ which are required to satisfy the finite difference equations

$$(4) \qquad \text{a) } L_\Delta \bar{\alpha}_j = \rho[\bar{\alpha}_j \bar{\gamma}_j + P x_j^2], \qquad \text{b) } L_\Delta \bar{\gamma}_j = \rho[x_j^2 - \bar{\alpha}_j^2], \qquad 1 \leqq j \leqq J-1.$$

Here the difference operator L_Δ is defined by

$$L_\Delta f_j \equiv \frac{x_j}{\Delta x^2}\left\{\left(\frac{x_{j+1}f_{j+1} - x_j f_j}{x_j + \Delta x/2}\right) - \left(\frac{x_j f_j - x_{j-1} f_{j-1}}{x_j - \Delta x/2}\right)\right\},$$

and so the equations (4) are approximations to the equations in (1). At the end-points $j=0$ and $j=J$ the net functions satisfy difference equivalents of the boundary conditions (2), say:

$$(5) \qquad \text{a) } \bar{\alpha}_0 = \bar{\gamma}_0 = 0, \qquad \text{b) } \bar{\alpha}_J = 0, \quad \bar{\gamma}_J = \left(\frac{1 + \nu\Delta x/2}{1 - \nu\Delta x/2}\right)\bar{\gamma}_{J-1}.$$

For Δx sufficiently small we expect that the solutions of these $2J+2$ nonlinear algebraic equations (4)–(5) will be close approximations to the exact solutions of (1)–(2) evaluated on the net.

The algebraic problem is solved by means of an accelerated iteration scheme which starts from some initial estimate $\{\bar{\alpha}_{j,0}, \bar{\gamma}_{j,0}\}$, of the solution for fixed (P, ρ). Then a sequence of iterates $\{\bar{\alpha}_{j,n}, \bar{\gamma}_{j,n}\}$, $n=1, 2, \cdots$ is defined by the recursions

$$(6) \qquad \begin{aligned} &\text{a) } L_\Delta \bar{\gamma}_{j,n+1} = \rho\,[x_j^2 - \bar{\alpha}_{j,n}^2], \qquad \text{b) } L_\Delta \overset{*}{\bar{\alpha}}_{j,n+1} = \rho\,[\bar{\alpha}_{j,n}\bar{\gamma}_{j,n+1} + P x_j^2], \\ &\qquad\qquad \text{c) } \bar{\alpha}_{j,n+1} = \omega \overset{*}{\bar{\alpha}}_{j,n+1} + (1-\omega)\bar{\alpha}_{j,n}, \end{aligned}$$

and the boundary conditions (5) are also imposed. This system is easily solved, in the order a), b), c), by simply solving linear systems with tridiagonal coefficient matrices. The parameter ω is adjusted to speed convergence (see [4] for details).

As in the power series case there is no difficulty in obtaining good initial estimates and rapid convergence for any ρ when P is small. Then, with ρ fixed, the numerical solution at load P is used as the initial iterate for the load $P+\delta P$ and the iterations generally converge (according to some numerical criterion) with perhaps some slight adjustment of the acceleration parameter ω. In this manner solutions are obtained for a sequence of loads $P \uparrow P_U$ corresponding to points on the unbuckled branch $\overline{OU}$ in Fig. 1.

However we finally reach a load say P' such that at $P'+\delta P$ the iterations

converge to a solution on the buckled branch, $\overline{LN}$. The difference between buckled and unbuckled solutions is quite obvious from the magnitude of the displacements and the sign changes in certain stresses. Thus the numerical solution automatically "jumps" from the unbuckled to the buckled branch and when this occurs we define P_U to be in the interval $P' < P_U < P' + \delta P$. After the numerical buckling we reduce P, use the buckled solution as an initial iterate and proceed as before to obtain a sequence of solution for loads $P \downarrow P_L$ on the buckled branch $\overline{LN}$. In this way P is finally reduced below P_L, which is defined in analogy with P_U, and the numerical solution "jumps" back to a previously obtained solution on the unbuckled branch (i.e. the solution unbuckles just as do some actual shells when the pressure is reduced). Thus we obtain close estimates of both P_U and P_L and for loads in $P_L < P < P_U$ two distinct sets of numerical solutions are determined which correspond to pairs of nonunique solutions of the boundary value problem (1)–(2).

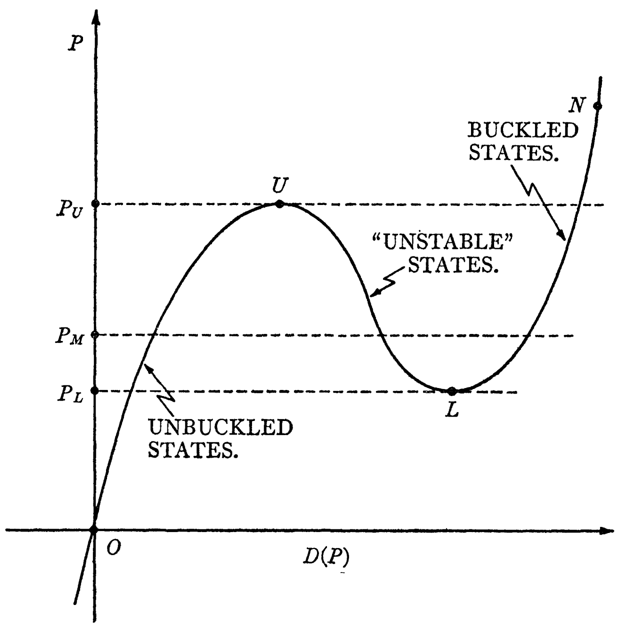


FIG. 1

These numerical experiments were much more successful than the power series tests and they were performed on a faster computer. In fact using an IBM 704 the upper and lower buckling loads (and the intermediate load, P_M) were obtained up to $\rho = 20$ and then slow convergence made it impractical to continue [4]. However with an IBM 7090 the same procedure was extended in [1] to above $\rho = 140$. Other iterative calculations for this problem have been carried out using difference equations obtained from integral equations [2, 10]. The

results in [1, 4, 10] seem to confirm the conjecture of von Karman and Tsien while those in [1, 4] also seem to verify Friedrichs' conjecture regarding P_M.

Perhaps the most recent and by far the most complete numerical study [6] of the boundary value problem has employed a standard method for solving such problems—reduction to initial value problems. Thus we consider the initial value problem

$$(7)\qquad \begin{aligned} &\text{a)}\quad \frac{dU(\xi)}{d\xi} = -\frac{1}{\xi}U(\xi) + V(\xi), && U(0) = 0;\\ &\text{b)}\quad \frac{dV(\xi)}{d\xi} = \frac{1}{\xi}U(\xi)Y(\xi) + P\xi, && V(0) = 2t;\\ &\text{c)}\quad \frac{dY(\xi)}{d\xi} = -\frac{1}{\xi}Y(\xi) + Z(\xi), && Y(0) = 0;\\ &\text{d)}\quad \frac{dZ(\xi)}{d\xi} = -\frac{1}{\xi}U^2(\xi) + \xi, && Z(0) = 2s; \end{aligned}$$

and denote a solution by: $U(P, s, t; \xi), \cdots, Z(P, s, t; \xi)$. For a given P if (s, t) can be found such that at $\xi = \lambda > 0$:

$$(8)\qquad U(P, s, t; \lambda) = \lambda, \qquad Z(P, s, t; \lambda) = \frac{1+\nu}{\lambda}Y(P, s, t; \lambda),$$

then a solution of (1)–(2) with $\rho = \lambda^2$ is given by

$$\alpha(x) = \frac{1}{\lambda}U(P, s, t; \lambda x), \qquad \gamma(x) = \frac{1}{\lambda}Y(P, s, t; \lambda x).$$

It can be shown that a unique solution of (7) exists and is entire in P, s, t and analytic in ξ. Then it follows that for any given (P, ρ) there are as many solutions of the boundary value problem (1)–(2) as there are distinct roots (s, t) of the system (8) with $\lambda = \sqrt{\rho}$. Thus the problem has been reduced (rigorously) to solving some, in general, transcendental system.

We do not describe here the iterative numerical methods used to solve (8) as they are rather complicated (see [6]). However, it should be observed that in order to evaluate the functions occurring in (8) it is necessary to solve the initial value problem (7). This is a standard numerical problem and it can be done with great accuracy (since the solution is analytic). However to find many roots over a large range of P and ρ the computations are extremely time consuming. These computations were in fact run on an IBM 704, on a 7090 and finally on a 7094 where the bulk of the results were obtained.

Similar initial value calculations were first used in [8] and verified the results in [4]. However, the much more extensive calculations in [6] have revealed the existence of at least nine distinct equilibrium states for some values

of (P, ρ). The critical loads P_L and P_U are found to be determined as in Fig. 1 only for special intervals of ρ values. Thus the conjecture of von Karman and Tsien is of limited validity. More general definitions of P_L and P_U are suggested by the calculations. In addition many other "critical pressures" were determined as values where the multiplicity of solutions changed. An examination of the properties of these solutions has suggested several possible mechanisms for the initiation of the buckling process. Detailed discussions of the results and some of their implications can be found in [5] and [6].

There is considerable scatter of the experimentally measured buckling loads but almost all of them lie between the computed P_U and P_L. Although experimental errors contribute to the spread of the data the recently computed multiplicity of solutions [6] suggests that such scatter is inherent to the problem. Thus the computations have provided new insight into the phenomena of shell buckling and indicate that it is far more complex than previously believed.

The preparation of this paper was supported jointly by the A.E.C. under contract AT(30-1)-1480 and by the O.N.R. under contract NONR-285(42).

References

1. F. Bauer, H. B. Keller and E. L. Reiss, Numerical studies of the deformation and buckling of spherical caps, submitted to J. Soc. Indust. Appl. Math.

2. B. Budiansky, Buckling of clamped shallow spherical shells, Symp. on Theory of Thin Elastic Shells, Delft (1959) 64–94.

3. K. O. Friedrichs, On the minimum buckling load for spherical shells, von Karman Anniv. Vol., Pasadena (1941) 258–272.

4. H. B. Keller and E. L. Reiss, Spherical cap snapping, J. Aero/Space Sci., 26 (1959) 643–652.

5. ———, Some recent results on the buckling mechanism of spherical caps, NASA Tech. Note D-1510, Collected Papers on Instability of Shell Structures (1962) 503–514.

6. H. B. Keller and A. W. Wolfe, On the nonunique equilibrium states and buckling mechanism of spherical shells, submitted to J. Soc. Indust. Appl. Math.

7. T. von Karman and H. S. Tsien, The buckling of spherical shells by external pressure, J. Aero. Sci., 7 (1939) 43–50.

8. F. J. Murray and F. W. Wright, The buckling of thin spherical shells, J. Aero/Space Sci., 28 (1961) 223–236.

9. E. L. Reiss, H. J. Greenberg and H. B. Keller, Nonlinear deflections of shallow spherical shells, J. Aero. Sci., 24 (1957) 533–543.

10. G. A. Thurston, A numerical solution of the nonlinear equations for axisymmetric bending of shallow spherical shells, J. Appl. Mech., 38 (1961) 557–562.

11. H. J. Weinitschke, On the stability problem for shallow spherical shells, J. Math. and Phys., 38 (1960) 209–231.

MATHEMATICAL PROGRAMMING

R. E. GOMORY, Thomas J. Watson Research Center, IBM

Mathematical programming is the name used for the study of a variety of maximization problems. In these problems, maxima can occur on the boundaries as well as in the interior of a region. In many cases the maximization is actually over a finite set of points. When this happens it is clear that: 1) the usual derivative criteria for a maximum, as used in the calculus, do not apply; and 2) that the emphasis must be on algorithms for actually obtaining maxima rather than on existence, for existence is trivial in these finite cases. Both these observations do, in fact, apply to most of mathematical programming and the second is the reason why the subject is intimately bound up with the use of computers.

In what follows we will discuss two of the most active areas of mathematical programming: first, linear programming, and secondly, and very sketchily, dynamic programming.

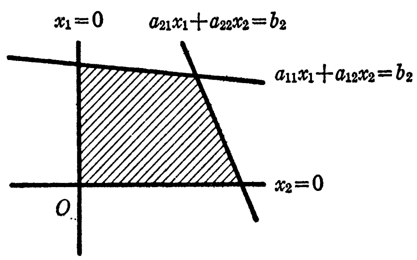


FIG. 1

Linear programming. The maximization problem here is to maximize a linear form subject to linear inequality constraints. The basic problem can be written

$$\max \sum_{j=1}^{j=N} c_j x_j, \qquad \sum_{j=1}^{j=N} a_{ij} x_j \leqq b_i, \qquad i = 1, \cdots, M, \tag{1}$$

$$x_j \geqq 0, \qquad j = 1, \cdots, N,$$

or vectorially,

$$\max \; c \cdot x, \quad a_i \cdot x \leqq b_i, \qquad i = 1, \cdots, M + N.$$

Here the last N a_i are negative unit row vectors and the last N b_i are 0. A geometrical way of looking at the problem is illustrated in Figure 1. Each inequality constraint confines the variables to a closed half space, the intersection of these half spaces gives a polyhedron of values satisfying the inequalities. This polyhedron is called the feasible region. The maximization problem is to find that point of the feasible region which maximizes the "objective function" $\sum c_j x_j$.

Maximization criterion. The usual condition for the free maximum of a function at an interior point is that the gradient of the function should vanish. If there are also equality conditions to be satisfied, the condition for a maximum is, roughly, that the gradient should point in a direction that is normal to the equality surfaces. More precisely, the gradient should be a linear combination of the normals to these constraint surfaces—this is the usual Lagrangian condition. When dealing with linear inequalities the condition is similar. The function will be maximal at $\bar{x}$ if the gradient at $\bar{x}$ points into a region forbidden by the inequalities. This means that it should be a nonnegative combination of the normals of those (and only those) inequality constraints that are satisfied as equalities at $\bar{x}$. See Figure 2. Thus at $\bar{x}$, the maximum point of (1), there will be an $M+N$ vector $\bar{y}=(\bar{y}_1, \cdots, \bar{y}_{M+N})$ satisfying

(2a) $$c = \sum_{i=1}^{i=M+N} a_i\bar{y}_i, \quad \bar{y}_i \geqq 0$$ and (2b) $$\bar{y}_i > 0 \Rightarrow \bar{a}_i \cdot \bar{x}_i = b_i$$

and, if at a feasible point $\bar{x}$ there is such a $\bar{y}$, then $\bar{x}$ is maximal.

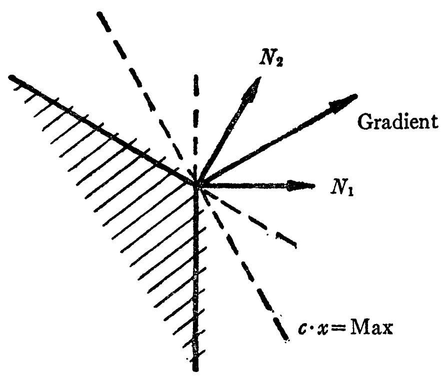


FIG. 2

Duality. Further examination of the maximization condition leads to the notion of duality due to Gale, Kuhn and Tucker [1].

Using (2a) and forming the scalar product with $\bar{x}$ yields, using (2b)

(3) $$c \cdot \bar{x} = \sum_{i=1}^{i=M+N} \bar{y}_i a_i \cdot \bar{x} = \sum_{i=1}^{i=M+N} \bar{y}_i b_i.$$

For any other $M+N$ vector y satisfying (2a)

$$c \cdot \bar{x} = \sum_{i=1}^{M+N} y_i a_i \cdot \bar{x} \leqq \sum_{i=1}^{M+N} y_i b_i.$$

Therefore $\bar{y}$ solves the minimization problem

$$(4) \qquad \min b \cdot y,\ c = \sum_{i=1}^{M+N} a_i y_i, \qquad y_i \geqq 0.$$

Since $a_{M+1}, \cdots, a_{M+N}$ are negative unit N-vectors, (4) is equivalent to the problem

$$(5) \qquad \min b \cdot y, \quad \sum_{i=1}^{i=M} a_i y_i \geqq c, \qquad y_i \geqq 0,$$

in which y is now an M-vector.

(5) is a problem very much like (1). The rows of (1) are the columns of (4), max in (1) has been replaced by min in (5), and the inequality sign reversed in the first set of inequalities. Problem (5) is called the dual of (1). As (3) shows, the max value of (4) and the min of (5), its dual, are numerically equal.

Clearly the dual of the dual is the original problem.

Once we have a solution $\bar{x}$ to (1), we need only represent the gradient at $\bar{x}$ as a nonnegative combination of the normals of the faces on which it lies to obtain a $\bar{y}$ minimizing the dual problem. In most methods of solving (1), this nonnegative representation is usually obtained in the course of recognizing the optimality of $\bar{x}$, so the solution to the dual is usually at hand once the original problem has been solved.

$$\begin{array}{ccl}
 & & \overbrace{\qquad\qquad\qquad\qquad}^{N \text{ Columns}} \\
 & p_1 & a_{1,1}, a_{1,2}, \cdots, a_{1,N} \\
 & p_2 & a_{2,1}, \cdots\cdots \\
 & \cdot & \\
M \text{ Rows} & \cdot & \\
 & \cdot & \\
 & p_M & a_{M,1}, \cdots\cdots, a_{M,N} \\
 & & q_1, q_2, \cdots, q_N
\end{array}$$

FIG. 3

The theory of zero-sum two-person games provides an interesting illustration of duality. The essential data for a zero sum two-person game is represented in Figure 3. The play of the game is as follows. One player, called the row player, chooses a row, say row i. The other player, called the column player, chooses a column, say column j, without knowing the choice of the row player. The row player then pays the column player an amount a_{ij}. Of course, the row player tries to minimize this amount, the column player to maximize it.

An allowable strategy for the row player is a probability vector $P = (p_1, \cdots, p_M)$. At each play he then chooses the ith row randomly and

with probability p_i. Similarly, the column player has a strategy $Q=(q_1, \cdots, q_N)$. The *expected* payment v, with strategies P and Q is

$$v = \sum_{i=1}^{i=M} \sum_{j=1}^{j=N} a_{ij} p_i q_j. \tag{6}$$

Let us suppose that the row player's object is to choose P so that $v \leqq \theta$ *no matter what his opponent's strategy Q is*, and, subject to this condition, to make θ as small as possible.

To find the strategy P that does this we reason as follows: if the column player plays one column only, say the jth, then the expected payment is $\sum_{i=1}^{i=M} p_i a_{ij}$. The best strategy against an opponent confined to single column strategies is the P that minimizes θ subject to

(7) $\sum_{i=1}^{i=M} p_i a_{ij} \leqq \theta \quad j = 1, \cdots, N$ and (8) $\sum_{i=1}^{i=M} p_i = 1, p_i \geqq 0, i = 1, \cdots, M.$

One can easily verify that (7) implies $v \geqq \theta$ even if a general Q is used by the opponent since the payment then is a weighted sum of numbers each $\leqq \theta$. So (7) and (8) are actually the row player's entire maximization problem.

For simplicity of exposition we will assume here that the smallest possible θ, $\theta_{\min} > 0$, so (7) and (8) can be solved only for $\theta \geqq \theta_{\min} > 0$. Introducing $p_i' = (1/\theta) p_i$, (7) becomes

(7a) $\sum_{i=1}^{i=M} p_i' a_{ij} \leqq 1,$ and (8) becomes (8a) $\sum_{i=1}^{i=M} p_i' = \dfrac{1}{\theta},$

where $p_i' \geqq 0$, $i=1, \cdots, M$, and $j=1, \cdots, N$. Thus the minimization problem becomes maximize $\sum_{i=1}^{i=M} p_i'$ subject to (7a) which is a problem of the form of (1).

Its dual is min $\sum_{j=1}^{j=N} q_j'$, subject to

$$\sum_{j=1}^{j=N} q_j' a_{ij} \geqq 1, \qquad i = 1, \cdots, M, \tag{9}$$

$q_j' \geqq 0$, $j=1, \cdots, N$. When $1/\beta = \sum_{j=1}^{j=N} q_j'$ and $q_j = \beta q_j'$ are introduced, (9) becomes

$$\begin{array}{ll} \multicolumn{2}{c}{\max \beta} \\ \sum_{j=1}^{j=N} q_j a_{ij} \geqq \beta & i = 1, \cdots, M, \\ q_j \geqq 0 & j = 1, \cdots, N, \\ \sum_{j=1}^{j=N} q_j = 1. & \end{array} \tag{10}$$

Now (10) is the column player's problem, i.e., choose a Q such that the expected payment is at least β, no matter what the row player's strategy is. Thus, with

the aid of a change of variables, we see that the two players' problems are dual to each other.

Since the max of (8a) equals the min in (9), $1/\theta_{\min}=1/\beta_{\max}$, so $\theta_{\min}=\beta_{\max}=\bar{v}$. No matter what the column player does, the row player's expected payment need never exceed $\bar{v}$, if he plays optimally. Similarly, no matter what the row player does, the column player's expectation, if he plays optimally, is always $\geqq \bar{v}$. If both play optimally, the expected payment is exactly $\bar{v}$ which is called the value of the game. Also, because of duality, the strategy Q is usually immediately available, once P is known.

Maximization method. It is geometrically clear, and easily proved, that the maximum in equation (1) will be obtained at one of the vertices of the feasible polyhedron. Thus we are dealing with a finite maximization problem. It is sufficient to search through the vertices of the polyhedron to find the point taking on the maximum value. The simplex method of G. B. Dantzig [2] the usual maximization method of linear programming, inspects only vertices. Instead of looking at all of them, however, it starts at one, moves on to a neighboring vertex that gives a larger value to the function being maximized, and so on until the maximum is reached.

To simplify the discussion of the simplex method, we introduce new variables called "slacks." Slacks represent the difference between the left-hand side of the inequalities of (1) and the right-hand side. When slacks are introduced these equations become

$$\text{(11)} \qquad \begin{aligned} &\max \sum_{j=1}^{j=N} c_j x_j \\ &\sum_{j=1}^{j=N} a_{ij}x_j + s_i = b_i \qquad i = 1, \cdots, M, \end{aligned}$$

$x_j \geqq 0$, $j=1, \cdots, N$, $s_i \geqq 0$, $i=1, \cdots, M$. If s_i is nonnegative, the x's satisfy the ith inequality in (1); consequently the inequalities in (1) can be replaced by equation (11) and the condition that *all* variables appearing are nonnegative. Let us then denote the matrix of all coefficients appearing in the equations (11) by A, and the $M+N$ vector of all variables by y. (11) becomes

$$\max c \cdot y, \; Ay = b, \; y \geqq 0.$$

A vertex is a point where N of the inequalities of (1) are satisfied as equalities. This means that N of the components of y are zero. To obtain the values of the remaining M variables (called basic variables), we simply strike out the columns corresponding to the zero value variables (the nonbasic ones) and solve the remaining M equations in M unknowns. That is, if $\bar{y}$ is the M-vector of basic variables, and B the square matrix formed from the corresponding columns, then $\bar{y}=B^{-1}b$. Thus all components of y are determined, and so in particular x, the

location of the vertex is known. If all $y_i \geqq 0$, the point x satisfies (1) and so is a vertex of the feasible polyhedron; otherwise, it is merely an intersection of N hyperplanes outside the feasible polyhedron.

In carrying out the simplex method we start with a feasible vertex. The next step is to examine neighboring vertices and find those that are: 1) on the polyhedron; 2) yield an improvement in the objective function.

It is clear from the geometry, Figure 4, that neighboring vertices have $N-1$ faces in common, hence their nonbasic sets of variables are the same except for one element. It follows that to go to a neighboring vertex, one variable must switch from the nonbasic to the basic set and one from basic to nonbasic.

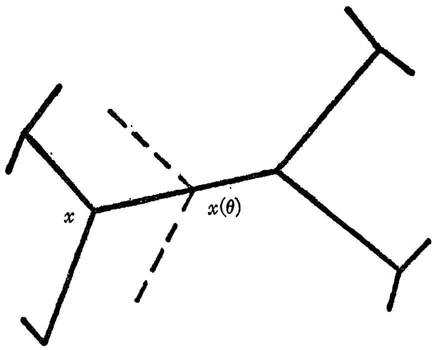


FIG. 4

Specifying the variable to enter the basic set already determines the neighboring vertex. For let us suppose that the variable y_i is to be added to the basic set. Let us give y_i the value θ and solve for the basic variables $\bar{y}$. Then $\bar{y}(\theta) = B^{-1}(b-\theta v_i)$ when v_i is the column associated with y_i. As θ increases from zero, the values of the components of $\bar{y}_i$ of $\bar{y}$ change, and there will be some last θ value at which they all are still nonnegative. Call this value $\theta_{\min}$. With $\theta=\theta_{\min}$, one of the $\bar{y}_i$, say $\bar{y}_{i_0}$, is zero. Geometrically, as θ increased, the solution point of the equations moved from the original vertex along an edge created by displacing the hyperplane corresponding to the one equation which previously had a 0 slack but now has a positive one. Finally another vertex is reached. Going beyond this point would violate the inequality represented by $\bar{y}_{i_0} \geqq 0$. Then $\bar{y}_{i_0}$ is the variable removed from the basis. The new basis consists of y_i and the remaining basic variables. With y_i at the value $\theta_{\min}$ and the remaining $\bar{y}_i$ components at their values in $\bar{y}(\theta_{\min})$ the new basic variables provide a solution to the equations. Thus we have found the coordinates of a neighboring vertex on the polyhedron.

To be able to repeat this process, we would need to find the B^{-1} corresponding to the new basic set. To do this it is not necessary to start over again. Using the old B^{-1} as a starting point, the new inverse can be obtained by a single Gaussian elimination step.

Next we turn to the question of finding a neighboring vertex that will yield an improvement. As we went up the edge from the old to the new vertex, was the objective function increasing or decreasing? Denoting by c' the row M-vector of costs corresponding to the basic variables $\bar{y}$, the change in the objective function is given by

$$c'(\bar{y}(\theta_{\min}) - \bar{y}(0)) = c'(B^{-1}(b - \theta_{\min}v_i) - B^{-1}b) = -\theta_{\min}(c'B^{-1})v_i = \theta_{\min}\pi\cdot v_i,$$

using π for the M-vector $-c'B^{-1}$.

The question of increase or decrease is determined by the sign of the scalar product $\pi\cdot v_i$. If this is positive, there is an increase, otherwise not. Note that the same π can be used in considering all possible v_i that might be introduced.

To carry out the simplex procedure, then, one starts with a vertex, and computes π. Using π, one tests for vertices that will yield an improvement by forming the scalar product with all possible columns v_i. If one of these yields a positive result, it is introduced into the basis, and one of the basic variables becomes nonbasic as described above. This process is iterated. Finally we reach a point where all the scalar products of the current π and the nonbasic vectors are zero or negative. This can easily be shown to be a maximum point.

The effectiveness of the method depends on the number of vertices that must be inspected. Each vertex requires one Gaussian elimination. Two empirically observed facts are: (i) it helps to choose as the entering column the one for which πv_i is maximal. (ii) If the rule (i) is followed, problems like (1) are usually solved in between $2M$ and $3M$ vertex inspections. Although the entire practical success of linear programming rests on observation (ii), there is, at present, no theoretical explanation.

Integer solutions. The simplex method consists of a succession of matrix inversions, the successive matrices being so related that there are great economies in passing from the inverse of one to the inverse of the next. One consequence of dealing with matrix inversions is that the calculation is not one where integral data will produce an integral answer.

Many combinatorial problems are susceptible of a formulation like equation (1), but integral answers are usually needed. For an example, consider the assignment problem. In this problem N men are to be assigned to N jobs. If the ith man is assigned to the jth job, the cost of this assignment is taken to be c_{ij}. The problem is to find that assignment which gives each man a job and each job a man, and minimizes the total cost. This can be formulated in terms of linear inequalities as follows:

$$\min \sum_{i=1}^{i=N} \sum_{j=1}^{j=N} c_{ij}x_{ij},$$

$$(12) \qquad \sum_{j=1}^{j=N} x_{i,j} \geqq 1,\ i = 1, \cdots, N, \quad \sum_{i=1}^{i=N} x_{i,j} \leqq 1, \quad j = 1, \cdots, N$$

$x_{i,j} \geqq 0$, $i=1, \cdots, N, j=1, \cdots, N$. If $x_{i,j}=1$, man i is assigned job j, if $x_{i,j}=0$

he is not assigned job j. If the $x_{i,j}$ all turn out either 0 or 1, the inequalities in (12) will force a solution in which each man gets one assignment and each job one man. The difficulty lies in restricting the variables to be 0 or 1, and the simplex method outlined above might appear to be unable to provide an answer. A closer study of the coefficient matrix of (12) shows, however, that every square submatrix will have a determinant either $+1$, -1, or 0. Therefore, the matrix inversions involve division by ± 1 only and so will provide integral values for all the variables. In the particular case (12) it is easy to see the only possible integral values for the x_{ij} are zero and one.

A matrix all of whose sub-determinants are either $+1$, -1, or 0 is said to have the unimodular property. It is an interesting and important problem to try to discover classes of matrices having this property. A typical result which includes the matrix of (12) is

If A is the incidence matrix of the vertices versus the edges of an ordinary linear graph G, then for A to have the unimodular property it is necessary and sufficient that G have no loops with an odd number of vertices [3].

Even if a matrix does not have the unimodular property one may still want to find the best integral solution. The problem of finding the integer x that maximizes in (1) is called the integer programming problem. The integral points satisfying (1) are, of course, all within the polyhedron of feasible x's. They, together with the points that lie between them, form another polyhedron P' included in the first. All the vertices of P' are integral points. If one could produce the inequalities that yield the faces of this P', the integer problem could be solved as an ordinary linear programming problem over P'. This approach has been developed in [4] and [5] and methods of generating the faces of p' do exist. Nothing like the computational effectiveness of the ordinary simplex method has been obtained so far, however, and much work remains to be done in this area.

Nonlinear systems of inequalities. In the system (1) the linear objective function could be replaced by a nonlinear one. If the replacement is quadratic and positive definite an elegant adaptation of the simplex procedure, due to Wolfe, [6] is possible. For the general nonlinear objective function or when the inequalities involve nonlinearities, gradient methods are often used [7].

Large systems. Although the straightforward simplex method outlined above can be applied to problems of as many as a thousand inequalities and several thousand variables, many practical problems are even larger.

Usually the very large problems involve matrices with special structures. For example, there is the problem of selecting the most profitable products to produce. One unit of the jth product requires an amount a_{ij} of the ith resource. Resources are such things as labor, raw materials, etc. If the amount of the ith resource available is b_i and the profit on a unit of the jth product is c_j, then the problem is precisely (1), x_j being the amount of the jth product produced to

	N	N	R	R	R	R	$\cdots$		
M	$a^1_{1,1}, \cdots, a^1_{M}$ $a^1_{M,1}, \cdots, a^1_{M,N}$								
M		$a^2_{1,1}, \cdots, a^2_{1,N}$ $a^2_{M,1}, \cdots, a^2_{M,N}$							
N	1 $1 \ddots$ 1		$-1\ -1\ -1$		$-1\ -1\ -1$		$-1\ -1\ -1$		
N		1 $1 \ddots$ 1		$-1\ -1\ -1$		$-1\ -1\ -1$		$-1\ -1\ -1$	
R			$1 \ddots 1$	$1 \ddots 1$					
R					$1 \ddots 1$	$1 \ddots 1$			
R							$1 \ddots 1$	$1 \ddots 1$	
R $\vdots$									

FIG. 5

maximize profit without using up more resources than are available. If we now envisage two plants and the possibility of supplying customers from either one plant or the other, depending on costs of production and transportation costs, we obtain a matrix with the structure shown in Figure 5. If either N, the number of products, or R, the number of customers, is large, the size of the matrix is largely determined by the highly structured part. Special methods for large structured problems have been developed, under the name of decomposition methods, during the past several years [8], and this is one of the most fruitful fields of current research. The spirit of these methods can be illustrated on another structured problem that leads us into the area of dynamic programming.

Imagine that identical (parent) rolls of material are to be cut up to make narrower rolls of the same diameter. We will consider all possible cutting patterns, the jth way of cutting a parent roll being described by the numbers $a_{i,j}$ of rolls of width w_i which it produces. We try to cut up the rolls to satisfy certain demands d_i for rolls of width w_i. The corresponding system of linear in-

equalities is

$$\text{(13)} \qquad \min \textstyle\sum_j x_j, \qquad \sum_j a_{ij}x_j \geqq d_i, \qquad x_j \geqq 0, \qquad i = 1, \cdots, M,$$

where j ranges over all possible cutting patterns and x_j is the number of parent rolls cut up using the jth pattern. One difficulty is that the x_j should be integers, but aside from this there is the fact that all the cutting patterns themselves are so numerous that this matrix could never really be written down, even for a problem in which M, the number of different widths to be produced is as small as 10 or 15.

To carry out the calculations in spite of this, [9] we start with some fairly arbitrary square submatrix A, involving M patterns, and invert that. The next stage of the calculation is where the difficulty comes in. To follow out the simplex procedure we would next form the scalar product of π with all nonbasic columns v_j and select the one maximizing $\pi \cdot v_j$ to enter the basic set. The difficulty is that there are too many columns to allow this.

This problem of finding the next column can, however, in a structured problem, be a solvable maximization problem itself. In our example (13) a column v_j is any set of nonnegative integers $y_1, \cdots, y_M$ satisfying

$$\text{(14)} \qquad \sum_{i=1}^{i=M} y_i w_i \leqq W.$$

Inequality (14) simply means that the width cut out of the roll should not total more than the width W of the parent roll. Thus the problem $\max_j \pi \cdot v_j$ becomes $\max \sum_{i=1}^{i=M} \pi_i y_i$ where the y_i are nonnegative integers satisfying (13). Fortunately, this problem can be solved easily by one of the other important methods of mathematical programming, dynamic programming.

Dynamic programming.

Knapsack Problem: For the dynamic programming or recursive approach we introduce the function $v_s(x)$, which is defined to be the value of the solution to the problem

$$\text{(15)} \qquad \max \sum_{i=1}^{i=s} \pi_i y_i, \qquad \sum_{i=1}^{i=s} w_i y_i \leqq x,$$

where the y_i are nonnegative integers. Problem (15) is the same as (14) but now the total width is x and only $y_1, \cdots, y_s$ can be used. $v_s(x)$ satisfies the recursion

$$\text{(16)} \qquad v_s(x) = \max \{v_{s-1}(x), \pi_s + v_s(x - w_s)\}$$

since $v_s(x) = v_{s-1}(x)$ if w_s was not used in the solution to (15) (i.e. $y_s = 0$) and equals π_s plus the best use of the remaining width $x - w_s$ if it was used (i. e. $y_s \geqq 1$). Now $v_1(x)$ can be obtained trivially for all $x \leqq W$. Also $v_s(0) = 0$ for all s. Once provided with $v_1(x)$ and $v_2(0)$ we can compute $v_2(1), v_2(2), \cdots, v_2(W)$ using (16). Then with $v_2(x)$ we compute $v_3(x)$, and so on. Finally we obtain $v_M(W)$ which is the maximum under the restriction (14) and solves the original problem.

The y_i of the solution are easily obtained from (16) and form the column chosen for the next stage of the simplex method applied to the problem (13). A Gaussian elimination is then performed and the process can be iterated. In this manner the vast assemblage of possible columns or cutting patterns is never written down. Instead, the dynamic programming recursion is a device to create each column as it is needed.

Dynamic programming has a large area of applications. The example above is an illustration of its use in a combinatorial problem. The combinatorial interpretation of the problem emerges if the w_i are taken to be weights. Then we have just found the most valuable collection of objects that a man can carry in a knapsack if he is constrained by a weight limitation W and each object has a weight w_i and worth π_i. This is the origin of the name "knapsack problem."

Inventory theory. Dynamic programming can be used in what appears to be a totally different area, that of inventory theory. To see this, consider the problem of a firm wishing to keep an inventory of some item. The firm can purchase a supply y at the beginning of each week. The supply is delivered at the end of the week. During the week customers buy, depleting the inventory. The probability that the customers will want to buy an amount z during the sth week is known and is denoted by $p_s(z)$. Let us suppose that after N weeks all items remaining in inventory are disposed of at a price π per unit. The firm wishes to balance off storage costs, loss through being out of stock, etc. What amount should it buy to be most economical?

If $V_s(x)$ is the maximal expected return to the firm for starting with a supply x and maintaining inventory for s weeks with the best possible buying decisions, then $V_s(x)$ satisfies the recursion

$$\begin{aligned} V_s(x) = \max_y \Big\{ - cy + \int_0^\infty [&p \min(x, z) - \tfrac{1}{2}s(x + \max(0, x - z)) \\ &+ V_{s-1}(\max(0, x - z) + y)]p_s(z)dz \Big\}, \end{aligned} \tag{17}$$

where c is the unit price of the items bought, s the unit storage cost, and p the sales price. Once $V_{s-1}(x)$ is known, the $V_s(x)$ values can be computed together with the y values that yield them. Since $V_0(x)$ can be taken as the value obtained by selling the remaining stock when there are 0 weeks to go, $V_0(x)$ is available, hence $V_1(x), \cdots, V_N(x)$ can be calculated using (17). If the present inventory is x, then the y that maximizes (17) in computing $V_N(x)$ is the amount to buy this week.

The recursive approach of dynamic programming is often useful in situations where a sequence of decisions must be made about a system whose state can be described by a small number of variables, and where the effect of the decision is determined by the current state alone. In the inventory example the (single)

state variable was the inventory on hand, in the knapsack example it was the amount x of unfilled space remaining.

Conclusions. A systematic discussion of dynamic programming is available in Bellman [10] and many applications are described in Bellman and Dreyfus [11]. Rather comprehensive treatments of linear programming and its applications are to be found in Dantzig [12] and Charnes and Cooper [13].

This discussion has emphasized what is known, but only a small portion of the maximization problems that one would like to solve can be attacked by the methods of either linear or dynamic programming. There are a variety of special techniques that can be used [14, 15, 16], but there are still many important maximization problems for which no computationally reasonable method is now known.

References

1. D. Gale, H. W. Kuhn, and A. S. Tucker, Linear programming and the theory of games, Chapter XIX of the Cowles Comm. for Res. in Economics Monogr. No. 13, *Activity Analysis of Production and Allocation*, Proc. Conf. on Linear Programming, Chicago, (June 20–24, 1949) Wiley, New York (1951) 317–329.

2. G. B. Dantzig, Maximization of a linear function of variables subject to linear inequalities, Chapter XXI, *ibid.* pp. 339–347.

3. A. J. Hoffman and J. B. Kruskal, Integral boundary points of convex polyhedra, Chapter in *Linear Inequalities and Related Systems*, Ann. Math. Studies No. 38, Princeton University Press, (1956) 223–246.

4. R. E. Gomory, An algorithm for integer solutions to linear programs, Chapter in *Recent Advances in Mathematical Programming*, McGraw-Hill, New York, (1963) 269–302.

5. ———, All-integer integer programming algorithm, Chapter in *Industrial Scheduling*, Prentice-Hall, Englewood Cliffs, N. J. (1963) 193–206.

6. P. Wolfe, The simplex method for quadratic programming, Econometrica, 27 (1959) 382–398.

7. W. S. Dorn, Non-linear programming, A Survey, Management Science, No. 2, 9 (June 1963) 171–208.

8. G. B. Dantzig and P. Wolfe, Decomposition principle for linear programs, Operations Res., 8 (1960) 101–111.

9. P. C. Gilmore and R. E. Gomory, A linear programming approach to the cutting stock problem—Part II, Operation Res. No. 6, 11 (1963) 863–888.

10. R. E. Bellman, Dynamic programming, Princeton University Press, 1957.

11. R. E. Bellman and S. E. Dreyfus, Applied dynamic programming, Princeton University Press, 1962.

12. G. B. Dantzig, Linear programming and extensions, Princeton University Press, 1963.

13. A. Charnes and W. W. Cooper, Management models and industrial applications, 2 vols., Wiley, New York, 1961.

14. P. C. Gilmore and R. E. Gomory, Sequencing a one-state variable machine, A solvable case of the Traveling Salesman Problem, Operations Res. No. 5, 12 (1964) 655–679.

15. J. D. C. Little, K. G. Murty, D. W. Sweeney and C. Karel, An algorithm for the Traveling Salesman Problem, Operations Res., No. 6, 11 (1963) 972–989.

16. M. Held and R. E. Karp, A dynamic programming approach to sequencing problems, J. Soc. Indust. Appl. Math., No. 1, 10 (March 1962) 196–210.

SIMULATION OF HUMAN PROCESSING OF INFORMATION

ALLEN NEWELL AND H. A. SIMON, Carnegie Institute of Technology

Confront a high school student with the beginning letters of a sequence:

$$ABMCDME \cdots$$

and ask him to fill in the next letters. Most students will rather quickly reply with the letters $FMGHM \cdots$. Next, give the starting sequence $DEFGEFGHF \cdots$. This time there will be a longer pause before the reply. Some, but not all, the students will presently propose $GHIGHIJ \cdots$ as the continuation.

Occasionally, a mathematically talented student, when presented with such problems, will reply that *any* letter is a legitimate continuation of *any* starting sequence. When we encounter such a student, we will compliment him on his acumen, and then pose a new kind of problem to him. We will point out to him that *in fact*, subjects presented with such sequences do *not* reply with "any" letter, but show a high degree of consensus as to what the "right" next letters are. Moreover, the sequences can clearly be ranked according to difficulty, on the basis of how much time subjects take to answer, and on the basis of how many find the "right" answer. Finally, subjects who score high on tests of general intelligence, and particularly on tests of mathematical aptitude, are more likely to give the "right" answers than subjects who score low on such tests.

Hence, if the problem we have posed is ill-defined from a mathematical standpoint, it raises a perfectly well-defined problem for psychology: how do human beings perform tasks like this one? What processes do they carry out, how do they find an answer, how do they judge it correct, how do they produce the extrapolating letters? Psychology needs a theory to explain behavioral phenomena like these; it needs a formal language for stating the theory; and it needs techniques for testing the theory empirically.

Information processing theories. One approach, developed over the past decade, toward the construction and testing of psychological theories of human cognitive behaviors—thinking, problem-solving, concept-finding—makes use of computer programming languages as the means for stating theories rigorously, and makes use of computers as the means for testing the theories, by comparing actual human behavior with the behavior predicted by the theory and simulated by the computer. Since the theory takes the form of a computer program, the theory must be wrong if a computer, operating under that program and given the same tasks as a human subject in the laboratory, does not emit the same behavioral responses. In particular, suppose we had such a theory (as, in fact, we do in [1]) that purported to explain how human subjects handle the letter sequence tasks. We would expect a computer, programmed to conform to the theory, to continue the sequence $ABMCDME \cdots$ with $FMGHM \cdots$.

Of course a computer program might predict particular behaviors without

being a correct theory—this is a danger that is unavoidable with any theory. Successful prediction is a necessary, but not a sufficient, condition that a theory be correct. Our usual inductive procedure for gradually strengthening our confidence in a theory is to confront it with an ever larger and more varied range of phenomena to predict.

With a theory of the letter series performances, we would expect the computer program to experience more difficulty with hard (for humans) problems than with easy ones. We might have some human subjects perform the task in the laboratory, asking them to think aloud while they worked. We could then demand that the computer simulation predict not only their final answers, but also the stages they went through in their thinking, as revealed by the protocols —what they noticed, the sequence in which they noticed it, the blind alleys they went down, and so on. Finally, we might look about for other tasks more or less resembling the letter sequence task, and demand that the theory, with only such modification as is required to handle the new task instructions, predict human behavior in these new situations as well. Thus, we might extrapolate from letter sequences to number sequences: 0 1 1 2 3 5 8 13 · · · ; or to so-called symbol analogy problems: *DOOM MOOD GOLF* ——. The performance of subjects on these tasks suggests strongly that humans use essentially the same processes for solving these problems as for extrapolating letter sequences. A theory to explain one of these performances cannot be regarded as satisfactory if it does not also explain the others.

Psychological theories expressed as computer programs and tested by simulation are usually called *information processing* theories. For the most part, the theories say little or nothing about underlying microscopic, neurophysiological events. Their primitive components, called elementary information processes, symbols, and lists, are much grosser than individual neural events or neural structures. Just as chemistry, for many purposes, takes the atom as its elementary unit, and certain simple assumptions about the combination of atoms in molecules, ignoring the detail of internal atomic structure; so information processing theories take units like "letter from familiar alphabet" and processes like "compare symbols for identity" as elements, ignoring the detail of how these structures and processes are realized by systems of neurons. There is no implication that the explanation cannot, at least in principle, be reduced to the more microscopic level. Regardless of whether such a reduction is possible, it is more convenient to have a division of labor between psychologists and neurophysiologists (as between chemists and nuclear physicists), and to use aggregative theories at the information processing level as the bridge between them. The strategy aims at understanding the very complex in terms of the minutely simple by dividing and conquering—the elementary information processes being the point of division.

Sequence extrapolation tasks. As an example of an information processing theory of human thinking, we shall describe briefly a computer program that

appears to account reasonably well for human behavior in the sequence extrapolation task. A number of versions of this program have been written in the programming language IPL-V (and one, recently, also in LISP). (These programming languages are examples of so-called list processing languages. We will have more to say about them later. See [2].) The program is organized as follows:

FIND AND STORE THE PATTERN
    DISCOVER THE PERIODICITY
    FIND THE RELATIONS FOR EACH POSITION
        IN THE PERIOD
EXTRAPOLATE, BY APPLYING THE STORED PATTERN

The two main subroutines for finding and storing the pattern, one to discover the periodicity of the sequence, the other to describe each position in the period, make use of a small set of common subprocesses for finding simple relations between symbols. The only two relations that, in fact, have proved essential are SAME (identity between two symbols) and NEXT (on a familiar alphabet).

Let us see how this system would operate on the simple sequence $ABM \cdots$. To discover the periodicity, the subroutine would search through the sequence, looking first for multiple occurrences, at regular intervals, of the same symbol. In this case, it would find that each third symbol is an M. (If there are no multiple occurrences, the subroutine searches for recurrent relations of NEXT and NEXT OF NEXT, and for subsequences of constant length. Thus, in $DEFGEFGH \cdots$ the subroutine might discover that the sequence divides into subsequences of length 4, in which each symbol is NEXT in the English alphabet to its predecessor.)

After the first subroutine has determined the periodicity of the sequence, a second subroutine develops a pattern description for each position in the period, using again the relations of SAME and NEXT, and the notion of "corresponding" positions in successive periods. Let x_{ij} be the ith symbol in the jth periodic repetition of the pattern, and let N stand for the relation *next* in some alphabet. Then, the description of $ABM \cdots$ reads:

Period: 3; Alphabet: Roman letters

$$x_{1i} = N(x_{2(i-1)})$$
$$x_{2i} = N(x_{1i})$$
$$x_{3i} = \text{“}M\text{”}$$

Similarly, the sequence $DEFGEFGH \cdots$ may be described:

Period: 4; Alphabet: Roman letters

$$x_{ji} = N(x_{(j-1)i}) \qquad \text{for } j = 2, 3, 4$$
$$x_{1i} = N(x_{1(i-1)})$$

A slight extension of the vocabulary permits the description of numerical sequences also. Consider 1 3 3 5 5 7 · · ·

Period: 2; Alphabet: Odd numbers

$$x_{1i} = x_{2(i-1)}$$
$$x_{2i} = N(x_{1i})$$

where N now means "next on the sequence of odd numbers."

What sequence is described by?

Period: 1; Alphabet: Positive integers

$$x_i = x_{i-1} + x_{i-2}$$

With appropriate initialization, this is simply Fibonacci's sequence, 0 1 1 2 3 5 8 13 · · · , mentioned earlier. To discover such sequences, the relations of SUM and PRODUCT must be added to SAME and NEXT, and processes included for detecting the former relations by differencing and dividing. Thus, Fibonacci's sequence is discovered by observing, through differencing of successive pairs of terms that DIFFERENCE [(Position), (Preceding Position)] =(Preceding Preceding Position).

In explaining the human performance, the theory assumes that the human subjects have certain familiar alphabets stored in memory. Some children, but few adults, can extrapolate *OTTFFSSE* · · · . It can only be done from the knowledge that the appropriate alphabet is *O*ne, *T*wo, *T*hree, *F*our, and so on. On the other hand, the alphabet for certain patterns can be inducted from the pattern itself. Consider the sequence: $&*$&*$&* · · · . This can be described as a sequence of period 1 on the (circular) alphabet comprised of the three symbols $, &, and *.

Finally, the symbol analogy problems can be handled by a slight extension of the same scheme. In *DOOM MOOD*, we notice that the symbols in the first subsequence, reading from the left, are the SAME as the corresponding symbols in the second subsequence, reading from the right.

Once a pattern has been discovered, inductively, and described in terms similar to those sketched above, it is a simple matter to construct an EXTRAPOLATE subroutine that will accept the pattern description and the initial conditions as input, and perform the extrapolation.

This computer program, which almost borders on triviality, appears from the data we have gathered so far to simulate important aspects of the behavior of intelligent human beings solving a class of pattern-finding problems. The program constitutes a theory to explain how humans perform the task—what processes they use, and how these processes are organized. Formally, the computer program is a system of difference equations that, starting from any given initial conditions, determines the behavior of the system through time. Although

these are difference equations of a rather strange kind, judged by classical standards, thinking of them as such helps to clarify the sense in which a program can serve as a theory.

The very simplicity of the program, of the processes it uses and the relation-finding abilities it postulates, is, of course, important. The ability of the program actually to solve pattern-finding problems demonstrates, at least, that no mechanisms, processes or capacities more complex than these need be postulated to account for this human capability. No undefined notions of "intuition," "insight," or "Gestalt" need be evoked. The sufficiency, though not, of course, the necessity, of the mechanisms incorporated in the program is demonstrated.

Other problem solving tasks. The range of problem solving and other thinking tasks for which information processing theories in the form of simulation programs have been constructed is considerable. These programs can be divided, roughly, into two categories: on the one hand, those, like the program for extrapolating sequences, designed explicitly as explanations of human performance; on the other hand, those designed primarily with an "artificial intelligence" objective, which nevertheless cast some light on the human processes. We shall make a few comments on the latter programs—those aimed at an artificial intelligence goal—then return to those that are avowedly psychological in intent.

Most existing theorem-proving programs, including the most powerful ones, were designed not to show how mathematicians discover proofs for theorems, but to discover proofs, by whatever means, as efficiently and powerfully as possible. This approach, exemplified by the work of Wang, Robinson, Lehmer, and others, is discussed in several other papers in this issue. The Geometry-Theorem Proving Machine of Gelernter and Rochester, and Slagle's Symbolic Automatic Integrator (SAINT) are additional examples. (For a good sample of artificial intelligence programs, see [3].) The interest of these programs for psychology, which is considerable, is in showing what kinds of mechanisms may be sufficient to give a program power to match, or exceed, human performance, in professional-level tasks for which we have a wealth of human experience. There is no claim that these programs simulate, in detail, the processes used by humans; but they demonstrate what kinds of search heuristics provide problem-solving power in these well-known task environments. The Geometry program, for example, shows how search for a proof can be aided by using a diagram as a source of conjectures about properties of the geometrical figure under consideration.

Most programs for playing chess, checkers, and other games should also be classified as investigations of artificial intelligence. Samuel, for example, in his very powerful checker-playing program, makes no pretense of simulating human play. His program almost certainly makes much more use of rapid numerical and nonnumerical calculation, and much less use of heuristics that would permit search to be highly selective, than do good human players.

The considerable number of programs that have now been written for processing natural language tend to fall between the artificial intelligence and psychological simulation categories. Our ignorance of how humans interpret and produce linguistic utterances—how they extract meaning from, and encapsulate meaning into such utterances—has been so profound that any program that performs any of these functions can hardly help but cast some light on the psychology of language. Many of the early language-handling programs, aimed at applications to translation, had a primarily artificial intelligence goal. Some more recent programs, designed to retrieve information and answer questions, come closer, implicitly or explicitly, to providing psychological theories. Two question-answering programs are described in Part I, Section 5, of [3].

Daniel Bobrow has constructed (in [4]) a program to do algebra word problems of the sort found in high-school algebra books. The program operates in two stages. In the first stage, it translates the sentences stating the problem into algebraic equations. In the second stage, it solves the resulting equations. The program indicates what kind of a sentence-parsing scheme is required for the translation process—and the evidence it provides is that a rather crude and incomplete scheme will do the trick. Thus, the program contributes a good deal to explaining how humans with a very sketchy knowledge of formal grammar can nevertheless interpret language.

Bobrow's scheme for translating word problems resembles rather closely some of the standard advice on "how to do word problems" that can be found in books on algebra and the teaching of algebra. We were interested in seeing whether this advice also describes the processes that persons use who are good at such problems. We are currently tape-recording the thinking-aloud protocols of subjects solving word problems, and comparing the protocols with the process described by the program. Although the work (by Mr. Jeffery Paige) is still in a very preliminary stage, it already suggests that the processes used by the best problem solvers may be different, in one fundamental respect, from the computer program. The good problem solvers tend *not* to translate directly from the English prose to the algebraic equations (except, perhaps, in problems that are very simple for them). Instead, they translate the language into some kind of internal representation of the physical or spatial situation described by the problem, and then translate this "physical" representation into the equations. Their procedure is reminiscent of the Geometry Theorem Prover's use of diagrams.

The program that has perhaps been furthest developed and tested as an explicit psychological theory of human problem solving is the General Problem Solver (GPS). Since it has been described in some detail elsewhere, we will consider it only briefly here. (A description at flow-diagram level of the main mechanisms of GPS can be found in Part II, Section 1 of [3].) GPS is "general" because it is not restricted to a single problem domain, but can go to work on any task that is posed to it in suitable form: discovering proofs for theorems in the sentential calculus, for example, or proving trigonometric identities. Its

basic mode of procedure is to compare the given situation (axioms, available theorems, etc.) with the desired situation (theorem to be proved, for example). The difference between these leads to the selection of an operator for transforming the given situation into a new one that, hopefully, will be closer to the desired one. In trying to prove a trigonometric identity, for example, GPS might detect that certain trigonometric functions occur in the left-hand expression that are absent from the right-hand expression. This would lead it to select a transformation that would eliminate one of the unwanted functions.

There is no guarantee in GPS, or in most of the other simulation programs, that the program will solve the problem given it, even if it is solvable. This is as it should be, for purposes of psychological theory, for human beings confronted with mathematical problems do not always solve these problems (to understate the matter!) even if the problem domain is decidable. The aim in simulation is not to get a computer to do things people can't do, but to construct a program that will have the same powers and weaknesses, use the same processes, and make the same mistakes as some class of human subjects. The aim of an artificial intelligence program, of course, may be quite different.

Information processing languages. It is not usually convenient to write simulation programs in machine language, nor are the algebraic languages like FORTRAN and ALGOL, so useful in numerical analysis, particularly suited to this purpose. A number of computer programming languages have been devised that are especially designed for the construction of complex, nonnumerical, information processing programs. IPL-V, the current version of the earliest family of such languages, has been widely used for psychological simulation. LISP has been used extensively in artificial intelligence programs. (The division of labor here is probably an historical accident.) COMIT was designed especially to make the processing of natural language strings convenient.

The information processing languages cannot be described in detail here (see [2]). The characteristic that most distinguishes them from typical algebraic languages is that they organize information in memory in *lists*—well-ordered sets of symbols. For this reason, they are often called list-processing languages. In such languages, information is accessed normally by coursing down lists (using a NEXT operation) until the desired symbols are located. In addition to lists (whose members may themselves be lists), list-processing languages also make extensive use of *descriptions*. Descriptions are ordered triads: for example, (apple, color, red), which is interpreted "the apple has the color red."

If we refer back to the sequence extrapolation task, we can see how natural a list language is for handling that task. The "familiar alphabets" the subject is assumed to know are simply stored in memory as lists. The relation of NEXT between two symbols is then directly representable by the NEXT operation on lists in the information processing language. A statement like "$x_{1i} = N(x_{1(i-1)})$" can simply be stored as the description: ($x_{1(i-1)}$, NEXT, x_{1i}).

It is far from clear to what extent the adaptability of list processing lan-

guages to simulation is purely a matter of programming convenience, and to what extent it arises from the parallelism between the elementary processes and structures in those languages, on the one hand, and the elementary processes and modes of storage in the central nervous system, on the other. Whether the central nervous system stores information in ways that are more or less isomorphic to lists and descriptions, and possesses processes for operating on such structures is an open question whose answer must be sought at a more microscopic level of theory than we have considered. It is a question of the relation of information processing theories to neurophysiology.

The work reported here was supported in part by Research Grant MH-07722-01 from the National Institutes of Health. We are also indebted to Kenneth Kotovsky and Donald Williams who have collaborated in the analysis of the sequence extrapolation task.

References

1. H. A. Simon and Kenneth Kotovsky, Human acquisition of concepts for sequential patterns, Psychological Review, 70 (1963) 534–546.

2. A. Newell *et al*, Information processing language—V Manual (2nd ed.) Prentice-Hall, Englewood Cliffs, N. J., 1964.

3. E. Feigenbaum and J. Feldman (eds.), Computers and thought, McGraw-Hill, New York, 1963.

4. D. Bobrow, 1 Natural Language Input for a Computer Problem Solving System, Mass. Inst. of Tech., Project MAC-TR-I, September 1964.

THE PROCESSING OF FILES, DATA AND INFORMATION

C. C. GOTLIEB, Institute of Computer Science, University of Toronto

Information processing. Computers can be regarded as devices for operating on streams of input data to produce output streams. The input may be a series of numbers undergoing Fourier analysis, a list of statements written in source language and being compiled into assembly language, or the file of records describing an inventory being updated by entering changes which have been made to stock. The term *information processing* is used for all of these transformations, or for any others carried out by programmed computers. *Business data processing*, or data processing, refers to transformations where the data originates in a business environment. Often the emphasis is on file processing, which is described in detail in the next section, but this is not the only type of business data processing. There has been good success in applying mathematics to some of the manufacturing and administrative operations of large business and when this happens the differences between business data processing and scientific or mathematical computing are by no means sharp. Almost all of the production scheduling for oil refineries is done on computers by numerical procedures which are now standard [1], but such data processing is generally considered to be part of mathematical programming or operations research. The widely-used techniques of Critical Path Scheduling and PERT [2], for monitoring on large construction projects involving many contractors and subcontractors, are really methods in data processing. This paper contains a survey of operations in business and file processing, with particular emphasis on problems and techniques which are mathematically interesting.

Files. Most computers have been installed to do business data processing, and in most installations this comes down to some variation of file processing. In *file processing* the essential technique is that a master file, containing descriptive records about the items of interest, is maintained up-to-date as various transactions involving the items arise. The master file may be a payroll register, an inventory of goods, or of seats in an airline reservation system, an insurance policy file, a set of grades for university students, a history of vehicle registrations, etc. It may be recorded on punched cards, punched paper tape, magnetic tape, magnetic drums or disks, or other media. The activities against the file may be processed periodically, when a batch of certain size has accumulated, or as they occur. Figure 1 illustrates the inputs and outputs for a typical file processing run when the transactions are batched. An essential feature is that the master file is arranged sequentially according to an identification number or key, and that previously the transactions must have been sorted according to this key. The updated-master-file will be the master file for the next run. In a separate computer run the contents of the transaction tape will be printed to prepare cheques, invoices, or other forms appropriate to the application.

The importance of sorting in processing serially arranged files has encouraged

the development of efficient sorting methods, and the result has been the evolution of fast, reliable, magnetic tape units, the production of vast and ingenious programs, and even fundamental contributions to the theory of combinations and permutations. In the first stage of a large sort a group of records is sorted internally, i.e. within the high speed memory of the computer. There are many ways of doing this, but most are variants of three basic methods: (1) where the items being sorted are placed in pockets according to the value of a digit in the key, (2) where the smallest (or largest) item is selected from the group, (3) where pairs of items are exchanged if necessary to form a sequenced pair, following which pairs are merged to form ordered strings of length four, strings of length four are merged to form strings of length eight, and so on. In the first method the number of key comparisons (and hence the sorting time) is proportional to $n \log_p K$, where n is the number of records in the group, K the largest key, and p the number of pockets (equal to the radix of the number system in which the keys are represented). In the second method the sorting time is proportional to n^2, and in the third to $n \log_2 n$. Thus theoretically the last method is best for large n, but other considerations also enter, especially the size of the working storage required by the program. Later stages of the sort consist of merging sequenced groups to produce tapes, each containing a single string, and then merging the tapes to yield a sorted file. Sort programs usually take the form of *generators*, in which sorting routines for a particular application are assembled by the computer from basic blocks of coding embodied in the generator. Reference 3 contains good accounts of recent developments in sorting.

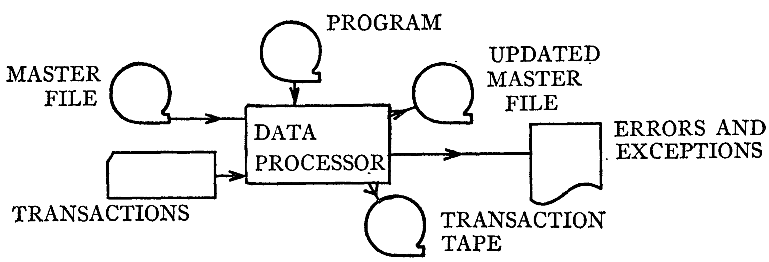


FIG. 1. File Processing

The emergence of large, random-access files, and more particularly magnetic disk stores in which the access time for any record is a fraction of a second, is focussing attention on methods for storing, searching and retrieving information in files. There are challenging problems in formulating these methods in mathematical terms. For example, suppose it is desired to subject a set of keys to a simple transformation which will yield numbers (to be interpreted as addresses in a disk memory) distributed uniformly over an assigned range. Note that this reduction to randomness would be regarded as a loss of vital information in most contexts but here it can be shown to result in a more efficient use of the store and a reduction of the access time for a record. The transformation can be con-

structed by a procedure available from the theory of error-constructing codes, which in turn is based on the algebra of fields [4]. Suppose that the key K is represented by n symbols $(a_1, \cdots, a_n)$ each of which is represented by q bits, and that the transformed address, A, has m symbols $(p_1, \cdots, p_m)$. The key K and address A are interpreted as n and m dimensional vector spaces over the field of integers mod 2^q. Let the weight w, for any key vector be defined as the number of nonzero symbols, and the distance, d, between two keys as the weight of $x-y$. For the addresses to be randomly distributed, it can be shown that the transformation should have the property that any two keys less than a given distance d from each other are to be mapped on different addresses. We also want d to be maximal for the given set of keys. In the references quoted the method of finding the maximal D is given and it is shown further that the transformation can be effected by dividing K, regarded as a polynomial,

$$K(x) = \sum_{i=1}^{n} a_i x^{i-1}$$

by the polynomial $g(x) = (x-\alpha) \cdots (x-\alpha^{D-1})$, where α is a primitive element of the Galois Field (2^q). That is, we find the remainder $R(t)$ defined by

$$K(x) = Q(x)g(x) + R(x) \qquad \text{degree of } R(x) < D - 1$$

and the digits of $R(x)$ form the transformed address, with the property that the minimum distance between two keys with the same R is at least D.

The most intensive research on files is taking place in the field of information retrieval where there is a whole family of problems on methods for abstracting and classifying documents, on methods for devising keys which will not only identify a record, but reveal the logical relation between important terms in it, and on methods for constructing and assessing search strategies. In one promising approach [5] the important terms (or descriptors) are represented as the vertices of a graph and connections between the terms as edges of the graph. The basic problem can then be stated as: given two graphs, determine whether one contains a subgraph isomorphic to the other. When represented as Boolean matrices, graphs are easily manipulated by computers, and since we have a finite system an algorithm for solving the problem clearly exists. On the other hand, it is well known that combinational expressions very soon become intractably large as the number of elements increases and it becomes necessary to restrict the basic problem in different ways if there is to be any hope of practical solutions. In spite of the necessity to work with restricted problems, it is easy to think of natural generalizations of the basic problem, for example where the vertices may be connected by edges of different types, or where the number of edges connecting a pair of vertices is irrelevant.

Systems. Any data processing run on a computer is but one operation in a set of activities being prosecuted by the organization under study. The definition and representation of these inter-related activities is called *systems analysis.*

A common way to present the results of system analysis is in the form of system flow charts in which boxes representing specific procedures are drawn to show the sequence of events. The procedure within one box may be specified further by a more detailed flow chart, and these charts may serve as flow diagrams for a computer program. Whereas in scientific data processing and mathematical computing flow diagrams can often be omitted, in business data processing they are almost always necessary because the processing is usually a sequence of elementary calculations and steps, but with complicated decision rules for determining which step should be done next.

Instead of describing a complex set of decisions and alternative actions by a flow diagram a tabular method based on so-called decision tables can be used with considerable effect. A decision table is a concise way of making a set of statements of the form:

If B is true then do S_1 else do S_2,

where B is a boolean expression evaluated by carrying out logical operations on defined operands, and S_1, S_2 are component statements, each indicating a number of actions to be taken. In the usual form decision tables are divided into four quadrants: 1) the condition stub where the operands are listed, 2) the condition entry containing values of the operands (True, False, or X, irrelevant), 3) the action stub listing the types of actions, and 4) the action entries listing the values which specify the action types (see Fig. 2). Systematic ways of constructing decisions and efficient algorithms for implementing them by computer programs are still being developed [6]. The problems resemble, in some respects, those arising in switching theory and the logical design of circuits where sophisticated and formal design techniques have been evolved.

	Stub	Entry		
Condition	β_1	T	F	T
	β_2	T	T	F
	β_3	X	X	F
Action	Set Net to	0	0	100
	Go To	Statement 7	Error Routine	Exit

FIG. 2. A Decision Table.

PERT (Program Evaluation and Review Technique) and CPA (Critical Path Analysis) are data processing techniques which were introduced during the development of large military projects where hundreds of contractors and sub-contractors must coordinate their activities. The techniques are finding successful application in other construction projects. The overall project is regarded as a set of discrete activities giving rise to events. For each event there are predecessors, and successors, and for each activity there are minimum,

expected, and maximum durations, and in some versions of PERT, costs. A representation of the network which may contain thousands of activities is stored in the computer.

There are routines to detect inconsistencies in the data (e.g. loops, or events without predecessors), trace the critical path (a chain of activities such that a delay anywhere along the chain causes a corresponding delay in the final event), calculate the float times for activities (the latest starting date which will not cause a delay, minus the earliest possible starting date) etc. Moreover, if reports on the progress of the work are fed to the computer periodically, then jobs which are not on schedule and which are affecting the critical path can be noticed immediately. It is obvious that PERT networks are graphs and in fact many of the subroutines which have been developed for PERT are manipulations of graphs represented as precedence matrices. In particular there are various methods of sequencing events (topological ordering), detecting components which are disconnected, merging components, etc. The number of different PERT type programs, and the size of some of these programs, are very large indeed.

Languages. There is no theory of data processing in the sense that there exists a unified collection of principles and theorems along with methods of applying these to obtain solutions to particular problems. There have been several attempts at rigorous definition of concepts and elementary operations so that procedures on data processing can be described concisely and unambiguously.

Sometimes the very difficult task of specifying the overall operation of a business has been undertaken [7, 8]. Essentially all the components of the business must be identified, the functions of each component stated explicitly, the flow of products and information described and so on, all expressed in a suitable terminology. No one of these languages for the overall description of a system has gained general acceptance. But more restricted languages, for describing algorithmic procedures of business in a form which is easily translated to a computer program, are used widely.

COBOL (Common Business Oriented Language), the result of a cooperative effort of computer manufacturers, is being considered as a standard language [9]. Statements are written so as to resemble English, a typical example being:

IF A EQUALS B AND X IS GREATER THAN Y ADD C TO D.

Particular care has been given to data definition, file description and input/output specification in COBOL.

The most comprehensive programming language is that proposed by Iverson [10]. This has a rigorously defined notation not only for ordinary computer instructions, but also for bit manipulation, microprogrammed instructions (describing the operations carried out by the computer hardware), file operations, logical and matrix computations, etc. Included also is a notation for describing data structures represented by trees and lists. Iverson's notation is precise and

powerful, but its acceptance has been slow because there are many strange forms in it, even for common matrix and arithmetic operations. As examples:

If $\mathbf{U}$ is a matrix, with elements u_{ij},

$$+//\mathbf{U} = \sum_i u_{ij},$$

and $j \uparrow \alpha^k(n)$ represents a vector with n components, of which the first $k-j$ and the last j are ones, and the rest are zeros.

Simulation languages such as SIMSCRIPT [11] and GPS [12] have been developed especially to describe the elements and procedures which make up a system. They are useful in setting up models of a complex operation and investigating the behaviour as system parameters are varied. In a sense they are alternatives to a theoretical analysis of a business environment, much as a Monte-Carlo calculation is an alternative to an analytical solution of a physical problem. Although the information gained from such models does not have the value of a general solution, it is useful when mathematical formulation of the problem is not possible, or when the solution of the governing equations is too difficult to obtain.

In summary, some of the concepts of modern algebra, discrete mathematics and combinatorial theory have already found application in data processing problems. There is undoubtedly a rich harvest awaiting mathematicians who are willing to examine these problems and are able to formulate them in ways which will allow the powerful ideas of modern mathematics to be brought to bear on them.

References

1. J. S. Arnofsky, Growing applications of linear programming, Comm. ACM, **7** No. 6, (June 1964) 325–332.

2. A. B. Kahn, Skeletal structure of PERT and CPA Computer Programs, Comm. ACM **6** No. 8 (August 1963) 473–479.

3. D. Teichrow (ed.), Papers presented at ACM Sort Symposium, Comm. ACM **6** No. 5 (May 1963) 194–281.

4. G. Schay and N. Raver, A method for key-to-address transformation, IBM J. Res. and Devel., 1 No. 2 (April 1963) 121–126. (See also Hanan and Palerno, An application of coding theory to a file address problem, *ibid.*)

5. G. Salton and E. H. Sussenguth, Jr., Some flexible information retrieval systems using structure matching procedures, AFIPS Conf. Proc., 15 (1964) 587–597.

6. J. R. Slagle, An efficient algorithm for finding certain minimum-cost procedures for making binary decisions, J. ACM 11 No. 3 (July 1964) 253–264.

7. J. W. Young and H. R. Kent, Abstract formulation of data processing problems, J. of Ind. Engineering, IX No. 6 (Nov. 1958) 471–479.

8. J. C. Miller, Conceptual models for determining information requirements, AFIPS Conf. Proc., 25 (1964) 609–620.

9. COBOL—1961, Revised specifications for a common business oriented language, Superintendent of Documents, Washington 25, D. C.

10. K. E. Iverson, A programming language, Wiley, New York, 1962.

11. J. M. Markowitz, B. Hardsner and H. W. Karn, SIMSCRIPT—A simulation programming language, Prentice-Hall, Englewood Cliffs, N. J., 1963.

12. General purpose simulation II, Reference Manual, B 20-6346, IBM, N. Y.

A PERSPECTIVE VIEW OF DISCRETE AUTOMATA AND THEIR DESIGN

C. C. ELGOT, IBM Watson Research Center
and Courant Institute of Mathematical Sciences, New York University

Abstract: Some relationships between the theory of Turing machines and the theory of finite automata are discussed. An attempt is made to indicate some relationships between these mathematical theories and the physical devices called "digital computers."

1. Introduction. In recent years a new branch of automata theory has been intensively developed which has been variously called "theory of finite automata," "theory of (finite memory) sequential machines," "theory of sequential circuits" and by many other similar phrases. The relationships between the older theory of Turing machines and this younger theory have often been confounded by people not working in these fields. Indeed, it is not unusual to find the theories identified. This identification finds support in some claims, on the one hand, that finite automata are "good" mathematical models of digital computers and, on the other, that Turing machines are. The identification is sometimes jarred by the realization that almost every question of the type "is there an algorithm such that . . . ," which deals with finite automata turns out to be algorithmically answerable while in the case of Turing machines, the questions turn out to be algorithmically unanswerable.

This note displays some relationships between the theory of Turing machines and the theory of finite automata and attempts to indicate how they are related to the physical devices called "digital computers." In order to facilitate the discussion of these relationships a nonstandard terminology is employed for the Turing machine discussion. The concepts and terminology are largely drawn from [7]. The kinds of questions discussed are oriented toward automata design.

2. Background. Algorithms have played a significant role in mathematics for a long period of time. An algorithm which antedates antiquity for finding the greatest common divisor of two positive integers is so fundamental it is still being taught in the schools. Another familiar algorithm is that for finding the ith digit in the decimal expansion of $\sqrt{n}$. Less familiar and more complicated is an algorithm, based upon an extension of Sturm's theorem, for deciding whether a given sentence of elementary algebra (or elementary geometry) is true or not, (cf. [3]).

Now suppose one seeks an algorithm—let's say for discovering whether a given polynomial with integral coefficients has integral solutions (Hilbert's 10-th problem)—and fails. What does one conclude? Before this century such a failure was wont to be interpreted as reflecting a deficiency in one's intellectual prowess or one's assiduity. It was a striking achievement of the twentieth century which produced a mathematical explication of the intuitive notion "algorithm" and which showed that there exist problems (the word problems

for semi-groups or for groups, for example) which are algorithmically unsolvable, (cf. [13], [10], [8]). One such explication, which in a certain precise sense, is equivalent to a host of others, was given by A. M. Turing [2] in 1936 via a notion which he called a "machine." A closely related formulation was independently obtained by E. L. Post [6] in the same year. Before discussing Turing machines, it is worth noting that the applications of the theory of algorithms (which may be taken as co-extensive with "theory of computability" or "theory of recursive functions") is not restricted to showing certain problems to be unsolvable but has some positive facets as well. For example, a much deeper understanding of Gödel's profound results of 1931 [9] and its relationship to Tarski's [4] is obtainable via this theory.

3. Turing machines. In order to point to some similarities and some dissimilarities between universal Turing machines and general purpose digital computers, a certain amount of detail in the brief discussion given here is necessary. Intuitively one thinks of a Turing machine (cf. e.g. [10]) as a device which consists of a two-way infinite tape which is ruled off into squares (also called cells). At any instant of time, each square contains a symbol from a finite subset of the alphabet $\{S_0, S_1, \cdots\}$. If the integers are used as names of the squares, then the *tape content* at any instant may be represented by a function k from the integers into the set $\mathcal{S}_r = \{S_0, S_1, \cdots, S_r\}$, say. At any instant of time a certain square n will be distinguished and called the *square under scan*.

In addition to its tape, a Turing machine possesses a fixed and finite part which will be called its *control*. The control is capable of assuming a finite number of *internal states*. The internal states will be thought of as consisting of a finite number of *instruction locations*, one of which is designated as the location of the *current* instruction, each instruction location containing an *instruction code*. The internal state of the control at a given instant is determined by the current instruction location. The instruction locations will be taken as elements of $Q = \{q_1, q_2, q_3, \cdots\}$. A *state* of a Turing machine will be defined as an ordered triple $\langle k, n, q\rangle = \langle\langle k, n\rangle, q\rangle$, with k a function as above and n an integer, $q \in Q$; $\langle k, n\rangle$ is the *tape state* and q the current instruction location.

Informally, the action of a Turing machine may be described as follows. The machine starts in a certain tape state and starts following the instruction at location q_1. Pictorially imagine there is an arrow pointing to the scanned square. Then the tape state may be pictured as consisting of the two way infinite tape marked with symbols from $\mathcal{S}_r$ together with an arrow pointing to one of the squares. The effect of an instruction is now described.

Depending on the symbol scanned the instruction specifies the next instruction location and one of the following acts (or outputs): (a) print S_i on the scanned square (after first erasing what was there), where i may be any number between 0 and r; (b) move the arrow one square to the right; (c) move the arrow one square to the left.

Thus the machine starts by executing the instruction at location q_1. The out-

put of the instruction brings about a change (in general) in tape state; the instruction also specifies a new instruction location. Now the instruction at this new instruction location is executed and the process is iterated.

To introduce a static element into this dynamic picture of the action of Turing machines, a fourth alternative is usually added to (a), (b), (c) above. This fourth alternative is "HALT" or "STOP".

This means that the state of the machine undergoes no further change. It is, however, not necessary to employ such an explicit HALT. If at a particular instant of time a machine is in state $\langle k, n, q\rangle$ and if execution of the instruction at location q results in the same state, then the state after this is also the same and so is the state after this, etc. Thus, "no change of state" may be made to play the role of HALT. The reader who pursues this matter further will easily verify for himself that this difference from the more usual treatment results in no change of "computing power." "Instruction" is now defined more mathematically.

4. Instructions. An *instruction* of a Turing machine will be understood to be a mapping from states $\langle k, n, q\rangle$ into states $\langle k', n', q'\rangle$. The particular instructions defined below correspond to Turing machines as treated in [10]. The mapping which takes $\langle k, n, q\rangle$ into $\langle k', n', q'\rangle$ for a machine with alphabet $\mathcal{S}_r$ may be described by a conjunction of $r+1$ statements P_i, $0 \leqq i \leqq r$. Each P_i is a statement of one of the following three forms: For all k, n, q, k', n', q', if $k(n) = S_i$ then

(a) $\quad q' = q^i,\ n' = n,\ k'(n) = S$ and for all $m \neq n$, $k'(m) = k(m)$,

(b) $\quad q' = q^i,\ n' = n + 1$ and $k' = k$,

(c) $\quad q' = q^i,\ n' = n - 1$ and $k' = k$.

Here $q^i \in Q$ and $S \in \mathcal{S}_r$. The possibility that $q^i = q$ or $S = S_i$ (or both) is not ruled out. The three forms may be abbreviated as follows:

(a) $\langle S, q^i\rangle$, (b) $\langle R, q^i\rangle$, (c) $\langle L, q^i\rangle$.

A particular Turing machine is determined by a finite function table of the adjoining kind. (See Table 1.)

In the table each x_j^i is a specific element of $\mathcal{S}_r \cup \{R, L\}$ and each q_j^i is a specific element of $Q_p = \{q_1, q_2, \cdots, q_p\}$. Thus, the table describes a function from $Q_p \times \mathcal{S}_r$ into $(\mathcal{S}_r \cup \{R, L\}) \times Q_p$. If one deletes from the table those rows which correspond to no change of state, one obtains a *contracted* table which contains the same information. Each row of the tables is a quadruple. The set of quadruples of a table is called a Turing machine in [10].

It should be noted, however, that Davis employs, in effect, a special "halt" instruction and does not identify "no change of state" with "halt" which, in effect, is done here.

TABLE 1

Current Instruction Location	Input or Symbol Scanned	Output	New Instr. Loc.
q_1	S_0	$\overset{1}{x}_0$	$\overset{1}{q}_0$
q_1	S_1	$\overset{1}{x}_1$	$\overset{1}{q}_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
q_1	S_r	$\overset{1}{x}_r$	$\overset{1}{q}_r$
q_2	S_0	$\overset{2}{x}_0$	$\overset{2}{q}_0$
q_2	S_1	$\overset{2}{x}_1$	$\overset{2}{q}_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
q_2	S_r	$\overset{2}{x}_r$	$\overset{2}{q}_r$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
q_p	S_0	$\overset{p}{x}_0$	$\overset{p}{q}_0$
q_p	S_1	$\overset{p}{x}_1$	$\overset{p}{q}_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
q_p	S_r	$\overset{p}{x}_r$	$\overset{p}{q}_r$

The matrix consisting of the first $(r+1)$ rows and last 3 columns of the original table may be regarded as an instruction code which is located at instruction location q_1. The matrix consisting of the next $(r+1)$ rows and last three columns is the instruction code located at q_2, etc.

It is worth noting that the machines of [6] are based upon simpler instructions. A table describing such a machine would have the output column depending only on the current instruction location and not on the symbol scanned. Indeed, the only instructions which take cognizance of the symbol scanned are ones which make no change in the tape state. In digital computer parlance these are "conditional transfer" instructions.

Remark: It is interesting to observe in passing that the outputs R, L may be interpreted alternatively as keeping the square scanned fixed and shifting the tape content one square relative to the tape. With this interpretation the square scanned may always be taken to be square 0. While this interpretation promotes simplicity (for example, in the notion tape state), the points we wish to make are more readily made with the interpretation adopted.

Turing machines become very much more like digital computers if one permits a broader instruction set. To facilitate making the point clearly, let it now be supposed that the integer which designates the scanned square is the content of a "special" cell s and let the tape content now be given by a function k on $Z\cup\{s\}$, where Z is the set of integers. (Now tape content and tape state are synonymous.) Then the output and new instruction location always depend on

$k(k(s))$. The content of only one cell is affected by an instruction and this cell is either s or $k(s)$. (Actually in Turing's original formulation, two cells would in general be affected, viz. s and $k(s)$.) On the other hand the effect of a digital computer instruction may depend on $k(x)$ and $k(y)$ and affect cell z where x, y, z may range through a large set of values. To put this matter succinctly: Turing machine instructions do employ instruction locations (or addresses) but do *not* employ "data" addresses; digital computer instructions typically employ both instruction and data addresses. (The set of addresses of digital computer instructions are often ordered. There are some variants of the Turing machine notion in the literature which also employ ordered (instruction) addresses.) Yet other differences involve the use in digital computers of a variety of "memories" with different characteristics. A discussion of these various memories is outside the scope of this note. It may occur to the reader familiar with digital computers that instructions of (general purpose) computers have an important facility that has not yet been mentioned, viz. the ability to modify themselves. This is more appropriately taken up after discussing "universal" Turing machines.

5. Computations. The *direct transition function* ν (for this Turing machine notion) is a function of two variables one of which varies over particular Turing tables and the other over states. If T is a Turing table and $\sigma = \langle k, n, q\rangle$ is a state and $\nu_T(\sigma) = \sigma'$ then $\sigma' = T_q(\sigma)$, where "T_q" denotes the instruction at location q. The *transition function* ν^* is a function of 3 variables; the first variable ranges over Turing tables, the second variable over the nonnegative integers N and the third over states. The function ν^* is defined inductively by the requirement that it satisfy

$$\overset{*}{\nu}_T(0, \sigma) = \sigma \quad \text{and} \quad \overset{*}{\nu}_T(n + 1, \sigma) = \nu_T(\overset{*}{\nu}_T(n, \sigma)).$$

The function from N into the set of states obtained from ν^* by fixing T, σ is called the (*Turing machine*) *computation* determined by T, σ and is denoted by "$\text{comp}_T(\sigma)$." A computation is *successful* if there exists an n such that $\nu_T^*(n+1, \sigma) = \nu_T^*(n, \sigma)$. If such an n exists, let n_0 be the smallest such. Finally a function i of two variables is introduced as follows: $i_T(\sigma)$ is defined if and only if $\text{comp}_T(\sigma)$ is successful; if $\text{comp}_T(\sigma)$ is successful then $i_T(\sigma) = \nu_T^*(n_0, \sigma)$. It is readily verified that if $i_T(\sigma)$ is defined, $i_T(\sigma) = i_T(i_T(\sigma))$. Thus $i_T(\sigma)$ may be described as the last state in the computation determined by T, σ provided a last (or more accurately, stable) state exists.

6. Functions computable by Turing machines. In order to compute functions on N via Turing machines it is necessary to have some means for representing the elements of N on the Turing machine tape. An r-tuple $\langle m_1, m_2, \cdots, m_r\rangle$ of nonnegative integers will be represented by any tape state $\langle k, n\rangle$ such that

$$k(n) = S_1 = k(n + 1) = \cdots = k(n + m_1),\ k(n + m_1 + 1) = S_0,\ k(n + m_1 + 2)$$
$$= S_1 = k(n + m_1 + 3) = \cdots = k(n + m_1 + m_2 + 2),$$

for the next argument k assumes the value S_0, for the next m_3+1 arguments k assumes the value S_1, etc., for all other arguments k assumes the value S_0. Thus an infinite number of tape states correspond to the same r-tuple. Had the alternative of the Remark of Section 4 been adopted, however, the correspondence between tape states and r-tuples would be 1-1.

A function f, say of two variables, defined on a subset of $N \times N$ with values in N is said to be *computable by a Turing machine* T if for all x, y, τ, σ, where τ is a tape state which represents $\langle x, y\rangle$ and $\sigma = \langle \tau, q_1\rangle$, (a) if $\langle x, y\rangle$ is in the domain of f, then $i_T(\sigma)$ is defined and if $i_T(\sigma) = \sigma' = \langle \tau', q'\rangle$, then τ' is a representation of $\langle x, y, f(x, y)\rangle$, (b) if $\langle x, y\rangle$ is not in the domain of f, then $i_T(\sigma)$ is not defined.

In a similar way one defines what is meant by "T computes f" where f is a function of one, three, four, etc. variables.

Now, the number of Turing machines is denumerably infinite since each Turing machine is describable by a finite table whose entries are chosen from a denumerably infinite set. Let $p(0) = 2$, $p(1) = 3$, $p(2) = 5$, $p(3) = 7$, $p(4) = 11$, $\cdots$ be an enumeration of the prime numbers. Given any positive integer n, let $d(n)$ be the function from N into N which takes i into the exponent of $p(i)$ in the (unique) prime decomposition of n. The mapping d is a 1-1 mapping from positive integers onto the functions from N into N of finite support, i.e. those functions which assume the value zero except for a finite number of arguments. By assigning numbers to the entries of the Turing tables and making use of the mapping d, one may assign numbers (cf. [10]) to Turing machines in a *uniform* way. Similarly, numbers may be assigned to tape states $\langle k, n\rangle$ where k is a function which assumes the value S_0 except for a finite number of arguments.

Turing proved, in effect, that there is a machine U which computes a function f of two variables that satisfies the following property: (a) if t is a number of the Turing machine T and T computes the function g of a single variable, then $f_t = g$, where f_t is the function of one variable obtained by fixing the value of the first variable of f to be t; (b) for any $n \in N$, f_n is computed by some Turing machine.

Such a machine has been called *universal.* It is profitable to view universal Turing machines in the image of digital computers. To this end consider the nature of a plan to design (i.e., specify the table of) a universal machine U.

7. Gross outline of plan. Starting with an initial state $\langle k, 0, q_1\rangle$, where the tape state $\langle k, 0\rangle$ represents $\langle t, x\rangle$ and where t is the number of a Turing machine T, the universal machine U determines the number of the tape state $\langle k_1, 0\rangle$ where $\langle k_1, 0\rangle$ represents the number x. By "decoding" t which is "stored" on U's tape, U determines what the instruction of T is, at location q_1. Then U determines and records on its tape the next instruction location of T and determines the number of the next tape state of T and records that on its tape. Now U seeks on its tape the last instruction location recorded, decodes t to find what the instruction is, etc. In order for U to determine whether or not T's computa-

tion is successful, U must test (and therefore have had recorded) two successive states in T's computation. Even though the computation is successful, T's last state may not represent an ordered pair whose first member is the given x. In this case, as well as the case that T's computation is unsuccessful, the function f of a single variable computed by T is not defined for the argument x.

In case $f(x)$ is defined, U determines this and records $f(x)$ on its tape in proper position.

8. Digital computer control. There is an aspect of the design of digital computers which bears a close resemblance to the aspect of universal Turing machine design discussed in the previous section. The number t of a Turing machine T is analogous to what is called a program for a digital computer. A program may be regarded as a coded description of a machine. The code used has a special property, viz. each cell contains a code word for an instruction. The contents of several cells taken together constitute a description of a machine. In the design of a general purpose digital computer one must, as in the case of Turing machines, decode the machine description, determine and record the new instruction location (frequently in a special "cell" called the instruction counter), compute and record the output of the instruction, then return to the coded machine description to find the new instruction, etc. Because of the addressing facility of the digital computer, the instruction code word at an instruction location may be "immediately" found. In the case of the universal Turing machine, the instruction code word at a given location has to be computed. Unlike, too, the plan of section 7, the general purpose digital computer computes directly on the "tape content" of the described machine and not on a number which represents the tape state of the described machine. It may be noted that the salient property of a general purpose digital computer that it permits alteration of (computation on) the machine description, is shared by a universal Turing machine.

If one attempts to simplify the design of a universal Turing machine by using as a coded description of a machine T a finite sequence of code words each of which is to occupy a single cell, one runs into the difficulty that this requires an infinite number of symbols S_i, since there are an infinite number of instructions. Of course codings simpler than the one indicated exist, but more than one cell per instruction code is required.

The matter of how instructions are to be "executed" has received little or no attention in the design of universal Turing machines. In the case of digital computers, however, the design of execution (arithmetic) units is of considerable importance.

9. Sequential machine equivalence. Let T, T' be two Turing (uncontracted) tables over the same alphabet $\mathcal{S}_r$ and let Q_T, $Q_{T'}$ be their respective sets of instruction locations. The tables T, T' will be said to be *sequential machine equivalent*, or equivalent in the sense of the theory of sequential machines, *via the*

binary relation R, $R \subseteq Q_T \times Q_{T'}$, provided that (a) the set of all first components of R is Q_T, (b) the set of all second components of R is $Q_{T'}$ and letting f_T (resp. g_T) be the function (f_T is the *next instruction location* function of T and g_T is the *output* function of T) which takes the first two elements of a row of T into the fourth (resp. third) element of the same row of T, (c) for any q, q' such that qRq' (i.e. $\langle q, q'\rangle \in R$) and for any $S \in \mathcal{S}_r$, $f_T(q, S) R f_{T'}(q', S)$ and $g_T(q, S) = g_{T'}(q', S)$.

The tables T, T' are *sequential machine equivalent* if there exists such an R. The binary relation R induces a binary relation D on machine states as follows: $\langle k, n, q\rangle D \langle k, n, q'\rangle$ if and only if $k = k'$, $n = n'$ and qRq'. It is readily verified that

THEOREM. *If T and T' are sequential machine equivalent via R and $\sigma D \sigma'$ then $\nu_T(\sigma) D \nu_{T'}(\sigma')$. Conversely, for any R satisfying* (a) *and* (b), *if for any states σ, σ', $\sigma D \sigma'$ implies $\nu_T(\sigma) D \nu_{T'}(\sigma')$, then T and T' are sequential machine equivalent via R.*

This simple theorem is important for understanding how the theory of Turing machines is related to the theory of sequential machines. In the theory of Turing machines, one is frequently interested in whether T and T' compute the same function (cf. Section 6), say, of one variable. If T and T' are sequential machine equivalent, then T and T' compute the same function but *not conversely.* Indeed, there is an algorithm for deciding given any T, T' whether they are sequential machine equivalent or not, but it is known (assuming one accepts that "algorithm" has been satisfactorily explicated) *no algorithm exists* for deciding given any T, T' whether they compute the same function.

It is worthwhile to pursue a consequence of the theorem. If the sequence $\sigma_0, \sigma_1, \sigma_2, \cdots$ is a computation of some Turing machine and $\sigma_i = \langle k_i, n_i, q^i\rangle$, call the sequence $\langle k_0, n_0\rangle, \langle k_1, n_1\rangle, \langle k_2, n_2\rangle, \cdots$ the *track* of the computation and the sequence $q^0, q^1, q^2, \cdots$ the *trace* of the computation. Then

COROLLARY. *If T and T' are sequential machine equivalent via R and $\sigma D \sigma'$ then the track of $comp_T(\sigma)$ equals the track of $comp_{T'}(\sigma')$.*

If sequential machine equivalence alone is being investigated it is not necessary to restrict oneself to Turing tables but one may include the broader class of *sequential machine tables.* These are tables like the Turing tables except that the restriction that the set of output symbols be a (particular) superset of the input symbols is removed. The set of output symbols is permitted to be any set whatever.

Thus a Turing table is a sequential machine table; the role of "R" and "L" is different, however, when a Turing table is "regarded as" a sequential machine table. See the definition of sequential machine computation given below.

Given a sequential machine table T with input alphabet $\mathcal{S}_r$, let $S^1, S^2, S^3, \cdots$ be any (finite or infinite) sequence of input symbols and let $q \in Q_T$. Then two sequences $q^1, q^2, q^3, \cdots$ and $O^1, O^2, O^3, \cdots$ are induced by the requirements: $f_T(q^i, S^i) = q^{i+1}$, $q^1 = q$, $g_T(q^i, S^i) = O^i$. It is easy to check that if T and T' have the same input alphabet and are sequential machine equivalent via R and

qRq' then the output sequence induced by T and q equals the ouput sequence induced by T' and q'.

The sequential machine notion also admits a description as a special kind of Turing machine. Again, given a sequential machine table T with input alphabet $\mathcal{S}_r$, let $S^1, S^2, S^3, \cdots, S^n$ be a finite sequence of input symbols and let $q' \in Q_T$. The direct transition function τ (for this sequential machine notion) is a function of two variables one of which varies over sequential machine tables and the other over states $\sigma = \langle k, n, q \rangle$. Let $k(x) \in \mathcal{S}_r \cup \{\beta\}$, where $\beta \notin \mathcal{S}_r$ and suppose $\tau_T(\sigma) = \sigma' = \langle k', n', q' \rangle$; τ_T is so defined that (1) if $k(n) = \beta$ then $\sigma' = \sigma$, (2) if $k(n) \in \mathcal{S}_r$, then $k'(n) = g_T(q, k(n))$, $k' = k$ elsewhere, $q' = f_T(q, k(n))$ and $n' = n+1$. We may now define τ^* from τ as ν^* was defined from ν. The function from N into the set of states obtained from τ^* by fixing T, σ is called the (*sequential machine*) *computation* determined by T, σ. If $\sigma = \langle k, 1, q^1 \rangle$ and $k(j) = S^j$, $1 \leqq j \leqq n$, and $k(n+1) = \beta$, and if $\tau_T^*(n, \sigma) = \langle k', n', q' \rangle$ then $\tau_T^*(n+1, \sigma) = \tau_T^*(n, \sigma)$, $k'(j) = O^j$, $1 \leqq j \leqq n$, $n' = n+1$, $q' = q^{n+1}$ and the trace of the (sequential machine) computation is $q^1, q^2, q^3, \cdots, q^{n+1}, q^{n+1}, q^{n+1}, \cdots$ (cf. previous paragraph).

10. Brief mention. If T is an arbitrary sequential machine table and q is a particular instruction location, then by the *behavior of the initialized table* T_q is meant the mapping which takes an arbitrary input sequence into its induced output sequence. Given a binary relation R between input and output sequences, an initialized table T_q is said to *satisfy* R if its behavior b fulfills the requirement: for any input sequence u, $uRb(u)$. Given a formal language in which R and T_q are expressible, questions of the following kind have been treated:

(a) is there an algorithm for deciding, given T_q and R, whether T_q satisfies R?

(b) is there an algorithm for deciding, given R, whether there exists a T_q which satisfies R?

(c) find an algorithm which produces, given R, a (resp. all, an optimal) T_q when one exists.

These problems have been construed as problems in the design of sequential machines. For a survey of results of this character (with the exception of the optimality alternative of (c)) see [1]. "Optimal sequential machine table" has been taken to mean "minimum number of instruction locations." This problem however has been treated only for the special case in which R is obtainable from a sequential machine table which may be "incomplete," cf. [12]. In this case the existence of at least one T_q satisfying the given R is always assured.

In the case of sequential machine tables with two output symbols, say 0, 1, the behavior T_q is characterized by the set of input sequences u of length n, $n > 0$, whose induced output sequence has 1 in position n. Call the class of all such sets $\mathfrak{J}$. The class $\mathfrak{J}$ has been characterized in an entirely different way, and *regular* (in the sense of Kleene [14]) *expressions* have been used to represent elements of $\mathfrak{J}$. The problem of finding "good" algorithms for passing from regu-

lar expressions to initialized tables which are "equivalent" has also been treated in the literature.

Sequential machines and sequential circuits are intimately related but the latter are outside the scope of this short note. The circuits are more detailed structures and there are many additional design problems characteristic of them. Some of these problems originate in the design of arithmetic units for executing instructions. The extensive bibliography of [5] mentions much of the pertinent literature. A recent interesting paper is [15]. At a still more detailed level, the design of combinational circuits which forms a part of the design of sequential circuits, has been treated in a vast literature. The problems most frequently treated are optimality ones with some attention to algorithm complexity.

This work was supported in part by AF 30(602)-3340, Rome Air Development Center.

References and additional reading

1. A. Church, Logic, arithmetic and automata, Proc. Intern. Congress Math., 1962.

2. A. M. Turing, On computable numbers, with an application to the Entscheidungs problem, Proc. London Math. Soc., 42 (1936–37) Section 2.

3. A. Tarski, A decision method for elementary algebra and geometry, Berkeley, 1951.

4. ———, Der Wahrheitsbegriff in den formalisierten Sprachen, Studia Philosophica, 1936. English translation in *Logic, Semantics, Metamathematics*, Oxford University Press, New York, 1956.

5. E. F. Moore, Sequential machines: Selected papers, Addison Wesley, Boston, Mass., 1964.

6. E. L. Post, Finite combinatory processes-formulation I, J. Symb. Logic, (1936) 103–105.

7. C. C. Elgot and A. Robinson, Random-access stored-program machines, An approach to programming languages, J. Assoc. Comp. Mach., 11 (1964) 365–399.

8. H. Hermes, Aufzahlbarkeit, Entscheidbarkeit, Berechenbarkeit, Springer-Verlag, 1961.

9. K. Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, Monatsh. Math. und Physik, 38 (1931) 173–198.

10. M. Davis, Computability and unsolvability, McGraw-Hill, New York, 1958.

11. R. C. NcNaughton, The theory of automata, A Survey in *Advances in Computers*, Vol. II, Academic Press.

12. S. Ginsburg, An introduction to mathematical machine theory, Addison-Wesley, Boston, Mass., 1962.

13. S. C. Kleene, An introduction to metamathematics, Van Nostrand, Princeton, N. J., 1952.

14. ———, Representation of events in nerve nets and finite automata, Automata Studies, Princeton University Press, 1956, pp. 3–41.

15. S. Winograd, On the time required to perform addition, IBM Report RC-1183.

LOGIC AND COMPUTERS

HAO WANG, Harvard University

1. Historical and philosophical background. Familiar connections between mathematical logic and automatic computers are the possibility of representing basic building blocks of computers by (sequential) Boolean functions and the close resemblance between programming languages and symbolisms of logic. As a result, both for the construction and for the use of computers, a certain degree of acquaintance with logic becomes indispensable. As early as 1656, Leibniz dreamed of a universal scientific language in his first published work; and many people today are actively seeking for a universal language for computers. Gottlob Frege wished to reduce arithmetic to logic from 1879 on, and in an oblique way the performance of arithmetic operations in computers by means of electronic circuits which are essentially logical functions may be said to accomplish the task in a particularly down-to-earth manner.

A more basic link between logic and computers is perhaps the common interest in algorithms. Although Charles Babbage conceived of and started to build in the 1830's the Analytical Engine which possessed most of the basic characteristics of modern computers, it was only in the 1940's that automatic computers began to appear through the efforts of Howard Aiken, John von Neumann, and others. The logicians had, on the other hand, made not only a highly successful abstract study of algorithms but even clarified the relation between machines and algorithms, largely through A. M. Turing's theory of idealized machines, all in the 1930's.

Traditionally the study of algorithms falls outside the domain of logic. The deepest source of the affinity of logic and computers is the preoccupation of logicians with formalization. The long evolution of attempts to formalize mathematical proofs, from Euclid to *Principia*, finally led to mechanizability as the ultimate criterion of complete success. This desire to make arguments formally precise and exact is concerned more with the product rather than with the theory of formalization. Only in the 1920's, D. Hilbert, P. Bernays and others began to study metamathematics: the theory of proofs, the theory of formal systems. The distinction may be illustrated by the method of "discarding 9's" for checking multiplications. It is a metamathematical result on the usual technique of multiplying particular integers that a number is divisible by 9 if the sum of its digits is. This example also brings out the fact that the distinction between mathematics and metamathematics is not sharp. For we can easily state and prove the above result as a simple theorem in number theory by the easy relation $10^n \equiv 1 \pmod 9$.

It was the concern with a theory of proofs which at first led J. Herbrand to an abstract definition of calculation processes, as a particularly simple type of proofs. Although Turing was said to have had formulated his theory of machines before he was familiar with much of the achievements in logic, he certainly did

his homework quickly and soon put his work in the main stream. The surprisingly simple solutions to the question of giving a general definition of algorithms were undoubtedly an important cause of the rapid developments in the area of abstract studies of calculations.

2. Between engineering and mathematics. Logic was a bastard of mathematics and philosophy; while actual computers first came into being as a great feat of engineering. This divergence in their ancestry presents serious sociological and scientific difficulties for those who are interested in the vaguely defined region referred to as "logic and computers." This is not the place to digress into sociology.

The trouble on the scientific side is that most ambitious people find dreary piecemeal engineering and idle intellectual gymnastics equally repulsive. And it seems as though there is little else to offer at the present stage, except it be irresponsible speculations. The origin of the problem goes further back. Every branch of applied mathematics embodies an intrinsic dilemma: each piece of work is either not sufficiently applied or not sufficiently mathematical. We have to present a patient defense in each case.

As Turing machines and actual computers were studied more or less independently of one another, there slowly developed a desire to bring about a marriage of theory and practice. This has proved to be an exceedingly difficult task. True, there are a number of basic results in very general terms. Turing machines are equivalent to actual computers if we disregard speed and the question of a potentially infinite supply of tape. In fact, there are alternative formulations of Turing machines which are more similar to actual computers, e.g., a representation of Turing machines by programs with a small number of basic instructions. Hence, since there are problems (e.g., the halting problem) which are unsolvable on Turing machines, the corresponding problems are unsolvable on actual computers. Another example is the result that in theory erasing is dispensable on Turing machines. Hence, magnetic tapes (in contrast with, say, paper tapes) are in theory not necessary for building computers.

Along with Turing machines, a simpler model of computers under the name "finite automata" has been extensively investigated and sometimes compared, in an expansive and speculative mood, to the human brain. This elegant model has also given rise to a number of amusing mathematical results.

In both cases, however, decisive steps remain to be taken before more significant applications can be made, probably with more realistic and less neat models. One basic difficulty is the transition from theoretical possibility to practical feasibility. For example, even though finite automata are closer to actual computers in being finite, the large size needed to represent an actual computer makes it seem likely that Turing machines will be a more useful model than finite automata even for practical purposes.

It must be emphasized that the mathematical theory of machines is a young discipline and, as such, it is doing very well so far. Moreover, it has the impor-

tant advantages that little equipment beyond native wits is required for its pursuit and that it promises great things to come. But great things are rare. What is needed at present is not quantity but rather pursuers of good quality.

On the more theoretical level, the study of impractical algorithms and abstract machines, not as isolated idealizations, but in relation to other parts of logic and mathematics, has led to many significant mathematical results. Moreover, these results, although not often their proofs, can usually be stated in quite simple terms. In terms both of their intrinsic intellectual merit and of their potential applications, they would seem to have as wide an appeal as, say, molecular biology.

3. Unsolvable problems. While engineering is primarily the study of how to make things, mathematics is more often concerned with showing that under certain general conditions, certain things can or cannot be done. There is a special appeal to show that certain things cannot be done, because such results involve, in a negative way, all the available resources of a given method. For example, in bisecting an angle, we use only a small part of the resources of ruler and compass; while in proving the impossibility of trisecting an arbitrary angle, we have to possess a clear conception of all the possible constructions which we can make with ruler and compass. It is in this area of demonstrating unsolvability that the abstract study of idealized machines has produced results of the greatest mathematical interest. In particular, the interplay of logic and the theory of computers is striking.

It is familiar to logicians that all mathematical theories can be formulated in the framework of elementary logic. Hence, if we could decide whether in general a statement is a theorem of logic, we would also be able to decide whether a statement is provable in any given mathematical theory. This situation explains why the Hilbert School regarded the Entscheidungsproblem, i.e., the problem of deciding whether a statement in logic is a theorem, as the main problem of logic.

From 1920 on, Post tackled this problem by formulating a more general one that deals with derivability in arbitrary production systems, of which the system of elementary logic is a special case. This turned out to yield a general concept of formal systems, and indirectly, one of calculation processes. Moreover, the abstract formulation renders possible experimentations on apparently simple cases, with a view to discovering some common pattern useful for the handling of the general case.

Unfortunately, as a means to get positive results, this attractive approach is, on the whole, quite powerless. One of the very first examples which Post studied in 1920–21, and reported publicly in 1943, remains unsettled today. Consider all (finite) strings made out of 0's and 1's and use two very simple rules: if a string begins with 0, delete the three symbols at the beginning and add 00 at the end; if it begins with 1, delete the three initial symbols and append 1101 at the end; (stop if a string contains less than three symbols). The problem is

simply: do we have a general method of deciding, for any two strings, whether the second can be obtained from the first by the above two rules?

On the other hand, Post's approach can be employed to establish negative results, once the step is taken to identify solvability with that by a production system or some other equivalent method, say by a Turing machine. In fact, in 1936 Turing proposed and argued for such an identification, and applied results on Turing machines to prove the unsolvability of the Entscheidungsproblem. Quite recently, this result has been sharply refined to a certain degree of finality, with the help of a picturesque auxiliary tool of "dominoes." This type of work exemplifies the rich possibilities of applying the theory of machines to establish basic results about logic.

Applications in other branches of mathematics include P. S. Novikov's proof (1955) that the word problem for groups is unsolvable, a result which has been applied by A. A. Markov to prove (1958) that the 4-dimensional homeomorphy problem is unsolvable. The 3-dimensional homeomorphy problem remains open. Impressive partial results have been obtained on Hilbert's tenth problem: whether there is a general method of deciding the question of solvability in integers of every polynomial equation with integer coefficients. To establish the unsolvability of this problem would be considered a major mathematical result.

The (formally) simplest example of an unsolvable problem is probably the following word problem formulated by G. S. Tsentin and D. Scott in 1955. Consider strings (words) made out of the five symbols, a, b, c, d, e and the following seven rules for mutual substitution: $ac \leftrightarrow ca$, $ad \leftrightarrow da$, $bc \leftrightarrow cb$, $bd \leftrightarrow db$, $adac \leftrightarrow abac$, $eca \leftrightarrow ae$, $edb \leftrightarrow be$. It is an unsolvable problem to decide whether any two words are equivalent by these rules.

4. Formalization. From the rich domain of mathematical logic, we have selected two aspects as specially relevant to computers, viz. the theory and the practice of formalization. The unsolvability results belong to the theory side, while formalizing individual mathematical proofs or "deriving mathematics from logic" belongs to the practice side. In this latter aspect the interplay of logic and computers is significant on a more concrete level: developments of logic combined with the great power of actual computers give rise to the hope of mechanizing mathematical arguments, not just in principle, but in practice as well.

As in engineering, there is little likelihood of general results in this positive enterprise. But, unlike an engineer, we are not concerned with actually making things and we do get exact results in each individual case.

The interest in mechanization implies a reorientation of formal logic with a view toward greater efficiency. In particular, this means that the need for economy of axioms and primitive concepts is to be supplemented with an exact formulation of a large body of concepts and rules, which make up the average mathematician's stock of trade.

This can best be illustrated by an example from elementary number theory.

Suppose we wish to prove:

(I) $$x > 1 \rightarrow (Ey)[Py \wedge (y \mid x)],$$

i.e., every integer greater than 1 has a prime divisor.

We assume given an organized stock of information SF with properties of $+$, $\cdot$, $<$ listed first, and then properties of P and $\mid$, which may involve the more basic concepts $+$, $\cdot$, $<$. The list SF is organized so that not too much searching is necessary to look up required properties.

The basic strategy is to assume the theorem false and try to derive a contradiction from the least counterexample, which embodies a mechanically convenient form of the principle of mathematical induction. The imagined least counterexample provides an "ambiguous constant" which possesses not only general properties true of all integers but also unusual properties arising from the assumption that it is a counterexample. In simple cases such as (I), after we draw on SF and use simple truth-functional deductions, we quickly get an ambiguous constant with contradictory properties. It should be emphasized that the proof below is merely a sketch of an illustration. More elaborate strategies of roughly the same type are necessary in order to prove more complex theorems.

To prove (I), we first assume it false and let m be the least counterexample:

(1) $$m > 1$$

(2) $$Pb \rightarrow b \nmid m$$

(3) $$1 < a < m \rightarrow Py_a \wedge (y_a \mid a).$$

The only ambiguous constant thus far is m. Substitute m for a and b above in order to get further properties of m.

(4) $$Pm \rightarrow m \nmid m$$

(5) $$1 < m < m \rightarrow Py_m \wedge (y_m \mid m).$$

Look up SF and try to derive simple consequences from (1), (4), (5) with the help of SF. Find $m \nless m$ in SF and delete the trivially true (5). Find $m \mid m$ in SF and infer from (4):

(6) $$\sim Pm.$$

Now (4) may be deleted since it is a direct consequence of (6). We have now (1), (2), (3), (6). Look up SF and get the defining property of P as applied to m:

$$\sim Pm \leftrightarrow (Ex)[1 < x < m \wedge (x \mid m)].$$

By (6), we get:

(7) $$1 < x_m < m$$

(8) $$x_m \mid m.$$

Since x_m is a new ambiguous constant, it is desirable to substitute it for the free variables in the general statements obtained so far, viz. (2) and (3) only.

(9) $$Px_m \rightarrow x_m \nmid m$$

(10) $$1 < x_m < m \rightarrow Py_{x_m} \wedge (y_{x_m} \mid x_m).$$

Derive truth-functional consequences (first without appealing to SF) from (1), (2), (3), (6)–(10).

(11) $\sim Px_m$, by (8) and (9).

(12) Py_{x_m}, by (7) and (10).

(13) $y_{x_m} \mid x_m$, by (7) and (10).

Now appeal to the list SF and use: $(a \mid b) \wedge (b \mid c) \rightarrow (a \mid c)$.

(14) $y_{x_m} \mid m$, by (8) and (13).

Substitute the ambiguous constant y_{x_m} for the free variables in (2) and (3):

(15) $$Py_{x_m} \rightarrow y_{x_m} \nmid m.$$

(16) $$1 < y_{x_m} < m \rightarrow Py_{y_{x_m}} \wedge (y_{y_{x_m}} \mid y_{x_m}).$$

By (14) and (15), we get:

(17) $\sim Py_{x_m}$, contradicting (12).

Obviously we have not listed all the blind alleys and the method has to be specified much more exactly before a machine program can be written. But, it is thought, the above outline makes it plausible that a fairly natural program can be written on existing machines to prove theorems like (1) and, for example, also $2x^2 \neq y^2$ (x, y range over positive integers).

A Short Reading List

1. B. A. Trakhtenbrot, Algorithms and automatic computing machines, D. C. Heath, Boston, 1963.

2. Martin Davis, Computability and unsolvability, McGraw-Hill, New York, 1963.

3. American Mathematical Society, Experimental arithmetic, high speed computing and mathematics, Proc. Symposia in Appl. Math., 15 (1963).

4. Brooklyn Polytech. Inst., Mathematical theory of automata (Proc. of Symposium held April, 1962), 1963.

COMPUTER LANGUAGES

M. GOLDSTEIN, Courant Institute of Mathematical Sciences, New York University

Digital computers have turned out to be useful in many more ways than their originators imagined. Machines which were developed primarily for solving long and complicated mathematical problems involving millions of separate calculations and which will diligently perform the tedious calculations of nuclear physics and meteorology will also do the bookkeeping and inventory chores of the business world and the tedious statistical analyses of social science. Even problems that seem essentially nonnumerical can be restructured to take advantage of the computers' untiring ability for manipulating digits.

Because computers spend much of their time calculating complicated mathematical formulas, it is commonly thought that computers are "giant brains" which can be mastered only by mathematical wizards. In fact, computers can be instructed by anyone who understands the series of operations that he wants performed. Digital computers are designed to perform only a specific number and type of operations. A computer is directed to perform each operation by an instruction. The instruction defines the basic operation to be performed and identifies the data, device or mechanism needed to carry out the operation. The entire series of instructions required to complete a given procedure is known as a program.

For example, the computer may have the operation of multiplication built into its circuits in much the same way that the ability to add is built into a simple desk calculator, but there must be some means of directing the computer to perform multiplication just as the desk calculator is directed by depressing keys. There must also be a way to instruct the computer where it can find the factors to multiply.

Further, the comparatively simple operation of multiplication implies other activity that must precede and follow the calculation. The multiplicand and multiplier may have been read into the computer by an input device, and stored in its storage unit—storage is somewhat like an electronic filing cabinet completely indexed and almost instantaneously accessible to the computer. Once the calculation is performed, the product must be returned to storage at a specified location, from which it may be written out by an output device or used again in another operation.

Any calculation, therefore, could involve reading, locating operands in storage, returning the result to storage, making logical decisions, and writing out the completed result. Even the simplest portion of a procedure involves a number of planned steps that must be spelled out to a computer if the procedure is to be accomplished.

An entire procedure is composed of these individual steps grouped in a sequence that directs the computer to produce a desired result. Thus, a complex problem must first be reduced to a series of basic machine operations before it can be solved. Each of these operations is coded as an instruction in a form that

can be interpreted by a computer and is placed in the main storage unit as a stored program. A mathematician can specify a sequence of operations to the computer to solve a partial differential equation, an educational psychologist who never heard of a differential equation can study, with the aid of a computer, the psychological concomitants of creativity, a medical doctor can analyze phonocardiograms on the same computer and a lawyer can instruct the computer to retrive information on cases of interest. A computer by itself can do nothing, but with suitable instructions it can solve equations, process data, and in general, augment human reasoning.

As we indicated above, a sequence of instructions suitable for solving a particular problem is called a program, and the process of devising such a sequence is called programming.

Programming a digital computer can be an exacting task, because the machine is such a relentless follower of instructions. At the present time, computers are machines without intelligence, in any sense of the word. But the programs which are written for the computer may cause the machine to exhibit behavior strikingly similar to intelligent behavior. While they obey highly intricate sets of commands, they do not possess human judgment. A computer does exactly what it is instructed to do, showing no common sense. Thus every possible contingency that might arise in the course of the calculation must be foreseen and provided for. Programming requires careful attention to details. The computer's hardware repertoire is so elementary in terms of steps required in solving a problem that a problem must be dissected into very small pieces in order to be programmed. The point is that an enormous amount of planning goes into even the simplest work done by a computer.

In most general purpose computers, information appears as combinations of bits or binary digits representing "0" or "1". These bits are organized as characters or words. A "character" typically consists of 6 or 8 bits. Six bit characters would provide enough combinations to represent 64 different characters—the alphabet, the decimal digits and several special characters. A word is usually made up of 5, 6, or 8 characters. Words may represent instructions, collections of characters to be used as data or numbers expressed in binary form to be used as data. The decision as to whether a word is to be interpreted as an instruction word or data word occurs when the program is being executed in the machine. Thus, a computer can treat the instructions stored in the memory as data and can modify these. This allows a program to be modified as it is running and is an important and powerful characteristic of present day computers.

An instruction word is subdivided to represent the elements of the instruction. These elements are an operation, an address and, usually, modifiers of either or both. Some machines have two or three addresses per instruction, some have two instructions per word, and some have instructions of variable length. The type of programming which uses the basic machine instructions is referred to as machine language programming. For example, if we wanted to compute $a \cdot b + c = d$ we could write the machine equivalent of these steps:

1) Load register with a;
2) Multiply by b;
3) Add c;
4) Store as d.

Deep in the innards of the machine each instruction would be a string of zeros and ones, and each variable would be represented by its specific location in storage. Clearly it is impractical for humans to deal with such strings of zeros and ones. Instead one uses standard mnemonic operations and symbolic addresses which are translated into true machine words and loaded into memory ready for execution. The translation is done by a program already in the computer. This type of program, now the minimum essential "software" for any computer, is called an assembly program. It is a set of instructions to the computer of how to operate upon symbolic input to produce machine instructions. In assembly language, our example might be written as:

LDQ	A
FMP	B
FAD	C
STO	D

It is this assembly language programming that has come to be called machine language rather than the primitive binary or octal coding. The assembler is one of the simplest examples of a class of amazingly elaborate programs, called processors, which are used to construct other programs.

Many of the difficulties and inconveniences of writing programs directly in machine coding can be eliminated or simplified by more advanced systems of program writing. These programming systems consist of two parts: a language and a processor. The language is similar to the programmer's language and can be translated into machine language by the processor. The required procedure is first written in the programming language; this is called the source program. Then the source program is translated automatically into machine language, the object program, by the processor, called a compiler.

When such a programming system is used, the computer actually operates as a translator or program assembly device. Statements in the programming language are translated into machine coded instructions. Storage areas are automatically assigned, constants or other reference factors are included and pre-written library routines (subroutines) for checking, input-output, etc. are assembled into a completed object program. This object program is then actually used by the computer to run the problem. In addition to translation and assembly, a processor performs the additional function of compiling the object program in the most efficient manner it can.

The purpose of the programming language is to state a procedure in a way that is convenient for the programmer but that transfers much of the clerical work of program writing to the computer. This language (sometimes called artificial to distinguish it from the natural languages), like any other, has established rules of grammar, punctuation, and expression. The terms of expression must be precise in the way that they describe any procedure to the computer.

These statements must convey to the computer exactly what it is to do, even though the aim of the language is to allow the programmer to state a procedure in a language nearly like his own. The programming language must then compromise between the easiest way for the programmer to write the design and organization of the computer system, and the requirements of the procedures to be expressed. The language of the programming system may be either problem-oriented or machine-oriented. If it is problem- or procedure-oriented, the language is independent of the computer and more closely approximates the everyday language of the user and can be translated into several different machine languages using appropriate processors. The term "computing language" most often refers to a problem-oriented one.

The compiler is usually incorporated in an automatic monitor system. The monitor carries out the routine tasks of loading programs in the machine, keeps track of machine time used, and normally provides the programmer with useful output such as a complete map of storage allocations for the subroutines used and with useful diagnostic output for various types of program failure.

In the early days of "automatic programming," as programming using a compiler is known, the most important criterion on which a compiler was judged was the efficiency of the object code. "After all, you only compile once, but you run the object program many times . . . " was the cry to justify a philosophy of compiler design that permitted the compiler to take as long as necessary to produce good object code. For some types of programs the coding produced is very good. Of course, there are some for which it is not so good. Experience led to a gradual change of philosophy with respect to compilers. During the code checking stage, known as debugging, compiling is done over and over. One of the major reasons for using a problem-oriented language is to make it easy to modify programs frequently on the basis of experience gained in running the problems. In many cases the total compiler time used by a job is much greater than the total time used running the code. Many of the newer compilers have emphasized compiling time rather than efficiency. Some have gone too far in that direction.

In most computing centers today our tiny example of $a \cdot b + c = d$ would almost certainly be written in FORTRAN or ALGOL. FORTRAN is one of the oldest and by far the most successful of computer languages. This language requires a processor which is considerably more complex than a machine language assembler. The initial system released in 1957 was probably the most complex programming system ever produced up to that time and comprised some 25,000 lines of machine language code. Developed originally by IBM for scientific computation, FORTRAN processors now exist for most major makes and models. The statement of our problem would be written in FORTRAN as

$$D = A * B + C$$

that is, source programs in this system are written in a language which closely resembles the ordinary language of mathematics. The language is intended to

allow expression of problems of numerical computation, particularly problems containing large sets of formulas and many variables. Because FORTRAN is intended primarily for problems which have a numeric meaning, the language is less satisfactory for the data-processing type of problem like commercial-type jobs, information-retrieval systems and general nonnumeric problems. Many logical operations not directly expressible in the FORTRAN language can be obtained, however, by making use of provisions for incorporating subroutines written in assembly language.

An important step in artificial language development centered around the idea that it is desirable to be able to exchange computer programs between different computer labs or at least between programmers on a universal level. In 1958, after much work, a committee representing an active European computer organization, GAMM, and a United States computer organization, ACM, published a report (updated two years later) on an algebraic language called ALGOL. The language was designed to be a vehicle for expressing the processes of scientific and engineering calculations of numerical analysis. Equal stress was placed on man-to-man and man-to-machine communication. It attempts to specify a language which included those features of algebraic languages on which it was reasonable to expect a wide range of agreement, and to obtain a language that is technically sound. In this respect, ALGOL set an important precedent in language definition by presenting a rigorous definition of its syntax. ALGOL compilers have also been written for many different computers. It is very popular among university and mathematically oriented computer people especially in Western Europe. For some time in the United States, it will remain second to FORTRAN, with FORTRAN becoming more and more like ALGOL.

The largest user of data-processing equipment is the United States Government. Prodded in part by a recognition of the tremendous programming investment and in part by the suggestion that a common language would result only if an active sponsor supported it, the Defense Department brought together representatives of the major manufacturers and users of data-processing equipment to discuss the problems associated with the lack of standard programming languages in the data processing area. This was the start of the conference on Data Systems Languages that went on to produce COBOL, the common business-oriented language. COBOL is a subset of normal English suitable for expressing the solution to business data processing problems. The language is now implemented in various forms on every commercial computer.

In addition to popular languages like FORTRAN and ALGOL, we have some languages used perhaps by only one computing group such as FLOCO, IVY, MADCAP and COLASL; languages intended for student problems, a sophisticated one like MAD, others like BALGOL, CORC, PUFFT and various versions of university implemented ALGOL compilers; business languages in addition to COBOL like FACT, COMTRAN and UNICODE; assembly (machine) languages for every computer such as FAP, TAC, USE, COMPASS; languages

to simplify problem solving in "artificial intelligence," such as the so-called list processing languages IPL V, LISP 1.5, SLIP and a more recent one NU SPEAK; string manipulation languages to simplify the manipulation of symbols rather than numeric data like COMIT, SHADOW and SNOBOL; languages for command and control problems like JOVIAL and NELIAC; languages to simplify doing symbolic algebra by computer such as ALPAK and FORMAC; a proposed new programming language tentatively titled NPL; and many, many, more. A veritable tower of BABEL!

In spite of the work involved, efforts to improve the communication with computers have led to the development of these numerous artificial computer languages. Some, such as FORTRAN, have found widespread productive use and others are valuable research tools.

In the last few years some attempt at standardization has been made. At the present, the American Standards Association (ASA) which is the authority for industrial standardization in the United States has assumed this responsibility in the area of computers and information processing. One of the subcommittees has its scope of responsibility, "Standardization of common programming languages of broad utility through standard methods of specification, with provision for revision, expansion and improvement, and for definition and approval of test problems." In addition it acknowledges the need for work in the peripheral areas of language theory, language specifications, and processor specifications, as well as programming languages' glossary and participation in international standardization.

The following bibliography is intended to be a very short list of publications that may help to document the text. Each reference contains many additional references of interest. A great deal of reference material is contained in manuals published by computer manufacturers. Although these manuals are not listed here, they exist for every commercially available programming language and system.

The work presented in this paper is supported by the AEC Computing and Applied Mathematics Center, Courant Inst. of Mathematical Sciences, New York University, under Contract AT(30-1)-1480 with the U. S. Atomic Energy Commission.

References

1. Victor H. Yngve and Jean E. Sammet, Toward better documentation of programming languages, Comm. Ass. Comp. Mach. (CACM), March (1963) 76–99. This is a series of papers describing the documentation of significant current programming languages and lists practically all programming languages and processors known to that date. It also contains additional references.

2. Saul Rosen, Programming systems and languages—A historical survey, AFIPS Conf. Proc. 25(1964) Spring Joint Comp. Conf. This article also contains many additional references.

3. Assoc. Comp. Mach., ACM Computer Symposium, CACM, January 1961, p. 33. The papers in this issue were contributed to an ACM Compiler Symposium and cover some formal and technical problems encountered in compiler design. On the cover of this issue there is an amusing drawing of the Tower of Babel, 1960, with most computer languages on it.

4. IEEE Transactions on Electronic Computers, The special issue on Computer Languages, vol. EC-13, August 1964, No. 4. This is a series of papers describing both formal methods for the study of programming languages and applied developments in this area.

COMPUTATIONAL LINGUISTICS

A. G. OETTINGER, Harvard University

Computers for languages and languages for computers are two points of contact of a modern synthesis of mathematics and linguistics with the computer sciences. The roots of mathematical linguistics reach back at least as far as Leibniz and the subject has attracted the attention of other mathematicians such as Grassmann. Its modern vogue is the product of a confluence of once unrelated developments in several fields. In logic, there was notably the work of Post, [1], Łukasiewicz, [2], and Turing [3]. There was the blossoming of information theory, following the publication of Shannon's *The Mathematical Theory of Communication* [4]. Formal linguistics in the style of Chomsky emerged [5, 6]. Finally, 20 years ago, Aiken brought forth (see [7]) the first practical realization of large scale automatically sequenced digital calculators, capable, it was soon realized, of manipulating not only numerals in accordance with the laws of arithmetic, but also any symbol system based on a discrete alphabet and obeying combination laws of the greatest generality.

Much of the concern of mathematical linguistics has been—and remains—with languages such as English or Russian which have, so to speak, a life of their own. These are conveniently called *natural* languages. Mathematical linguistics is also concerned with specialized technical jargons, mathematical or chemical notations, the formal symbol systems of logic, or the various systems for instructing computers which, as products of more self-conscious and deliberate human creation, are called *artificial* languages. The distinction between natural and artificial languages should be viewed as a rhetorical convenience rather than an absolute dichotomy.

The scope of mathematical linguistics includes the application of statistical techniques to the central linguistic problems of describing the structure of languages and their development, problems which are the domains of conventional historical and descriptive linguistics. Surveys and bibliographies concentrated on these areas have been published by Guiraud [8], Plath [9], and Oettinger [10].

Recent work has been influenced far more, however, by algebra and, generally, by discrete mathematics or logic rather than by statistics or analysis. An excellent over-view of this aspect of mathematical linguistics, as directed to the study of natural languages, is to be found in a collection of 20 papers presented at the Symposium on the Structure of Language and Its Mathematical Aspects, sponsored in 1960 by the American Mathematical Society, which was published under the editorship of Jakobson [11].

Computational linguistics, as the subset of mathematical linguistics of greatest concern here has come to be labeled, deals with the application of computers to linguistic problems and with the application of linguistics to computer prob-

lems. Its scope also overlaps with the general problems of information retrieval, currently including such matters of primarily industrial or military concern as command and control.

Automatic language translation, whose possibility was suggested by Warren Weaver [12] as early as 1949, but whose realization remains in the future in spite of repeated enthusiastic but not factual claims to the contrary, is one of the more notorious branches of computational linguistics. A critical survey of the state of the art of automatic translation may be found in Oettinger [13].

The scope of computational linguistics also includes such matters as automatic fact retrieval systems, question answering systems, the production of computer generated abstracts, indexes and catalogs, and the analysis of information content of literature by statistical, syntactic and logical methods. A survey of European efforts in this realm has recently been given by Salton [14], and one of the best sources of information on current work, primarily in the United States but also abroad, is given in the series *Current Research and Development in Scientific Documentation* published semi-annually by the National Science Foundation's Office of Science Information Service [15].

Contemporary interest in what is vaguely described as man-machine interaction, which may be characterized as a vision of machines serving as direct human appendages rather than as more remote and less accessible tools, has further spurred interest in natural languages, since communication with machines in the course of interaction is, according to the current view, to be conducted best in something approximating the vernacular. Studies in the syntax of natural languages, already begun for purposes of automatic language translation, have been further spurred by this interest, especially concerning English syntax. The proceedings of the 1963 Fall Joint Computer Conference include a survey and some details on this topic [16, 17, 18].

Syntax is also a major concern of the designers of languages for computers, whose need to interpret and translate statements expressed in a source notation congenial to people or convenient for describing a class of problems has led them to independent studies of syntactic structures. Although the literature in this area [19, 20] is still almost wholly disjoint from that on natural languages, there is, in fact, a strong underlying kinship owing to the similarity of the mathematical techniques used in describing syntactic structures of either kind.

It is in the realm of syntactic studies that the most important, elegant and extensive original contributions or novel adaptations of known mathematical or logical results have been made to date. The mathematical tools employed are almost exclusively discrete and in the spirit of algebra, set theory, the theory of groups and of semi-groups, and the theory of computable functions. For example, Davis' *Computability and Unsolvability* [21] provides a better foundation for work in this area than a standard calculus text. Bar-Hillel [22], Chomsky [23], Evey [24], Greibach [25], Hartmanis [26], Rabin [27], and Schutzenberger [28] are among those who have contributed to developing this field.

Much of the theoretical interest of their results lies in their discovery of a hierarchy of abstract languages of graded complexity of structure, and graded difficulty of mutual translatability or structural analysis. Significant advances have been made in correlating these language families and their structures with a hierarchy of abstract machines, ranging from finite-state automata through push-down store automata to unrestricted Turing machines.

In addition, this correlation has done much to explain problems encountered in the practical handling of both natural languages and computer languages and, therefore, it stands also as a very interesting chapter of applied mathematics by providing an excellent example of the value of discrete mathematical techniques in the process of solving significant engineering problems in language data processing.

References

1. E. Post, A variant of a recursively unsolvable problem, Bull. Amer. Math. Soc., 52 (1946) 264–268.

2. J. Łukasiewicz, Aristotle's syllogistic from the standpoint of modern formal logic, Clarendon Press, Oxford, 1951.

3. A. Turing, On computable numbers, with an application to the Entscheidungsproblem, Proc. London Math. Soc., 42 (1936–37) 230–265.

4. C. E. Shannon and W. Weaver, The mathematical theory of communication, University of Illinois Press, Urbana (1949).

5. N. Chomsky, Syntactic structures, Mouton and Co., 'S Gravenhage, 1957.

6. ———, Formal properties of grammars, in *Handbook of Mathematical Psychology*, vol. II, Wiley, New York, 1963.

7. A. G. Oettinger and T. C. Bartee, A note on the twentieth anniversary of the Mark I Computer, IEEE Spectrum, vol. I, No. 8, August 1964.

8. P. Guiraud, Bibliographie Critique de la Statistique Linguistique, Comité de la Statistique Linguistique, Conseil International de la Philosophie et des Sciences Humaines, Editions Spectrum, Utrecht, 1954.

9. W. Plath, Mathematical linguistics, in *Trends in European and American Linguistics*, Spectrum Publ., Utrecht, 1961, pp. 21–57.

10. A. G. Oettinger, Linguistics and mathematics, in *Studies presented to Joshua Whatmough*, Mouton and Co., The Hague, 1957, pp. 179–186.

11. R. Jakobson (ed.), Structure of language and its mathematical aspects, Proc. Symp. Appl. Math., vol. XII, Amer. Math. Soc., 1961.

12. W. Weaver, in Locke-Booth, Machine translation of languages, Fourteen essays, Wiley, New York, 1955, pp. 15–23.

13. A. G. Oettinger, The state of the art of automatic language translation: An appraisal, *Sprachkunde und Informationsverarbeitung*, No. 2, Oldenbourg Verlag, Munich, Nov. 1963, pp. 17–33.

14. G. Salton, Automatic information processing in Western Europe, Science, 144 (May 1964) 626–632.

15. National Science Foundation, Current research and development in scientific documentation, Office of Sci. Inf. Service, NSF, Washington 25, D. C.

16. D. G. Bobrow, Syntactic analysis of English by computer—A survey, AFIPS Conf. Proc., vol. 24, Spartan Books, Baltimore, 1963, pp. 365–387.

17. S. Kuno and A. G. Oettinger, Syntactic structure and ambiguity of English, *ibid.* pp. 397–418.

18. J. L. Dolby, H. L. Resnikoff and E. Mac Murray, A tape dictionary for linguistic experiments, *ibid.* pp. 419–423.

19. Report of Proc. of a working Conference on Mechanical Language Structure, Comm. ACM, 7 (February 1964) 51–136.

20. IEEE Transactions on Electronic Computers, vol. EC-13, No. 4, August 1964.

21. M. Davis, Computability and unsolvability, McGraw-Hill, New York, 1958.

22. Y. Bar-Hillel, A quasi-arithmetical notation for syntactic description, Language, 29 (January–March 1953) 47–58.

23. N. Chomsky, Three models for the description of language, IRE Trans. on Information Theory, IT-2 (September 1956) 11–124.

24. J. Evey, The theory and applications of pushdown store machines, Ph.D. Thesis, Rep. to the NSF 10, Comp. Lab., Harvard University, 1963.

25. S. Greibach, Undecidability and the ambiguity problem for minimal linear grammars, Information and Control, 6 (1963) 119–125.

26. J. Hartmanis and R. E. Stearns, On the computational complexity of algorithms, G. E. Res. Lab, Rep. No. 63-RL-3292. April 1963. Also to be published in the annals of the Amer. Math. Soc.

27. M. Rabin and Scott, Finite automata and their decision problems, IBM J. Res. Develop., 3 (1959) 114–125.

28. M. P. Schützenberger, On an application of semi-group methods to some problems in coding, IRE Transl., On Information Theory, IT-2 (1956) 47–60.

COMPUTER SYSTEMS

C. H. WEISERT, Applied Physics Laboratory, Johns Hopkins University

Background. It is assumed that the reader is acquainted with the general concept of a stored-program sequential *processor* (von Neumann machine) having a repertoire of *instructions.* A *routine* is a sequence of instructions and routines which causes a processor to perform some purposeful action. "Program" is almost a synonym for "routine," but is sometimes used in the sense of the complete array of routines invoked for an application. If routine α belongs to routine β, then α is called a *subroutine* of β and β is said to *call* (or *use*) α. The act of creating a routine is called *coding.*

The purpose of a computer system is to serve as a tool to produce "answers" or to perform some other useful action. In a general-purpose system this is most often accomplished by effecting the execution of routines supplied in some manner by the users; such routines are called *object programs.*

Three very important points purposely obscured above are insufficiently germane to the aims of this article to justify the detail required for precise statement. First, an instruction or a routine is by no means the same thing as the act of performing or *executing* the instruction or routine, but ambiguities pervade current terminology. Secondly, this definition of "subroutine" depends

upon a vague use of "belong to"; where a distinction is required "n-th level subroutine" or "to call directly" will be used in the hope that the intended interpretation is obvious. Thirdly, the statement of an algorithm in some programming language is not the same thing as a routine, and in some cases (interpretive processors) no direct mapping is performed; however, the two will be regarded as equivalent.

Throughout this article the word "user" denotes anyone who communicates with the system without regard to whether he is a problem originator or an intermediary such as a professional programmer.

Computer systems. A general-purpose computer system consists of the following:

1. Hardware—an array of processors and *devices* for storage or communication.
2. Conventions—a set of standard procedures for using the various parts of the system.
3. Software—a collection of routines concerned with the system itself rather than with applications.
4. Library—a collection of applications-oriented routines to be used as building blocks in the construction of object programs.

Whether a routine is called a "library routine" or an "object program" depends only upon whether it is permanently known to the system; object programs having general usefulness usually find their way into the library. Similarly the distinction between "software" and "library," like that between "pure" and "applied" mathematics depends on one's point of view and does not justify much attention.

The ordering of items in the above list is not a chronology of system specification; the most successful systems are those for which design was carried on with a high degree of integration. It should also be emphasized that item 2 refers both to conventions observed by routines and to those observed by human beings involved in system operation.

Problem solving. To solve a problem with a computer system usually requires, in roughly chronological order, the following steps:

1. Statement of the problem.
2. Analysis of the problem, derivation of algorithms, and general design of the program.
3. Detailed design and coding of routines.
4. Testing and correcting of routines ("checkout" or "debugging").
5. Execution of the resulting object program.

Until a few years ago disproportionate attention was given to step 3, leading to great interest in programming languages and to some neglect of the role of the whole system in the problem-solving process. As recently as 1962 Mealy [1] found it necessary to warn that

" . . . the current emphasis on programming languages, as opposed to other aspects of the programmer's approach to the machine, is misplaced. The actual amount of time the programmer spends coding is small as compared to the time he spends in problem analysis, checkout, and setting up runs. An advance in the art such as the introduction of recursive procedures must be reckoned as small when compared with advances in symbolic modification and checkout methods or in our understanding of input-output systems. It is entirely possible that, in our current enthusiasm for so-called common languages, we may be settling for a common level of mediocrity rather than a genuine advance. . . . "

The user may become aware of the system as early as step 2; for example, operations involving large matrices would be done quite differently depending on whether the operands would fit all at once in the main high-speed memory, would be stored on a *serial-access* device such as a magnetic tape, or would be stored on a *direct-access* device such as a magnetic drum. An understanding of the more subtle characteristics of whatever programming language processors are available in the system is needed if the user is to take account of peculiarities and restrictions imposed by the languages or their processors on inter-routine communication and data structures. An appreciation of the organizational facilities of the *assembly system* [**3**, **4**], insofar as they can be exploited in the languages to be used, can greatly influence the fundamental structure of the object program.

By the time the user gets to step 4 he must come face-to-face with the system. In all too many problems, perhaps in a majority of the non-trivial ones, debugging has been the step most consuming of elapsed calendar time. Responsibility for this unfortunate situation is shared by users who have not really learned to use the system and system designers who have embedded "easy to learn" languages in minimal systems that present formidable obstacles to debugging and to system access. The growing awareness during the last few years of the importance of such aspects of computer systems as man-machine communication, debugging, input-output, and file maintenance illustrates that the lesson has been well learned by system designers. The moral for the prospective computer user is that learning a programming language, however powerful and elegant, is but a part of learning how to use a computer system as an effective problem solving tool.

Functions of computer systems. Early computer systems as well as a few of today's smaller systems have relied heavily on the manual performance of many essential functions by operational personnel as well as by the very users for whom the system was supposed to be a time-saving tool. As the cost of hardware has increased and as the problems to be solved have become more complex, so have the penalties for human error and human delay become higher. Similarly, early systems presented "bare" hardware to the users; each object program (and each major software component) had to cope with unpleasant, low level hardware peculiarities irrelevant to the problem to be solved. This led to a cha-

otic proliferation of contradictory conventions. The necessary automation of system management functions and coordination of special hardware constitute major factors in the development of hardware, software, and even programming languages.

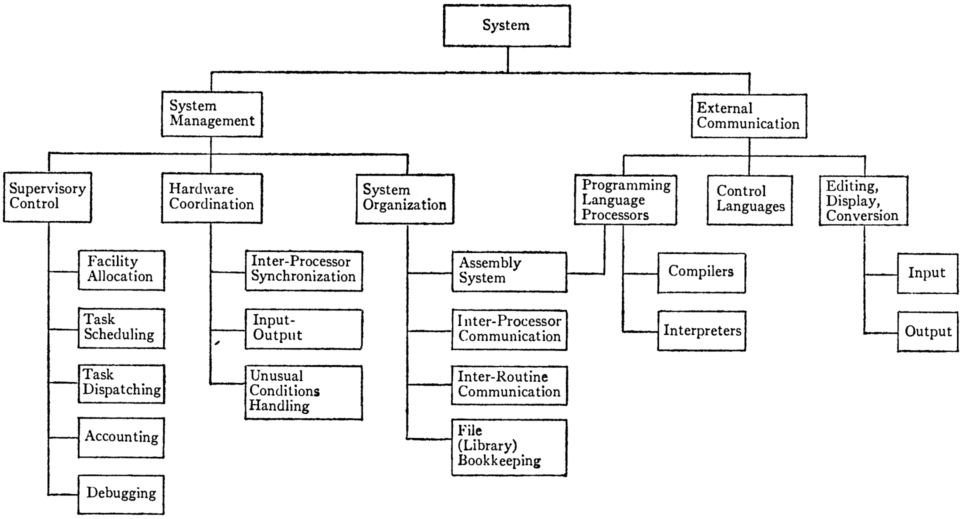


Fig. 1

Figure 1 shows a general outline of system functions. Of course, the relative importance of each function depends upon the size of the system, especially upon the cost of the hardware. At the current "state of the art" every function on the chart, for a large scale system, is implemented in software or in coding conventions with minimum requirements for manual intervention. Note that the somewhat arbitrary hierarchy shown is one of function rather than of control—i.e., the chart says nothing about the existence of specific subroutines or who is a subroutine of whom.

Tasks and jobs. A task can be thought of as a pair $\langle S, \alpha \rangle$, where S is a set of ordered collections of data (files) and α is a routine that operates on the members of S. Clearly, what one regards as a task depends upon what one regards as a file, to some extent a matter of taste; tasks are meant to be the units of work assigned to the system. The files operated on may be *read* (input) or *written* (output) by α. If α is a programming language processor then one or more of its files is itself a routine in some form.

A *job* is an array of related tasks. For example, a typical job might consist of these tasks:

1. Use compiler α to translate algorithm β_1, giving routine β_1'.
2. Do the same for β_2.

3. Use global assembler [5] to create an object program consisting of β_1', β_2', previously compiled β_3', and library routines $\gamma_1, \cdots, \gamma_n$.
4. Effect the execution of the object program.
5. Display (convert to external form and print) the answers, if any.

Access to the system. Of central importance to the user's approach to program organization, coding, and debugging are the modes of communication between him and the system. These modes can be classified as follows:

1. *Direct access*—users make appointments to take over exclusive control of the hardware.
2. *Remote access*—users submit jobs to the system, which selects tasks to be performed and eventually sends results back to the submitters.
3. *On-line access*—users communicate with the system in "real time," without exercising complete control over it.

In practice a combination of methods is used; for example, before keeping his appointment with the hardware, the direct access user might submit his routines in hand-written form for conversion to a form readable by the hardware, such as punched cards.

Direct access is, of course, characteristic of the smaller systems mentioned earlier, in which software for system management is almost non-existent. Although these systems provide to the user the ability to stop at any point to interact in any way he pleases, they require him to understand the hardware in greater detail than is reasonable, considering current programming languages and user populations. In addition, they are economically feasible only for slow hardware where delays for individual set-up and human decision are not of duration comparable with that of the tasks to be done.

The user's tactics in communicating with a system by remote access are very different. He does not make decisions during or between tasks, but plans his whole job in advance. He does not tell the system what to do by pushing buttons and mounting tapes or by writing notes to an operator, but rather by means of a formal *control language.* He is more mindful of programming conventions so as not to interfere with other jobs. These courtesies on his part are not unrewarded by well-conceived systems. The user can devote his full attention to the problem to be solved, remaining ignorant of hardware details if he so chooses, but can exercise control where required. He is by no means required to provide for every possible occurrence before submitting his job; the control language, the language processors, and the various debugging aids all can provide "normal case" actions in the absence of detailed instructions (the actions which a majority of users would choose). A system is *non-multiprogrammed* if, once a task has been started, it is completed before any other task is started. A classic example of this kind of system is the SHARE 709 system [1, 6–10].

According to a considerable body of opinion on-line access overcomes most of the shortcomings of other methods [11]. It depends, for the same economic reasons stated above, on a multiprogramming task scheduler, and on suitable

hardware *terminals*. Given a single routine to be implemented, remote access systems present a *turn-around time* problem—i.e., it may require several hours to discover one error; this problem is of less importance to those users who work on several routines concurrently. The main feature of on-line access is the ability of the user to interact with the system dynamically during coding and checkout. A great deal of attention is being given to the development of easy-to-use, flexible terminals—not everyone is a skilled typist. Also, it is not apparent that those programming languages that are suitable for one mode of access are suitable for others; languages have been created, in fact, specifically for the on-line user [**12**], revealing new classes of problems to be solved with computer systems.

The typical, large scale system of the next few years will certainly provide both remote and on-line access, hopefully to the same features.

Files. The need to effect the transmission of data between the main high-speed memory of a processor and such external media as punched cards and magnetic tapes arises in the following situations:

1. A file is too big to fit in the main memory.
2. A file is originated by or is to be sent to some external entity such as another computer system or a human being.
3. A file is to be retained for later use.

To a large extent (1) can be programmed implicitly; i.e., the user pretends that the main memory is infinite and the system allocates secondary storage, keeps track of where information is, and initiates transmissions automatically. Although this relieves the user of a large burden, it has seldom worked well in practice, because techniques used to analyze what the object program is doing are, so far, insufficiently developed to provide strategies having efficiency comparable to that of custom-tailored input-output. This problem is also encountered by the system itself; many language processors have finite capacities. It is extremely annoying to the user to work for many hours developing an elegant algorithm only to have it rejected by the system as too taxing—a surprisingly frequent occurrence, especially with many of today's list processors and macro-assemblers. Recent research promises a significant improvement here in the near future.

Most programming languages provide explicit facilities for *input-output* operations—i.e., the writing, reading, and positioning of files. Few, if any, provide the means to achieve the necessary initializing and bookkeeping actions that must be done before a routine can begin to perform input-output operations. This is quite proper; such functions belong in the system management area, and the user would communicate, where necessary, through the control language or through some interface to the control language.

The availability of inexpensive, large scale storage devices, such as disk files, makes possible and necessary the automation of many awkward, error-prone system functions having to do with file maintenance. Especially in large systems, where files are stored on a wealth of media, both detachable (cards, tape,

smaller disk units) and not (large disks, drums, cores), the job of keeping track of what is where, what belongs to whom, who may read what, and who may destroy what is so complicated as to preclude manual performance. Routines themselves ordinarily constitute a large part of the files in a system. The user wants to be able to retrieve and modify his routines without having to rummage through huge filing cabinets of punched cards; this is especially true of the on-line user whose routines are not likely ever to have existed in a hard-copy, machine-readable form.

Conclusion. Current research in the theory and the technology of hardware and of software promises to continue the trend toward ever increasing power, flexibility, and ease of use in computer systems. Ease of use, however, is not to be confused with ease of learning; as Iverson observes in the preface to his excellent book [13], the " . . . power of an adequate programming language amply repays the considerable effort required for its mastery." Of course, it would be unrealistic to expect a user to commit every detail of a computer system to memory; this is true of any other branch of applied mathematics. If he is to make effective use of the potential power of a computer system, however, the user must understand many of its basic concepts and know its organization well enough to know how to find answers to specific questions as they arise.

References

1. George H. Mealy, Operating Systems, RAND Corporation P2584, Santa Monica, 1962.

2. Saul Rosen, Programming systems and languages—A historical survey, AFIPS Conf. Proc., vol. 25, Spartan Books, Baltimore, 1964.

3. M. D. McIlroy, Macro-extensions of compiler languages, Comm. ACM, vol. 3, no. 4, April 1960.

4. George H. Mealy, A generalized assembly system, RAND Corporation RM-3646-PR, Santa Monica, 1963.

5. F. J. Corbato, M. Merwin-Daggett, and J. McCarthy, The linking segment subprogram language and linking loader, Comm. ACM, vol. 6, no. 7, July, 1963.

6. Donald L. Shell, The SHARE 709 system: A cooperative effort, J. ACM, vol. 6, no. 2, April, 1959.

7. Irwin D. Greenwald, and Maureen Kane, The SHARE 709 system: Programming and modification, *ibid.*

8. Vincent DiGri, and Jane E. King, The SHARE 709 system: Input-Output translation *ibid.*

9. Owen Mock, and Charles J. Swift, The SHARE 709 system: Programmed Input-Output buffering, *ibid.*

10. Harvey Bratman, and Ira V. Boldt, Jr., The SHARE 709 system: Supervisory control, *ibid.*

11. F. J. Corbato, M. Merwin-Daggett, and R. C. Daley, An experimental time-sharing system, AFIPS Conf. Proc., vol. 21, National Press, Palo Alto, 1962.

12. C. L. Baker, JOSS: Scenario of a filmed report, RAND Corp. RM-4162-PR, Santa Monica, 1964.

13. K. E. Iverson, A programming language, Wiley, New York, 1962.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 3

CONTENTS

MARCH 1965

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

General Office: SUNY at Buffalo, Buffalo, N. Y. 14214. *Executive Director:* H. M. GEHMAN

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $8.00 ($10.00 effective August–September 1965)

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

A POLYGON PROBLEM

E. R. BERLEKAMP, E. N. GILBERT, AND F. W. SINDEN,
Bell Telephone Labs., Murray Hill, N. J.

Let P_0 be a closed polygon. Let P_1 be the polygon obtained by joining the midpoints of P_0's sides in order (see Figure 1). We write

$$P_1 = TP_0.$$

Let $P_2 = TP_1$, $P_3 = TP_2$, etc. We will call $P_1, P_2, P_3, \cdots$ the *descendants* of P_0.

We consider the question: for what polygons P_0 does there exist a *convex* descendant P_n?

It is easy to show that all descendants of convex polygons are convex. Hence, the existence of one convex descendant implies the existence of infinitely many.

For polygons with fewer than five sides the results are immediate:

2-gons: all descendants have coincident vertices.

3-gons: all descendants are similar to the parent P_0, which is necessarily convex.

4-gons: all descendants are parallelograms, hence convex, though the parent P_0 may be nonconvex.

(To see the last result observe that the sides of P_1 are parallel to the diagonals of P_0.)

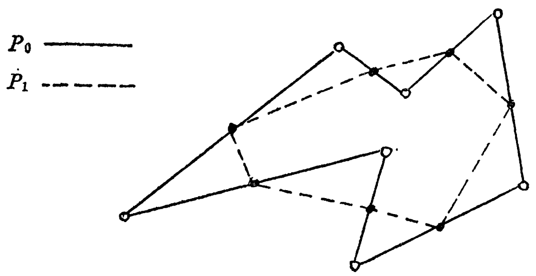


FIG. 1

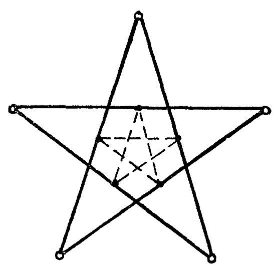


FIG. 2

Polygons with five or more sides are more complicated, but in many examples we found that repeated application of T led quickly to convex polygons. This did not seem surprising, for intuitively T is a smoothing operator. We began with the conjecture that every polygon ultimately has convex descendants.

Figure 2 shows this conjecture to be false. The regular pentagram begets only its own kind.

This example, though, may seem unfair since the pentagram intersects itself. Figure 3 shows a non-selfintersecting polygon without convex descendants.

This polygon, in fact, is less innocent than it looks. Not only are its decendants nonconvex but infinitely many of them are self-intersecting. (To see

this, note that the relation

$$\frac{X_2 - X_1}{X_3 - X_2} = \frac{1 + \sqrt{5}}{2}$$

between vertex projections X_1, X_2, X_3 holds for all descendants as well as for the polygon itself. P_i is self-intersecting whenever P_{i-1} is not.)

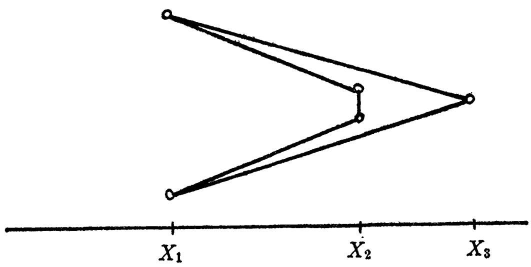


FIG. 3. $(X_2-X_1)/(X_3-X_2)=(1+\sqrt{5})/2$

In spite of these examples, the conjecture that all polygons have convex descendants is not entirely empty. It turns out that *almost* all polygons have convex descendants.

This problem originated with G. R. MacLane, to whom we are indebted for some helpful suggestions. We have recently heard of an independent solution by A. M. Gleason.

Planar polygons. To analyze planar polygons it is convenient to use complex numbers. Let the N-tuple $\mathbf{z}=(z_1, \cdots, z_N)$ represent the vertices of a closed polygon in order. We will refer to the polygon simply as $\mathbf{z}$. The first descendant of $\mathbf{z}$ is a polygon $\mathbf{w}$ with vertices

$$w_1 = \tfrac{1}{2}(z_1 + z_2), \cdots, w_N = \tfrac{1}{2}(z_N + z_1).$$

This may be written $\mathbf{w}=T\mathbf{z}$, where

$$T = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & & \cdots & 1 & 1 \\ 1 & & \cdots & 0 & 1 \end{bmatrix}$$

The eigenvalues and eigenvectors of T, i.e., the solutions of $T\mathbf{u}=\lambda\mathbf{u}$, are of particular interest, for if $\mathbf{u}$ is an eigenvector then the polygon represented by $\mathbf{u}$ is merely rotated and uniformly diminished by T. Specifically it is rotated about the origin through the angle arg λ and diminished by the factor $|\lambda|$.

The eigenvalues and eigenvectors can be written down explicitly. The eigenvalues are

$$\lambda_k = \tfrac{1}{2}(1 + e^{ik(2\pi/N)}), \qquad k = 0, 1, \cdots, N-1;$$

(see Figure 4) and the corresponding eigenvectors are

$$\mathbf{u}_k = (e^{i1k(2\pi/N)}, e^{i2k(2\pi/N)}, \cdots, e^{iNk(2\pi/N)}).$$

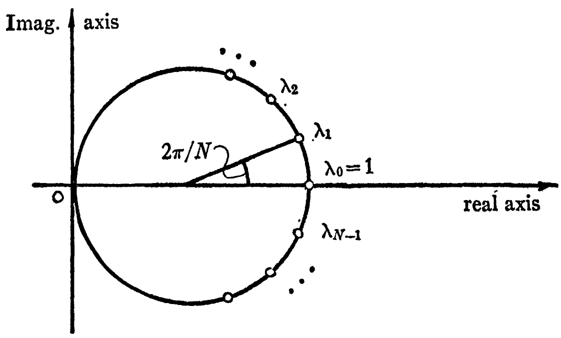


FIG. 4. Eigenvalues

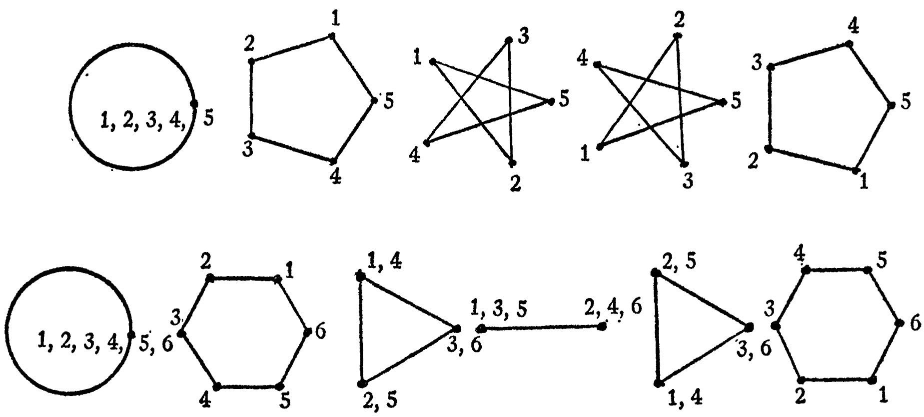


FIG. 5. Eigenpolygons

The normalization $\mathbf{u}_k \cdot \mathbf{u}_k^* = N$ seems a little more convenient in the present context than the usual $\mathbf{u}_k \cdot \mathbf{u}_k^* = 1$. Note that the components of the eigenvectors are all roots of unity. Thus the polygons represented by the eigenvectors (eigenpolygons) are all inscribed in the unit circle. The kth one is obtained by joining in order the points on the unit circle with arguments

$$k1\phi,\ k2\phi,\ \cdots,\ kN\phi, \qquad \phi = 2\pi/N.$$

The eigenpolygons for $N=5, 6$ are shown in Figure 5. It is evident geometrically

that these polygons are only rotated and diminished by the transformation T.

We will call any polygon that is similar to the kth eigen-N-gon a *regular N-gon of order k*. Thus, we include among regular polygons self-intersecting, multiply-traversed and degenerate polygons.

We observe that regular N-gons have the following properties: (1) The regular N-gons of order 0 are totally degenerate, i.e., all vertices coincide. (2) The regular N-gons of orders 1 and $N-1$ are the regular convex N-gons, counterclockwise and clockwise, respectively. We will regard all self-intersecting, multiply-traversed and degenerate polygons as nonconvex. (3) The regular N-gons of orders k and $N-k$ are the same except for the sense in which they are traversed. (4) The regular N-gon of order k is multiply-traversed if and only if k and N are not relatively prime.

The crucial fact is that an arbitrary N-gon can be written as a sum of regular N-gons. This is the meaning of the following eigenvector expansion, whose validity follows readily (see [1]) from the fact that the eigenvalues are distinct.

$$\mathbf{z} = \sum_{k=0}^{N-1} a_k \mathbf{u}_k \quad (a_k \text{ complex}) \tag{1}$$

or explicitly:

$$z_j = \sum_{k=0}^{N-1} a_k e^{ijk(2\pi/N)}.$$

The coefficients are given by the formula

$$a_k = \frac{1}{N} \sum_{j=1}^{N} z_j e^{-ijk(2\pi/N)}.$$

This formula can be given a geometric interpretation as follows. Consider the transformation $\mathbf{w} = W\mathbf{z}$, where

$$w_j = z_j \cdot e^{ij(2\pi/N)}.$$

The effect of W is to rotate the jth vertex about the origin through the angle $-j(2\pi/N)$. The vertex z_N makes a complete revolution, returning to its starting position, while the other vertices turn through a fraction j/N of a revolution proportional to their indices. This diminishes the angle between successive vertices by the constant amount $2\pi/N$. We will say that W *winds* polygon $\mathbf{z}$ about the origin. Winding carries the kth eigenpolygon into the $(k-1)$-st mod N. In particular it carries the convex eigenpolygon $\mathbf{u}_1$ into the degenerate eigenpolygon $\mathbf{u}_0 = (1, 1, \cdots, 1)$ and it carries $\mathbf{u}_0$ into the convex eigenpolygon $\mathbf{u}_{N-1}$. Applied to an arbitrary polygon, winding simply permutes cyclically the coefficients of the component eigenpolygons:

$$a_1 \rightarrow a_0,\ a_2 \rightarrow a_1,\ \cdots,\ a_0 \rightarrow a_{N-1}.$$

The centroid of a polygon $\mathbf{z}$ is $\sum z_j/N$. All of the eigenpolygons except $\mathbf{u}_0$ have centroid O. Hence, the centroid of an arbitrary polygon is the centroid of its zeroth component, namely a_0. It follows that the centroid of the polygon obtained by winding $\mathbf{z}$ is a_1.

The coefficients a_k, then, can be given this geometric interpretation:

The coefficient a_k is the centroid of the polygon obtained by winding $\mathbf{z}$ k times.

We return now to the midpoint-joining transformation T. Consider a polygon $\mathbf{z}$ in decomposed form:

$$\mathbf{z} = \sum_{k=0}^{N-1} a_k \mathbf{u}_k.$$

Since $T\mathbf{u}_k = \lambda_k \mathbf{u}_k$, the first descendant of $\mathbf{z}$ is

$$\mathbf{z}_1 = T\mathbf{z} = \sum_{k=0}^{N-1} a_k \lambda_k \mathbf{u}_k$$

and the nth descendant is

$$\mathbf{z}_n = T^n \mathbf{z} = \sum_{k=0}^{N-1} a_k \lambda_k^n \mathbf{u}_k.$$

As n increases, the components with the largest eigenvalues in absolute value become relatively dominant. As can be seen from Figure 4, the eigenvalues are ordered as follows:

$$1 = \lambda_0 > |\lambda_1| = |\lambda_{N-1}| > |\lambda_2| = |\lambda_{N-2}| > \cdots$$

Since $\lambda_0 = 1$, the centroids (zeroth coefficients) of $\mathbf{z}_n$ remain fixed. The other components die out. Thus all vertices of $\mathbf{z}_n$ converge to the centroid. Relatively, though, the convex components ($k=1$, $N-1$) die out least rapidly.

The normalized polygons

$$\mathbf{z}_n' = \frac{1}{|\lambda_1|^n} (\mathbf{z}_n - a_0 \mathbf{u}_0)$$

have centroids O and convex components of constant magnitude. As n increases all nonconvex components go to zero.

The sum of the two convex components of $\mathbf{z}$ is the polygon with vertices

$$w_j = a_1 e^{ij(2\pi/N)} + a_{N-1} e^{i(N-1)j(2\pi/N)}.$$

This polygon is the affine image of a regular convex N-gon. Its vertices all lie on the ellipse

$$w(t) = a_1 e^{it} + a_{N-1} e^{-it}$$

at the points $w_j = w(t_j)$, $t_j = j(2\pi/N)$. The ellipse $w(t)$ circumscribes not only the

sum $\mathbf{w}$ of the two convex components of $\mathbf{z}$, but also the sums $\mathbf{w}_n'$ of the two convex components of $\mathbf{z}$'s normalized descendants $\mathbf{z}_n'$. As n increases, $\mathbf{z}_n'$ approaches $\mathbf{w}_n'$. If the circumscribing ellipse $w(t)$ is nondegenerate, then for sufficiently large n $\mathbf{z}_n'$ is convex. If, on the other hand, $w(t)$ is degenerate, then all $\mathbf{z}_n'$ are self-intersecting or degenerate, hence by our definition nonconvex. A necessary and sufficient condition for $\mathbf{z}$ to have convex descendants, therefore, is that $w(t)$ be nondegenerate.

The semimajor axis of $w(t)$ has length $|a_1|+|a_{N-1}|$ and the semiminor axis has length $||a_1|-|a_{N-1}||$. The ellipse degenerates if and only if $|a_1|=|a_{N-1}|$, i.e., if and only if the two convex components of $\mathbf{z}$ are of equal magnitude.

We can now state the answer to the question posed at the beginning of the paper as follows:

Represent the vertices of polygon P_0 by the complex numbers $z_1, \cdots, z_N$. Let $a_1, \cdots, a_N$ be the complex numbers given by:

$$a_k = \frac{1}{N}\sum_{j=1}^{N} z_j e^{-ijk(2\pi/N)}.$$

P_0 has a convex descendant P_n if and only if

$$|a_1| \neq |a_{N-1}|. \tag{2}$$

Stated geometrically the condition is this: P_0 has a convex descendant if and only if the centroids of the polygons P_+ and P_-, obtained by winding P_0 positively and negatively about the origin, lie at different distances from the origin.

Comments on planar polygons.

1. In many cases the descendants of a polygon display a curious alternation. The descendants of a rectangle, for example, are alternately rhombuses and rectangles. In the general case, the even descendants $P_2, P_4, P_6, \cdots$, if normalized in size, approach a fixed polygon P and the odd descendants approach another fixed polygon P'. P and P' are affine images of regular polygons and are inscribed in the same ellipse.

2. Without essentially complicating the problem the transformation T, which had the formula $w_j=\frac{1}{2}(z_j+z_{j+1})$ can be replaced by the more general transformation T' with the formula

$$w_j' = A_0z_j + A_1z_{j+1} + \cdots + A_{N-1}z_{j+N-1}$$

where $A_0, A_1, \cdots$ are any constants and where the N subscripts on the z_{j+k} are to be computed modulo N. The eigenvectors are the same $\mathbf{u}_k$ as before but the eigenvalues are now

$$\lambda_k = \sum_r A_r e^{irk2\pi/N}.$$

If $|\lambda_1|$ or $|\lambda_{N-1}|$ exceeds all of $|\lambda_2|, \cdots, |\lambda_{N-2}|$ then, for almost all polygons P, all but a finite number of descendants of P are convex. Two examples of this

kind are:

(i) $$w_j' = \alpha z_j + \beta z_{j+1} \qquad (\alpha > 0, \beta > 0, \alpha + \beta = 1)$$

and

(ii) $$w_j' = \tfrac{1}{3}(z_j + z_{j+1} + z_{j+2}).$$

The midpoints of the sides are replaced by points which subdivide the sides in ratio α/β in (i) and by the centroids of successive vertex triples in (ii).

3. There exist orphan polygons which have no ancestors. Any quadrilateral that is not a parallelogram is such a one. Orphan polygons are those even-sided polygons containing a nonzero component of order $N/2$. The vanishing eigenvalue $\lambda_{N/2} = 0$ wipes out this component in the first generation. Thus no polygon containing the $N/2$ component can be the descendant of another polygon.

Similar remarks apply to the more general transformation T' of the preceding paragraph. If T' has no zero eigenvalue (as in example (i) if $\alpha \neq \beta \neq \frac{1}{2}$) then all polygons have unique ancestors. In this case the family tree may be traced backwards as well as forwards. It is interesting to note that almost all convex polygons have remote ancestors which are not convex.

Nonplanar polygons. The nonplanar case, surprisingly, is hardly more complicated than the planar case. To make the generalization it is convenient to write the eigenpolygon decomposition in a new form. Combining the kth and $(N-k)$th terms in (1) we obtain a sum of $[N/2]+1$ terms each representing an affine image of a regular polygon; here $[N/2]$ means the integer part of $N/2$. For d-dimensional space this sum generalizes as follows. If X is a closed N-gon in d-space then X can be written

$$X = \sum_{k=0}^{[N/2]} W_k,$$

where again each component polygon W_k is an affine image of a regular polygon. *This differs from the planar case only in that the W_k need not all lie in the same plane.*

This may be derived explicitly as follows. Let

$$\mathbf{x}(t) = \sum_{k=0}^{[N/2)} \mathbf{u}_k \cos kt + \mathbf{v}_k \sin kt,$$

where the symbols $\mathbf{x}$, $\mathbf{u}$, $\mathbf{v}$ stand for vectors with d real components. It is always possible [2] to choose $\mathbf{u}_k$ and $\mathbf{v}_k$ so that $\mathbf{x}(t)$ assumes given values at N given points $t_1, \cdots, t_N$, in particular so that

$$\mathbf{x}\left(j\,\frac{2\pi}{N}\right) = \mathbf{x}_j,$$

where $\mathbf{x}_j$ are the vertices of the given polygon X. Then X can be written as

$$X = \sum_{k=0}^{[N/2]} W_k,$$

where W_k is the polygon with vertices

$$\mathbf{w}_{kj} = \mathbf{u}_k \cos kj\frac{2\pi}{N} + \mathbf{v}_k \sin kj\frac{2\pi}{N} \cdot$$

Polygon W_k lies in the plane determined by $\mathbf{u}_k$ and $\mathbf{v}_k$ and is inscribed in the ellipse

$$\mathbf{w}(t) = \mathbf{u}_k \cos t + \mathbf{v}_k \sin t.$$

W_k may be regarded as the sum of a clockwise and a counterclockwise regular N-gon of order k as was done in the first section. Alternatively W_k may be regarded as the projection onto a plane of a regular N-gon of order k. In general, a d-dimensional polygon may be regarded as the projection onto d-space of a $(d+1)$-dimensional polygon all of whose components are regular.

The midpoint-joining transformation T may be analyzed in terms of the decomposition in the same way as before. The only new element is the dimension of the descendants. (A polygon has the dimension of the smallest space it can be imbedded in.) The following example shows that T can reduce the dimension: Let X be a 3-dimensional polygon with vertices alternately in the planes $z=1$ and $z=-1$. The first descendant of X lies in the x, y-plane. This, however, is as far as the dimension reduction can go. Further application of T leaves the dimension unchanged. In general the situation is this:

All descendants $X_1, X_2, \cdots$ of a d-dimensional polygon X_0 have the same dimension. X_1 may be of dimension one less than X_0, but this can occur only if X_0 is an orphan, i.e., only if X_0 contains a nonzero component of order $N/2$.

Proof. Suppose X_n has dimension d and X_{n+1} has dimension $d'<d$. Let $\overline{X}_n$ and $\overline{X}_{n+1}$ be the projections of X_n and X_{n+1} onto a line orthogonal to the d'-space containing X_{n+1}. All vertices of $\overline{X}_{n+1}$ lie in a single point p. Since p must be the midpoint of all sides of $\overline{X}_n$, the vertices of $\overline{X}_n$ must lie alternately in two points p_1, p_2 equidistant from p. $\overline{X}_n$, then, is a regular N-gon of order $N/2$. If X_n has the decomposition

$$X_n = \sum_{k=0}^{[N/2]} W_k$$

then $\overline{X}_n$ has the decomposition

$$\overline{X}_n = \sum_{k=0}^{[N/2]} \overline{W}_k.$$

But in the latter decomposition only $\overline{W}_{N/2}$ can be nonzero. If $\overline{W}_{N/2}$ is nonzero

then so is $W_{N/2}$. In this case X_n is an orphan, hence $n=0$. This completes the proof.

In conclusion, we may say that almost all nonplanar polygons lack planar descendants. If the first descendant is nonplanar then so are all the rest.

On the other hand, all nonplanar polygons have descendants which differ arbitrarily little (relative to their size) from planar polygons. And *almost* all nonplanar polygons have descendants which differ arbitrarily little (relative to their size) from planar *convex* polygons.

The first author is now at the University of California, Berkeley.

References

1. F. R. Gantmacher, Theory of Matrices, vol. 1, Chelsea, New York, 1959, p. 72.
2. Ch. de la Vallée Poussin, Leçons sur l'approximation des fonctions d'une variable réelle, Gauthier-Villars, Paris, 1919, p. 93.

ENUMERATION OF CYCLIC PAIRED-COMPARISON DESIGNS

H. A. DAVID, Virginia Polytechnic Institute (Now at Dept. of Biostatistics, University of North Carolina at Chapel Hill)

1. Introduction. Suppose that n "objects" $0, 1, 2, \cdots, n-1$, are to be compared in pairs. Examples of this situation are given by chess tournaments of n players, field experiments involving n treatments applied in blocks of two plots, and n pure lines of plants to be evaluated on the basis of their crosses. In each of these cases it will often be impracticable to make all possible $\frac{1}{2}n(n-1)$ pairings. Cyclic designs provide a convenient and flexible way for reducing the number of comparisons without serious imbalance. They consist of suitable combinations of cyclic sets, a typical set $\{x\}$ containing n pairs as follows:

$$\{x\} = 0, x \quad 1, x+1 \cdots t, x+t \cdots n-1, x+n-1,$$

where x is a positive integer less than $\frac{1}{2}n$ and $x+t$ has to be reduced modulo n when necessary. If n is odd there are $m=\frac{1}{2}(n-1)$ such sets; if n is even there are $\frac{1}{2}n-1$ sets of n pairs and one set of $\frac{1}{2}n$ pairs, viz.

$$\{\tfrac{1}{2}n\} = 0, \tfrac{1}{2}n \quad 1, \tfrac{1}{2}n+1 \cdots \tfrac{1}{2}n-1, n-1.$$

Cyclic paired-comparison designs are a useful cross-section of partially balanced incomplete block designs of block-size two with up to $\frac{1}{2}n$ associate classes. They were introduced by Kempthorne [6] and further studied by Zoellner and Kempthorne [7], David [4], and Curnow [2]. Here we are concerned rather with the systematic enumeration, and thereby construction, of distinct (i.e. nonisomorphic) designs involving n objects each of which is to be compared r times ($r=1, 2, \cdots, n-2$; for n odd r must be even). Among such designs of

size (n, r) the one which is "best" for some particular purpose may then be selected.

2. n prime. When n is prime (>2) the sets $\{x\}$ $(x=1, 2, \cdots, m)$ can be shown to be equivalent (see [4]) in the sense that any set can be changed into any other by suitable renumbering of the objects. For example, the renumbering (or permutation) $R(7, 2)$ in which the number of each of 7 objects is multiplied by 2 (mod 7) transforms

$$\{1\}: \quad 01 \quad 12 \quad 23 \quad 34 \quad 45 \quad 56 \quad 60$$

to

$$02 \quad 24 \quad 46 \quad 61 \quad 13 \quad 35 \quad 50$$

which is set $\{2\}$. Applying $R(7, 2)$ again, we obtain

$$04 \quad 41 \quad 15 \quad 52 \quad 26 \quad 63 \quad 30$$

which might be designated $\{4\}$ in the first instance but is equivalent to $\{3\}$. In general, the permutations $R(n, 1), R(n, 2), \cdots, R(n, m)$ form a group which is isomorphic with the multiplicative group G of the integers

$$G: \quad 1, 2, \cdots, m$$

when products of these are reduced not only modulo n but also for equivalences E of the type $i \equiv n-i$.

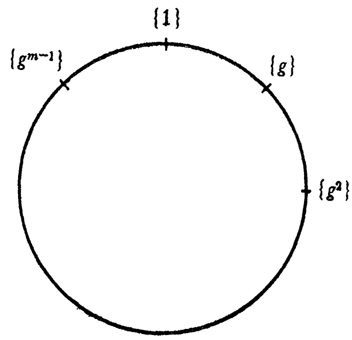


FIG. 1

The group G may also be expressed as

$$G: \quad 1, g, g^2, \cdots, g^{(n-3)/2}, \text{ (mod } n; E),$$

where g is a primitive root mod$(n; E)$ of n; i.e., g is an integer such that $g^y \not\equiv 1$ mod $(n; E)$ for $y=1, 2, \cdots, m-1$ but $g^m \equiv 1$ (mod n; E). We may now imagine the sets $\{x\}$ to be arranged around a circle in the order shown in Figure 1. Each application of the permutation $R(n, g)$ then produces a single clockwise turn.

A design of size (n, r) is obtained by selecting $s=\frac{1}{2}r$ of the m points along the circumference. Let us represent a set chosen by a white bead and a set omitted by a black bead. If the circle is regarded as a necklace (which may not be turned over) distinct designs will correspond to distinct arrangements of white and black beads along a necklace. The number of cyclic paired-comparison designs of size (n, r) which remain distinct under renumbering is thus equal to the number of arrangements remaining distinct under rotation of s white and $s'=m-s$ black beads on a necklace. Finding the latter number is a special case of an old problem commonly attributed to Jablonski [5], where the beads may have more than two colors (see e.g., F. N. David and Barton [3]). If $\phi(x)$ is Euler's function, the number of positive integers less than and prime relative to the integer x, the required number of distinct designs of size (n, r) is

$$N(s, s') = \frac{1}{m} \sum \phi(d) \frac{(m/d)!}{(s/d)!(s'/d)!}, \tag{1}$$

the summation being over all positive integers d which are divisors of both s and s'.

3. Necklaces. Before proceeding to the more complicated case when n is not prime we shall examine equation (1) more closely (cf. [5]). If s and s' are relative primes $N(s, s')$ reduces to

$$\frac{1}{m}\binom{m}{s}$$

since each possible arrangement gives rise to $m-1$ others on rotation of the necklace. The sum in (1) augments $\binom{m}{s}$ when s and s' have a common divisor (>1) to take account of those arrangements which due to symmetry can be reproduced in fewer than m turns. We may speak of the *period* τ of an arrangement and will now find an expression for the number of arrangements with $\tau=m/d$.

If d equals $g(s, s')$, the greatest common divisor (g.c.d.) of s and s', arrangements with $\tau=m/d$ must consist of d like parts, each with s/d white and s'/d black beads. Their number is accordingly

$$\frac{d}{m}\binom{m/d}{s/d}. \tag{2}$$

If d is a divisor of s, s' smaller than $g(s, s')$, however, then (2) has to be adjusted to allow for arrangements with higher symmetry, i.e., corresponding to a larger value of d. The resulting expression is seen to be

$$\frac{d}{m}\left[\binom{m/d}{s/d} - \sum_1 \binom{m/d'}{s/d'} + \sum_2 \binom{m/d'}{s/d'} - \cdots\right],$$

where $\sum_1$ extends over all divisors d' of $g(s, s')$ which are multiples of d and

for each term in $\sum_1$ there are contributions to $\sum_2$ corresponding to all those d' which are multiples of any d' occurring in that term; and so on, by inclusion and exclusion.

Example. $m=12$, $s=6$. Possible values of d are 6, 3, 2, 1. Correspondingly, the numbers of distinct arrangements are

$$\frac{1}{2}\binom{2}{1}=1,\ \frac{1}{4}\left[\binom{4}{2}-\binom{2}{1}\right]=1,\ \frac{1}{6}\left[\binom{6}{3}-\binom{2}{1}\right]=3,$$

and

$$\frac{1}{12}\left\{\binom{12}{6}-\left[\binom{6}{3}+\binom{4}{2}+\binom{2}{1}\right]+\left[\binom{2}{1}+\binom{2}{1}\right]\right\}=75.$$

As a check the total number is $N(6, 6)=80$.

For $m \leqq 15$ Table 1 gives the number $N_\tau(s, s')$ of distinct arrangements by periods. Clearly $N_\tau(1, m-1)=1$. All results for values of $s>[\frac{1}{2}m]$ follow immediately by complementation: $N_\tau(s, s')=N_\tau(s', s)$. The totals $N(s, s')$ have also been tabulated by Barton and F. N. David [1] in a different context.

TABLE 1

Number $N_\tau(s, s')$ of distinct arrangements with period τ of s white and $s'=m-s$ black beads on a necklace.

m	s	$\tau:N_\tau$	$\tau:N_\tau$	$\tau:N_\tau$	$\tau:N_\tau$	Total $N(s, s')$
4	2	4: 1	2:1			2
5	2	5: 2				2
6	2	6: 2	3:1			3
	3	6: 3	2:1			4
7	2	7: 3				3
	3	7: 5				5
8	2	8: 3	4:1			4
	3	8: 7				7
	4	8: 8	4:1	2:1		10
9	2	9: 4				4
	3	9: 9	3:1			10
	4	9: 14				14
10	2	10: 4	5:1			5
	3	10: 12				12
	4	10: 20	5:2			22
	5	10: 25	2:1			26

m	s	$\tau:N_\tau$	$\tau:N_\tau$	$\tau:N_\tau$	$\tau:N_\sigma$	Total $N(s, s')$
11	2	11: 5				5
	3	11: 15				15
	4	11: 30				30
	5	11: 42				42
12	2	12: 5	6:1			6
	3	12: 18	4:1			19
	4	12: 40	6:2	3:1		43
	5	12: 66				66
	6	12: 75	6:3	4:1	2:1	80
13	2	13: 6				6
	3	13: 22				22
	4	13: 55				55
	5	13: 99				99
	6	13:142				142
14	2	14: 6	7:1			7
	3	14: 26				26
	4	14: 70	7:3			73
	5	14:143				143
	6	14:212	7:5			217
	7	14:245	2:1			246
15	2	15: 7				7
	3	15: 30	5:1			31
	4	15: 91				91
	5	15:200	3:1			201
	6	15:333	5:2			335
	7	15:429				429

4. n composite. Corresponding to each of the $\frac{1}{2}\phi(n)$ positive integers less than $[\frac{1}{2}n]$ and prime to n there is a connected set. By "connected" we mean, as usual, that the objects cannot be divided into subsets with no comparison made between objects in different subsets. These connected sets are equivalent as before. Thus the number of designs of size (n, r) made up entirely of connected sets is

$$N(s, \tfrac{1}{2}\phi(n) - s), \qquad s \leqq \tfrac{1}{2}\phi(n).$$

If n has divisors $d_1(=1)$, d_2, d_3, $\cdots$, then the integers $1, 2, \cdots, [\frac{1}{2}n]$ can be arranged in *bundles* I, II, III, $\cdots$ having respectively d_1, d_2, d_3, $\cdots$ as greatest divisors. Corresponding to these bundles there are $m_1=\frac{1}{2}\phi(n)$, $m_2=\frac{1}{2}\phi(n/d_2)$, $m_3=\frac{1}{2}\phi(n/d_3)$, $\cdots$ sets consisting of d_1, d_2, d_3, $\cdots$ connected pieces; if n is even, the set $\{\frac{1}{2}n\}$ consists of $\frac{1}{2}n$ rather than n pairs. A design built up from a selection of such sets will be connected as long as the g.c.d. of the d's for the sets included is unity. To enumerate all designs of size (n, r) we

must therefore consider all ways of dividing s into s_1, s_2, s_3, $\cdots$ parts, s_1 of type I etc., the parts being distinguishable. Any unconnected designs can be removed at sight. The necessary procedure is best illustrated by an example.

Example. To enumerate all distinct connected designs of size (15, 6). There are three bundles

$$1247 \mid 36 \mid 5$$

with $m_1=4$, $m_2=2$, $m_3=1$. A total selection of $s=3$ can be achieved, with obvious positional notation, in the following ways:

$$300,\ 210,\ 120,\ 201,\ 021,\ 111.$$

The first of these, corresponding to selecting 3 from the first bundle and none from the other two, gives one design. A little care is required with some of the others. Thus 210 can come about in the following four distinguishable ways conveniently represented by the analogy of two necklaces. Note that although there are $N(2, 2)=2$ ways of arranging two white and two black beads and $N(1, 1)=1$ way of arranging one white and one black bead, the total number of distinct combined arrangements is not just $N(2, 2)\ N(1, 1)$. The reason why two combined arrangements can be distinguished on the left of Fig. 2 is, of course, that the periods of the two necklaces have a common divisor 2. On the right we have $\tau_1=2$, $\tau_2=2$, which results again in the multiplier 2. Proceeding in like manner we find that the total number of distinct designs is

$$1+4+1+2+1+2=11.$$

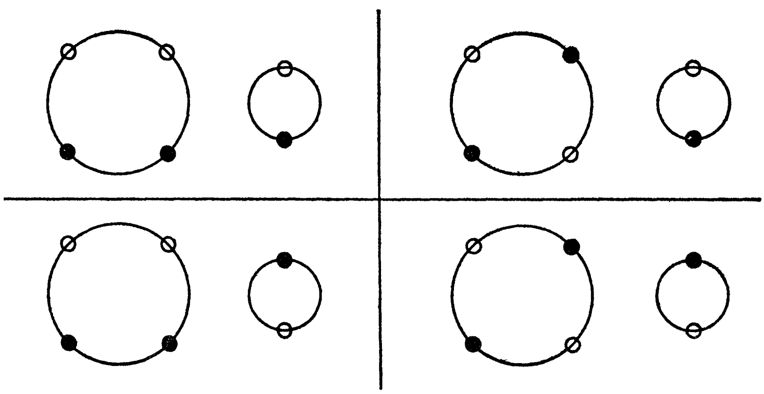


FIG. 2

If $\{i, j, k\}$ denotes the design consisting of sets $\{i\}$, $\{j\}$, $\{k\}$, the eleven designs may be taken respectively as

$$\{1, 2, 4\};\ \{1, 2, 3\},\ \{1, 2, 6\},\ \{1, 4, 3\},\ \{1, 4, 6\};\ \{1, 3, 6\};$$
$$\{1, 2, 5\},\ \{1, 4, 5\};\ \{3, 6, 5\};\ \{1, 3, 5\},\ \{2, 3, 5\}.$$

The general approach to enumeration is now clear. If there are k bundles we consider first a particular choice of the s_i $(i=1, 2, \cdots, k;\ 0\leqq s_i\leqq m_i,\ \sum s_i=s)$. This gives

$$\prod_{i=1}^{k} N(s_i, s_i') \qquad (s_i' = m_i - s_i)$$

different combinations of k necklaces without taking into account their relative orientations. Of these there will be

$$\prod_{i=1}^{k} N_{\tau_i}(s_i, s_i')$$

combinations for each of which the period of the arrangement on the ith necklace is τ_i $(i=1, 2, \cdots, k)$. Such a combination of k necklaces will first return to its original position after $l(\boldsymbol{\tau})$ turns, where $l(\boldsymbol{\tau})$ is the lowest common multiple of the τ_i. Correspondingly there are

$$D_{\boldsymbol{\tau}} = \prod \tau_i/l(\boldsymbol{\tau}) \tag{3}$$

distinct arrangements.

To sum up: A general expression for the number of cyclic paired-comparison designs of size (n, r) is

$$\sum_{s} \sum_{\boldsymbol{\tau}} \mathrm{D}_{\boldsymbol{\tau}} \prod_{i=1}^{k} N_{\tau_i}(s_i, s_i'), \tag{4}$$

where $0\leqq s_i\leqq m_i$, $\sum s_i=s=\frac{1}{2}r$, $\sum m_i=\frac{1}{2}(n-1)$, $N_{\tau_i}(s_i, s_i')$ is the number of distinct arrangements with period τ_i of s_i white and s_i' black beads on a necklace, $D_{\boldsymbol{\tau}}$ is defined by (3), $\sum_{\boldsymbol{\tau}}$ is the sum over all τ_i compatible with a given set of the s_i, $\sum_{s}$ is the sum over all permissible sets of the s_i, and may at choice include or exclude sets leading to disconnected designs.

When $n(=2m+1)$ is a prime up to 31 Table 1 gives the number of distinct designs of size (n, r) as $N(s, s')$ with $s=\frac{1}{2}r$, $s'=\frac{1}{2}(n-1-r)$. For n composite formula (4) may be computed with the help of the values of N_τ given in the Table. In fact, the present approach essentially constructs the designs in the course of enumeration. For $n\leqq 15$ the actual (connected) designs are given in [4] and the list can now easily be extended.

Work supported by U. S. Army Research Office (Durham), Grant no. DA-ORD-29.

References

1. D. E. Barton and F. N. David, Runs in a ring, Biometrika, 45 (1958) 572–578.

2. R. N. Curnow, Sampling the diallel cross, Biometrics, 19 (1963) 287–306.

3. F. N. David and D. E. Barton, Combinatorial chance, Griffin, London, 1962.

4. H. A. David, The structure of cyclic paired-comparison designs, J. Austral. Math. Soc., 3 (1963) 117–127.

5. E. Jablonski, Théorie des permutations et des arrangements circulaires complets, J. Math. Pure Appl., 4th series, 8 (1892) 331–349.

6. O. Kempthorne, A class of experimental designs using blocks of two plots, Ann. Math. Statist., 24 (1953) 76–84.

7. J. A. Zoellner and O. Kempthorne, Incomplete block designs with blocks of two plots, Iowa Agric. Expt. Sta., Res. Bull., 418 (1954).

NEW RESULTS IN EXTENDED ANALYTIC GEOMETRY

R. S. UNDERWOOD, Texas Technological College

1. Introduction. The MONTHLY has published four papers by R. S. Underwood [1, 2, 3, 4] and one by D. C. B. Marsh [7] on, or leading up to, the subject called "extended analytic geometry." The first three exploratory papers by Underwood missed the simplified approach finally found, and the paper by Marsh dealt with algebraic criteria rather than plotting techniques. Consequently, most of the basic theorems having to do with "rapid sketching" of loci have not appeared in the MONTHLY. Part of what follows has appeared in a monograph [5], and part is completely new. To conserve space, proofs which may be found in the references are omitted.

2. Résumé. The major application of the subject to date is a technique for solving simultaneous algebraic equations, including complete disposal of linear-quadratic pairs in n variables. The key to the method is a scheme for mapping equations on the XY plane—often rapidly, given the theorems below—by associating a specific point (X, Y) with each set of the variables involved. One locus of an equation is the totality of points designated by the equation *and* a chosen plotting rule. As will be shown in this paper the pictures obtained are in effect three-dimensional; and the indication of depth suggests in turn alternate views also obtainable.

To illustrate and give immediate point to the idea, consider the equations

$$(2x - z)^2 + (3y + 2u)^2 = 25, \tag{1}$$

and

$$\frac{x^2}{3} + \frac{y^2}{2} + \frac{z^2}{4} + \frac{u^2}{1} = 1. \tag{2}$$

Do these have common real solutions? How would one find out without spending an embarrassing amount of time? The answers are respectively "No" and "By sketching the loci in a matter of seconds, given a little practice in the method." The locus of (1) is obviously the circle $X^2+Y^2=25$ by the indicated L.P.R. (linear plotting rule):

$$X = 2x - z, \qquad Y = 3y + 2u, \tag{3}$$

and that of (2), using (3) and Theorem 1 below, is the ellipse

$$\frac{X^2}{16}+\frac{Y^2}{22}=1 \tag{4}$$

plus the points inside. Even without the sketch it is clear that the circle lies outside the ellipse, and hence that no common solutions exist.

The circle (1) and filled ellipse (2), as graphed by (3), illustrate the *degenerate* and *normal* types of loci, respectively. In the first the G. C. (generalized coordinates) of the point (X, Y) as obtained from (3), namely $(x, y, 2x-X, (Y-3y)/2)$, satisfy (1) with x and y arbitrary. In the second, x and y are restricted to the values allowed when the G. C. are substituted in (2). In other words, some but not all of the sets of 4-coordinates of a point (X, Y) on a normal locus satisfy the equation, whereas *all* sets work on a degenerate locus.

The foregoing definitions show implicitly that when at least one of a pair of loci is degenerate, the corresponding equations are consistent if and only if the loci overlap. When both loci are normal, nonoverlapping is still decisive, showing inconsistency; but overlapping leaves the matter in doubt, as when we see two clouds apparently merged. If they actually do not touch, this would be apparent with a proper change of viewing position. An analogous result is obtained mathematically by altering the plotting rule, as may be demonstrated precisely in the case of three variables. When $n>3$ the stereoscopic impression still directs plotting-rule (or viewpoint) operations usefully (Sections 6, 7, and 8). In accord with this pragmatic approach, a degenerate locus may be considered a cylindrical surface turned edgewise to the viewer which always penetrates (has solutions in common with) any "cloud" (normal locus) in the line of sight.

The general method of graphing a quadratic equation is explained in Section 7 of [4]. The method involves the solution of $n-2$ linear equations in $n-2$ unknowns, and is obviously a last resort, though dependable. If shortcuts were not available, such importance and usefulness as the subject may have would be drastically reduced; but fortunately this is not the case.

3. Short-cuts for equations lacking cross-product terms. The following result appears as Theorem 5.1 in [5], and the proof may be found there.

THEOREM 1. *The locus of the equation*

(5) $\displaystyle\sum_1^n \frac{x_i^2}{a_i^2}=1$ *by the L.P.R.* (6) $\displaystyle X=\sum_1^m A_ix_i,\quad Y=\sum_{m+1}^n A_ix_i,$

is the ellipse

$$\frac{X^2}{M}+\frac{Y^2}{N}=1 \tag{7}$$

plus the points inside, in which $M=\sum_1^m a_i^2A_i^2$ *and* $N=\sum_{m+1}^n a_i^2A_i^2$. *Furthermore,*

the sole n-coordinates which satisfy (5) *and locate a point* (X, Y) *on* (7) *are as follows*:

$$x_i = a_i^2 A_i X/M \qquad (i = 1, 2, \cdots, m),$$
$$x_i = a_i^2 A_i Y/N \qquad (i = m + 1, \cdots, n). \tag{8}$$

THEOREM 2. *The locus by L.P.R.* (6) *of equation* (5) *with* 1 *replaced by* -1 *is imaginary* (*or blank*) *on the* XY *plane.*

Note that the vacuous locus is a *silhouette*, if we define the latter as a locus which leaves an uncovered area on the XY plane.

THEOREM 3. *The locus by L.P.R.* (6) *of the equation*

$$\sum_1^n \frac{x_i^2}{a_i} = 1, \tag{9}$$

in which some but not all of the a_i *are negative, will, if it is a silhouette, be bounded by the curve* (7), *in which* M *and* N *are as in Theorem 1 except that* a_i^2 *is replaced by* a_i. *Furthermore, given the silhouette, the sole n-coordinates which satisfy* (9) *and locate a point* (X, Y) *on* (7) *are those in* (8) *with* a_i^2 *replaced by* a_i.

Since one or both of M and N in (7) may be negative or zero, (7) may be imaginary or nonexistent, in which case the locus will cover the plane. If (7) is a real ellipse or hyperbola, this serves as the *test curve.* The substitution in the original equation of the G.C. of two points on opposite sides of the test curve, when this is necessary, will show whether or not the locus is a silhouette. Often this "two-points method" is not necessary, as shown in Section 4.

THEOREM 4. *The locus of the equation*

$$\sum_1^m \frac{x_i^2}{a_i^2} = \sum_{m+1}^n A_i x_i + A \tag{10}$$

by the semi-arbitrary L.P.R.

$$X = \sum_1^m A_i x_i, \qquad Y = \sum_{m+1}^n A_i x_i, \tag{11}$$

in which the constants A_i $(i > m)$ *are fixed by* (10), *is the parabola*

$$\frac{X^2}{M} = Y + A \tag{12}$$

plus the points inside, in which M *is as in Theorem* 1. *Furthermore, the first* m *of the n-coordinates of a point* (X, Y) *on* (12) *which satisfy* (10) *must be as in the first line of* (8), *but the remaining n-coordinates are restricted only by the second equation in* (11).

THEOREM 5. *The locus by* (11) *of the equation*

$$\frac{x_1^2}{a_1^2} - \sum_2^m \frac{x_i^2}{a_i^2} = \sum_{m+1}^n A_i x_i + A, \tag{13}$$

in which the constants A_i $(i>m)$ *are fixed by* (13) *and* A_1 *is chosen large enough so that* $M=a_1^2A_1^2-\sum_2^m a_i^2A_i^2>0$, *is the parabola* (12) *plus the points outside.*

Of course if the left side of (13) contains one negative instead of one positive term, Theorem 5 still holds after signs and notation are changed. Marsh has proved that no L.P.R. will yield a silhouette if the left side of (13) is changed to contain at least two positive and at least two negative terms.

Note that the "filled parabola" of Theorem 4 and the "surrounded parabola" of Theorem 5 are respectively the silhouettes of elliptic and hyperbolic paraboloids in 3-space.

Finally, the nonhomogeneous cases in which squares may be completed and the x_i^2 terms in (5), (9), (10) and (13) are replaced by $(x_i-b_i)^2$, may be reduced to the cases already treated by use of the "translation" $x_i-b_i=x_i'$. This completes the disposal, without resort to calculus or long algebra, of the loci of quadratic equations lacking cross-product terms.

4. Dependable silhouettes. Silhouettes always appear under the conditions cited in Theorems 1, 2, 4, and 5. Theorem 3 is less definite, since the test curves there obtained may be *spurious*, or false boundaries of loci which really cover the plane. However, in this case the L.P.R. indicated below will yield silhouettes dependably, and the "two-points method" discussed below Theorem 3 is not necessary.

Consider equation (9), in which m positive terms are assumed to precede the $n-m$ negative ones, with $m>0$ and $n>m$. The equation then has index m and rank n. Four cases are involved, as shown. It is assumed that $n\geqq 3$.

Case 1: $m=1$. The L.P.R. is $X=x_1$, $Y=\sum_2^n A_ix_i$, and the silhouette is that of a hyperboloid of two sheets. (It is proved in [5] that no other type of silhouette can be obtained in this case.)

Case 2: $m=n-1$. The L.P.R. is $X=\sum_1^{n-1} A_ix_i$, $Y=x_n$, and the silhouette is that of a hyperboloid of one sheet, seen from the side.

Case 3: $m=2$. The L.P.R. is $X=A_1x_1+\sum_3^k A_ix_i$, $Y=A_2x_2+\sum_{k+1}^n A_ix_i$, in which $3\leqq k\leqq n$ and A_1 and A_2 are chosen large enough to make both $M=a_1A_1^2-\sum_3^k a_iA_i^2$ and $N=a_2A_2^2-\sum_{k+1}^n a_iA_i^2$ positive. The silhouette is that of a hyperboloid of one sheet seen from the end (otherwise described as an "empty but surrounded" ellipse).

Case 4: $m>2$ and $n-m>2$. No L.P.R. will yield a silhouette, as shown by Marsh. (Note that on the whole this enhances rather than impairs the usefulness of the method in solving simultaneous equations. In this case the equation is consistent with any other equation in the n variables, quadratic or not, which has a degenerate, nonvacuous locus.)

When $n=3$, Cases 2 and 3 both apply, and indeed both views of a hyperboloid of one sheet can be obtained from the same equation.

5. Classification by silhouettes. The remark under Case 1, Section 4, suggests that classification of equations by silhouettes may be feasible, as it definitely is when $n=3$. Figure 88c [6] shows that each silhouette in this case reveals unambiguously the corresponding quadric surface, if we list all imaginary loci under one heading. When $n>3$ this classification would be limited, however, by the nonappearance of silhouettes in some cases. Other aspects of the investigation are more useful.

6. New features of loci when $n>3$. Superficially, loci of quadratic equations when $n=3$ and $n>3$ are the same. That is, when they are nonvacuous silhouettes they are either conic sections (including straight lines and points) or areas bounded by them. However, at least two new features appear when $n>3$.

The first feature is the "cliff effect." Consider the equation

$$\frac{(x+z)^2}{7}+\frac{y^2}{3}+\frac{u^2}{2}=1 \tag{14}$$

whose graph by L.P.R.

$$X=x+z, \qquad Y=y+u \tag{15}$$

is the filled ellipse

$$\frac{X^2}{7}+\frac{Y^2}{5}=1. \tag{16}$$

But here the G. C. of a point (X, Y) on the bounding curve (16) has an "unassigned and free" small letter, namely x in $(x, 3Y/5, X-x, 2Y/5)$ so that the boundary suggests a cylindrical surface just as degenerate loci do. In effect, we are looking at the end view of a *filled elliptic cylinder*—never met when $n=3$. The side view might be expected, and sure enough it appears inside the lines $(X+Y)^2=12$ when conjured by the L.P.R.

$$X=x+y, \qquad Y=z+u. \tag{17}$$

Solid figures suggested by the "cliff effect" are those bounded in various ways by elliptic, parabolic, and hyperbolic cylinders, and by parallel and crossing planes, all turned edgewise to the observer.

The two views of (14), as well as a wealth of other examples suggest the following

Conjecture. If a silhouette provides a "depth view," either as a degenerate locus or one with a "cliff effect," and if this view permits an alternate silhouette, then an L.P.R. exists which yields that alternate.

Since it has been shown [4] that different L.P.R. present in effect merely different viewing positions for the quadric surfaces of 3-space, the conjecture

is correct when $n=3$. No counter-example has been found when $n>3$, but the proof would probably be tedious.

The second new feature is the appearance of two loci for the same equation which are intuitionally contradictory when we conceive depth in the picture as previously explained. For example, the locus of the equation

$$(x+z)^2+(y+u)^2-x^2=1 \tag{18}$$

by the respective L.P.R. $X=x+z$, $Y=y+u$ and $X=x+y+u$, $Y=z$, present end views of an "empty but surrounded" right circular cylinder $(X^2+Y^2=1)$ and a hyperbolic cylinder $(2XY=1)$ filled in the center. Two *different* solid figures are obtained, or at least visualized, from the same equation. Note, however, that this does not invalidate the conjecture.

7. Manipulating the view-point. It is well known that if $P(x_0, y_0, z_0)$ is a regular point on a quadric surface Q, (or $f=0$) one form of the equation of the tangent plane to Q at P is

$$\left.\frac{\partial f}{\partial x}\right]_P (x-x_0)+\left.\frac{\partial f}{\partial y}\right]_P (y-y_0)+\left.\frac{\partial f}{\partial z}\right]_P (z-z_0)=0, \tag{19}$$

in which the coefficients are the values of the partial derivatives at P. Extending this in the obvious manner to "quadric surfaces" (the quotes meaning $n>3$) and "turning the tangent planes on edge" by choosing L.P.R. which make them lines, we get interesting and useful results. For example:

(a) If the tangent plane has exactly one point of contact, this *must* appear in the silhouette; in fact, if the silhouette shows two points of contact then two distinct sets of variables satisfy both equations, contradicting the premise in (a).

(b) If a silhouetted tangent plane and its corresponding surface show exactly one point of contact, the plane may touch the surface in a single point or in "a line seen endwise." In the latter case an infinite set of n-coordinates of the point will satisfy both equations.

(c) The generalization of the "plane-to-point" formula can be shown to have geometric significance by "turning the plane on edge." Calling $(ax+by+cz+du)/\sqrt{(a^2+b^2+c^2+d^2)}$ the "distance" from the "plane" $ax+by+cz+du=0$ to the "point" (x, y, z, u), as a sample of the natural algebraic extension, we find that when a "plane" is silhouetted as a line and a "point" P is always k times as far from a "point" Q off the "plane" as it is from the "plane," the silhouetted locus of P is, as might be expected, that of an ellipsoid, paraboloid, or hyperboloid of two sheets according as $k<1$, $k=1$, or $k>1$. Also, when $k=1$ and Q is on the "plane," the locus is a straight line. These results hold in all cases examined, but the detailed proof is not complete.

8. Identifying equations which yield degenerate loci. In [7], Marsh obtained algebraic criteria for equations yielding degenerate loci by proper choice of the L.P.R. An alternate "intuitional" method identifies such equations and often turns up Diophantine sidelights in the process. Let us start with

THEOREM 6. *If two sets of real numbers satisfy a quadratic equation $f=0$ and yield nonparallel tangent "planes" $F=C_1$ and $G=C_2$, C_1 and C_2 being constants, the L.P.R. $X=F$ and $Y=G$ gives a degenerate locus for the equation in case such a locus exists.*

In less precise but more illuminating language, we "stand *both* planes on edge and thus get an end view of the cylindrical surface which makes the degenerate locus." If this had been suggested previously by "standing *one* plane on edge" and getting a silhouette with straight line boundaries, we could not at that point deduce that a degenerate locus must exist. The locus, as in (14), might be merely *bounded* by a cylindrical surface.

Proof. Assume that the locus of $f=0$ is degenerate when a proper L.P.R. is selected, —say $X=\sum_1^n a_ix_i+a$, $Y=\sum_1^n b_ix_i+b$. Then we may write

$$f \equiv AX^2 + BXY + CY^2 + DX + EY + F = 0. \tag{20}$$

Let one of the sets of real numbers mentioned in the theorem, namely, $(c_1, \cdots, c_n)$ be the n-coordinates of a tangent point P. Then, using the extension of (19) as the equation of the "tangent plane at P" by definition, we get as that equation

$$\left.\frac{\partial f}{\partial x_1}\right]_P (x_1 - c_1) + \cdots + \left.\frac{\partial f}{\partial x_n}\right]_P (x_n - c_n) = 0, \tag{21}$$

in which

$$\begin{aligned} \left.\frac{\partial f}{\partial x_i}\right]_P &= \left(\frac{\partial f}{\partial X}\frac{\partial X}{\partial x_i} + \frac{\partial f}{\partial Y}\frac{\partial Y}{\partial x_i}\right)_P \\ &= (2AX_1 + BY_1 + D)a_i + (BX_1 + 2CY_1 + E)b_i. \end{aligned} \tag{22}$$

Here X_1 and Y_1 are the values of X and Y at P.

Hence the equation of the "tangent plane," from (21) and (22), is

$$\begin{aligned} &(2AX_1 + BY_1 + D)\sum_1^n a_i(x_i - c_i) \\ &\qquad + (BX_1 + 2CY_1 + E)\sum_1^n b_i(x_i - c_i) = 0, \end{aligned} \tag{23}$$

or

$$H_1\sum_1^n a_ix_i + H_2\sum_1^n b_ix_i = H_3' \tag{24}$$

in which H_1, H_2, and H_3' are constants, or

$$H_1X + H_2Y = H_3' + H_1a + H_2b = H_3. \tag{25}$$

Similarly, at the second point mentioned in the theorem, we get, for the "tangent plane" not parallel to the first one,

$$K_1X + K_2Y = K_3. \tag{26}$$

Then an L.P.R. which yields a degenerate locus is

$$X' = H_1X + H_2Y, \qquad Y' = K_1X + K_2Y. \tag{27}$$

For since the "planes" are nonparallel

$$\begin{vmatrix} H_1 & H_2 \\ K_1 & K_2 \end{vmatrix} \neq 0,$$

and X and Y are expressible in terms of X' and Y'. Thus (20) by L.P.R. (27) yields an equation in X' and Y' with none of the small letters appearing explicitly. Note that X' and Y' are found by getting the tangent planes (25) and (26) determined by the granted points, and then by "turning *both* on edge" via (27). Advance knowledge of the values of X and Y in terms of the small letters (which in itself would show the locus of (20) as degenerate) is not necessary.

Applying the theorem to the equation

$$4x^2 + 5y^2 + 2z^2 + u^2 + 4xy + 4xz + 6yz + 4yu + 2zu = 25, \tag{28}$$

and using the "points" (0, 0, 0, 5) and (0, $\sqrt{5}$, 0, 0), we get $X' = 2y+z+u$ and $Y' = 2x+5y+3z+2u$, whence the G.C. of (X', Y') are (x, y, $Y'-2X'-2x-y$, $3X'-Y'+2x-y$), and the left side of (28) collapses to $5X'^2-4X'Y'+Y'^2$. Since we had started out sneakily with the left side of (28) in the form $(2x+y+z)^2 + (2y+z+u)^2$, we have learned that this is identical with

$$5(2y + z + u)^2 - 4(2y + z + u)(2x + 5y + 3z + 2u) + (2x + 5y + 3z + 2u)^2.$$

Interesting identities of this nature are spawned by the method, and the algebraic work is much shorter if we start with an equation in the form for which the degenerate locus is obvious. Doing that with (28) and using the "points" (1, 1, 0, 2) and (1, 0, 2, 1) we find that another version of the left side of (28) is $(25X'^2-48X'Y'+25Y'^2)/49$, in which

$$X' = 6x + 11y + 7z + 4u \quad \text{and} \quad Y' = 8x + 10y + 7z + 3u.$$

Of course, it may be impossible to find two "points" as in the theorem. This would occur in 3-space when the locus of the quadratic is imaginary, a straight line, a point, or two coincident planes. Corresponding cases when $n > 3$ would be exposed at once in the first three cases by the normal plotting technique. In the case of two coincident "planes" whose equation is $f^2 = 0$, or two intersecting "planes" ($FG = 0$), the "tangent plane" equation reveals one factor at once, showing incidentally one sure method of factoring a reducible quadratic in n variables.

References

1. R. S. Underwood, An analytic geometry for N variables, this MONTHLY, 52 (1945) 253–262.
2. ———, Some applications of extended analytic geometry, this MONTHLY, 56 (1949) 158–164.
3. ———, Functions of N variables in extended analytic geometry, this MONTHLY, 59 (1952) 453–460.
4. ———, Extended analytic geometry as applied to simultaneous equations, this MONTHLY, 61 (1954) 525–542.
5. ———, Silhouette mathematics, Texas Tech. Bookstore, Lubbock, Texas, 1961.
6. R. S. Underwood and F. W. Sparks, Analytic geometry, 3rd ed., Houghton Mifflin, Boston, Mass., 1963.
7. D. C. B. Marsh, Silhouettes and degenerate loci of quadratic forms, this MONTHLY, 69 (1962) 867–876.

A CHARACTERIZATION OF THE GEOMETRIC DISTRIBUTION

T. S. FERGUSON, University of California, Los Angeles

1. Introduction. Certain distributions may be characterized by conditions asserting the independence of suitable statistics. The most renowned of these characterization theorems is that first proved by Kac [5] which states: *if random variables X and Y are independent, then $X+Y$ and $X-Y$ are independent if, and only if, both X and Y have normal distributions with a common variance.*

A similar characterization of the gamma distribution has been proved by Hogg [4] and Lukacs [6]. It is the purpose of this paper to investigate the extent to which the geometric distribution may be characterized by analogous assumptions of independence.

The starting point for this investigation was the observation in [1] that if X and Y are independent random variables each following the same geometric law with probability mass function

$$f(k) = (1-p)p^k \qquad k = 0, 1, 2, \cdots, \tag{1}$$

where $0<p<1$, then the random variables $\min(X, Y)$ and $X-Y$ are stochastically independent. It was also observed, however, that if X and Y are independent random variables having the same exponential distribution with density

$$\begin{aligned} f(z) &= \lambda e^{-\lambda(z-\theta)} && \text{if } z > \theta \\ &= 0 && \text{if } z < \theta, \end{aligned} \tag{2}$$

where $\lambda>0$, then again the random variables $\min(X, Y)$ and $X-Y$ are stochastically independent. Thus, the independence of $\min(X, Y)$ and $X-Y$ for independent X and Y will not by itself characterize either the geometric or exponential distributions, so that a complete analogue of the result of Kac cannot be valid. Therefore, in order to obtain a characterization of either the geometric

or the exponential distribution, it is necessary to restrict the distributions in some way. In [2], the following result was obtained: *if X and Y are independent random variables with absolutely continuous distributions, then* $\min(X, Y)$ *and* $X-Y$ *are independent if, and only if, both X and Y have exponential distributions of the form* (2) *with a common value of θ (but possibly different values of λ).*

For a formulation of the problem which contains the normal, the gamma, the exponential, the geometric, and other distributions in a single statement, see [1] or [2]. In this paper the distributions of X and Y are restricted to be discrete.

The term *geometric distribution* signifies in this paper the distribution of any random variable X for which there is a real number θ and a positive real number c such that the random variable $(X-\theta)/c$ has the distribution of formula (1). The parameters θ and c are respectively location and scale parameters of the distribution of X. The parameter p will be referred to as the geometric parameter. A random variable X is said to be degenerate at x_0 if $P(X=x_0)=1$. In the next section, it is shown that if X and Y are independent, nondegenerate, discrete random variables, then $\min(X, Y)$ and $X-Y$ are independent if, and only if, X and Y both have geometric distributions with common location and scale parameters, but with possibly different geometric parameters.

With such a theorem, the study of the independence of $\min(X, Y)$ and $X-Y$ for independent, discrete X and Y is easily completed with the aid of the following simple theorem which takes care of the degenerate cases.

THEOREM 1. *If X is degenerate at x_0, then* $\min(X, Y)$ *and* $X-Y$ *are independent if, and only if, either* (i) *Y is degenerate or* (ii) $P(Y\geqq x_0)=1$.

Proof. The "if" part of the theorem is immediate, since degenerate random variables are independent of any random variables, and since both (i) and (ii) imply that $\min(X, Y)$ is degenerate. To prove the "only if" part, suppose that Y is nondegenerate and that $P(Y\geqq x_0)<1$. Then, there exists a number $m<x_0$ such that $0<P(Y<m)<1$. Using the independence of $\min(x_0, Y)$ and x_0-Y, we have

$$\begin{aligned} P(Y<m) &= P(\min(x_0, Y)<m) = P(\min(x_0, Y)<m \mid x_0-Y>x_0-m) \\ &= P(Y<m \mid Y<m) = 1. \end{aligned}$$

This contradiction completes the proof.

The analogous theorem where Y is degenerate is also valid, since if $\min(X, Y)$ is independent of $X-Y$, it is automatically independent of $Y-X$.

2. The characterization theorem. We first present two lemmas. The first lemma may be found as a special case of a lemma of Ghurye [3, Lemma 1] proved by means of Fourier transforms. For the sake of completeness and simplicity we give a proof of Lemma 1 involving only elementary notions.

LEMMA 1. *If X and Y are independent, and if X and $X-Y$ are independent, then X is degenerate.*

Proof. Suppose that X is not degenerate. Then, there exist four points $x_1<x_2<x_3<x_4$ such that $P\{x_1\leqq X\leqq x_2\}>0$ and $P\{x_3\leqq X\leqq x_4\}>0$. For any other variable Y and all numbers y

$$P\{Y - X \leqq y, x_1 \leqq X \leqq x_2\} \leqq P\{Y \leqq y + x_2, x_1 \leqq X \leqq x_2\} \tag{3}$$

and

$$P\{Y - X \leqq y, x_3 \leqq X \leqq x_4\} \geqq P\{Y \leqq y + x_3, x_3 \leqq X \leqq x_4\}. \tag{4}$$

Inequality (3) and the independence assumptions imply

$$P\{Y - X \leqq y\} \leqq P\{Y \leqq y + x_2\} \tag{5}$$

for all y. Similarly, from inequality (4),

$$P\{Y \leqq y + x_3\} \leqq P\{Y - X \leqq y\} \tag{6}$$

for all y, which in conjunction with (5) implies that Y has no mass anywhere. This contradiction completes the proof.

LEMMA 2. *Let X and Y be independent nondegenerate random variables and suppose that $U=\min(X, Y)$ and $V=X-Y$ are independent. If $P(X=x)>0$ and $P(Y>x)>0$, then $P(Y=x)>0$. Similarly, if $P(Y=y)>0$ and $P(X>y)>0$, then $P(X=y)>0$.*

Proof. Suppose $P(Y=x)=0$. Then if $P(X=x)>0$ and $P(Y>x)>0$,

$$P(V \leqq 0) = P(V \leqq 0 \mid U = x) = P(X \leqq Y \mid X = x, Y \geqq x) = 1. \tag{7}$$

Thus, $P(X\leqq Y)=1$, so that U is equal to X with probability 1. Since now X is independent of $X-Y$, Lemma 1 will provide a contradiction, proving that $P(Y=x)>0$. The last statement follows by symmetry since if U is independent of V it is also independent of $-V$.

A number x is said to be a possible value of a discrete random variable X, if $P(X=x)>0$. Using this terminology, the conclusions of Lemma 2 may be stated for discrete variables as: *Every possible value of X less than some possible value of Y is a possible value of Y, and every possible value of Y less than some possible value of X is a possible value of X.*

THEOREM 2. *Let X and Y be independent, nondegenerate, discrete random variables. Then, $U=\min(X, Y)$ and $V=X-Y$ are independent if, and only if, both X and Y have geometric distributions with the same location and scale parameters.*

Proof. First, suppose that U and V are independent; then for all u and all $v\geqq 0$,

$$\begin{aligned} P(U = u)P(V = v) &= P(\min(X, Y) = u, X - Y = v) \\ &= P(X = u + v)P(Y = u). \end{aligned} \tag{8}$$

Lemma 2 implies that $P(V=0)\neq 0$, so that by putting $v=0$ in equation (8) we

may solve for $P(U=u)$. Substituting this value into equation (8) gives for all u and all $v \geqq 0$

$$P(X = u)P(Y = u)P(V = v) = P(X = u + v)P(Y = u)P(V = 0). \tag{9}$$

Lemma 2 and nondegeneracy imply that there are at least two distinct numbers, $x_0 < x_1$, which are possible values simultaneously for X and Y. Two equations may be obtained by letting u assume the values x_0 and x_1 in (9), from which $P(V=v)$ may be eliminated, yielding for all $v>0$,

$$P(X = x_1 + v) = \left[\frac{P(X = x_1)}{P(X = x_0)}\right] P(X = x_0 + v). \tag{10}$$

This implies, by induction, that

$$P(X = n(x_1 - x_0) + x_0) = \left[\frac{P(X = x_1)}{P(X = x_0)}\right]^n P(X = x_0). \tag{11}$$

Since the total mass of the distribution of X must be finite, $P(X=x_0) > P(X=x_1)$. But since x_0 and x_1 were arbitrary numbers which were possible values of both X and Y, we see that there can be at most a finite number of possible values of X which are less than x_1. We now suppose that x_0 is the smallest of the possible values of X and that x_1 is the next smallest. Then equation (10) inductively implies that the only possible values of X are $n(x_1-x_0)+x_0$, $n=0, 1, 2, \cdots$ and equation (11) implies that X has a geometric distribution on these values. By symmetry, Y also has a geometric distribution, and Lemma 2 implies that Y must have the same set of possible values, and hence the same location and scale parameters.

Now, suppose that X and Y have geometric distributions with the same location and scale parameters, and geometric parameters p_1 and p_2 respectively. A change of location and scale will not affect the independence or nonindependence of U and V, so we may choose the variables to be defined on the nonnegative integers as in equation (1). Then, for all u and all $v \geqq 0$,

$$P\{U = u, V = v\} = P\{X = u + v\}P\{Y = u\} = (1 - p_1)(1 - p_2)p_1^{u+v}p_2^u,$$

while for all u and $v<0$,

$$P\{U = u, V = v\} = P\{X = u\}P\{Y = u - v\} = (1 - p_1)(1 - p_2)p_1^u p_2^{u-v}$$

Thus $P\{U=u, V=v\}$ obviously factors into a product of a function of u, say $P\{U = u\} = (1 - p_1p_2)(p_1p_2)^u$, and a function of v, say $P\{V = v\} = (1-p_1)(1-p_2)(1-p_1p_2)^{-1}p_1^v$ for $v \geqq 0$, and

$$P\{V = v\} = (1 - p_1)(1 - p_2)(1 - p_1p_2)^{-1}p_2^{-v} \quad \text{for} \quad v < 0,$$

proving the independence of U and V.

References

1. T. S. Ferguson, Location and scale parameters in exponential families of distributions, Ann. Math. Statist., 33 (1962) 986–1001.

2. ———, A characterization of the exponential distribution, Ann. Math. Statist., 35 (1964) 1199–1207.

3. S. G. Ghurye, Random functions satisfying certain linear relations, Ann. Math. Statist., 25 (1954) 543–554.

4. R. V. Hogg, On ratios of certain algebraic forms, Ann. Math. Statist., 22 (1951) 567–572.

5. M. Kac, On a characterization of the normal distribution, Amer. J. Math., 6 (1939) 726–728.

6. E. Lukacs, A characterization of the gamma distribution, Ann. Math. Statist., 26 (1955) 319–324.

THE LAMBDA NUMBER OF A MATRIX: THE SUM OF ITS n^2 COFACTORS

MARJORIE R. BICKNELL, San Jose State College (Now a teacher at Adrian Wilcox High School, Santa Clara, Calif.)

This paper contains a brief exposition of the effects of adding a given number to each element of a matrix and of properties of the sum of the cofactors of a matrix. A novel method of evaluating a determinant, a quick computational check for the elements of the matrix inverse, and a new proof of a well-known property of the binomial coefficients are presented.

1. The lambda number. The lambda number $\lambda(A)$ of an $n\times n$ matrix A is defined as the change in the value of the determinant of A when the number one is added to each element of A. Three methods of evaluating $\lambda(A)$ and general properties of the lambda number are given below.

THEOREM 1. *The lambda number of the $n\times n$ matrix $A=[a_{ij}]$ is given by $\lambda(A)=\det L$, where $L=[c_{ij}]$ is the matrix of order $n-1$ given by*

$$c_{ij} = a_{ij} + a_{i+1,j+1} - a_{i,j+1} - a_{i+1,j}.$$

Further, if A^* is the matrix formed from A by adding x to each element of A, then

$$\det A^* = \det A + x\lambda(A). \tag{1}$$

COROLLARY 1.1. *For any scalar k, $\lambda(kA)=k^{n-1}\lambda(A)$.*

COROLLARY 1.2. *For the matrices of Theorem 1, $\lambda(A^*)=\lambda(A)$.*

THEOREM 2. *The lambda number of the $n\times n$ matrix A is given by $\lambda(A) = \sum_{i=1}^{n} \det A_i$, where A_i is the matrix formed from A by replacing the i-th row (column) by a row (column) of 1's.*

COROLLARY 2.1. *$\lambda(A)$ equals the sum of the cofactors of the n^2 elements of A.*

THEOREM 3. *Let $B=[b_{rs}]$ be the matrix of order $n+1$ formed by bordering the $n\times n$ matrix $A=[a_{ij}]$ above and to the left, so that $b_{11}=0$, $b_{1s}=b_{r1}=1$, and $b_{i+1,j+1}=a_{ij}$ for $i, j=1, 2, \cdots, n$. Then* det $B=-\lambda(A)$.

COROLLARY 3.1. *If A' is the transpose of A, then $\lambda(A')=\lambda(A)$.*

COROLLARY 3.2. *If C is the matrix formed from A by exchanging the r-th and s-th rows (columns) of A, then $\lambda(C)=-\lambda(A)$.*

COROLLARY 3.3. *If there is a constant difference between corresponding elements of any two rows (or columns) of A, then $\lambda(A)=0$.*

Theorems 1 and 2 are proved by expanding the determinant of the matrix A^* of Theorem 1. Theorem 3 is obtained by expanding det B by its first row (or column) and applying Theorem 2. Theorems 1, 2, and 3 have been given by Stancliff [1] and by Muir [2], [3], [4]; for proofs of the main theorems above, the reader is directed to Muir [2]. Corollary 3.3, given by Trigg [5], is interesting, since when $\lambda(A)=0$, any number x can be added to each element of A without changing the value of det A. Theorem 1 is illustrated in the example following.

At this point, it is worthwhile to notice that Equation (1) affords an alternative method of evaluating a determinant, for if $\lambda(A)$ is known, any number may be subtracted out of A to form A^*. If several zeroes can be introduced or if the size of the elements can be reduced appreciably by adding some x, the method suggested by Equation (1) is practical. For an example, consider

$$M=\begin{bmatrix} 1000 & 998 & 554 \\ 990 & 988 & 554 \\ 675 & 553 & 554 \end{bmatrix}.$$

Here, if $x=-554$, det $M^*=0$, since each element in its third column will be zero. By Theorem 1,

$$\lambda(M)=\begin{vmatrix} 1000+988-990-998 & 998+554-988-554 \\ 990+553-988-675 & 988+554-553-554 \end{vmatrix}$$

$$=\begin{vmatrix} 0 & 10 \\ -120 & 435 \end{vmatrix}=1200.$$

Hence, det $M=(1200)(554)$.

The next theorem, conceived by Stancliff, provides a computational check upon the elements of a matrix inverse.

THEOREM 4. *If the $n\times n$ matrix $A=[a_{ij}]$ is nonsingular, then $\lambda(A)$ equals* det *A times the sum of the elements of A^{-1}.*

Proof. Since the element in the ith row and jth column of A^{-1} is given by $A_{ji}/\det A$, where A_{ji} is the cofactor of a_{ji}, Theorem 4 is a consequence of Corollary 2.1.

2. The ultimate lambda number. The ultimate lambda number is obtained by repeatedly applying Theorem 1. If A is an $n\times n$ matrix, let $\lambda(A)=\det L_1$, where L_1 is the matrix of order $n-1$ defined in Theorem 1. Then, find $\lambda(L_1)=\det L_2$ and $\lambda(L_2)=\det L_3$ by reapplying Theorem 1, continuing the process until the matrix L_{n-1} of order 1 is obtained. We define $\det L_{n-1}$ as the ultimate lambda number of A, denoted by $U(A)$.

Two forms of the ultimate lambda number of the identity matrix provide a new proof of a well-known property of the binomial coefficients. First, we introduce the following lemma:

LEMMA. *If $A=[a_{ij}]$ is an $n\times n$ symmetric matrix, and if $\lambda(A)=\det L$ where $L=[c_{ij}]$ is constructed as in Theorem* 1, *then L is also a symmetric matrix.*

Proof. By hypothesis, $A=A'$ so that $a_{ij}=a_{ji}$ for all i and j. By substitution in Theorem 1, $c_{ij}=a_{ji}+a_{j+1,i+1}-a_{j+1,i}-a_{j,i+1}=c_{ji}$ for all i and j so that $L=L'$.

THEOREM 6. *The ultimate lambda number of the identity matrix I of order n is given by the binomial coefficient*

$$\binom{2n-2}{n-1}.$$

Proof. The proof is by mathematical induction. To apply the definition of ultimate lambda number, let $L_k=[c_{ij}]$ be the matrix formed from I after k steps. We show that

$$c_{ij} = (-1)^{i+j}\binom{2k}{k+i-j}, \tag{6}$$

where we note that $\det L_{n-1}=c_{11}$ for $k=n-1$. Since, by the lemma, the L_k are symmetric, without loss of generality we assume that $j\geqq i$. Also, define $\binom{m}{p}=0$ whenever $p>m$ or $p<0$. We observe that Equation (6) holds when $k=0$, or for I itself. Assume that Equation (6) holds for L_s and consider $L_{s+1}=[c_{ij}^*]$. Then, apply the recursion formula for the binomial coefficients,

$$\binom{n}{m}+\binom{n}{m+1}=\binom{n+1}{m+1},$$

and Theorem 1. By the inductive hypothesis,

$$\begin{aligned} c_{ij}^* &= (-1)^{i+j}\left[\binom{2s}{s+i-j}+\binom{2s}{s+i-j}+\binom{2s}{s+i-j-1}+\binom{2s}{s+i+1-j}\right] \\ &= (-1)^{i+j}\left[\binom{2s+1}{s+i-j}+\binom{2s+1}{s+i+1-j}\right] = (-1)^{i+j}\binom{2s+2}{s+1+i-j}. \end{aligned}$$

Then, Equation (6) holds for all $k \geqq 0$, so that the theorem follows.

THEOREM 7. *The ultimate lambda number of the $n \times n$ diagonal matrix $A = [a_{ij}]$ is given by*

$$U(A) = \sum_{m=1}^{n} \binom{n-1}{m-1}^2 a_{mm}.$$

Proof. The proof by induction is a generalization of the proof of Theorem 6, applying only Theorem 1 and the recursion formula for the binomial coefficients. Here, we let $L_k = [c_{ij}]$ be the matrix formed from A after k steps and show that

$$c_{ij} = (-1)^{i+j} \sum_{m=j}^{k+j} \binom{k}{m-i} \binom{k}{m-j} a_{mm}.$$

Since the proof follows exactly the pattern of the proof of Theorem 6, we omit the details.

THEOREM 8.

$$\binom{2n-2}{n-1} = \sum_{m=1}^{n} \binom{n-1}{m-1}^2.$$

Proof. Let $a_{ii} = 1$ in Theorem 7 and equate the expressions for $U(I)$ as given by Theorems 6 and 7.

3. The lambda number of A^n. In this section we study a type of matrix which contains the lambda number of one of its powers as an element. Also, a relationship between the lambda number and the characteristic equation of a matrix is derived.

THEOREM 9. *Let $A = [a_{ij}]$ be any nonsingular $n \times n$ matrix such that every element in the r-th row and every element in the p-th column equals* 1. *Then,* $\lambda(A^{-k})$ *equals* $\det A^{-k}$ *times the element appearing in the r-th row and p-th column of A^{k+2} for all integers k.*

Proof. Let k be any integer. Let $A^k = [b_{ij}]$, $A^{k+1} = [c_{ij}]$, and $A^{k+2} = [d_{ij}]$. Since $A^{k+1} = AA^k$,

$$c_{rj} = \sum_{p=1}^{n} a_{rp} b_{pj} = \sum_{p=1}^{n} b_{pj}$$

by definition of matrix multiplication and since $a_{rp} = 1$ for all p. Similarly, since $A^{k+2} = A^{k+1}A$,

$$d_{rp} = \sum_{s=1}^{n} c_{rs} a_{sp} = \sum_{s=1}^{n} c_{rs}, = \sum_{p=1}^{n} \sum_{r=1}^{n} b_{pr}.$$

That is, the element in the rth row and pth column of A^{k+2} equals the sum of

the elements of A^k. The theorem follows by Theorem 4.

COROLLARY 9.1. *Let A be the matrix of Theorem 9, where $A^k=[b_{ij}]$. If* $\det A=\pm 1$, *then the cofactor B_{rp} equals $(-1)^{r+p}$ times $\lambda(A^{k+2})$.*

Proof. B_{rp} equals $(-1)^{r+p}$ or $(-1)^{k+r+p}$ times the element in the rth row and pth column of A^{-k} as $\det A=1$ or -1. In either case, the corollary follows from substitution in Theorem 9.

THEOREM 10. *Let A be any nonsingular $n\times n$ matrix. If c_i is the coefficient of x^i in the characteristic equation of A, then, for all integers k,*

$$\lambda(A^{-k}) = \sum_{i=0}^{n-1} - c_i(\det A^{i-n})\lambda(A^{n-k-i}).$$

Proof. By the Hamilton-Cayley Theorem, A satisfies its own characteristic equation. Thus,

$$A^k = \sum_{i=0}^{n-1} - c_i A^{i+k-n}.$$

Since the corresponding elements of the A^k must have the same recursive property as do the matrices A^k, so must the sums of the elements of the A^k. We obtain the theorem by applying Theorem 4 and dividing by $\det A^k$.

In conclusion, two interesting extensions of Theorem 10 follow.

COROLLARY 10.1. *If* $\det A=1$, *then* $\lambda(A^{-k})=\sum_{i=0}^{n-1} -c_i\lambda(A^{n-k-i})$.

COROLLARY 10.2. *The ultimate lambda numbers of the A^k are related by the recursion formula* $U(A^k)=\sum_{i=0}^{n-1} -c_iU(A^{i+k-n})$.

Proof. The proof is based upon showing that, if $\lambda(A^k)=\det L_k$, where L_k is formed as in Theorem 1, then the L_k satisfy the same recursion formula as do the A^k.

The lambda number arose in the unpublished notes of the late Fenton S. Stancliff, a professional musician who had a lifelong creative interest in mathematics. Mr. Stancliff's notes provided the background material and the inspiration for further study of the topic. I wish to thank Dr. Verner E. Hoggatt for his help and encouragement.

References

1. From the unpublished notes of Fenton S. Stancliff.
2. Thomas Muir, A treatise on the theory of determinants, (Revision by William H. Metzler), Dover, New York, 1960, pp. 97–99.
3. ———, The generating functions of certain special determinants, Proc. Roy. Soc. Edinburgh, 24 (1902) 387–392.
4. ———, The sum of the signed primary minors of a determinant, Proc. Roy. Soc. Edinburgh, 25 (1904) 372–382.
5. Charles W. Trigg, A theorem on determinants, Math. Mag., 34 (1961) 328.

IDEALS IN ORDERED GROUPS

JAN F. ANDRUS, University of Florida (Now with General Electric Co., Huntsville, Alabama)
and ALTON T. BUTSON, University of Miami

1. Introduction. Consider the mathematical system P satisfying the postulates:

(i) *P is a group under the operation* $+$;

(ii) *P is a partially ordered set under the relation* $\geqq$;

(iv) *For any ideal J of P, $x+J+y$ is an ideal or a dual ideal when x and y are any elements of P.*

(We defer the definition of an *ideal* until later in the paper.)

Axiom (iv) here replaces the familiar axiom:

(iii) *If a and b are any elements of P such that $a \geqq b$, then $x+a+y \geqq x+b+y$ for any pair of elements x and y in P.*

This paper shows that with a minor qualification, postulate (iv) may be phrased in terms of the more fundamental and easily visualized concept expressed in the following postulate:

(v)′ *For each component C of P and each pair of elements x and y in P, either it is true that $x+a+y \geqq x+b+y$ for all elements a and b in C such that $a \geqq b$ or it is true that $x+a+y \leqq x+b+y$ for all pairs of elements a and b in C such that $a \geqq b$.*

A *component* C of P is defined to be a minimal set of elements in P such that no element of C is related by $\geqq$ to any element not in C. The component is the fundamental concept to which we have referred. It is our intention to prove the two following assertions:

ASSERTION 1. Disregarding a class of trivial systems to be characterized in the proof of the assertions, postulate (iv) is equivalent to the axiom:

(v) *For any ideal or dual ideal J of P, $x+J+y$ is either an ideal or a dual ideal.*

ASSERTION 2. Postulates (v) and (v)′ are equivalent.

In an earlier paper [1] the authors analyzed the systems defined by (i), (ii) and (v)′. Therefore, the two assertions relieve us of the task of analyzing systems defined by (i), (ii) and (iv). For examples and other related information, refer to [1].

2. Preliminary definitions. We say that two elements of P are *connected* if and only if they are contained in the same component of P.

The symbol F^* represents the set of all elements of P equal to or greater than all elements of the subset F of P, and F^+ stands for the set of those elements equal to or less than all elements of F. The symbols f^* and f^+, where f is an element of P, are defined as if f represents the set composed of f alone.

A subset J of P is defined to be an *ideal* (closely following Frink in [2]) if

for every finite nonvoid subset F of J, F^* is nonvoid and $(F^*)^+ \subset J$. A *dual ideal* is defined by interchanging the symbols * and +.

The *length* of a set S is the largest integer n for which there exist $n+1$ elements $x_1, x_2, \cdots, x_{n+1}$ of S such that $x_1 > x_2 > \cdots > x_{n+1}$.

3. Proof of the assertions. The proof that postulates (i), (ii) and (v)′ together imply (v) is long but straightforward. Consequently, we do not present it here except to say that the proof employs the equalities

$$x + F^{*+} = (x + F)^{*+} \quad \text{(if } x \text{ preserves order)}$$

$$x + F^{*+} = (x + F)^{+*} \quad \text{(if } x \text{ reverses order)}$$

which may be derived from (i), (ii) and (v)′.

Assume now that P is an algebraic system satisfying (i), (ii) and (iv).

THEOREM 1. *One and only one of the following statements is true of P:* (1) *P has infinite length and no minimal elements,* (2) *P has length one and no isolated elements,* (3) *P is totally unordered.*

Proof. If a is a minimal element of P, the set consisting of a alone is an ideal. Hence any translation $x+a$ of a is an ideal or a dual ideal. It can be translated into any element of P, in particular into an element which is neither maximal nor minimal if such exists. A set composed of an element such as this, however, can be neither an ideal nor a dual ideal because it contains neither $(x+a)^+$ nor $(x+a)^*$. Thus the existence of minimal elements is incompatible with that of chains of length greater than one.

LEMMA 1. *If a is connected to b, then $x+a+y$ is connected to $x+b+y$.*

Employing Lemma 1, which follows from (iv) and the transitivity of the order relation, one can obtain

THEOREM 2. *The component C_0 containing the zero element* 0 *is a normal subgroup of P whose cosets are the components of P.*

In accordance with Theorem 1, the investigation will be divided into a study of systems of length one and systems of infinite length.

LEMMA 2. *If P is of length one and $a > b$, then $x+a$ and $x+b$ are not both maximal.*

Proof. Assume that $a > b$ and $x+a$ and $x+b$ are maximal. According to the definition of an ideal (dual ideal) J, every nonvoid subset F of J must have an upper (lower) bound. Clearly $x+a$ and $x+b$ have no upper bound. Since $x+a^+$ is either an ideal or a dual ideal containing $x+a$ and $x+b$, there exists an element $x+c \in \{x+a, x+b\}^+$. Since $-x+(x-a)^+$ is either an ideal or a dual ideal, either $\{a, c\}^+ \neq \emptyset$ or $\{a, c\}^* \neq \emptyset$, where $\emptyset$ is the null set. Likewise, the fact that $-x+(x+b)^+$ is an ideal or a dual ideal implies that either $\{b, c\}^+ \neq \emptyset$ or $\{b, c\}^* \neq \emptyset$, which is impossible unless $c > b$ or $c < a$.

It can be shown that the figures of Fig. 1 are incompatible. Likewise, the configuration shown in Fig. 2 is impossible. It is possible to show that either of the cases, $c>b$ or $c<a$, gives rise to an impossible situation like that of Fig. 1 or Fig. 2. This contradicts the initial assumption and proves Lemma 2.

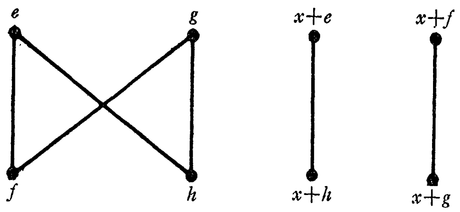


FIG. 1

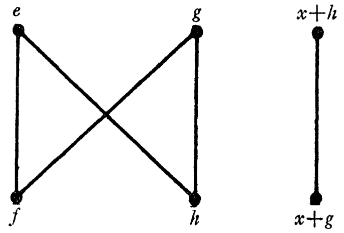


FIG. 2 ($x+e$ and $x+f$ assumed to be maximal)

LEMMA 3. *If P is of length one and $a\geqq b$ implies $x+a+y$ and $x+b+y$ are comparable, then P satisfies postulate* (v).

Proof. Let c and c' be maximal elements of a component C of P. There exists a sequence $c=x_1, x_2, \cdots, x_n=c'$ of distinct elements of C such that x_i is comparable to x_{i+1}. Furthermore, x_i is maximal if and only if i is odd. In particular, n is odd. The sequence $x+c=x+x_1, x+x_2, \cdots, x+x_n=x+c'$ is another sequence of distinct elements such that $x+x_i$ is comparable to $x+x_{i+1}$. If $x+c$ is maximal (minimal), then $x+x_i$ is maximal if and only if i is odd (even). Since n is odd, $x+c'=x+x_n$ is maximal if and only if $x+c$ is maximal. Thus x preserves or reverses order (when added to the left of comparable elements) in C according to whether or not $x+c'$ is maximal. The remainder of the proof is obvious.

THEOREM 3. *If P does not satisfy Postulate* (v), *then P is* (1) *a partially ordered set of length one in which* (2) *the components are the cosets of C_0 and* (3) *every component is a principal ideal (i.e. every component can be expressed as a^+ for some $a\in P$) containing more than two distinct elements. Conversely, any group P satisfying* (1)–(3), *for some normal subgroup C_0, satisfies axioms* (i), (ii) *and* (iv) *but not* (v).

Proof. The converse statement will be proven first. Assume that P satisfies (1)–(3) for some normal subgroup C_0 and let J be an ideal of P. It is contained in some component C. It is simple to show that $J=C$. Since $x+J+y=x+C+y$ is a coset of C_0 and hence a principal ideal, $x+J+y$ is an ideal. This proves that P satisfies (iv). It is not difficult to construct a dual ideal J for which $x+J$ is neither an ideal nor a dual ideal.

Now let P satisfy axioms (i), (ii) and (iv) but not (v), and assume it is of length one. By the contrapositive statement of Lemma 3, there are elements a, b and x such that $a>b$ but $x+a$ and $x+b$ are not comparable. By Lemma 2 and the fact that $\{x+a, x+b\}$ is contained in $x+a^+$, there is an element $x+c$ properly over $x+a$ and $x+b$. By the use of Lemma 2, it is possible to show that $c<a$.

The set $x+a^+$ is an ideal, because if it were a dual ideal, it would contain two minimal elements, $x+a$ and $x+b$. It cannot contain a maximal element distinct from $x+c$ or a minimal element not under $x+c$. It follows that $x+a^+=(x+c)^+$. Employing the latter equality and Lemma 2, it can be shown that any element e, such that $e>c$, is equal to a. From this it follows that if $f>d$, for some $f\in P$ and any $d\in P$ such that $d<a$, then $f=a$. Therefore, the particular component C_a containing a is the set a^+. It can be shown from this that any component C of P is a principal ideal.

The proof of Theorem 3 (and Assertion 1) would be complete if it were known that all systems satisfying axioms (i), (ii) and (iv) which are of infinite length must satisfy postulate (v). This result will be obtained in the remainder of the paper. The proof of Assertion 2 is a by-product.

Since $x+a\in x+b^+$ implies $a\leqq b$, we have

LEMMA 4. *If $a>b$ and $x+b^+$ is an ideal, then $x+a\nleqq x+b$. If $a>b$ and $x+b^+$ is a dual ideal, then $x+a\geqq x+b$.*

LEMMA 5. *If $x+b^+$ is an ideal and b is not maximal, then $x+b^+=(x+b)^+$.*

Proof. If $-x+(x+b)^+$ were a dual ideal, we would have $b^*\subset -x+(x+b)^+\subset b^+$ which is possible only if b is maximal. Thus $-x+(x+b)^+$ is an ideal. Since $b\in -x+(x+b)^+$, $b^+\subset -x+(x+b)^+$ and $x+b^+\subset(x+b)^+$.

The proof of the following Lemma is based upon Lemma 5, the contrapositive of Lemma 4, and the fact that every finite subset of a dual ideal has a lower bound.

LEMMA 6. *If a is not maximal, $a>b$, $-x+(x+a)^+$ is a dual ideal, and $x+b^+$ is an ideal, then $x+a>x+b$.*

LEMMA 7. *If a is not maximal, $a>b$, and P is of infinite length, then $x+a$ and $x+b$ are comparable.*

Proof. We assume that $-x+(x+a)^+$ is a dual ideal and $-x+(x+b)^+$ is an ideal, for otherwise the lemma is obvious. We may also assume that $x+b$ is maximal, because otherwise Lemmas 5 and 6 imply $x+a>x+b$.

Now if $x+b^+$ is an ideal, according to Lemma 6, $x+a$ and $x+b$ are comparable. Assume, therefore, that $x+b^+$ is a dual ideal. There is an element $d<b$ since P is of infinite length. Since $-x+(x+b)^+$ is an ideal containing b and d, $x+d\in(x+b)^+$ and $x+d<x+b$. Now $a>d$, where a and $x+d$ are not maximal. It follows from Lemmas 5 and 6 that $x+a$ and $x+d$ are comparable. If $x+a<x+d$, then $x+a<x+b$. Therefore, assume that $x+a>x+d$. Since $d\in b^+$, $x+d\in x+b^+$. Thus $(x+d)^*\subset x+b^+\subset(x+b)^+$. This implies $x+a\in(x+b)^+$, completing the proof.

Suppose that a is maximal and that P is of infinite length. There exists a chain $a>e>f>g$ and elements t, b and c in P such that $t+a=f$, $t+b=e$ and $t+c=g$. The chain may be written as $a>t+b>t+a>t+c$. Since $t+b>t+a$, it

follows from Lemma 7 that a and b are comparable. Moreover, $a>b$, because a is maximal.

By Lemma 4, $-t+(t+a)^+$ is a dual ideal. Now $c\in -t+(t+a)^+$. Thus $c^*\subset -t+(t+a)^+$ and $t+c^*\subset(t+a)^+$. If $b\in c^*$, then $t+b\in(t+a)^+$, which is impossible since $t+b>t+a$. Hence $b\notin c^*$. However, by Lemma 7, b and c are comparable. Thus $b<c$. The elements a and c are also comparable because $t+a>t+c$ and $t+a$ is not maximal. Since a is maximal, $a>c$. From Lemma 4 and the relations $t+a>t+c$ and $a>c$, it follows that $-t+(t+c)^+$ is an ideal. Lemma 5 implies $-t+(t+c)^+=c^*$. Since $b\in c^+$, $t+b\in(t+c)^+$ and $t+b<t+c$, a contradiction. We have proven the two following theorems:

THEOREM 4. *A system satisfying axioms* (i), (ii) *and* (iv) *and having infinite length has no maximal or minimal elements.*

THEOREM 5. *Comparability is preserved under addition in a system satisfying* (i), (ii) *and* (iv) *and having infinite length.*

Theorems 5 and 6 and Lemmas 4 and 5 may be used to prove the following lemma.

LEMMA 8. *If $a>b$ in P, where P is of infinite length, then $x+a^+$ is an ideal if and only if $x+a>x+b$, $x+a^+$ is a dual ideal if and only if $x+a<x+b$, $x+b^+$ is an ideal if and only if $x+a>x+b$, and $x+b^+$ is a dual ideal if and only if $x+a<x+b$.*

An immediate consequence of Lemma 8 is

LEMMA 9. *Let a and b be comparable elements of P, where P has infinite length. Then $x+a^+$ is an ideal if and only if $x+b^+$ is an ideal, and $x+a^+$ is a dual ideal if and only if $x+b^+$ is a dual ideal.*

The first paragraph of Section 3 and application of finite induction to Lemmas 8 and 9 complete the proof of Assertion 1 and the proof of Assertion 2 for the case in which P is of infinite length. It remains to be shown that if P is of length one and satisfies (i), (ii), and (v), then it must also satisfy (v)′. This relatively simple proof is left to the reader.

References

1. J. F. Andrus and A. T. Butson, Ordered groups, this MONTHLY, 70 (1963) 619–628.
2. O. Frink, Ideals in partially ordered sets, this MONTHLY, 61 (1954) 223–234.

SOLUTION OF THE INITIAL VALUE PROBLEM FOR THE RICCATI EQUATION

W. C. HOFFMAN, Boeing Scientific Res. Laboratories, Seattle, Washington

As is well known the general Riccati differential equation

$$Y' + p(\xi)Y^2 + q(\xi)Y(\xi) + r(\xi) = 0 \tag{1}$$

is reduced by the transformations

$$Y = \exp\left[-\int^{\xi} q d\xi'\right] y(x), \qquad x = \int^{\xi} p(\xi') \exp\left[-\int^{\xi} q d\xi''\right] d\xi'$$

to the canonical form

$$y' + y^2 + R^2(x) = 0, \tag{2}$$

where

$$R^2(x) = \frac{r(\xi)}{p(\xi)} \exp\left[2\int q d\xi'\right].$$

Cioranescu [1] has shown that the following linear system is equivalent to (1):

$$\begin{aligned} u' &= Au - rv, \\ v' &= pu + (A + q)v, \end{aligned}$$

where $Y = u/v$ and A is an arbitrary function. The corresponding linear system for the canonical form (2) is

$$\begin{aligned} u' &= A(x)u - R^2(x)v \\ v' &= u(x) + A(x)v. \end{aligned} \tag{3}$$

In matrix form (3) reads as follows

$$\frac{d}{dx}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A(x) & -R^2(x) \\ 1 & A(x) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \tag{4}$$ or $$\frac{d\mathbf{u}}{dx} = \mathfrak{A}\mathbf{u}, \tag{4'}$$

where the definitions of $\mathbf{u}$ and $\mathfrak{A}$ are clear from (4). The eigenvalues and eigenfunctions of $\mathfrak{A}$ are, respectively,

$$\lambda_{1,2} = A \pm iR \qquad w_{1,2} = (1 - R^2)^{-1/2}\begin{bmatrix} \pm iR \\ 1 \end{bmatrix}. \tag{5}$$

The initial value problem associated with (4′) is solved by the matrizant solution [2]

$$\mathbf{u}(x) = \underset{x_0}{\overset{x}{\Omega}} \{\mathfrak{A}\}\mathbf{u}_0, \tag{6}$$

where

$$\mathbf{u}_0 = \mathbf{u}(x_0) \tag{7}$$

and the matrizant $\Omega_{x_0}^{x}\{\mathfrak{A}\}$ is given by the series (uniformly and absolutely convergent in an interval for real x or a "star" region for complex x)

$$\underset{x_0}{\overset{x}{\Omega}}\{\mathfrak{A}\} = I + \int_{x_0}^{x} \mathfrak{A}(x_1)dx_1 + \int_{x_0}^{x} \mathfrak{A}(x_1) \int_{x_0}^{x_1} \mathfrak{A}(x_2)dx_2dx_1 + \cdots . \tag{8}$$

We shall adopt here the transformation of this solution due to Keller and Keller [2]. This results in a more rapidly convergent solution and one which lends itself more readily to physical interpretation. This form of the solution rests upon the identities [2] (valid for non-singular analytic matrices $\mathfrak{P}(x)$)

$$\underset{x_0}{\overset{x}{\Omega}}\{\mathfrak{A}(\xi)\} = \mathfrak{P}(x) \underset{x_0}{\overset{x}{\Omega}}\left\{\mathfrak{P}^{-1}(\xi)\mathfrak{A}(\xi)\mathfrak{P}(\xi) - \mathfrak{P}^{-1}(\xi)\frac{d\mathfrak{P}}{d\xi}\right\}\mathfrak{P}^{-1}(x_0), \tag{9}$$

$$\underset{x_0}{\overset{x}{\Omega}}\{\mathfrak{D}\} \equiv \exp \int_{x_0}^{x} \mathfrak{D}(\xi)d\xi \quad \text{for} \quad \mathfrak{D} \text{ a diagonal matrix;} \tag{10}$$

$$\underset{x_0}{\overset{x}{\Omega}}\{\mathfrak{B} + \mathfrak{C}\} = \underset{x_0}{\overset{x}{\Omega}}\{\mathfrak{B}\} \underset{x_0}{\overset{x}{\Omega}}\left\{\underset{x_0}{\overset{\xi-1}{\Omega}}\{\mathfrak{B}\}\mathfrak{C}(\xi)\underset{x_0}{\overset{\xi}{\Omega}}\{\mathfrak{B}\}\right\}. \tag{11}$$

If we now choose $\mathfrak{P}$ so that it diagonalizes the matrix $\mathfrak{A}$ (which is possible since the linear divisors of $\mathfrak{A}$ are simple), i.e.

$$\mathfrak{P} = (1 - R^2)^{-1/2}\begin{bmatrix} iR & -iR \\ 1 & 1 \end{bmatrix} \tag{12}$$

then it follows from (6), (9), (10), and (11) that

$$\begin{aligned} \mathbf{u}(x) &= \mathfrak{P}(x) \exp\left[\int_{x_0}^{x} \mathfrak{L}(\xi)d\xi\right] \\ &\cdot \underset{x_0}{\overset{x}{\Omega}}\left\{-\exp\left[-\int_{x_0}^{x} \mathfrak{L}(\xi)d\xi\right]\mathfrak{P}^{-1}\frac{d\mathfrak{P}}{dx}\exp\left[\int_{x_0}^{x} \mathfrak{L}(\xi)d\xi\right]\right\}\mathfrak{P}^{-1}(x_0)\mathbf{u}, \end{aligned} \tag{13}$$

where

$$\mathfrak{L} = \mathfrak{P}^{-1}\mathfrak{A}\mathfrak{P} = \begin{bmatrix} A + iR & 0 \\ 0 & A - iR \end{bmatrix}, \tag{14}$$

and

$$\mathfrak{P}^{-1}\frac{d\mathfrak{P}}{dx} = \frac{1}{2}\frac{d \ln R}{dx}\begin{bmatrix} \dfrac{1 + R^2}{1 - R^2} & -1 \\ -1 & \dfrac{1 + R^2}{1 - R^2} \end{bmatrix}. \tag{15}$$

Now introduce the matrices

$$
(16)\quad \begin{aligned}
\Lambda &= \mathfrak{L} + \operatorname{diag}\left[\mathfrak{P}^{-1}\frac{d\mathfrak{P}}{dx}\right] \\
&= \begin{bmatrix} A + iR - \dfrac{1}{2}\dfrac{d\ln R}{dx}\dfrac{1+R^2}{1-R^2} & 0 \\ 0 & A - iR - \dfrac{1}{2}\dfrac{d\ln R}{dx}\dfrac{1+R^2}{1-R^2} \end{bmatrix},
\end{aligned}
$$

$$
(17)\quad \mathfrak{M} = \mathfrak{P}^{-1}\frac{d\mathfrak{P}}{dx} - \operatorname{diag}\left[\mathfrak{P}^{-1}\frac{d\mathfrak{P}}{dx}\right] = -\frac{1}{2}\begin{bmatrix} 0 & \dfrac{d\ln R}{dx} \\ \dfrac{d\ln R}{dx} & 0 \end{bmatrix}.
$$

Then (13) becomes

$$
(18)\quad \begin{aligned}
\mathbf{u}(x) &= \mathfrak{P}(x)\exp\left[\int_{x_0}^{x}\Lambda d\xi\right] \\
&\cdot \underset{x_0}{\overset{x}{\Omega}}\left\{-\exp\left[-\int_{x_0}^{\xi}\Lambda d\eta\right]\mathfrak{M}(\xi)\exp\left[\int_{x_0}^{\xi}\Lambda d\eta\right]\right\}\mathfrak{P}^{-1}(x_0)\mathbf{u}_0.
\end{aligned}
$$

Since Λ is a diagonal matrix, the matrix formula $\exp(\lambda_i\delta_{ij}) = (\exp(\lambda_i)\delta_{ij})$ yields

$$
\begin{aligned}
\exp\left[\pm\int_{x_0}^{x}\Lambda(\xi)d\xi\right] &= \exp\left\{\pm\int_{x_0}^{x}\left[A - \frac{1}{2}\frac{d\ln R}{d\xi}\frac{1+R^2}{1-R^2}\right]d\xi\right\} \\
&\cdot\begin{bmatrix} \exp\left\{\pm i\displaystyle\int_{x_0}^{x}Rd\xi\right\} & 0 \\ 0 & \exp\left\{\mp i\displaystyle\int_{x_0}^{x}Rd\xi\right\} \end{bmatrix},
\end{aligned}
$$

whence

$$
(19)\quad \begin{aligned}
\mathfrak{P}\exp\left[\int_{x_0}^{x}\Lambda d\xi\right] &= [1 - R^2(x)]^{-1/2}\exp\left\{\int_{x_0}^{x}\left[A - \frac{1}{2}\frac{d\ln R}{d\xi}\frac{1+R^2}{1-R^2}\right]d\xi\right\} \\
&\cdot\begin{bmatrix} iR\exp\left[i\displaystyle\int_{x_0}^{x}Rd\xi\right] & -iR\exp\left[-i\displaystyle\int_{x_0}^{x}Rd\xi\right] \\ \exp\left[i\displaystyle\int_{x_0}^{x}Rd\xi\right] & \exp\left[-i\displaystyle\int_{x_0}^{x}Rd\xi\right] \end{bmatrix}
\end{aligned}
$$

We now form the argument of the matrizant in (18). Since

$$\mathfrak{M}(\xi)\exp\left[\int_{x_0}^{\xi}\Lambda d\eta\right] = -\frac{1}{2}\frac{d\ln R}{d\xi}\exp\left\{\int_{x_0}^{\xi}\left[A-\frac{1}{2}\frac{d\ln R}{d\eta}\frac{1+R^2}{1-R^2}\right]d\eta\right\}$$
$$\cdot\begin{bmatrix} 0 & \exp\left[-i\int_{x_0}^{\xi}Rd\eta\right] \\ \exp\left[i\int_{x_0}^{\xi}Rd\eta\right] & 0 \end{bmatrix},$$

we have

$$-\exp\left[-\int_{x_0}^{\xi}\Lambda d\eta\right]\mathfrak{M}(\xi)\exp\left[\int_{x_0}^{\xi}\Lambda d\eta\right] = \frac{1}{2}\frac{d\ln R}{d\xi}\mathfrak{N}(\xi), \tag{20}$$

where $\mathfrak{N}(\xi)$ is the matrix

$$\mathfrak{N}(\xi) = \begin{bmatrix} 0 & \exp\left[-2i\int_{x_0}^{\xi}Rd\eta\right] \\ \exp\left[2i\int_{x_0}^{\xi}Rd\eta\right] & 0 \end{bmatrix}. \tag{21}$$

The matrizant itself then has the expansion

$$\begin{aligned}\underset{x_0}{\overset{x}{\Omega}}\left\{-\exp\left[-\int_{x_0}^{\xi}\Lambda d\eta\right]\mathfrak{M}(\xi)\exp\left[\int_{x_0}^{\xi}\Lambda d\eta\right]\right\} &= I+\int_{x_0}^{x}\frac{1}{2}\frac{d\ln R}{dx_1}\mathfrak{N}(x_1)dx_1 \\ &+\int_{x_0}^{x}\frac{1}{2}\frac{d\ln R}{dx_1}\mathfrak{N}(x_1)\int_{x_0}^{x_1}\frac{1}{2}\frac{d\ln R}{dx_2}\mathfrak{N}(x_2)dx_2dx_1+\cdots\end{aligned} \tag{22}$$

Now introduce the notation

$$\mathfrak{N}_{\alpha+1} = \int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_\alpha}\exp\left[2i\sum_{j=1}^{\alpha+1}(-1)^j\int_{x_0}^{x_j}Rd\xi\right]\prod_{1}^{\alpha+1}\frac{1}{2}\frac{d\ln R}{dx_j}dx_j, \tag{23}$$
$$(\alpha = 0, 1, 2, \cdots).$$

The matrizant (22) can then be written as

$$\underset{x_0}{\overset{x}{\Omega}}\left\{-\exp\left[-\int_{x_0}^{\xi}\Lambda d\eta\right]\mathfrak{M}(\xi)\exp\left[\int_{x_0}^{\xi}\Lambda d\eta\right]\right\} = \sum_{n=0}^{\infty}\begin{bmatrix}\mathfrak{N}_n & 0 \\ 0 & \mathfrak{N}_n^*\end{bmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}^n$$

or

$$\mathfrak{S}(x_0, x) = \sum_{k=0}^{\infty}\left\{\begin{bmatrix}\mathfrak{N}_{2k} & 0 \\ 0 & \mathfrak{N}_{2k}^*\end{bmatrix}\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}+\begin{bmatrix}\mathfrak{N}_{2k+1} & 0 \\ 0 & \mathfrak{N}_{2k+1}^*\end{bmatrix}\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}\right\}, \tag{24}$$

with $\mathfrak{N}_0 = 1$,

$$(25)\quad \mathfrak{R}_n = \int_{x_0}^{x}\int_{x_0}^{x_1}\cdots\int_{x_0}^{x_{n-1}} \exp\left[2i\sum_{j=1}^{n}(-1)^j\int_{x_0}^{x_j}Rd\xi\right]\prod_{j=1}^{n}\frac{1}{2}\frac{d\ln R}{dx_j}dx_j, \qquad (n = 1, 2, \cdots).$$

Note: $x_{n-1}=x$ for $n=1$.

Returning now to formula (18) for $\mathbf{u}(x)$, we see from (12), (19) and (24) that

$$(26)\quad \begin{aligned} \mathbf{u}(x) &= \frac{1}{2}\left[\frac{1-R^2(x_0)}{1-R^2(x)}\right]^{1/2}\exp\left\{\int_{x_0}^{x}\left[A-\frac{1}{2}\frac{d\ln R}{d\xi}\frac{1+R^2}{1-R^2}\right]d\xi\right\}\Theta(x_0, x) \\ &\cdot\sum_{k=0}^{\infty}\left\{\begin{bmatrix}(-i\mathfrak{R}_{2k}/R_0) & \mathfrak{R}_{2k}\\ (-i\mathfrak{R}_{2k}/R_0)^* & \mathfrak{R}^*_{2k}\end{bmatrix}+\begin{bmatrix}(i\mathfrak{R}_{2k+1}/R_0) & \mathfrak{R}_{2k+1}\\ (i\mathfrak{R}_{2k+1}/R_0)^* & \mathfrak{R}^*_{2k+1}\end{bmatrix}\right\}\mathbf{u}_0, \end{aligned}$$

where we have used the notation $R=R(x)$, $R_0=R(x_0)$, $\theta(x_0, x)=\int_{x_0}^{x}Rd\xi$, and

$$(27)\quad \Theta(x_0, x) = \begin{bmatrix} iRe^{i\theta} & (iRe^{i\theta})^* \\ e^{i\theta} & (e^{i\theta})^* \end{bmatrix}.$$

Performing the final matrix multiplication, and, in the elements of the product of Θ and the infinite sum in (26), combining complex exponentials according to the appropriate trigonometric formulas, one obtains

$$(28)\quad \begin{aligned} \mathbf{u}(x) &= \left[\frac{1-R^2(x_0)}{1-R^2(x)}\right]^{1/2}\exp\left\{\int_{x_0}^{x}\left[A-\frac{1}{2}\frac{d\ln R}{d\xi}\frac{1+R^2}{1-R^2}\right]d\xi\right\} \\ &\cdot\sum_{k=0}^{\infty}\left\{|\mathfrak{R}_{2k}|\begin{bmatrix}(R/R_0)\cos\phi_{2k} & -R\sin\phi_{2k}\\ R_0^{-1}\sin\phi_{2k} & \cos\phi_{2k}\end{bmatrix}\right. \\ &\qquad \left. - |\mathfrak{R}_{2k+1}|\begin{bmatrix}(R/R_0)\cos\phi_{2k+1} & R\sin\phi_{2k+1}\\ R_0^{-1}\sin\phi_{2k+1} & -\cos\phi_{2k+1}\end{bmatrix}\right\}\begin{bmatrix}u_0\\ v_0\end{bmatrix}, \end{aligned}$$

where $\phi_m=\theta(x_0, x)+\arg\mathfrak{R}_m$. Now perform the final postmultiplication into $\mathbf{u}_0$ to obtain

$$(29)\quad \begin{aligned} \mathbf{u}(x) &= \left[\frac{1-R^2(x_0)}{1-R^2(x)}\right]^{1/2}\exp\left\{\int_{x_0}^{x}\left[A-\frac{1}{2}\frac{d\ln R}{d\xi}\frac{1+R^2}{1-R^2}\right]d\xi\right\} \\ &\cdot\sum_{k=0}^{\infty}\begin{bmatrix} R\dfrac{u_0}{R_0}\left[|\mathfrak{R}_{2k}|\cos\phi_{2k}-|\mathfrak{R}_{2k+1}|\cos\phi_{2k+1}\right] - v_0R\left[|\mathfrak{R}_{2k}|\sin\phi_{2k}+|\mathfrak{R}_{2k+1}|\sin\phi_{2k+1}\right] \\ \dfrac{u_0}{R_0}\left[|\mathfrak{R}_{2k}|\sin\phi_{2k}-|\mathfrak{R}_{2k+1}|\sin\phi_{2k+1}\right] + v_0\left[|\mathfrak{R}_{2k}|\cos\phi_{2k}+|\mathfrak{R}_{2k+1}|\cos\phi_{2k+1}\right] \end{bmatrix}. \end{aligned}$$

By definition $y=u/v$. Forming the quotient of the first row by the second in (29), and subsequently dividing numerator and denominator by v_0, yields the

following expression for the solution of the initial value problem for the Riccati equation:

$$(30)\quad y(x) = R(x)\frac{\sum_{k=0}^{\infty}\left\{\frac{y_0}{R(x_0)}\left[\left|\mathfrak{R}_{2k}\right|\cos\phi_{2k} - \left|\mathfrak{R}_{2k+1}\right|\cos\phi_{2k+1}\right] - \left[\left|\mathfrak{R}_{2k}\right|\sin\phi_{2k} + \left|\mathfrak{R}_{2k+1}\right|\sin\phi_{2k+1}\right]\right\}}{\sum_{k=0}^{\infty}\left\{\frac{y_0}{R(x_0)}\left[\left|\mathfrak{R}_{2k}\right|\sin\phi_{2k} - \left|\mathfrak{R}_{2k+1}\right|\sin\phi_{2k+1}\right] + \left[\left|\mathfrak{R}_{2k}\right|\cos\phi_{2k} + \left|\mathfrak{R}_{2k+1}\right|\cos\phi_{2k+1}\right]\right\}}.$$

References

1. N. Cioranescu, Les linéarizations de l'équation de Riccati, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 2 (50) (1958) 127–131.

2. H. B. Keller and J. B. Keller, On systems of linear ordinary differential equations, Res. Rep. EM-33, New York Univ., Inst. Math. Sci., 1951. Cf. Exponential-like solutions of systems of linear ordinary differential equations, J. SIAM 10, No. 2 (June 1962) 246–259.

3. R. Zurmühl, Matrizen, Springer, Berlin, 1950.

AN EXTENSION OF THE FIBONACCI NUMBERS

JOSEPH ARKIN, Spring Valley, N.Y.

1. Introduction. A table of Fibonacci extensions

	0	1	2	3	4	5	6	·	·	·
0	0	0	0	0	0	0	0	·	·	·
1	1	1	2	3	5	8	13	·	·	·
2	1	2	5	10	20	38	71	·	·	·
3	1	3	9	22	51	111	233	·	·	·
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
k	1	k	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	·	·	·	·	·

may be constructed as follows:

(1.1) To get the kth element in the nth column add the kth element in the $(n-1)$-st column and the kth element in the $(n-2)$-nd column together with the $(k-1)$-st element in the nth column.

We write the kth element in the nth column as $C_n^{(k)}$, so that a restatement of (1.1) reads

$$C_n^{(k)} = C_{n-1}^{(k)} + C_{n-2}^{(k)} + C_n^{(k-1)}, \tag{1.2}$$

where $C_0^{(k)}=1$, $C_1^{(k)}=k$, $0=C_0^{(0)}=C_1^{(0)}=C_2^{(0)}=C_3^{(0)}= \cdots$, $n=2, 3, \cdots$, and $k=1, 2, 3, \cdots$. In particular we have

$$C_n = C_{n-1} + C_{n-2} + C_n^{(0)}, \tag{1.3}$$

where $C_n=C_n^{(1)}$ is the nth Fibonacci number.

In this paper we establish the following four relations:

I. $$nC_n^{(k)} = (k+n-1)C_{n-1}^{(k)} + (2k+n-2)C_{n-2}^{(k)},$$

where $C_0^{(k)}=1$, $C_1^{(k)}=k$, $n=2, 3, \cdots$, and $k=2, 3, \cdots$.

II. $$\frac{nC_n^{(k)}}{C_{n-1}^{(k)}} = (k+n-1) + \frac{(n-1)(2k+n-2)}{(k+n-2)+} \; \frac{(n-2)(2k+n-3)}{(k+n-3)+} \; \frac{(n-3)(2k+n-4)}{(k+n-4)+} \cdots \frac{2(2k+1)}{(k+1)+} \; \frac{2k}{k},$$

where $n=2, 3, \cdots$, $k=2, 3, \cdots$, $C_0^{(k)}=1$, and $C_1^{(k)}=k$.

III. $$5kC_{n-1}^{(k+1)} = (4k+2n-2)C_{n-1}^{(k)} + nC_n^{(k)},$$

where $C_0^{(k)}=1$, $C_1^{(k)}=k$, $n=1, 2, 3, \cdots$, and $k=1, 2, 3, \cdots$.

IV. $$\sum_{v=1}^{n} C_{n-v}^{(v)} = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \qquad (n = 1, 2, 3, \cdots).$$

2. Proof of I, II, III. We use (1.2) to get

$$\begin{aligned}\sum_{n=2}^{\infty} C_n^{(k)} x^n &= \sum_{n=2}^{\infty} C_{n-1}^{(k)} x^n + \sum_{n=2}^{\infty} C_{n-2}^{(k)} x^n + \sum_{n=2}^{\infty} C_n^{(k-1)} x^n \\ &= x\sum_{n=1}^{\infty} C_n^{(k)} x^n + x^2 \sum_{n=0}^{\infty} C_n^{(k)} x^n + \sum_{n=2}^{\infty} C_n^{(k-1)} x^n,\end{aligned} \tag{2.1}$$

for $k=1, 2, \cdots$. Then

$$\sum_{n=0}^{\infty} C_n^{(k)} x^n - C_0^{(k)} - C_1^{(k)} x = x\sum_{n=0}^{\infty} C_n^{(k)} x^n - C_0^{(k)} x + x^2 \sum_{n=0}^{\infty} C_n^{(k)} x^n + \sum_{n=0}^{\infty} C_n^{(k-1)} x^n - C_0^{(k-1)} - C_1^{(k-1)} x,$$

and therefore

$$(1 - x - x^2) \sum_{n=0}^{\infty} C_n^{(k)} x^n = C_0^{(k)} - C_0^{(k-1)} + x(C_1^{(k)} - C_0^{(k)} - C_1^{(k-1)}) + \sum_{n=0}^{\infty} C_n^{(k-1)} x^n.$$

Now

$$C_0^{(k)} - C_0^{(k-1)} = \begin{cases} 1 - 0 = 1 & \text{if } k = 1 \\ 1 - 1 = 0 & \text{if } k \geqq 2 \end{cases}$$

and

$$C_1^{(k)} - C_0^{(k)} - C_1^{(k-1)} = \left\{ \begin{matrix} 1 - 1 - 0 = 0 & \text{if } k = 1 \\ k - 1 - (k-1) = 0 & \text{if } k \geqq 2 \end{matrix} \right\} = 0,$$

for $k=1, 2, 3, \cdots$, and $\sum_{n=0}^{\infty} C_n^{(0)} = 0$. Therefore

$$\sum_{n=0}^{\infty} C_n^{(k)} x^n = (1 - x - x^2)^{-1} \left(\sum_{n=0}^{\infty} C_n^{(k-1)} x^n \right) \qquad (k = 2, 3, \cdots),$$

and $\sum_{n=0}^{\infty} C_n^{(1)} x^n = (1-x-x^2)^{-1}$. From this we have, at once,

$$(2.2) \qquad (1 - x - x^2)^{-k} = \sum_{n=0}^{\infty} C_n^{(k)} x^n \qquad (k = 1, 2, 3, \cdots).$$

Differentiation of (2.2) leads to

$$k(2x + 1) \left(\sum_{n=0}^{\infty} C_n^{(k+1)} x^n \right) = \sum_{n=1}^{\infty} n C_n^{(k)} x^{n-1},$$

and by comparing coefficients we conclude that

$$(2.3) \qquad k(C_{n-1}^{(k+1)} + 2C_{n-2}^{(k+1)}) = nC_n^{(k)} \qquad (k = 1, 2, 3, \cdots, n = 2, 3, \cdots).$$

Combining (2.3) with (1.2), we get

$$(2.4) \qquad nC_n^{(k)} = (k + n - 1)C_{n-1}^{(k)} + (2k + n - 2)C_{n-2}^{(k)}$$

for $k=2, 3, \cdots, n=2, 3, \cdots, C_0^{(k)}=1$, and $C_1^{(k)}=k$; *this completes the proof for I.*

When we divide (2.4) by $C_{n-1}^{(k)}$, we have

$$\frac{nC_n^{(k)}}{C_{n-1}^{(k)}} = (k + n - 1) + \cfrac{(2k + n - 2)(n - 1)}{\cfrac{(n - 1)C_n^{(k)}}{C_{n-2}^{(k)}}} \qquad (n = 2, 3, \cdots; k = 2, 3, \cdots),$$

which in turn, along with $C_0^{(k)}=1$ and $C_1^{(k)}=k$, *implies II.*

The identity $5=4(1-x-x^2)+(2x+1)^2$ may be written as

$$(2.5) \qquad \frac{5}{(1 - x - x^2)^k} = \frac{4}{(1 - x - x^2)^{k-1}} + \frac{(2x + 1)^2}{(1 - x - x^2)^k} \qquad (k = 1, 2, 3, \cdots);$$

differentiation leads to

$$\frac{5kx}{(1-x-x^2)^{k+1}} = \frac{4kx}{(1-x-x^2)^k} + (2x+1)\left(\sum_{n=1}^{\infty} nC_n^{(k)} x^n\right).$$

By comparing coefficients we conclude that

$$5kC_{n-1}^{(k+1)} = (4k+2n-2)C_{n-1}^{(k)} + nC_n^{(k)}. \tag{2.6}$$

where $C_0^{(k)}=1$, $C_1^{(k)}=k$, $n=1, 2, 3, \cdots$, and $k=1, 2, 3, \cdots$, *which proves III.*

We observe that equations (II) and (III) immediately give an expression for

$$\frac{5kC_{n-1}^{(k+1)}}{C_{n-1}^{(k)}}$$

in the form of a continued fraction, for $n=2, 3, \cdots$, and $k=2, 3, \cdots$.

REMARK. $\text{Lim}_{n=\infty} C_n/C_{n-1}=(1+\sqrt{5})/2$ (see [1] with $i=1$); in (2.6), therefore, when $k=1$ and we divide by C_{n-1}, we have

$$5\left(\lim_{n=\infty}\frac{C_{n-1}^{(2)}}{nC_{n-1}}\right) = \lim_{n\to\infty}\left(\frac{2n+2}{n}+\frac{C_n}{C_{n-1}}\right) = 2+\frac{1+\sqrt{5}}{2} = \frac{5+\sqrt{5}}{2}.$$

3. Proof of IV. By (1.2) we have $C_n^{(k)}=C_{n-1}^{(k)}+C_{n-2}^{(k)}+C_n^{(k-1)}$, so that

$$\sum_{v=1}^{n} C_{n-v}^{(v)} = \sum_{v=1}^{n-1} C_{n-v-1}^{(v)} + \sum_{v=1}^{n-2} C_{n-v-2}^{(v)} + \sum_{v=2}^{n} C_{n-v}^{(v-1)} \qquad (n=2, 3, \cdots). \tag{3.1}$$

We see that $\sum_{v=1}^{n-1} C_{n-v-1}^{(v)}=\sum_{v=2}^{n} C_{n-v}^{(v-1)}$, and write (3.1) as

$$\sum_{v=1}^{n} C_{n-v}^{(v)} = 2\left(\sum_{v=1}^{n-1} C_{n-v-1}^{(v)}\right) + \sum_{v=1}^{n-2} C_{n-v-2}^{(v)}. \tag{3.2}$$

We let

$$u_n = \sum_{v=1}^{n} C_{n-v}^{(v)}; \tag{3.3}$$

then $u_{n-1}=\sum_{v=1}^{n-1} C_{n-v-1}^{(v)}$ and $u_{n-2}=\sum_{v=1}^{n-2} C_{n-v-2}^{(v)}$, so that (3.2) becomes

$$u_n = 2u_{n-1}+u_{n-2}. \tag{3.4}$$

Replacing n by $n+2$ in (3.4), we have

$$u_{n+2} = 2u_{n+1}+u_n, \tag{3.5}$$

where $u_1=C_0=1$, $u_2=C_0^{(2)}=C_1=2$, and $n=1, 2, 3, \cdots$.

We solve (3.5) for u_n by continued fractions (see [2] 10.13.13 with $a=b=2$ and $c=1$), and get

$$u_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} = \sum_{v=1}^{n} C_{n-v}^{(v)}, \tag{3.6}$$

for $n = 1, 2, 3, \cdots$, *which proves IV.*

REMARK. In (3.6), when $n = 4q+3$ is a prime, $2n-1$ is a prime if and *only* if $2u_n \equiv 0 \pmod{2n-1}$, (see [3]).

The author is grateful to Professor F. A. Ficken and to the referee for their valuable suggestions.

References

1. L. E. Dickson, Theory of Numbers, vol. I, Chelsea, New York, 1952, p. 402.
2. G. H. Hardy and E. M. Wright, Theory of Numbers, 4th ed. reprinted (with corrections) Oxford University Press, 1962, pp 146–147.
3. L. E. Dickson, Theory of Numbers, vol. I, Chelsea, New York, 1952, pp. 398–399.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146.

A CHARACTERISTIC PROPERTY OF THE MÖBIUS FUNCTION

T. M. APOSTOL, California Institute of Technology

In a recent issue of the *Mathematical Gazette* [5] Satyanarayana has shown that the Möbius μ function is characterized by its inversion property. More precisely, he has proved

THEOREM 1. *If f, g, h are three arithmetical functions satisfying*

$$g(n) = \sum_{d|n} f(d), \qquad f(n) = \sum_{d|n} g(d)h\left(\frac{n}{d}\right), \qquad f(1) \neq 0, \tag{1}$$

for all n, then $h = \mu$.

As usual, the Möbius function μ is defined as follows:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } a^2 \mid n \text{ for some } a > 1, \\ (-1)^s & \text{if } n = p_1p_2 \cdots p_s, \text{ (the } p_i \text{ distinct primes).} \end{cases} \tag{2}$$

The purpose of this note is to show that this theorem can be deduced as an immediate consequence of the simplest properties of the Dirichlet convolution.

Moreover, the use of the Dirichlet convolution seems to place this theorem in a natural setting.

If α and β are arithmetical functions (complex-valued functions defined on the positive integers) their Dirichlet convolution $\alpha * \beta$ is another arithmetical function defined by the equation

$$(\alpha * \beta)(n) = \sum_{d|n} \alpha(d)\beta\left(\frac{n}{d}\right) \qquad (n = 1, 2, 3, \cdots).$$

It is well known that the set of all arithmetical functions α with $\alpha(1) \neq 0$ forms an abelian group with respect to $*$, the identity element being the function I defined by the equation

$$I(n) = \left[\frac{1}{n}\right] = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

(See, for example, [1], [2], [3], or [4].) The inverse of α will be denoted by α^{-1}.

The hypotheses (1) of Theorem 1 may now be written as follows:

$$g = f * J, \tag{3}$$

$$f = g * h, \tag{4}$$

$$f(1) \neq 0, \tag{5}$$

where J is the arithmetical function such that $J(n) = 1$ for all n. Since $g(1) = f(1)$, (5) implies $g(1) \neq 0$ so g has an inverse. From (4) we obtain

$$h = f * g^{-1} \tag{6}$$

and from (3) we find $g^{-1} = f^{-1} * J^{-1}$, so (6) becomes

$$h = f * (f^{-1} * J^{-1}) = (f * f^{-1}) * J^{-1} = J^{-1}.$$

Therefore, to complete the proof of Theorem 1 we must show that $J^{-1} = \mu$, or equivalently, that

$$J * \mu = I. \tag{7}$$

But this follows easily from the definition in (2). In fact, for $n = 1$ we have $(J * \mu)(1) = \mu(1) = 1 = I(1)$; for $n > 1$ we write $n = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$, $n' = p_1 p_2 \cdots p_s$, and we obtain

$$(J * \mu)(n) = \sum_{d|n} \mu(d) = \sum_{d|n'} \mu(d) = \sum_{r=0}^{s} \binom{s}{r} (-1)^r = (1 - 1)^s = 0 = I(n).$$

This completes the proof of Theorem 1.

In the same way we can prove the following more general theorem:

THEOREM A. *Let f, g, h, k be four arithmetical functions with $f(1) \neq 0$, $k(1) \neq 0$. Then the two equations*

$$g(n) = \sum_{d|n} f(d)k\left(\frac{n}{d}\right), \qquad f(n) = \sum_{d|n} g(d)h\left(\frac{n}{d}\right)$$

hold for all n, if and only if, $h=k^{-1}$.

Satyanarayana has also characterized the Möbius function by means of the following theorem.

THEOREM 2. *Let F, G be complex-valued functions defined on the positive real axis, $(0, +\infty)$, with $F(x)=G(x)=0$ for $0<x<1$, $F(1)\neq 0$. Assume that for all real $x\geqq 1$ we have*

$$G(x) = \sum_{n\leq x} F\left(\frac{x}{n}\right) \tag{8}$$

and let h be an arithmetical function such that

$$F(x) = \sum_{n\leq x} h(n)G\left(\frac{x}{n}\right) \tag{9}$$

for all $x\geqq 1$. Then $h=\mu$.

This theorem can be proved by a method similar to that used above for Theorem 1. Let $\mathfrak{A}$ denote the set of all arithmetical functions and let $\mathfrak{F}$ denote the set of all complex-valued functions F defined on the positive real axis $(0, +\infty)$ with $F(x)=0$ for $0<x<1$. If $\alpha\in\mathfrak{A}$ and if $F\in\mathfrak{F}$ we define $\alpha \circ F$ to be that element in $\mathfrak{F}$ defined for all real $x>0$ by the equation

$$(\alpha \circ F)(x) = \sum_{n\leq x} \alpha(n)F\left(\frac{x}{n}\right). \tag{10}$$

This operation includes Dirichlet convolution as a special case. In fact, if $F(x)=0$ for nonintegral values of x then $(\alpha \circ F)(n)=(\alpha * F)(n)$ for integral n. The operation $\circ$ is, in general, neither commutative nor associative. However, as a substitute for the associative law we have the following:

LEMMA 1. *If α and β are in $\mathfrak{A}$ and if $F\in\mathfrak{F}$, we have $\alpha \circ (\beta \circ F)=(\alpha * \beta) \circ F$.*

Proof. For any real $x>0$ we have

$$\begin{aligned}\{\alpha\circ(\beta\circ F)\}(x) &= \sum_{n\leq x}\alpha(n)\sum_{m\leq x/n}\beta(m)F\left(\frac{x}{mn}\right) = \sum_{mn\leq x}\alpha(n)\beta(m)F\left(\frac{x}{mn}\right)\\ &= \sum_{k\leq x}\left(\sum_{n|k}\alpha(n)\beta\left(\frac{k}{n}\right)\right)F\left(\frac{x}{k}\right) = \sum_{k\leq x}(\alpha*\beta)(k)F\left(\frac{x}{k}\right)\\ &= \{(\alpha*\beta)\circ F\}(x).\end{aligned}$$

This proves Lemma 1.

To prove Theorem 2 we shall also make use of

LEMMA 2. *Let* $I(n) = [1/n]$ *for* $n=1, 2, 3, \cdots$. *Then for every* F *in* $\mathfrak{F}$ *we have*

$$I \circ F = F. \tag{11}$$

Moreover, if

$$\alpha \circ F = F \tag{12}$$

for some α *in* $\mathfrak{A}$ *and some* F *in* $\mathfrak{F}$ *with* $F(1) \neq 0$, *then* $\alpha = I$.

Proof. Equation (11) follows at once from the definition (10). To prove the second statement, assume (12) holds with $F(1) \neq 0$. This means that

$$\sum_{n \leq x} \alpha(n) F\left(\frac{x}{n}\right) = F(x) \tag{13}$$

for all real $x > 0$. When $x = 1$ this gives us $\alpha(1)F(1) = F(1)$. Since $F(1) \neq 0$ this implies $\alpha(1) = 1 = I(1)$. Assume now that $\alpha(n) = I(n)$ for all $n < m$. Taking $x = m$ in (13) we find $\alpha(m) = I(m)$, so the lemma is proved by induction.

Theorem 2 is now an easy consequence of Lemmas 1 and 2. The hypotheses of Theorem 2 may be written as follows: $F(1) \neq 0$,

$$G = J \circ F, \tag{14}$$

where $J(n) = 1$ for all n, and

$$F = h \circ G. \tag{15}$$

Substituting (14) in (15) and using Lemma 1 we obtain

$$F = h \circ (J \circ F) = (h * J) \circ F.$$

Therefore, by Lemma 2 we have $h * J = I$, or $h = J^{-1} = \mu$.

In the same way we can prove the following more general theorem:

THEOREM B. *Let* F, G *be functions in* $\mathfrak{F}$ *with* $F(1) \neq 0$, *and let* α *and* β *be arithmetical functions with* $\alpha(1) \neq 0$. *Then the two equations*

$$G(x) = \sum_{n \leq x} \alpha(n) F\left(\frac{x}{n}\right), \qquad F(x) = \sum_{n \leq x} \beta(n) G\left(\frac{x}{n}\right)$$

hold for all $x \geqq 1$ *if, and only if,* $\beta = \alpha^{-1}$.

References

1. E. T. Bell, An arithmetical theory of certain numerical functions, University of Washington Publ. in Math. and Phys. Sci., No. 1, vol. 1 (1915) 1–44.

2. ———, On a certain inversion in the theory of numbers, Tohoku Mathematical Journal, vol. 17 (1920) 221–231.

3. ———, Outline of a theory of arithmetical functions in their algebraic aspects, J. Indian Math. Soc., 17 (1927/28) 249–260.

4. J. Popken, On convolutions in prime number theory, Proc. Ned. Akad. van Wetensch., 58(A) (1955) 10–15.

5. U. V. Satyanarayana, On the inversion property of the Möbius μ-function, Math. Gaz., No. 359, Vol. XLVII (1963) 38–42.

FENCHEL'S THEOREM AS A CONSEQUENCE OF SCHUR'S

ALFRED AEPPLI, University of Minnesota

In this note a proof of Fenchel's theorem [2] on the total curvature of a closed curve is given, using a theorem of A. Schur [1]. Various other proofs can be found in [2], [3], [5], [7], [8] and elsewhere (cf. also [4], [6]).

A curve $C\subset R^n$, $n\geqq 2$, is a finite, connected, oriented (or directed), piecewise smooth curve in euclidean n-space, i.e., C is twice continuously differentiable except at a finite number of points $p_1, p_2, \cdots, p_N\in C$, where there are corners with angles $\alpha_1, \alpha_2, \cdots, \alpha_N$ between the "arriving" tangent and the "leaving" tangent ($0\leqq\alpha_k\leqq\pi$). The tangents in a corner are supposed to exist, and likewise the tangent in an initial or in an end point of C (initial and end points have finite coordinates in R^n).

If C is given by $\mathfrak{X}=\mathfrak{X}(s)$ then the curvature is defined by $\kappa=|\ddot{\mathfrak{X}}|$ (where it exists), the dot indicating differentiation with respect to the "natural" curve parameter s (curve length parameter). We formulate according to [1]:

(I) THEOREM OF A. SCHUR. *Let C be a curve in the plane R^2 with initial point a and end point b such that C together with the interval ab (on the straight line through a and b if $a\neq b$) forms a closed convex curve in R^2. Denote by $p_1, p_2, \cdots, p_N$ the corners of C with the angles $\alpha_1, \alpha_2, \cdots, \alpha_N$. Let C' be a curve in R^n, $n\geqq 2$, with initial point a' and end point b', of the same length as C and related with C by the same natural parameter s in such a way that* (i) *C' is smooth wherever C is,* (ii) *$\kappa(s)\geqq\kappa'(s)$ for the curvatures κ and κ' of C and C' (whenever both κ and κ' are defined), and* (iii) *$\alpha_k\geqq\alpha_k'$ for the angles $\alpha_1, \alpha_2, \cdots, \alpha_N$ of C and the corresponding angles $\alpha_1', \alpha_2', \cdots \alpha_N'$ of C'. Conclusion:* (a) *$|ab|\leqq|a'b'|$ where $|ab|$ denotes the distance between a and b;* (b) *$|ab|=|a'b'|$ if and only if $C\cong C'$, this congruence being given by the parameter s (i.e. $\mathfrak{X}(s)\leftrightarrow\mathfrak{X}'(s)$ is a congruence).*

From (I)(a) we obtain an immediate consequence:

COROLLARY. *Under the hypotheses of* (I), *$a\neq b$ implies $a'\neq b'$.*

[2] contains the following theorem:

(II) THEOREM OF W. FENCHEL. (a) *For a closed curve $\Gamma\subset R^n$, $n\geqq 2$, with the curvature κ and with the angles $\beta_1, \beta_2, \cdots, \beta_N$ at the corners $q_1, q_2, \cdots, q_N$, the inequality*

$$\int_\Gamma \kappa ds + \sum_{k=1}^{N}\beta_k \geqq 2\pi \tag{1}$$

holds. (b) *Equality in* (1) *holds if and only if Γ is a plane convex curve.*

Proof of (II)(a). Assume that (1) is not true:

$$\int_\Gamma \kappa ds + \sum_{k=1}^{N}\beta_k < 2\pi. \tag{2}$$

We construct a plane curve C (unique up to a congruence): $\mathfrak{X}=\mathfrak{X}(s)$ represents $C\subset R^2$, determined by $|\ddot{\mathfrak{X}}(s)|=\kappa(s)$, starting at some regular point $a'\in\Gamma$ (i.e. a' is not a corner), and C possesses angles $\alpha_k=\beta_k$ at the corners $p_1, p_2, \cdots, p_k, \cdots, p_N$ corresponding to $q_1, q_2, \cdots, q_k, \cdots, q_N$ on Γ. Let a be the initial point of C, b its end point. We assert that

$$a \neq b. \tag{3}$$

If $a=b$ then (2) implies that C is a closed convex curve in R^2 with a corner at a, and if we put $C'=\Gamma$, C' related to C by the same parameter s, then $\kappa'(s)=\kappa(s)$, $\alpha_k'=\beta_k=\alpha_k$, and $a'=b'$ corresponds to $a=b$. Thus (i), (ii), (iii) in (I) are fulfilled, and (I)(b) yields $C\cong C'=\Gamma$, the congruence being given by the parameter s. But such a congruence is impossible since C has a corner at a whereas C' does not have a corner at a'. This proves (3).

We distinguish two cases to finish the proof of (II)(a):

Case 1. C together with the interval ab forms a closed convex curve in R^2. Take $C'=\Gamma$, related with C by s. (i), (ii), (iii) hold, and (I) can be applied. (I)(a) (or the Corollary) implies $a=b$, contradicting (3) and disproving (2) in this case.

Case 2. C together with the interval ab is a closed curve which is not convex. Then there is a first point $x\in C$ ("first" with respect to the orientation of C induced by the linear order of the parameter s; $s=0$ at a, $s=L>0$ at b) where the oriented tangent is parallel to $\overrightarrow{ba}$, or else there is a corner $x\in C$ where the angle between its arriving and leaving tangent contains the direction $\overrightarrow{ba}$. We denote by C_{ax}, C_{xb} the arcs on C between a and x, x and b respectively. We translate C_{xb} by the vector $\overrightarrow{ba}$ onto an arc $\bar{C}_{\bar{x}\bar{b}}$ (with $\bar{b}=a$). The new curve $\hat{C}=\bar{C}_{\bar{x}a}\cup C_{ax}$ forms now together with the interval $x\bar{x}$ a closed convex curve, and $\hat{C}$ is related with $C'=\Gamma$ like C and C' in (I) (there is a well defined point $x'\in C'$ corresponding to both x and $\bar{x}$). (I)(a) (or the Corollary) applied to this situation yields $x=\bar{x}$; hence $a=b$, contradicting (3) and disproving (2) in Case 2.

Proof of (II)(b). If equality holds in (1), then the same construction as in the proof of (II)(a), Case 1 and Case 2, leads to a plane curve C which is closed ($a=b$) and convex (one applies (I)(a)); then (I)(b) implies immediately $C\cong C'=\Gamma$. Conversely, there is equality in (1) for a plane convex curve Γ. This completes the proof of (II).

References

1. A. Schur, Über die Schwarzsche Extremaleigenschaft des Kreises unter den Kurven konstanter Krümmung, Math. Ann., 83 (1921) 143–148.

2. W. Fenchel, Über Krümmung und Windung geschlossener Raumkurven, Math. Ann., 101 (1929) 238–252.

3. H. Liebmann, Elementarer Beweis des Fenchelschen Satzes über die Krümmung geschlossener Raumkurven, S.-B. Pr. Akad. Wissensch., Phys.-math. Klasse (1929) 392–393.

4. M. I. Fáry, Sur la courbure totale d'une courbe gauche faisant un noeud, Bull. Soc. Math. France, LXXVII (1949) 128–138.

5. J. Milnor, On the total curvature of knots, Ann. of Math., 52 (1950) 248–257.

6. W. Fenchel, On the differential geometry of closed space curves, Bull. Amer. Math. Soc., 57 (1951) 44–54.
7. K. Voss, Eine Bemerkung über die Totalkrümmung geschlossener Raumkurven, Arch. Math., 6 (1955) 259–263.
8. S. Sasaki, On the total curvature of a closed curve, Japan J. Math., 29 (1959) 118–125.

ON THE ASYMPTOTIC BEHAVIOR OF LINEAR DIFFERENTIAL EQUATIONS

J. W. BEBERNES AND N. X. VINH, University of Colorado

In this note we shall prove a theorem concerning the behavior of solutions of the linear system

$$x' = [F(t) + G(t)]x \tag{1}$$

as $t \to \infty$, where $F(t)$ and $G(t)$ are $n \times n$ matrix functions on $[0, \infty)$ and x is an n-dimensional vector.

This theorem generalizes to systems Trench's result [2] for second-order linear differential equations.

Let $X(t)$ be a fundamental matrix solution of

$$x' = F(t)x, \tag{2}$$

that is, a matrix of which the column vectors are linearly independent solutions of (2).

THEOREM. *If $G(t)$ is continuous on $[0, \infty)$ and if*

$$\int_0^\infty \|X^{-1}(s)G(s)X(s)\| ds < \infty, \tag{3}$$

then every solution $y(t)$ of (1) *can be expressed as*

$$y(t) = X(t)c(t), \tag{4}$$

where $c(t)$ is an n-vector function which is a solution of

$$c' = X^{-1}(t)G(t)X(t)c, \tag{5}$$

is bounded on $[0, \infty)$, and lim $c(t)$ *as $t \to \infty$ exists.*

Furthermore, given any n-vector ξ there is a unique $c(t)$ such that lim $c(t) = \xi$ *as $t \to \infty$.*

The specific matrix norm used in the proof is the sum of the absolute values of the elements.

Proof. Let $y(t)$ be any solution of (1), then because $X(t)$ is nonsingular, there exists a vector $c(t)$ such that

$$y(t) = X(t)c(t). \tag{6}$$

Differentiating both sides of (6) we get $y' = X'c + Xc'$. By (1), $Fy + Gy = X'c + Xc'$, so $c' = X^{-1}GXc$, and thus $c(t)$ is a solution of (5).

Hence,

$$c(t) = c(0) + \int_0^t X^{-1}GX \cdot c(s)ds \tag{7}$$

and $\|c(t)\| \leqq \|c(0)\| + \int_0^t \|X^{-1}GX\| \cdot \|c(s)\|ds$. By Bellman's lemma [**1**, p. 35],

$$\|c(t)\| \leqq \|c(0)\| \cdot \exp\left[\int_0^t \|X^{-1}GX\|ds\right]. \tag{8}$$

By (3) and (8), it follows that $c(t)$ is bounded on $[0, \infty)$ by

$$M \equiv \|c(0)\| \cdot \exp\left[\int_0^\infty \|X^{-1}GX\|ds\right].$$

From (3), given any $\epsilon > 0$ there exists $T \geqq 0$ such that, for any t_1, t_2 with $T \leqq t_1 < t_2$,

$$\int_{t_1}^{t_2} \|X^{-1}GX\|ds < \epsilon/M.$$

Hence $\|c(t_2) - c(t_1)\| < \epsilon$ and, by the Cauchy criterion, lim $c(t)$ as $t \to \infty$ exists.

Let $\Phi(t)$ be a fundamental matrix solution of (5). The column vectors $c_i(t)$ of $\Phi(t)$ tend to constant vectors c_i, $i = 1, 2, \cdots, n$, as $t \to \infty$ and, hence, $\Phi(t) \to \Phi_0$ as $t \to \infty$, where Φ_0 is a nonsingular constant matrix. We can prove that Φ_0 is nonsingular by recalling that $(\Phi^*)^{-1}$, where * denotes the conjugate transpose, is a fundamental matrix solution of

$$z' = -(X^{-1}GX)^*z, \tag{9}$$

the adjoint of (5). Thus, (3) and (9) imply that Φ^{-1} is bounded and lim $\Phi^{-1}(t)$ as $t \to \infty$ exists. Let $\Phi_1 \equiv \lim_{t\to\infty} \Phi^{-1}(t)$, then

$$\begin{aligned} \|\Phi_1\Phi_0 - I\| &\leqq \|\Phi_1\Phi_0 - \Phi_1\Phi(t)\| + \|\Phi_1\Phi(t) - \Phi^{-1}(t)\Phi(t)\| \\ &\leqq \|\Phi_1\| \cdot \|\Phi_0 - \Phi(t)\| + \|\Phi_1 - \Phi^{-1}(t)\| \cdot \|\Phi(t)\|. \end{aligned}$$

If $\|\Phi_0 - \Phi(t)\| < \epsilon/2 \cdot \|\Phi_1\|$ for $t > T_1$ and $\|\Phi_1 - \Phi^{-1}(t)\| < \epsilon/2Mn$, for $t > T_2$ then $\|\Phi_1 \cdot \Phi_0 - I\| < \epsilon$ for $t > \max[T_1, T_2]$. Since ϵ is arbitrary, $\Phi_1 \cdot \Phi_0 = I$. Therefore, Φ_0 is nonsingular.

Thus, given any n-vector ξ there is a unique $c(t)$ given by

$$c(t) = \Phi(t) \cdot \Phi_0^{-1} \cdot \xi$$

such that lim $c(t) = \xi$ as $t \to \infty$. The proof of the uniqueness of $c(t)$ is straightforward.

COROLLARY 1 (Trench's Theorem [2, p. 12]). *Given the second order scalar differential equation*

$$u'' = (a(t) + b(t))u, \tag{10}$$

where $b(t)$ *is continuous on* $[0, \infty)$, *let* $z_1(t)$ *and* $z_2(t)$ *be linearly independent solutions of* $u'' = a(t)u$, *and assume*

$$\int_0^\infty |b| y ds < \infty,$$

where $y(t) \equiv \max[z_1^2(t), z_2^2(t)]$ *on* $[0, \infty)$, *then, if* A *and* B *are arbitrary constants, there is a solution* $u(t)$ *of* (10) *which can be written in the form*

$$u(t) = \alpha(t)z_1 + \beta(t)z_2$$

with $\lim \alpha(t) = A$ *and* $\lim \beta(t) = B$ *as* $t \to \infty$.

Proof. Simply write (10) in its equivalent vector form and observe that

$$\int_0^t \|X^{-1}GX\| ds \leqq 4 |\det X(0)|^{-1} \int_0^t |b| y ds$$

for all $t > 0$. The conclusion follows from the theorem.

Several other well-known theorems follow almost directly from the theorem. For example,

COROLLARY 2 (Generalized Dini-Hukuhara Theorem [1, p. 47]). *If all solutions of* (2) *are bounded, if*

$$\operatorname{Re}\left[\int_0^t \operatorname{tr} F(s) ds\right] \geqq d > -\infty \tag{11}$$

for all $t > 0$, *and if* $\int_0^\infty \|G(s)\| ds < \infty$, *then all solutions of* (1) *are bounded on* $[0, \infty)$.

Proof. (11) implies X^{-1} is bounded on $[0, \infty)$, and thus

$$\int_0^\infty \|X^{-1}GX\| ds \leqq N \int_0^\infty \|G\| ds.$$

The result follows from the theorem.

References

1. L. Cesari, Asymptotic behavior and stability problems in ordinary differential equations, Springer-Verlag, Berlin, 1959.

2. W. Trench, On the asymptotic behavior of solutions of second order linear differential equations, Proc. Amer. Math. Soc., 14 (1963) 12–14.

ON CERTAIN RINGS OF POLYNOMIALS

SHERWOOD EBEY, Mercer University, Macon, Georgia

The purpose of this note is to use the geometry of the projective plane to obtain a description of certain rings of polynomials. Let K be an algebraically closed field, t an indeterminate over K and $f(t)$, $g(t)$ two polynomials in $K[t]$. We propose to examine the ring $K[f(t), g(t)]$ when f and g satisfy certain conditions. First we assume that f and g satisfy this condition A: If t_0 and t_1 are distinct elements of K, then $f(t_0)=f(t_1)$ implies $g(t_0)\neq g(t_1)$.

For any t_0 in K let $P(t_0)$ be the point $(f(t_0), g(t_0))$ in the affine plane A^2. Let C denote the irreducible algebraic curve which is the locus of $P(t)$. Condition A guarantees that $t_0\neq t_1$ implies $P(t_0)\neq P(t_1)$. The following proposition describes the curve C.

PROPOSITION 1. *If the degree of f is n and the degree of g is m with $n<m$, then C is a curve of order m.*

Proof. Embed C in the projective plane S^2 with projective coordinates (X, Y, Z). The line at infinity $(Z=0)$ meets C only at the point $(0, 1, 0)$.

If $L: aX+bY+cZ=0$ meets C at $(0, 1, 0)$, $b=0$. Then the points of intersection of L and C in the affine plane correspond to the solutions of the equation $af(t)+c=0$. Since degree of f is n, line L meets C in $\leqq n$ distinct points in the affine plane. Hence C meets L in $\leqq(n+1)$ distinct points in S^2. (Note: $n+1\leqq m$.)

Consider any line $M: aX+bY+cZ=0$ with $b\neq 0$. The only points of intersection of M and C are in the affine plane and correspond to solutions of the equation

$$F(t) = af(t) + bg(t) + c = 0.$$

This is a polynomial equation in t of degree m. It is possible to choose (a, b, c) so that $F(t)=0$ has no multiple roots and thus $F(t)=0$ will have m distinct roots.

Hence there exist lines in S^2 meeting C in exactly m distinct points and this number is maximal. Thus the order of the curve C is m and the irreducible polynomial $h(x, y)$ in $K[x, y]$ that defines C is of degree m.

At this point we will add condition B: The degrees of f and g are relatively prime. Then the following proposition gives further information about the irreducible polynomial h.

PROPOSITION 2. *If f and g satisfy conditions* A *and* B *then*

$$h(x, y) = y^n - r(x, y),$$

where $r(x, y)$ is a polynomial of degree less than n in y.

Proof. Define the weight of a monomial x^iy^j as the integer $in+jm$. Let $h_w(x, y)$ be the part of $h(x, y)$ consisting of nonzero terms of weight w where w is maximal.

If $h_w(x, y)$ consisted of a single monomial, then

$$h(f(t), g(t)) = ct^w + \text{terms of lower degree in } t, \qquad c \neq 0.$$

Since $h(f(t), g(t)) = 0$, $h_w(x, y)$ must have at least two nonzero terms. But since $(n, m) = 1$, $h_w(x, y)$ must be of the following form:

$$a_0x^iy^j + a_1x^{i-m}y^{j+n} + \cdots + a_qx^{i-qm}y^{j+qn}$$

with $a_0 \neq 0$, $a_q \neq 0$. Since $h(x, y)$ is of degree m, $h_w(x, y)$ must be $a_0x^m + a_1y^n$ with a_0, a_1 nonzero. We may assume $a_1 = 1$.

Since the maximum weight for h is mn, it follows that y^n is the only term of h of degree n in y. Therefore

$$h(x, y) = y^n - r(x, y),$$

where $r(x, y)$ is as required.

The description of the ring $K[f(t), g(t)]$ when f and g satisfy conditions A and B is now easily obtained. Proposition 2 implies $g^n(t) = r(f(t), g(t))$, where r is of degree less than n in y. The proof of the following proposition is immediate.

PROPOSITION 3. *Any element $h(t)$ in the ring $K[f(t), g(t)]$ is expressible in the form*

$$h(t) = h_0(f(t)) + h_1(f(t))g(t) + \cdots + h_{n-1}(f(t))g^{n-1}(t), \tag{1}$$

where the h_i are polynomials in $K[X]$.

Since any two monomials x^iy^j and x^ky^l with both j and l less than n cannot have the same weight, the $h_i(X)$ in expression (1) are uniquely determined. Thus we have proved the following propositions:

PROPOSITION 4. *The set of power products $\{f^i(t)g^j(t)\}$, where $0 \leqq j \leqq n-1$ and i is arbitrary, forms a basis of the K-vector space $K[f(t), g(t)]$.*

PROPOSITION 5. *In $K[f(t), g(t)]$ the minimal degree of any nonconstant polynomial is n.*

A NOTE ON THE HAUSDORFF SEPARATION PROPERTY IN FIRST COUNTABLE SPACES

ARNOLD J. INSEL, University of California, Berkeley

It is a well-known fact that every compact subset of a Hausdorff space is closed. In this note we prove the converse for the case in which the space satisfies the first axiom of countability. We are also provided with an example by the referee, to whom the author is deeply grateful, of a T_1 space for which every compact subset is closed but which is not Hausdorff (and hence does not satisfy the first axiom of countability). We establish the following theorem.

THEOREM. *Let X be any topological space which satisfies the first axiom of countability. If every compact subset of X is closed, then X is Hausdorff.*

To expedite the proof of this theorem we first establish the following lemma.

LEMMA. *Let X be a topological space and let $\{s_n\}$ be a sequence in X which converges to a point p. Let $S = \{s_n: n$ is a positive integer$\}$, i.e., the range of the sequence. Then $S \cup \{p\}$ is compact.*

Proof. Let $\mathfrak{U}$ be an open cover of $S \cup \{p\}$. Choose a member $U_0 \in \mathfrak{U}$ for which $p \in U_0$. Since $\{s_n\}$ converges to p there exists a positive integer m such that for $n \geqq m$, $s_n \in U_0$. For each $1 \leqq i < m$ choose a member $U_i \in \mathfrak{U}$ such that $s_i \in U_i$. Then the collection $\{U_0, U_1, \cdots, U_{m-1}\}$ is a finite subcover of $\mathfrak{U}$.

Proof of the theorem. We suppose that every compact subset of X is closed. Since X satisfies the first axiom of countability, to establish that X is Hausdorff it is sufficient to show that no sequence in X converges to more than one point. Let $\{s_n\}$ be a sequence in X which converges to the points p and q. We show that $p = q$. We divide the proof into three cases.

Case 1. $\{s_n\}$ is frequently in the set $\{p\}$.

Then $\{s_n\}$ has a subsequence $\{t_n\}$ such that $t_n = p$ for all n. Since $\{p\}$ is finite it is compact and hence by hypothesis closed. Since $\{s_n\}$ converges to q, $\{t_n\}$ converges to q. We conclude that $q \in \{p\}$, and therefore $p = q$.

Case 2. $\{s_n\}$ is frequently in $\{q\}$.

The proof is similar to that of case 1.

Case 3. $\{s_n\}$ is neither frequently in $\{p\}$ nor frequently in $\{q\}$.

Then there exists a positive integer m such that for $n \geqq m$, $s_n \notin \{p, q\}$. Let $\{t_n\}$ be the subsequence of $\{s_n\}$ defined by $t_n = s_{m+n}$ for all n. Then $\{t_n\}$ converges to p and q. Let T be the range of $\{t_n\}$. Then by the definition of $\{t_n\}$, $p \notin T$ and $q \notin T$. Consider the set $W = T \cup \{p\}$. By the lemma W is compact. Hence by hypothesis W is closed. Since $\{t_n\}$ is a sequence in the closed set W converging to q we conclude that $q \in W$. Since $q \notin T$, $q = p$. This establishes the third case and hence the theorem.

The following counter-example, due to the referee, consists of a T_1 space X which does not satisfy the first axiom of countability, and such that every compact subset of X is closed although X is not Hausdorff.

Example. Let X be any uncountable set. Consider the topology on X for which the closed sets consist of the countable subsets of X and of X itself. It it easily seen that no two distinct elements in X can be separated by open sets, and hence X is not Hausdorff. We next observe that the only compact subsets of X are the finite sets. For let K be any infinite subset of X. Then there exists a sequence $\{p_n\}$ of distinct elements in K. For each positive integer n define $U_n = X - \{p_j: j \geqq n\}$. Then the collection $\{U_n: n$ is a positive integer$\}$ is an open cover of K which has no finite subcover. Since every finite subset of X is countable, and hence closed, every compact subset of X is closed. By the theorem it necessarily follows that X does not satisfy the first axiom of countability.

AN ARITHMETIC CONGRUENCE

D. E. DAYKIN, Reading University, England

Let R be a fixed positive integer. Given an arithmetic function $g(n)$ and integers α, β we put

$$G(n) = \sum_{de=n} g(d) \quad \text{for} \quad n \geqq 1 \tag{1}$$

and

$$H(\alpha, \beta, n) = \sum_{\substack{de=n \\ (d,R)=1}} g(d)\{\alpha^e - \beta^e\} \quad \text{for} \quad n \geqq 1.$$

The object of this note is to prove Theorem 1 below. For the case $R=1, \beta=0$ of the theorem, sufficiency was established by L. Gegenbauer (cf. [2], p. 86) and necessity by L. Carlitz ([1], p. 271).

THEOREM 1. *We have*

$$H(\alpha, \beta, n) \equiv 0 \pmod{nR} \quad \textit{for} \quad n \geqq 1,\ \alpha \equiv \beta \pmod{R} \tag{2}$$

if and only if

$$G(n) \equiv 0 \pmod{n} \qquad \text{for } n \geqq 1,\ (n, R) = 1. \tag{3}$$

Proof. Let the function $K(\alpha, n)$ be defined by the relations

$$\alpha^n = \sum_{de=n} K(\alpha, e) \quad \text{for} \quad n \geqq 1. \tag{4}$$

Then we have the well-known result

$$K(\alpha, n) = \sum_{de=n} \mu(d)\alpha^e \equiv 0 \pmod{n} \quad \text{for} \quad n \geqq 1, \tag{5}$$

where $\mu(n)$ is the Möbius function. Given an integer $h \geqq 1$ we factorise it in the form $h=mr$, where $(m, R)=1$ and every prime factor of r divides R, and then using (1) and (4) we get

$$H(\alpha, \beta, h) = \sum_{de=m} g(d)\{(\alpha^r)^e - (\beta^r)^e\} \tag{6}$$

$$= \sum_{bc=m} G(b)\{K(\alpha^r, c) - K(\beta^r, c)\}. \tag{7}$$

Now suppose that (3) holds. Then by Gegenbauer's result it follows from (6) that $H(\alpha, \beta, h) \equiv 0 \pmod{m}$. If p is a prime divisor of R and p^s, p^t are the highest powers of p in r, R respectively, then since $\alpha \equiv \beta \pmod{p^t}$ we have $\alpha^{p^s} \equiv \beta^{p^s} \pmod{p^{s+t}}$. Therefore $(\alpha^r)^e \equiv (\beta^r)^e \pmod{rR}$ for each positive integer e, and $H(\alpha, \beta, h) \equiv 0 \pmod{rR}$ by virtue of (6). Since $hR = mrR$ and $(m, rR)=1$ we have thus shown that (2) holds.

Next we assume that (2) holds and prove by induction on n that (3) holds. Trivially (3) holds for $n=1$. Suppose that $(m, R)=1$ and (3) holds for $1 \leqq n < m$, $(n, R)=1$. Then we consider equation (7) with $h=m$, $r=1$, and by our induction hypothesis $G(b) \equiv 0 \pmod{b}$ for each $b < m$ in the sum (7). Moreover (5) shows that $K(\alpha^r, c) - K(\beta^r, c) \equiv 0 \pmod{c}$ for each c in (7), and so $G(m)\{K(\alpha^r, 1) - K(\beta^r, 1)\} \equiv 0 \pmod{m}$. We put $\alpha = R$ and $\beta = 0$ in this last congruence to obtain $G(m) \equiv 0 \pmod{m}$, and the theorem follows by induction.

The author would like to thank P. J. McCarthy for suggestions which generalised an earlier version of this note.

References

1. L. Carlitz, An arithmetic function, Bull. Amer. Math. Soc., 43 (1937) 271–276.
2. L. E. Dickson, History of the theory of numbers, 3 vols. Washington, 1919–27. (See vol. I.)

GERSHGORIN'S THEOREM AND THE ZEROS OF POLYNOMIALS

HOWARD E. BELL, Harpur College

1. Introduction. In a recent paper [7], H. S. Wilf makes use of the Perron-Frobenius theorem to obtain certain known root-location theorems for polynomial equations. As we shall show, these results, and other classical theorems as well, can be conveniently established by using the following easily-proved theorem of Gershgorin [3, 5].

THEOREM 0. *Let $A = [a_{ij}]$ be an $n \times n$ complex matrix, and let R_i be the sum of the moduli of the off-diagonal elements in the i-th row. Then each eigenvalue of A lies in the union of the circles*

$$(1) \qquad |z - a_{ii}| \leqq R_i, \qquad i = 1, 2, \cdots, n.$$

The analogous result holds if columns of A are considered.

In most of our applications we shall use the larger circles

$$|z| \leqq |a_{ii}| + R_i, \qquad i = 1, 2, \cdots, n.$$

2. The Ballieu-Cauchy Theorem. We begin by proving a theorem which generalizes a classical result of Cauchy.

THEOREM 1 (BALLIEU [1]). *Let $A_1, A_2, \cdots, A_{n-1}$ be arbitrary positive numbers, $A_0 = 0$, and $A_n = 1$. Then all the zeros of*

$$(2) \qquad f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

satisfy the inequality

$$(3) \qquad |z| \leqq \max\left\{\frac{A_i}{A_{i+1}} + \frac{|a_i|}{|a_n| A_{i+1}}\right\}, \qquad i = 0, 1, \cdots, n-1.$$

We start with the matrix

(4)
$$C = \begin{bmatrix} 0 & 0 \cdots 0 & -a_0/a_n \\ 1 & 0 \cdots 0 & -a_1/a_n \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 0 & 0 \cdots 1 & -a_{n-1}/a_n \end{bmatrix},$$

the companion matrix of f; and taking

(5)
$$P = \operatorname{diag}(A_1, A_2, \cdots, A_n),$$

we form the matrix

(6)
$$P^{-1}CP = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0/a_nA_1 \\ A_1/A_2 & 0 & \cdots & 0 & -a_1/a_nA_2 \\ 0 & A_2/A_3 & \cdots & 0 & -a_2/a_nA_3 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & & \cdots & A_{n-1}/A_n & -a_{n-1}/a_nA_n \end{bmatrix}.$$

Since the eigenvalues of (6) are the zeros of f, a direct application of Theorem 0 yields the conclusion of Theorem 1.

We note that if $A_1 = A_2 = \cdots = A_{n-1} = 1$ the bound (3) is the classical one of Cauchy,

$$|z| \leqq \max \left\{ \left| \frac{a_0}{a_n} \right|, 1 + \left| \frac{a_1}{a_n} \right|, \cdots, 1 + \left| \frac{a_{n-1}}{a_n} \right| \right\};$$

if $A_i = |a_i/a_n|$, $i = 1, \cdots, n-1$, then (3) becomes

$$|z| \leqq \max \left\{ \left| \frac{a_0}{a_1} \right|, 2\left| \frac{a_1}{a_2} \right|, \cdots, 2\left| \frac{a_{n-1}}{a_n} \right| \right\},$$

a bound due to Kojima [4, p. 107].

3. Theorems of Cauchy, Walsh and Fujiwara. In this section we discuss three well-known theorems, which have similar proofs.

THEOREM 2 (CAUCHY). *The moduli of the zeros of* (2) *do not exceed the positive zero of* $g(z) = |a_n| z^n - |a_{n-1}| z^{n-1} - \cdots - |a_0|$.

THEOREM 3 (FUJIWARA [2]). *If* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are positive numbers with* $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, *then the zeros of* (2) *satisfy*

$$|z| \leqq \max \left| \frac{1}{\lambda_j} \frac{a_{n-j}}{a_n} \right|^{1/j}.$$

THEOREM 4 (WALSH [6]). *The zeros of* (2) *all lie in the circle*

(7)
$$\left| z + \frac{a_{n-1}}{2a_n} \right| \leqq \left| \frac{a_{n-1}}{2a_n} \right| + \left| \frac{a_{n-2}}{a_n} \right|^{1/2} + \left| \frac{a_{n-3}}{a_n} \right|^{1/3} + \cdots + \left| \frac{a_0}{a_n} \right|^{1/n}.$$

If in (5) we take $A_i=\rho^{n-i}$, $i=1, 2, \cdots, n-1$, the matrix (6) takes the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0/a_n\rho^{n-1} \\ \rho & 0 & \cdots & 0 & -a_1/a_n\rho^{n-2} \\ \cdot & \rho & & & \\ \cdot & & \cdots & 0 & -a_{n-2}/a_n\rho \\ 0 & 0 & \cdots & \rho & -a_{n-1}/a_n \end{bmatrix}.$$

To prove Theorem 2 we apply Theorem 0 to the columns and choose ρ to be equal to the sum of the moduli of the elements in the last column.

To prove Theorem 3, we take

$$\rho = \max\left|\frac{1}{\lambda_j}\,\frac{a_{n-j}}{a_n}\right|^{1/j}, \qquad j = 1, 2, \cdots, n,$$

and note that

$$|a_0/a_n\rho^{n-1}| + |a_1/a_n\rho^{n-2}| + \cdots + |a_{n-1}/a_n| \leqq (\lambda_1+\lambda_2+\cdots+\lambda_n)\rho = \rho.$$

To prove Theorem 4, we let

$$\rho = \max\left|\frac{a_{n-j}}{a_n}\right|^{1/j}, \qquad j = 2, 3, \cdots, n,$$

and note that the circles (1) are in this case all contained in the circle (7).

4. A theorem on lower bounds. Clearly, similar methods will yield lower bounds for the moduli of the zeros of f. We conclude our discussion by proving as an example a theorem of Landau and Markovitch [4, p. 99].

THEOREM 5. *For a given positive number t, let $M=\max |a_i|t^i$, $i=1, 2, \cdots, n$. Then all the zeros of* (2) *satisfy*

$$|z| \geqq \frac{|a_0|t}{|a_0|+M}.$$

Beginning with the inverse K of (4) and the matrix $R=\operatorname{diag}(1, t, \cdots, t^{n-1})$, we form the matrix

$$RKR^{-1} = \begin{bmatrix} -a_1/a_0 & 1/t & 0 & \cdots & 0 \\ -a_2t/a_0 & 0 & 1/t & \cdots & 0 \\ \cdot & & & & \\ -a_{n-1}t^{n-2}/a_0 & 0 & 0 & \cdots & 1/t \\ -a_nt^{n-1}/a_0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

the eigenvalues of which are the reciprocals of the zeros of f. By Theorem 0, these eigenvalues satisfy

$$|z| \leqq \max \left\{ \frac{|a_0| + |a_i| t^i}{|a_0| t} \right\} \leqq \frac{|a_0| + M}{|a_0| t},$$

and the conclusion of Theorem 5 follows immediately.

References

1. R. Ballieu, Sur des limitations des racines d'une équation algébrique, Acad. Roy. Belgique Bull. Cl. Sci. (5) 33 (1947) 743–750.

2. M. Fujiwara, Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung, Tôhoku Math. J., 10 (1916) 167–171.

3. S. Gershgorin, Über die Abgrenzung der Eigenwerte einer Matrix, Izv. Akad. Nauk SSSR, 7 (1931) 749–754.

4. M. Marden, The geometry of the zeros of a polynomial in a complex variable, Mathematical Surveys, Amer. Math. Soc., 1949.

5. O. Taussky, A recurring theorem on determinants, this MONTHLY, 56 (1949) 672–676.

6. J. L. Walsh, An inequality for the roots of an algebraic equation, Ann. Math., 25 (1924) 285–286.

7. H. S. Wilf, Perron-Frobenius theory and the zeros of polynomials, Proc. Amer. Math. Soc., 12 (1961) 247–250.

CORRECTIONS FOR "A NOTE ON THE RELATION BETWEEN PERIODIC AND ORTHOGONAL FUNDAMENTAL SOLUTIONS OF LINEAR SYSTEMS, II"

F. S. VAN VLECK, University of Kansas

J. J. Schäffer has kindly pointed out to the writer that there are two errors in the paper: A Note on the Relation between Periodic and Orthogonal Fundamental Solutions of Linear Systems, II (this MONTHLY, 71 (1964) 774–776). The first error is that Theorem 2, as stated, is false. The theorem is true if, in addition to the given hypotheses, it is assumed that $A(t)$ is periodic of period σ. So Theorem 2 should read as follows:

THEOREM 2. *If n is odd, $A(t)$ is periodic of period σ, and $X(t)$ is orthogonal for all time t, then there is a periodic solution of $x' = A(t)x$ of period σ.*

The second mistake occurs in connection with the statement of Theorem 3, where we say "Hence the characteristic roots of R are simple . . . ," and again, on page 776, where we say "Now σS is a real skew-symmetric matrix, so its roots are simple . . . ." What is meant is that the elementary divisors are linear or, equivalently, that the matrices in question have simple structure in the terminology of Gantmacher (Matrix Theory, vol. 1, Chelsea, New York, p. 72).

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

THE TANGENT OF A HALF-ANGLE

STEPHEN M. ROBINSON, Stanford University

In standard texts on trigonometry (e.g., [1], [2]) the formula for the tangent of a half-angle is given as

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}, \qquad \theta \neq \pi, \tag{1}$$

with the sign to be chosen according to the value of θ. This expression is inconvenient to compute, as it involves both a root extraction and a choice of sign.

A very much simpler formula has been known for a long time, namely:

$$\tan\frac{\theta}{2} = (1-\cos\theta)/\sin\theta, \qquad 0 \neq \theta \neq \pi. \tag{2}$$

This expression is more convenient to use than is (1), both for practical computation and for analytical uses (e.g., integration). It therefore appears strange that (1) should continue to be taught to students.

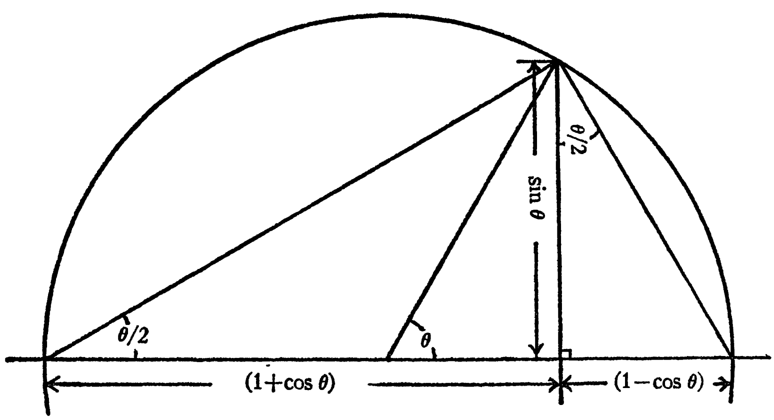


FIG. 1

The proof of (2) is immediate: we simply note that

$$\cos 2\psi = \cos^2\psi - \sin^2\psi$$
$$\sin 2\psi = 2\sin\psi\cos\psi,$$

so that $(1-\cos 2\psi)/\sin 2\psi = 2\sin^2\psi/2\sin\psi\cos\psi = \tan\psi$, which is equivalent to (2).

A simple geometrical illustration of (2) and its equivalent form,

$$\tan\frac{\theta}{2} = \sin\theta/(1+\cos\theta), \tag{3}$$

is given in Figure 1. It does seem that, as (2) and (3) are geometrically obvious and are relatively easy to use, they ought to be presented in place of (1), instead of being exiled (if given at all!) to remote exercise sections.

References

1. A. P. Hillman, and G. L. Alexanderson, Functional trigonometry, Allyn and Bacon, Boston, 1961.

2. A. Hooper and A. L. Griswold, A modern course in trigonometry, Holt, Rinehart and Winston, New York, 1954.

A COMPLEMENT TO CAUCHY'S THEOREM

R. BOJANIC, Ohio State University

The following problem is considered here: given a sequence $\{a_n\}$ of real numbers, what are the necessary and sufficient conditions for the existence of a sequence $\{b_n\}$ such that

$$a_n \leqq b_n,\ n = 1, 2, \cdots \quad \text{and} \quad \sum b_n \text{ converges?} \tag{1}$$

We shall prove here the following theorem:

A sequence $\{a_n\}$ *has the property* (1) *if and only if for any* $\epsilon > 0$ *there is a number* N *such that*

$$m \geqq n \geqq N \Rightarrow \sum_{\nu=n}^{m} a_\nu < \epsilon. \tag{2}$$

In a first course in analysis students quite often believe that (2) is a necessary and sufficient condition for the convergence of the series $\sum a_n$. Actually, it turns out that the condition (2) implies the convergence of the series $\sum a_n$, but in the extended real number system. More precisely, the condition (2) implies that

$$\sum_{\nu=1}^{\infty} a_\nu = A, \qquad (-\infty \leqq A < +\infty).$$

Thus the main difficulty in the proof of our theorem lies in the case $A = -\infty$!

As in the proof of Cauchy's theorem, the necessity of the condition (2) for the validity of (1) is obvious. Thus it remains only to show that (2) is sufficient.

Let

$$A_n = \sup_{p \geqq n} \sum_{\nu=n}^{p} a_\nu \text{ and } S_n = \sup_{p \geqq n} \sum_{\nu=1}^{p} a_\nu.$$

From (2) it follows that

$$\limsup_{n\to\infty} A_n \leqq 0. \tag{3}$$

Since $\{S_n\}$ is a nonincreasing sequence we have either

$$\lim_{n\to\infty} S_n = A, \qquad (-\infty < A < +\infty), \tag{4}$$

or

$$\lim_{n\to\infty} S_n = -\infty. \tag{5}$$

Assume first that (4) is true. Then from

$$A_n = \sup_{p \geqq n} \sum_{\nu=n}^{p} a_\nu = \sup_{p \geqq n} \sum_{\nu=1}^{p} a_\nu - \sum_{\nu=1}^{n-1} a_\nu = S_n - \sum_{\nu=1}^{n-1} a_\nu \tag{6}$$

it follows that $A_n \geqq S_n - S_{n-1}$ and so

$$\liminf_{n\to\infty} A_n \geqq 0. \tag{7}$$

From (3) and (7) it follows that $\lim_{n\to\infty} A_n = 0$, which by (6) and (4) implies that

$$\sum_{\nu=1}^{\infty} a_\nu = \lim_{n\to\infty} \sum_{\nu=1}^{n} a_\nu = A, \qquad (-\infty < A < +\infty),$$

and (1) follows obviously by choosing $b_n = a_n$, $n = 1, 2, \cdots$.

Next, suppose that (5) is true. Then from $\sum_{\nu=1}^{n} a_\nu \leqq S_n$, $n = 1, 2, \cdots$ and (5) it follows that

$$\sum_{\nu=1}^{\infty} a_\nu = \lim_{n\to\infty} \sum_{\nu=1}^{n} a_\nu = -\infty. \tag{8}$$

If $a_n > 0$ for finitely many n only and if a_m is the last positive term, (1) follows immediately by choosing $b_n = a_n$, $n = 1, \cdots, m$ and $b_n = 0$, $n > m$. Thus we can assume that $a_n > 0$ for infinitely many n. We shall show in this case that the sequence $\{b_n\}$ can be chosen such that

$$a_n \leqq b_n, \; n = 1, 2, \cdots \quad \text{and} \quad \sum_{n=1}^{\infty} b_n = 0.$$

Let a_{n_1} be the first positive term in the sequence $\{a_n\}$.

If $n_1 > 1$, let $b_n = 0$ for $1 \leqq n < n_1$. We have then $a_n \leqq b_n$ and $\sum_{\nu=1}^{n} b_\nu = 0$ for $1 \leqq n < n_1$.

Next, or if $n_1 = 1$, let a_{n_2} be the first negative term with $n_2 > n_1$ such that

$$a_{n_1} + \cdots + a_{n_2} < 0; \tag{9}$$

such a negative term clearly exists by (8). Let a_{n_3} be the first positive term in the sequence $\{a_n\}$ with $n_3 > n_2$. Let

$$b_n = \begin{cases} a_n & , \quad n_1 \leqq n < n_2 \\ -(a_{n_1} + \cdots + a_{n_2-1}), & \quad n = n_2 \\ 0 & , \quad n_2 < n < n_3. \end{cases}$$

We have clearly $a_n = b_n$ for $n_1 \leqq n < n_2$; by (9) we have

$$a_{n_2} < -(a_{n_1} + \cdots + a_{n_2-1}) = b_{n_2}$$

and $a_n \leqq 0 = b_n$ for $n_2 < n < n_3$; and it is easy to see that

$$\sum_{\nu=n_1}^{n} b_\nu = \sum_{\nu=n_1}^{n} a_\nu \geqq 0 \quad \text{for} \quad n_1 \leqq n < n_2, \qquad \sum_{\nu=n_1}^{n} b_\nu = 0 \quad \text{for} \quad n_2 \leqq n < n_3.$$

This defines b_n for $n_1 \leqq n < n_3$ and we have

$$a_n \leqq b_n \quad \text{and} \quad \sum_{\nu=n_1}^{n} b_\nu \geqq 0 \quad \text{for} \quad n_1 \leqq n < n_3.$$

Next, continue this procedure beginning with n_3 instead of n_1, find n_4 and n_5, define b_n for $n_3 \leqq n < n_5$ and so on, until the sequence $\{b_n\}$ is completely defined.

From the construction of the sequence $\{b_n\}$ it follows that $a_n \leqq b_n$, $n = 1, 2, \cdots$ and we have only to prove that the series $\sum b_n$ converges and that its sum is 0.

Given $\epsilon > 0$, we can, by (3), choose n_{2m-1} such that

$$n \geqq n_{2m-1} \Rightarrow A_n < \epsilon. \tag{10}$$

Take any $n \geqq n_{2m-1}$ and choose k such that $n_{2m-1} \leqq n_{2k-1} \leqq n < n_{2k+1}$. Then we have

$$\sum_{\nu=1}^{n} b_\nu = \sum_{l=1}^{k-1} \left(\sum_{\nu=n_{2l-1}}^{n_{2l+1}-1} b_\nu \right) + \sum_{\nu=n_{2k-1}}^{n} b_\nu = \sum_{\nu=n_{2k-1}}^{n} b_\nu,$$

since each of the sums on the right hand side, with the exception of the last one, is zero by the definition of $\{b_n\}$. If $n_{2k-1} \leqq n < n_{2k}$ we have by (10)

$$0 \leqq \sum_{\nu=n_{2k-1}}^{n} b_\nu = \sum_{\nu=n_{2k-1}}^{n} a_\nu \leqq A_{n_{2k-1}} < \epsilon;$$

if $n_{2k} \leqq n < n_{2k+1}$ we have

$$\sum_{\nu=n_{2k-1}}^{n} b_\nu = 0 < \epsilon.$$

Thus $n \geqq n_{2m-1} \Rightarrow \left| \sum_{\nu=1}^{n} b_\nu \right| < \epsilon$, i.e.

$$\sum_{\nu=1}^{\infty} b_\nu = \lim_{n\to\infty} \sum_{\nu=1}^{n} b_\nu = 0$$

and our theorem is completely proved.

AN N-TH ORDER SECOND MEAN VALUE THEOREM

DONALD H. TRAHAN, University of Pittsburgh

Although generalized mean value theorems have been studied, this subject has received surprisingly little attention. Yet this is a topic which should be readily available to the undergraduate. M. P. Drazin [1] gives an elegant proof of a Mean Value Theorem which is more general than the Corollary given in this paper. However, the determinant type proof given here should be of general interest and perhaps easier to adapt to the classroom.

In contrast to the usual statement of the theorem, the Second Mean Value Theorem is given here without any assumption about non-vanishing terms. Also, it seems interesting to prove the Mean Value Theorem as a corollary of the Second Mean Value Theorem. (One may prove the Mean Value Theorem using determinants, and the proof is similar to the proof of the Theorem below.)

DEFINITION. *The symbol* $n!!$ *represents* $1!2! \cdots n!$.

THEOREM. *If* f, g *are functions which are continuous on* $[a, a+nh]$, $h>0$, *the* n*-th derivatives of* f *and* g *exist on* $(a, a+nh)$, *and* $c_k = (-1)^k\binom{n}{k}$, *then there exists* $\xi \in (a, a+nh)$ *such that*

$$f^{(n)}(\xi)[c_0 g(a) + \cdots + c_n g(a+nh)] = g^{(n)}(\xi)[c_0 f(a) + \cdots + c_n f(a+nh)].$$

Proof. Consider the function

$$\phi(x) = \begin{vmatrix} 1 & x & \cdots & x^{n-1} & g(x) & f(x) \\ 1 & a & \cdots & a^{n-1} & g(a) & f(a) \\ 1 & a+h & \cdots & (a+h)^{n-1} & g(a+h) & f(a+h) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a+nh & \cdots & (a+nh)^{n-1} & g(a+nh) & f(a+nh) \end{vmatrix}$$

then $\phi(x)$ vanishes at $a, a+h, \cdots, a+nh$. Therefore by repeated application of Rolle's Theorem there exists a point $\xi \in (a, a+nh)$ such that $\phi^{(n)}(\xi) = 0$. That is,

$$\begin{vmatrix} 0 & 0 & \cdots & 0 & g^{(n)}(\xi) & f^{(n)}(\xi) \\ 1 & a & \cdots & a^{n-1} & g(a) & f(a) \\ 1 & a+h & \cdots & (a+h)^{n-1} & g(a+h) & f(a+h) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & a+nh & \cdots & (a+nh)^{n-1} & g(a+nh) & f(a+nh) \end{vmatrix} = 0.$$

The minor of $f^{(n)}(\xi)$ can be calculated by expanding down its last column, for the subsequent minors of $g(a+kh)$, $0 \leqq k \leqq n$, are Vandermonde determinants. The minor of $f^{(n)}(\xi)$ is

$$[h^{n(n-1)/2}(n-1)!!][c_0 g(a) + \cdots + c_n g(a+nh)].$$

Since the minor of $g^{(n)}(\xi)$ is similar, it follows that

$$f^{(n)}(\xi)[c_0 g(a) + \cdots + c_n g(a+nh)] - g^{(n)}(\xi)[c_0 f(a) + \cdots + c_n f(a+nh)] = 0.$$

COROLLARY. *If f is a continuous function on $[a, a+nh]$, $h>0$, the n-th derivative of f exists on $(a, a+nh)$, and $c_k = (-1)^k\binom{n}{k}$, then there exists $\xi \in (a, a+nh)$ such that $h^n f^{(n)}(\xi) = c_0 f(a+nh) + \cdots + c_n f(a)$.*

Proof. In the theorem let $n!g(x) = x^n$, then

$$c_0 g(a) + \cdots + c_n g(a+nh) = \frac{1}{n!}[c_0 a^n + c_1(a+h)^n + \cdots + c_n(a+nh)^n].$$

The Corollary follows from the last equation and the equations below:

$$\begin{aligned}
c_0 + c_1 + c_2 + \cdots + c_n &= 0\\
c_1 + 2c_2 + \cdots + nc_n &= 0\\
c_1 + 2^2 c_2 + \cdots + n^2 c_n &= 0\\
\cdots\cdots\cdots\cdots\cdots&\\
c_1 + 2^{n-1}c_2 + \cdots + n^{n-1}c_n &= 0\\
c_1 + 2^n c_2 + \cdots + n^n c_n &= (-1)^n n!
\end{aligned}$$

These equations are known. In fact, one can either solve the equations directly or use the evaluation of $D_x^{(k)}(1-x)^n$ at $x=1$, $0 \leqq k \leqq n-1$, and $D_x^{(n)}(1-x)^n$. For a direct solution (for the moment treating the c_k's as unknowns) let Δ be the determinant of the coefficients. After expanding on the first column, factor out $n!$; a Vandermonde determinant remains, so that $n!! = \Delta$. Similarly, $c_0\Delta = n!!$ and

$$c_k\Delta = (-1)^k n!\left[\frac{n!}{k}\right]\left[\frac{(n-1)!!}{(n-k)!(k-1)!}\right] \qquad k \neq 0.$$

Acknowledgment. I would like to thank the referee for his suggestions.

Reference

1. M. P. Drazin, Some inequalities arising from a generalized mean value theorem, this MONTHLY, 62 (1955) 226–232.

AN EXAMPLE IN THE PROBLEM OF MOMENTS

S. W. DHARMADHIKARI, University of California, Berkeley
(Now at the Indian Statistical Institute, Calcutta)

1. Let $\{\mu_n\}$ be the sequence of moments of a distribution function F. A sufficient condition that F be the only distribution function with $\{\mu_n\}$ as its moment-sequence is that the series $\sum \mu_n x^n/n!$ have a positive radius of convergence. (See [1], page 110.) We show here by an example that this condition is not necessary.

2. Let the random variables X and Y be independently distributed according to the negative exponential distribution. Let $Z=X\log(1+Y)$. Then the nth moment μ_n of Z is $\mu_n=n!\nu_n$, where

$$\nu_n = \int_0^\infty [\log(1+x)]^n e^{-x}dx. \tag{1}$$

It will be shown below that, for $n \geqq 1$,

$$e^{-1}\log(1+n) < \nu_n^{1/n} < C\log(1+n), \tag{2}$$

where C is a finite constant. Thus

$$\left(\frac{\mu_n}{n!}\right)^{1/n} = \nu_n^{1/n} > e^{-1}\log(1+n) \to \infty,$$

which shows that the series $\sum \mu_n x^n/n!$ does not converge for any nonzero x.

Now $(n!)^{1/n} \sim e^{-1}n$. Therefore, from (2),

$$\mu_n^{1/n} \sim e^{-1}n\nu_n^{1/n} < Ce^{-1}n\log(1+n).$$

It follows that

$$\sum_{n=1}^\infty \mu_{2n}^{-1/2n} = \infty.$$

Carleman's condition ([2], page 19) now shows that the sequence $\{\mu_n\}$ determines the distribution uniquely.

3. *Proof of (2).* We assume throughout that $n \geqq 1$. To simplify writing, put $\alpha_n(x)=[\log(1+x)]^n$. We have, from (1),

$$\nu_n = \int_0^n \alpha_n(x)e^{-x}dx + \int_n^\infty \alpha_n(x)e^{-x}dx. \tag{3}$$

Neglecting the first term of (3), we get

$$\nu_n > \int_n^\infty \alpha_n(x)e^{-x}dx > \alpha_n(n)\int_n^\infty e^{-x}dx = \alpha_n(n)e^{-n},$$

which immediately implies the first inequality in (2).

If $0 \leqq x < n$ then $\alpha_n(x) < \alpha_n(n)$, whence

$$(4) \qquad \int_0^n \alpha_n(x)e^{-x}dx < \alpha_n(n) \int_0^n e^{-x}dx < \alpha_n(n).$$

Let λ_n denote the second term in (3). Integration by parts gives

$$\begin{aligned}\lambda_n &= e^{-n}\alpha_n(n) + \int_n^\infty \frac{n}{1+x}\,\alpha_{n-1}(x)e^{-x}dx \\ &< \alpha_n(n) + \int_{n-1}^\infty \alpha_{n-1}(x)e^{-x}dx = \alpha_n(n) + \lambda_{n-1}.\end{aligned}$$

Since $\lambda_0 = 1$, it follows that

$$\lambda_n < 1 + \sum_{k=1}^n \alpha_k(k).$$

Since the $\alpha_k(k)$ increase with k and since $2\alpha_n(n) > 1$, we must have

$$(5) \qquad \lambda_n < (n+2)\alpha_n(n).$$

Combining (4) and (5), we see that

$$\nu_n < (n+3)\alpha_n(n),$$

which implies the second inequality in (2) as soon as it is noted that $(n+3)^{1/n} \to 1$.

References

1. M. G. Kendall and A. Stuart, The advanced theory of statistics, vol. I, Stechert-Hafner, New York, 1958.
2. J. A. Shohat and J. D. Tamarkin, The problem of moments, Amer. Math. Soc., 1943.

DISTRIBUTIONS OF PRODUCT AND QUOTIENT OF CAUCHY VARIABLES

PAUL R. RIDER, Aerospace Res. Labs., Wright-Patterson Air Force Base

1. Introduction. The Cauchy distribution,

$$(1) \qquad f(x) = \frac{1}{\pi(1+x^2)}, \qquad -\infty < x < \infty,$$

has several peculiar properties. The distribution of arithmetic means of samples from it is identical with the distribution itself. The reciprocal of a Cauchy variable has the same distribution as the variable itself. The distribution of harmonic means is, consequently, the same as the original distribution.

This note derives the distributions of product and quotient of two independent Cauchy variables.

2. Distribution of product. The probability element of the joint distribution of two independent variables, x and y, from the Cauchy distribution is

$$\frac{dxdy}{\pi^2(1+x^2)(1+y^2)}. \tag{2}$$

In order to find the distribution of the product $u=xy$ we substitute $y=x^{-1}u$, $dy=x^{-1}du$ in (2), obtaining

$$\frac{|x|\,dxdu}{\pi^2(1+x^2)(u^2+x^2)},$$

which separates into partial fractions as follows:

$$\frac{1}{\pi^2(u^2-1)}\left[\frac{|x|}{1+x^2}-\frac{|x|}{u^2+x^2}\right]dxdu.$$

We now integrate with respect to x between the limits $-\infty$ and ∞. (It is convenient to integrate between 0 and ∞ and double the result.) For the distribution of the product u, we find

$$f(u)=\frac{\log u^2}{\pi^2(u^2-1)}. \tag{3}$$

As a check we note [1, p. 192, formula (6)] that $\int_{-\infty}^{\infty}f(u)du=1$.

3. Distribution of quotient. The distribution of the quotient of two independent Cauchy variables can be found by the same process. It turns out to be identical with the distribution of the product, which of course must be the case, since, as already stated, the distribution of the reciprocal of a Cauchy variable is identical with the distribution of the variable itself.

It may be noted that (3) is also the distribution of the product or quotient of the means (arithmetic or harmonic) of two independent samples from the distribution (1).

4. Nature of frequency curve. The function $f(u)$ becomes infinite as u approaches zero. Its value is indeterminate for $u=\pm 1$, but by l'Hôpital's rule it can be shown that $f(u)$ approaches $1/\pi^2$ as u approaches ± 1.

To investigate the appearance of the frequency curve further we find the derivative

$$f'(u)=\frac{2u}{\pi^2(u^2-1)^2}\left[\frac{u^2-1}{u^2}-\log u^2\right]. \tag{4}$$

To show that this derivative is nonnegative we consider first the case $u>0$. If we set $u^{-2}=V$ the expression in brackets in (4) can be written $1+\log V-V$. Considering the ratio $(1+\log V)/V$, we find by usual methods that its maximum value is 1, given by $V=1$. Thus,

$$(1 + \log V)/V \leqq 1, \quad \text{or} \quad 1 + \log V \leqq V.$$

In terms of u^2 we have $1 - \log u^2 \leqq 1/u^2$, which shows that the bracketed expression in (4) is (for $u > 0$) always negative or zero. It can be zero for $u = 1$. But l'Hôpital's rule shows that $f'(u)$ approaches $-1/\pi^2$ as u approaches 1. Consequently $f'(u)$ is always negative and, as u increases from 0 to ∞, $f(u)$ decreases from ∞ to 0.

A similar argument shows that for $u < 0$, $f'(u)$ is always positive and, as u increases from $-\infty$ to 0, $f(u)$ increases from 0 to ∞.

Combining results, we see that the frequency curve resembles the graph of $y = 1/x^2$.

All moments of the distribution are infinite.

Thanks of the author are hereby expressed to Dr. Robert S. DeZur of the Martin Company, Denver Division, for valuable suggestions during the preparation of this note.

Reference

1. D. Bierens de Haan, Nouvelles Tables d'Intégrales Définies, Stechert Hafner, New York, 1929.

ANOTHER PROOF OF THE INFINITE PRIMES THEOREM

M. Wunderlich, University of Colorado
(Now at the State University of New York at Buffalo)

Let F_n be the nth Fibonacci number. It is a well-known and easily proved result [1] that

$$F_{(m,n)} = (F_m, F_n), \tag{1}$$

where (m, n) as usual denotes the greatest common divisor. This property yields another proof of the infinite prime theorem.

Theorem. *There are infinitely many primes.*

Proof. Suppose $p_1, p_2, \cdots, p_k$ are all the prime numbers. Then consider

$$F_{p_1}, F_{p_2}, \cdots, F_{p_k}. \tag{2}$$

From (1), the numbers in (2) are pairwise relatively prime, and since there are only k primes, each of the numbers in (2) has only one prime factor. But this contradicts the fact that

$$F_{19} = 4181 = 113 \cdot 37.$$

Reference

1. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell, New York, 1961, p. 30.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (other than proposers') should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to A. E. Livingston, Dept. of Math. University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before July 31, 1965.

E 1766. *Proposed by Stanton Philipp, Long Beach, California*

For positive integers n and x define $N_n(x)$ to be the number of positive integers d such that (1) d divides x and (2) $x \leqq d^2 \leqq n^2$. Prove that as $n \to \infty$, $n^{-2} \sum_{x=1}^{n^2} N_n(x)$ approaches $\frac{1}{2}$.

E 1767. *Proposed by H. W. Gould, West Virginia University*

If μ is the Möbius function and ϕ is Euler's totient function, show that $\sum_{d|n} \mu(d)\phi(d) = 0$, if and only if n is even.

E 1768. *Proposed by Jean M. Quoniam, Saint-Etienne, France*

In the triangle ABC let D, E, F be the points of contact of the sides with the incircle of radius r. Let H be the orthocenter of the triangle DEF, s the area of DEF, and I its incenter. Prove that

$$\sum \cos A = 3/2 - \overline{IH}^2/2r^2, \qquad \sum \sin A = 2s/r^2.$$

E 1769. *Proposed by I. J. Schoenberg, University of Pennsylvania*

Let $F_n(x)$ be the cyclotomic polynomial, i.e. the monic polynomial whose zeros are the primitive nth roots of unity. Show that all coefficients of $F_n(x)$ are nonnegative if and only if n is a power of a prime.

E 1770. *Proposed by G. P. Graham and T. A. Porsching, Carnegie Institute of Technology*

Let C be a planar simple closed curve enclosing a region R which is starlike in point P. Then a necessary and sufficient condition that C be radially symmetric in P is that every line through P divides R into subregions of equal area. Also, every line dividing R into two equal areas passes through P.

E 1771. *Proposed by Andy Vince, Stanford University*

Given a set of positive integers $(a_1, a_2, \cdots, a_n)$, it is always possible to choose a subset of these integers such that the sum of its elements is divisible by n.

E 1772. *Proposed by Henry Cox, David Taylor Model Basin, Washington, D. C.*

Let A and B be arbitrary matrices of dimensions $m\times n$ and $n\times m$ respectively. Does the existence of $[I_m+AB]^{-1}$ imply the existence of $[I_n+BA]^{-1}$, where I_m and I_n are the identity matrices of dimension m and n respectively? If so, determine the latter in terms of the former. If not give a counterexample.

E 1773. *Proposed by Michael Lieber, AVCO Corporation and Harvard University*

Let T be a triangle whose sides are consecutive integers and whose area is an integer. Prove that one of the altitudes divides the triangle into two Pythagorean right triangles. Furthermore, this altitude divides the base into segments whose lengths differ by 4. (The 1-2-3 "triangle" and the 3-4-5 right triangle are the only degenerate cases.)

E 1774. *Proposed by Wayne E. Smith, University of California, Los Angeles*

Let $0<a<b$. Prove that $\log(ax+1)/\log(bx+1)$ is a monotonic increasing function for all $x>0$.

E 1775. *Proposed by George Purdy, University of Reading, England*

Under what conditions do real $x_1, \cdots, x_n$ satisfy the equation $x_1^2+\cdots+x_n^2=(x_1+\cdots+x_n)^2/n$ for $n\geqq 1$?

SOLUTIONS OF ELEMENTARY PROBLEMS

E 1648 [1964, 919]. *Correction*

In the description of ring 8 the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & 0 & 0 \end{bmatrix} \quad \textit{should read} \quad \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & x & 0 \end{bmatrix}.$$

I receive you loud and clear

E 1681 [1964, 429]. *Proposed by Azriel Rosenfeld, The Budd Company, Silver Spring, Maryland*

Find the unique solution to the cryptarithm

$$
\begin{array}{r}
T\ H\ I\ S \\
I\ S\ A \\
G\ R\ E\ A\ T \\
T\ I\ M\ E \\
\hline
W\ A\ S\ T\ E\ R
\end{array}
$$

Solution by Stanton Philipp, Seal Beach, California. A solution is

$$
\begin{array}{r}
5\ 6\ 2\ 8 \\
2\ 8\ 0 \\
9\ 7\ 4\ 0\ 5 \\
5\ 2\ 3\ 4 \\
\hline
1\ 0\ 8\ 5\ 4\ 7
\end{array}
$$

That this is the only solution may be demonstrated as follows: It is clear that T, I, G, and W are not 0 and then that $W=1$ and $G=8$ or 9. But A must be 0 or 1, and $A\neq 1$ since $W=1$. We may therefore write

$$
\begin{aligned}
S+T+E &= R+10\alpha \\
\alpha+I+S+M &= E+10\beta \\
\beta+H+E+2I &= T+10\gamma \\
\gamma+R+2T &= S+10\delta \\
\delta+G &= 10
\end{aligned}
$$

where $0\leqq\alpha\leqq 2$, $0\leqq\beta\leqq 2$, $0\leqq\gamma\leqq 3$, and $1\leqq\delta\leqq 2$. It follows that

$$3T+E\leqq 10(\alpha+\delta)-\gamma$$

a central fact in our argument.

1. $\alpha=0\Rightarrow\{S, T, E\}=\{2, 3, 4\}\Rightarrow R=9\Rightarrow G=8\Rightarrow\delta=2\Rightarrow 3T+E=20-\gamma\geqq 17$; but $\{S, T, E\}=\{2, 3, 4\}\Rightarrow 3T+E\leqq 15$. Thus, $\alpha\neq 0$. $\alpha=2\Rightarrow S+T+E\geqq 22$ $\Rightarrow\{S, T, E\}=\{6, 7, 9\}\Rightarrow G=8\Rightarrow\delta=2\Rightarrow 3T+E=40-\gamma\geqq 37$; but $\{S, T, E\}$ $=\{6, 7, 9\}\Rightarrow 3T+E\leqq 34$. Thus, $\alpha\neq 2$. Hence $\alpha=1$ and then $\beta\neq 0$.

2. $\beta=2\Rightarrow I+S+M\geqq 21\Rightarrow\{I, S, M\}=\{5, 7, 9\}$ or $\{6, 7, 8\}$ or $\{6, 7, 9\}$. Since $\{I, S, M\}=\{5, 7, 9\}$ or $\{6, 7, 9\}\Rightarrow E=2$ or $3\Rightarrow T=9$, we must have $\{I, S, M\}=\{6, 7, 8\}$. But then $E=2$ and $G=9$, $\gamma=3$ and $T=5$, and $H+2I=31$, an impossibility. Consequently, $\beta=1$.

3. Since $\beta+H+E+2I\geqq 12$, $\gamma\neq 0$. $\gamma=3\Rightarrow H+E+2I\geqq 32\Rightarrow I=9$, $\{H, E\}$ $=\{6, 7\}$, and $G=8\Rightarrow T=2\Rightarrow 3+4+R\geqq 22$, an impossibility. Thus, $\gamma=1$ or 2.

4. $\delta=2\Rightarrow G=8\Rightarrow T=9$, $E=2$, and $\gamma=1\Rightarrow I+S+M=11$, contrary to $\{I, S, M\}\subset\{3, 4, 5, 6, 7\}$. Hence, $\delta=1$ and $G=8$.

5. $\gamma=2\Rightarrow T=5$ and $E=3$ $(S+T+E=R+10$ and $2+R+2T=S+10\Rightarrow 3T+E=18!)\Rightarrow H+2I=21\Rightarrow H=9$ and $I=6\Rightarrow S+M=6\Rightarrow\{S, M\}=\{2, 4\}\Rightarrow R=0$ $(S=2\Rightarrow 2+R+10=2+10!)$ or $R=2$ $(S=4\Rightarrow 2+R+10=4+10!)$, an impossibility. Therefore, $\gamma=1$.

We now have

$$\begin{aligned} S+T+E &= R+10 \\ I+S+M &= E+9 \\ H+E+2I &= T+9 \\ R+2T &= S+9 \end{aligned}$$

The first and fourth of these equations give $3T+E=19$ and, hence, $(T, E)=(5, 4)$ or $(4, 7)$. But $(T, E)=(4, 7)$ leads to $H+2I=6$, which is impossible. It is then a simple matter to deduce that $H=6$, $I=2$, $S=8$, $R=7$, and $M=3$.

Also solved by A. N. Aheart, Shair Ahmad, Joseph Arkin, Thelma Balbes, Merrill Barnebey, Philip Bellman, Jeanette Bickley, Z. E. Birnbaum and U. J. Schild (jointly), R. L. Breisch, Maxey Brooke, S. I. Brown, H. W. Buck, R. A. Carlson, Allan Chuck and Peter Goldstein (jointly), W. C. Daugherity, Barbara H. Desind, C. L. Dotton, David Eliezer and Kenneth Eliezer (jointly), P. G. Engstrom, David Finkel, Stephen Gehrett, Murray Geller, Anton Glaser, R. M. Grassl, S. H. Greene, Arthur Greenspoon, D. M. Hancasky, M. C. Henry, R. R. Herbert and L. J. Schneider (jointly), Stephen Hoffman, Steve Imbeau, R. A. Jacobson, J. E. Jeans, Jr., Milton Johnson, Roman Kaluzniacki, J. V. Kamer, Edgar Karst, J. D. E. Konhauser, Mary Ann Konsin, R. R. Korfhage, L. C. Larson, Murray Lieb, Carl Lindh, D. K. McCarley, D. C. B. Marsh, J. L. Matucha, Franklin Mohr, Gregory Moore, Sam Newman, Wahin Ng, Wayne Pfeiffer, J. B. Phillips, W. E. Philpott, Donald Quiring, S. G. Roth, Mark Samuel, P. A. Scheinok, J. F. Schreckengost, W. R. Scott, Delores A. Shaw, Patricia Shaw, J. R. Silva, D. R. Simpson, F. M. Stein, Terry Stephenson, W. B. Stovall, Jr., Jillian Strang, Don Stuber, J. F. Sullivan, Elias Toubassi and John Lee Lebbert (jointly), Simon Vatriquant, Hazel S. Wilson, and Stephen Wilson. (Thirty-nine of these solutions did not consider the uniqueness question.)

A Hilbert Inequality

E 1682 [1964, 429]. *Proposed by V. R. Rao Uppuluri, Oak Ridge National Laboratory*

Show that

$$\left(\sum_{i=1}^{n} a_i/i\right)^2 \leqq \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j/(i+j-1).$$

I. *Solution by J. Williams, Institute of Science and Technology, University of Michigan.* The Schwarz inequality yields the more general result: if $a_1, a_2, \cdots, a_n$ are complex numbers, then

$$\begin{aligned} \left|\sum_{i=1}^{n} \frac{a_i}{i}\right|^2 &= \left|\int_0^1 \left(\sum_{i=1}^{n} a_i x^{i-1}\right) dx\right|^2 \leqq \int_0^1 \left|\sum_{i=1}^{n} a_i x^{i-1}\right|^2 dx \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i \bar{a}_j/(i+j-1) \end{aligned}$$

with equality if and only if $\sum_{i=1}^n a_i x^{i-1} \equiv$constant (i.e., if and only if $a_2 = a_3 = \cdots = a_n = 0$).

II. *Solution by P. A. Scheinok, Hahneman Medical College and Hospital of Philadelphia.* For complex $a_1, a_2, \cdots, a_n$, we have

$$\sum_{i=1}^n \sum_{j=1}^n a_i \bar{a}_j/(i+j-1) - \left| \sum_{i=1}^n a_i/i \right|^2 = \sum_{i=1}^n \sum_{j=1}^n b_i \bar{b}_j/(i+j-1)$$

with $b_i = (i-1)a_i/i$. With $B = (b_1 b_2 \cdots b_n)$, this becomes $BH\bar{B}^T$, where H is the n by n Hilbert matrix $\|1/(i+j-1)\|$. But the Hilbert matrix is known to be positive-definite, so that $BH\bar{B}^T \geqq 0$ with equality if and only if $b_1 = b_2 = \cdots = b_n = 0$. We therefore obtain the inequality of the problem with equality if and only if $a_2 = a_3 = \cdots = a_n = 0$.

Editor's comment: Since

$$\sum_{i=1}^n \sum_{j=1}^n b_i b_j/(i+j-1) = \int_0^1 \left(\sum_{i=1}^n \sum_{j=1}^n b_i \bar{b}_j x^{i+j-2} \right) dx = \int_0^1 \left| \sum_{i=1}^n b_i x^{i-1} \right|^2 dx \geqq 0$$

with equality if and only if $b_1 = b_2 = \cdots = b_n = 0$, knowledge of the positive-definiteness of H is not required.

III. *Solution by William J. Hartman, National Bureau of Standards.* Writing

$$f(x) = x^{-1}\left(\sum_{i=1}^n a_i x^i/i \right)^2 - \sum_{i=1}^n \sum_{j=1}^n a_i a_j/(i+j-1),$$

we see that $f(0+) = 0$ and $f'(x) = -\left[\sum_{i=1}^n (i-1)a_i x^{i-1}/i\right]^2 \leqq 0$ for $x > 0$ with equality if and only if $a_2 = a_3 = \cdots = a_n = 0$. Thus, $f(1) \leqq f(0+)$ with equality if and only if $a_2 = a_3 = \cdots = a_n = 0$.

Also solved by S. H. Greene, J. H. Holton, R. F. Jackson, E. S. Langford, J. Lehner, D. C. B. Marsh, Henry Ricardo, M. S. R. K. Sastry, and the proposer.

There were two solutions to this problem under the additional assumption that each $a_i \geqq 0$, and 27 purported solutions in which the submitters inferred from $a \leqq b$ that $ac \leqq bc$.

Sum of Ratios of Sines

E 1683 [1964, 429]. *Proposed by Eugène Ehrhart, Strasbourg, France*

Let $\alpha = \pi/7$. Show that

$$\sum_{n=1}^6 \sin n\alpha/\sin 5n\alpha = 4, \qquad \sum_{n=1}^6 \sin 5n\alpha/\sin n\alpha = 2.$$

I. *Solution by N. J. Fine, Pennsylvania State University.* Let k be a positive integer, $0 < p < 2k$, $0 < q < 2k$, p odd, $(q, 2k) = 1$. We shall consider the sum

$$S_k(p, q) = \sum_{n=1}^{k-1} \sin (np\pi/k)/\sin (nq\pi/k).$$

Let $w=\exp(\pi i/k)$. It is easy to see that

$$\sum_{n=0}^{k-1} w^{2jn} = \begin{cases} 0 & (j \not\equiv 0 \pmod{k}), \\ k & (j \equiv 0 \pmod{k}). \end{cases}$$

Since $(q, 2k)=1$, there is a positive integer m and an integer r, $0 \leqq r \leqq q-1$, such that $p+2kr=mq$. Because p and q are both odd, $m=2\mu+1$. Also $(2\mu+1)q = mq = p+2kr < 2k+2k(q-1) = 2kq$, so that $2\mu+1<2k$, and $\mu<k$. Now

$$S_k(p, q) = \sum_{n=1}^{k-1} \frac{w^{pn} - w^{-pn}}{w^{qn} - w^{-qn}} = \sum_{n=1}^{k-1} \frac{w^{(p+2kr)n} - w^{-(p+2kr)n}}{w^{qn} - w^{-qn}} = \sum_{n=1}^{k-1} \frac{w^{(2\mu+1)qn} - w^{-(2\mu+1)qn}}{w^{qn} - w^{-qn}}$$

$$= \sum_{n=1}^{k-1} \sum_{j=-\mu}^{\mu} w^{2jqn} = -(2\mu+1) + \sum_{n=0}^{k-1} \sum_{j=-\mu}^{\mu} w^{2jqn} = -m + \sum_{j=-\mu}^{\mu} \sum_{n=0}^{k-1} w^{2jqn}.$$

The only value of j between $-\mu$ and μ for which $jq \equiv 0 \pmod{k}$ is $j=0$, since $(q, k)=1$ and $-k<-\mu \leqq j \leqq \mu < k$. Hence $S_k(p, q)=k-m$.

For the special case of this problem, we have $S_7(1, 5)=4$ since $1+14\cdot 1=3\cdot 5$, and $S_7(5, 1)=2$ since $5+14\cdot 0=5\cdot 1$.

II. *Solution by Stephen Hoffman, Trinity College, Hartford, Connecticut.* For $(k, v)=1$, let

$$T_k(u, v) = \sum_{n=1}^{k-1} \sin(nuv\pi/k)/\sin(nv\pi/k).$$

Since $\sin(ux)/\sin x = 1+2\sum_{j=1}^{w} \cos 2jx$ for $u=2w+1$ an odd positive integer, we have

$$\begin{aligned} T_k(u, v) &= \sum_{n=1}^{k-1} \sin(nu\alpha)/\sin n\alpha \\ &= k-1+2\sum_{j=1}^{w}\sum_{n=1}^{k-1} \cos 2jn\alpha \\ &= k-1-w+\sum_{j=1}^{w} \sin(2k-1)j\alpha/\sin j\alpha, \end{aligned}$$

where $\alpha = v\pi/k$. Now $(2k-1)j\alpha + j\alpha = 2kj\alpha$, so that $\sin(2k-1)j\alpha = -\sin j\alpha$, and $T_k(u, v) = k-1-2w = k-u$. For the first sum given, $\sin n\pi/7 = \sin 15n\pi/7$ and we have $\alpha = 5\pi/7$, $u=3$, $v=5$, and $k=7$, so that the sum in question has the value $T_7(3, 5)=4$. For the second sum, $\alpha=\pi/7$, $u=5$, $v=1$, and $k=7$, and we get $T_7(5, 1)=2$.

Editor's comment: With S_k and T_k as in the solutions above, it is clear that $S_k(0, q) = T_k(0, v) = 0$, both S_k and T_k have period $2k$ in their first argument as well as in their second, and each is defined when and only when the second argument is prime to k. It is also apparent that

$$T_k(u, v) = S_k(uv, v).$$

In view of these observations, we may first of all rephrase Hoffman's result above as follows: *If u is odd, then $T_k(u, v) = k - u^*$, where u^* is the least positive residue of u modulo $2k$.* Next, the change of summing index $n \to k - n$ shows that $T_k(u, v) = 0$ if u is even and v is odd, and it is easy to see that if $(q, k) = 1$ and $(q, 2k) > 1$, then $(q+k, 2k) = 1$. Since also

$$S_k(p + k, q + k) = S_k(p, q),$$

we may now combine Fine's and Hoffman's results to obtain the

THEOREM. *Let k be a positive integer greater than 1 and q an integer for which $(q, k) = 1$, and define*

$$S_k(p, q) = \sum_{n=1}^{k-1} \sin (np\pi/k)/\sin (sq\pi/k),$$

for each integer p. (i) *If p is even and $(q, 2k) = 1$ or if p is odd and $(q, 2k) > 1$, then $S_k(p, q) = 0$.* (ii) *If p is odd and $(q, 2k) = 1$, then $S_k(p, q) = k - (pq')^*$, where r' is any inverse of r modulo $2k$ (and u^* is the least positive residue of u modulo $2k$).* (iii) *If p is even and $(q, 2k) > 1$, then*

$$S_k(p, q) = k - [(p + k)(q + k)']^*.$$

Also solved by J. C. Abad, A. N. Aheart, Jim Campbell, D. I. A. Cohen, Huseyn Demir, M. S. Demos, G. C. Dodds, Michael Goldberg, K. K. Gorowara, S. H. Greene, J. H. Halton, Eldon Hansen, Ned Harrell, R. F. Jackson, R. A. Jacobson, A. J. Keeping, Lew Kowarski, Kenneth Kramer and Steven Minsker (jointly), W. R. McEwen, D. C. B. Marsh, Stanton Philipp, Donald Quiring, J. M. Quoniam, Simeon Reich, M. S. R. K. Sastry, U. J. Schild, P. A. Scheinok, Elias Toubassi, Simon Vatriquant, and the proposer.

Hansen found the formula

$$\sum_{n=0}^{k-1} \sin\left[(2m+1)\left(x + \frac{np\pi}{k}\right)\right] \Big/ \sin\left(x + \frac{np\pi}{k}\right) = k,$$

where $(p, k) = 1$ and $m = 0, 1, 2, \cdots, n-1$, as Exercise 18, p. 272, in T. J. Bromwich, *Introduction to the Theory of Infinite Series*, 2nd rev. ed., Macmillan (1949). Taking the limit as $x \to 0+$ leads to Hoffman's result $T_k(2m+1, p) = k - 2m - 1$.

Characteristic Values of $\|a_{ij}\| = \|a_{ik}a_{kj}\|$

E 1685 [1964, 430]. *Proposed by F. D. Parker, State University of New York at Buffalo*

A square matrix M of order n has the properties that $a_{ii} = 1$ and $a_{ik}a_{kj} = a_{ij}$ for all i, j, k. What are the characteristic values of M?

I. *Solution by John A. Burslem, St. Louis University.* In the determinant for the characteristic polynomial $C(x)$, subtract the $(j+1)$-st column from $a_{1,j+1}/a_{ij}$ times the jth $(j = 1, 2, \cdots, n-1)$ and add $a_{1,i+1}/a_{ij}$ times the ith row of the result to the new $(i+1)$-st row to get $C(x) = a_{1n} \det (b_{ij})$, where $b_{ii} = -a_{1,i+1}x/a_{1i}$ $(i = 1, 2, \cdots, n-1)$, $b_{nn} = n - x$, and all other b_{ij} are 0 except those in the nth column. Thus, $C(x) = (-1)^{n-1}(n-x)x^{n-1}$, so that 0 is an $(n-1)$-fold characteristic value and n a simple one.

II. *Solution by Roman Kaluzniacki, University of Santa Clara, California.* The ith row is the first row multiplied by the reciprocal of its ith element. Since $1 = a_{ii} = a_{ij}a_{ji}$ for all i and j, $a_{ij} \neq 0$. Hence the rank of M is one, and the values

of all minors of order greater than one are zero. In the characteristic polynomial $f(\lambda)$ of M, the coefficient c_r of λ^r is $(-1)^r$ times the sum of the principal minors of order $n-r$. Now $c_n=(-1)^n$, c_{n-1} is the trace and equals n; for $r<n-1$, $c_r=0$. Hence $f(\lambda)=(-1)^n[\lambda^n-n\lambda^{n-1}]$ has 0 and n as zeros (0 of multiplicity $n-1$).

III. *Solution by Theodore R. Smith, La Sierra, California.* Let the first row of M be $r=(1, a_{12}, a_{13}, \cdots, a_{1n})$. The properties of M then imply row j to be of the form $(1/a_{1j})r$. With this as the matrix M, substitution shows that $(1, 1/a_{12}, 1/a_{13}, \cdots, 1/a_{1n})$ is an eigenvector with eigenvalue n. Likewise, the $n-1$ vectors

$$v_k = a_{1n}e_k - a_{1k}e_n, \qquad (k = 1, 2, 3, \cdots, n-1),$$

where e_k is an element of the standard basis for E_n, are seen to be linearly independent eigenvectors with eigenvalue 0.

IV. *Solution by Gary B. Klatt and Aaron Strauss, University of Wisconsin.* Let a_{ii} $(i=1, 2, \cdots, n)$ be arbitrary and suppose that $a_{ik}a_{kj}=a_{kk}a_{ij}$ for all i, j, k. Then $M^2=(\text{trace } M)\, M$, and it follows that the characteristic roots of M satisfy the equation x^2-x trace $M=0$. Thus, the characteristic roots are 0 and trace M. Since the sum of the characteristic roots must be trace M, 0 is of multiplicity $n-1$ if trace $M\neq 0$ (which is certainly the case for the problem proposed).

Also solved by R. D. Ahrend, S. Baron, Randy Barron, Joel Brawley, Jr., Leonard Carlitz, David Carlson, Charles Chouteau, D. I. A. Cohen, D. E. Crabtree, C. G. Cullan, Frank Dapkus, Emmeline G. de Pillis, N. J. Fine, J. A. Fuchs, Seymour Geisser, R. W. Gilmer, Jr., S. H. Greene, Eldon Hansen, J. R. Hatcher, J. C. Hickman, Lee Hill, R. H. Hines, Yasuhiko Ikebe, R. F. Jackson, J. P. Jordan, A. J. Keeping, P. G. Kirmser, G. B. Klatt and Aaron Strauss (second joint solution), C. D. LaBudde, E. S. Langford, G. M. Leibowitz, C. R. MacCluer, D. C. B. Marsh, J. J. Moore, M. G. Murdeshwar, Stanton Philipp, T. J. Rao, V. K. Rohatgi, P. A. Scheinok, U. J. Schild, R. E. Showalter, Robert Singleton, Simon Vatriquant, J. E. Wilkins, Jr., J. Williams, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before September 30, 1965.

5270. *Proposed by Michael Fried, Bell Aerosystems, Wheatfield, N. Y.*

Consider decimal expansions of the form $x=.x_0x_1x_2\cdots$, where $0\leqq x_i\leqq 9$, and suppose that there exists some ordered pair of positive integers (j, k) such that $x_{j+lk}=x_{j+(l+1)k}$, $l=0, 1, 2, \cdots$. Call the set of such expansions I. Is $S=[0, 1]-I$ countable?

5271. *Proposed by A. J. Goldman, National Bureau of Standards*

A groupoid is a nonempty set closed under a binary operation written multiplicatively. The direct product $G=G_1\times G_2$ of two groupoids consists of the Cartesian product of their sets together with component-wise multiplication.

Suppose G admits such a factorization in which G_1 has the property that, if $x, y \in G_1$ then $xy = y$, while G_2 has a two-sided identity. Let $L(G)$ consist of the left identities of G. Prove that $G_1 \cong L(G)$ and that $G_2 \cong Ge$ for some $e \in L(G)$.

5272. *Proposed by S. M. Robinson, Union College*

A topological space X is called a door space if for each subset $A \subseteq X$, either A or $X - A$ is open. We shall call a space X a semi-door space if for each subset $A \subseteq X$, there is an open set O with the property that either $O \subseteq A \subseteq \text{cl}[O]$ or $O \subseteq X - A \subseteq \text{cl}[O]$. Find an example of a semi-door space which is not a door space. Also, prove or disprove the following statement: If X is both a Hausdorff space and a semi-door space, then X is a door space.

5273. *Proposed by T. J. Head, Iowa State University*

Show that for a commutative integral domain R to have the property that each of its finitely generated torsionfree modules is free it is necessary and sufficient that each finitely generated ideal of R be principal.

5274. *Proposed by J. J. Malone, Jr., University of Houston*

Let $(G, +)$ be a finite group of order g such that if $(N, +)$ is a proper normal subgroup of $(G, +)$ of order n there exists no subgroup $(K, +)$ of $(G, +)$ of order k such that $K \cap N = \{0\}$ and $kn = g$. An example of such a group is the cyclic group of order 4. What is the characterization of such groups? In particular, if $(G, +)$ is not cyclic, is $(G, +)$ simple?

5275. *Proposed by Hewitt Kenyon, George Washington University*

Using the terminology of Problem 5148 [1964, 1054], suppose that f and g are functions mapping the set X onto the sets Y and Z, respectively; suppose that H is an ultrafilter eventually in X and let F and G be the ultrafilters consisting of maps of f and g, respectively, of subsets of X belonging to H. If F is equivalent to G, is it necessarily true that H is eventually in $\{t \mid f(t) = g(t)\}$?

5276. *Proposed by Leonard Carlitz, Duke University*

The following formula is a consequence of known results when a is a power of a prime and e is not a divisor of $a^t - 1$ for $0 < t < k$:

$$\sum_{cd=k} \mu(c) a^d = \sum_{e \mid a^k - 1}{}' \phi(e) \qquad (k > 1).$$

(See L. E. Dickson, *Linear Groups*, p. 20.) Prove that the result holds for all $a > 1$, $k > 1$.

5277. *Proposed by Kwangil Koh, University of North Carolina*

It is known that in a Noetherian ring, every irreducible ideal is primary (see, e.g., N. H. McCoy, *Rings and Ideals*, Carus Monograph No. 8, p. 200). Let R

be a ring (not necessarily commutative) which satisfies the ascending chain condition on right ideals. Prove that if S is a two-sided ideal in R such that it is irreducible, in the sense that if J_i, $i=1, 2$, is a pair of right ideals such that $J_i \supset S$ then $J_1 \cap J_2 \supset S$, then each divisor of zero in the ring R/S is nilpotent (where $\supset$ means "contains strictly").

5278. *Proposed by Eric Nix, Norwich, Vermont*

Given a convex metric space S, does S as a topological space necessarily also admit a bounded convex metric?

5279. *Proposed by S. M. Robinson, Union College*

Let X be a completely regular space, X^* its one-point compactification, and $C[X]$ its ring of continuous real-valued functions. For each compact subset F of X, consider the ideal $0_F = \{f \in C[X]: f$ is constantly zero in the neighborhood of $F\}$. Show that X^* is metrizable if and only if there exists a sequence $\{f_n: n=1, 2, \cdots\}$ contained in $C[X]$, such that each ideal 0_F is generated by a subset of $\{f_n\}$ and each function f_n belongs to no free ideal of $C[X]$. (An ideal I in $C[X]$ is free if there exists no point $x \in X$ with the property that $f(x)=0$ for all $f \in I$.)

SOLUTIONS OF ADVANCED PROBLEMS

Idempotent Element

3133 [1925, 261; 1962, 237]. *Proposed by A. A. Bennett*

(Rephrased.) Prove that every finite semigroup contains an idempotent element.

II. *Solution by William E. Coppage, Dayton Campus of Miami University and Ohio State University.* The result is obvious for semigroups of order 1, so we let S be a semigroup of order $n>1$ and proceed by induction on n. Let $x \in S$. The semigroup

$$A = \{x^k \mid k = 2, 3, \cdots\}$$

either does not contain x and hence, by the inductive hypothesis, contains an idempotent element, or else $x=x^k$ for some $k \geqq 2$. If $k=2$, then x is idempotent. If $k>2$, then

$$x^{k-1}x^{k-1} = x^{2k-2} = x^k x^{k-2} = x\, x^{k-2} = x^{k-1}$$

and x^{k-1} is idempotent.

Five Circles Inclined at Same Angle

5142 [1963, 1013]. *Proposed by J. G. Mauldon, Oxford University*

Do there exist five coplanar oriented circles, not necessarily real, mutually inclined at the same angle?

Solution by the proposer. Yes. The angles must be arc $\cos(-\frac{1}{4})$. Let O be the midpoint of a side of a square inscribed in a circle S. Let T be a spiral similarity with center O, ratio $e^{2i\alpha}$ and angle α, where $\alpha=2\pi/5$. Then the five circles T^mS ($m=0, 1, 2, 3, 4$) have the required property.

The problem may be generalized to elliptic and hyperbolic space. See J. G. Mauldon, *Sets of equally inclined spheres*, Canad. J. Math., 14 (1962).

Hermitian-like Decompositions of Square Matrices

5179 [1964, 217]. *Proposed by David Carlson, Oregon State University*

It is well known that, for any square matrix A with complex elements, there is a unique decomposition $A=B+C$, where $B=(A+A^*)/2$ is Hermitian (and has all roots along the real axis) and $C=(A-A^*)/2$ is skew-Hermitian (and has all roots along the imaginary axis). Given any two distinct lines through the origin in the complex plane, prove an analogous result for a unique decomposition of A into two "Hermitian-like" matrices, each with roots along one of the two lines.

Solution by Hwa S. Hahn, Pennsylvania State University. We use the fact that if $c_1, \cdots, c_n$ are the characteristic roots of an $n\times n$ matrix A, then $uc_1, \cdots, uc_n$ are the characteristic roots of uA. For $\det(uA-ucI)=u^n\det(A-cI)$.

Take two nonzero complex numbers u and v lying on two given lines separately. For any square matrix A now it is sufficient to find matrices B which is Hermitian and C which is skew-Hermitian satisfying

$$A = uB - ivC. \tag{1}$$

Then it is easily seen that uB has all its characteristic roots on the line where u lies and $-ivC$ has all its characteristic roots on the line where v lies. Since $B^*=B$ and $-C^*=C$, we have from (1)

$$A^* = \bar{u}B + i\bar{v}C. \tag{2}$$

From (1) and (2) we obtain $B=(\bar{v}A-vA^*)/(u\bar{v}-\bar{u}v)$ and $C=(\bar{u}A-uA^*)/(u\bar{v}-\bar{u}v)$ where $u\bar{v}-\bar{u}v\neq 0$ because $u\neq 0$ and $v\neq 0$ lie on two distinct lines through the origin.

Therefore we have a desired decomposition which is clearly unique for fixed u and v

$$A = u\,\frac{\bar{v}A - vA^*}{u\bar{v}-\bar{u}v} - iv\,\frac{\bar{u}A - uA^*}{u\bar{v}-\bar{u}v}.$$

Notice that a particular case when $u=1$ and $v=i$ in this formula is the case stated in the problem.

Also solved by D. Ž. Djoković (Yugoslavia), A. S. Householder, C. Donald La Budde, J. G. Mauldon (England), M. G. Murdeshwar, R. F. Rinehart, and the proposer.

Abel Limit of Nearly Periodic Sequences

5180 [1964, 217]. *Proposed by A. E. Livingston, University of Alberta*

Find $\lim_{x\to 1-0}(1-x)\sum_{n=0}^{\infty} a_n x^n$ if each set $\{a_{kN}, a_{kN+1}, \cdots, a_{(k+1)N-1}\}$ is a permutation of the set $\{a_0, a_1, \cdots, a_{N-1}\}$ for $k=1, 2, \cdots$, where N is a fixed positive integer.

Solution by J. Boersma, Mathematical Institute, Groningen, Netherlands. We consider the polynomials

$$P(x) = \sum_{j=0}^{N-1} a_j x^j, \qquad P^*(x) = \sum_{j=0}^{N-1} a_j^* x^j,$$

where the set of coefficients $\{a_0^*, a_1^*, \cdots, a_{N-1}^*\}$ will be a permutation of the set of coefficients $\{a_0, a_1, \cdots, a_{N-1}\}$. Because $P(1)=P^*(1)$, corresponding to each $\epsilon>0$, a number $\delta>0$ may be found, such that

$$|P(x) - P^*(x)| < \epsilon, \quad \text{for} \quad 1-\delta < x \leqq 1.$$

Because the number of permutations is finite, δ can be chosen in such a way that the inequality holds for every permutation-polynomial $P^*(x)$. Now we have

$$(1-x)\sum_{n=0}^{\infty} a_n x^n = (1-x)P(x)\sum_{k=0}^{\infty} x^{kN} + R(x) = \frac{1-x}{1-x^N}P(x) + R(x),$$

with

$$|R(x)| < \epsilon(1-x)\sum_{k=0}^{\infty} x^{kN} = \epsilon\frac{1-x}{1-x^N} \leqq \epsilon, \qquad 1-\delta < x < 1.$$

Hence we have

$$\lim_{x\to 1-0}(1-x)\sum_{n=0}^{\infty} a_n x^n = \lim_{x\to 1-0}\frac{1-x}{1-x^N}P(x) = \frac{1}{N}\sum_{j=0}^{N-1} a_j.$$

Also solved by Robert Breusch, H. E. Chrestenson, M. J. Cohen, Robert Cohen, D. Ž. Djoković (Yugoslavia), J. E. H. Elliott, N. J. Fine, R. L. Forbes, Ralph Greenberg, H. S. Hahn, E. S. Langford, Lee Lorch, J. G. Mauldon (England), M. D. Mavinkurve (India), D. E. Myers, Stanton Philipp, Dennis C. Russell (England), Richard Sinkhorn, N. R. Wagner, L. J. Wallen, W. C. Waterhouse, J. E. Wetzel, Oswald Wyler, and the proposer.

Several solvers noted that the given sequence was $(C, 1)$ summable because of its periodic nature and that the Abel limit is an immediate consequence. We observe, in addition, that $(C, 1)$ summability follows from the Abel limit proved above by noting the boundedness of the sequence and applying a well-known Tauberian theorem.

Derivatives of Functions Discontinuous on the Rationals

5181 [1964, 325]. *Proposed by E. J. Burr, University of New England, Australia*

Let $f(x)=0$ for x irrational, $f(0)=\epsilon_1>0$, and $f(m/n)=\epsilon_n>0$ for m, n coprime integers with $n>0$. Find, or disprove the existence of, sequences $\{\epsilon_n\}$ with

$\lim \epsilon_n = 0$ such that (a) $f'(x)$ exists nowhere, (b) $f'(x)$ exists for some x, (c) $f'(x)$ exists for all irrational x.

Solution by M. D. Mavinkurve, Siddharth College, Bombay, India. For every sequence $\{\epsilon_n\}$ of the stated kind, the function f will be continuous at irrational points and discontinuous at rational points. f' can therefore exist only at irrational points and, where it exists, can have only the value zero.

(a) If we take $\epsilon_n = 1/n$, the derivative will not exist at any point. For if ξ be any irrational, $|\xi - m/n| < 1/n^2\sqrt{5}$ eventually by Hurwitz's theorem, and $|f(\xi) - f(m/n)| / |\xi - m/n| > n\sqrt{5}$.

(b) If we take $\epsilon_n = 1/n^4$, the derivative exists at all quadratic irrationalities ξ. For then $|\xi - m/n| > 1/n^3$ eventually and $|f(\xi) - f(m/n)| / |\xi - m/n| < 1/n \to 0$.

(c) No sequence $\{\epsilon_n\}$ of the stated kind can ensure the existence of f' at all irrational points. For by a theorem due to M. K. Fort, the points of continuity of a real function (on any interval) which itself is discontinuous on a dense subset, form a set of the first category, and the set of irrationals is not of the first category. For if it were, together with the countable set of rational numbers, the complete real line would be of the first category.

Also solved by Eugene Allgower, J. O. Herzog, G. A. Heuer, Solomon Marcus (Rumania), J. G. Mauldon (England), I. J. Schoenberg, W. C. Waterhouse, and the proposer.

The substance of this problem has occurred in a variety of papers. The earliest reference (furnished by Waterhouse) is to W. D. A. Westfall, a *Class of Functions Having a Peculiar Discontinuity*, Bull. A.M.S., 15 (1909). Other references containing solutions to this problem include:

M. K. Fort, Jr., *A theorem concerning functions discontinuous on a dense set*, this MONTHLY, 58 (1951).

I. J. Schoenberg, *The integrability of certain functions and related summability methods*, this MONTHLY, 66 (1959).

S. Marcus, *Sur les propriétés différentielles des fonctions dont les points de continuité forment un ensemble frontière partout dense*, Ann. Sc., École Norm. Sup. 79 (1962).

G. A. Heuer, *Functions continuous at irrationals and discontinuous at rationals*, (abstract), this MONTHLY, 71 (1964) 349.

Integer Solutions of a Pair of Diophantine Equations

5184 [1964, 326]. *Proposed by A. Oppenheim, University of Malaya, Kuala Lumpur.*

Find the complete solution in rational integers of

$$x^3 + y^3 + z^3 + t^3 = 2$$
$$x + y + z + t = 2.$$

Solution by Jack Diamond, Queens College, Flushing, N. Y. If x, y, z, t is a solution in rational integers, then two (or all four) of the numbers are odd. Suppose that x and y are odd, whence $x+y=2w$ is even, giving $y=2w-x$, and $t=2-2w-z$. Substituting for y and t in the cubic equation gives a quadratic equation in z if $w\neq 1$ and gives $x^2-2x+1=0$ if $w=1$. In order for z to be an

integer it is necessary that $w[(x-w)^2+w-1]/(w-1)=v^2$, where v is a rational integer if $w\neq 1$; and necessary that $x=1$ if $w=1$.

In either case, letting $u=x-w$:

$$wu^2-(w-1)v^2+w(w-1)=0, \qquad (u, v, w \text{ rational integers}), \tag{1}$$

$$x=w+u,\ y=w-u,\ z=1-w+v,\ t=1-w-v. \tag{2}$$

Conversely, if u, v, and w are any three integers satisfying (1) and x, y, z, and t are defined as in (2), then the given equations are satisfied. For each w $(=0,\ \pm 1,\ \pm 2, \cdots)$ (1) becomes a Pell equation which can be solved by standard methods.

Also solved by Joseph Arkin, W. J. Blundon, H. Flanders and D. Hertzig, R. Venkatachalam Iyer (India), John B. Kelly, Stanton Philipp, E. Rosenthall, and the proposer.

Editorial Note. A Pell equation from which solutions can be obtained is found more easily as follows. Eliminate t from the given two equations:

$$(x+y)(y+z)(z+x)=2(x+y+z-1)^2.$$

Put $a=y+z-1$, $b=x+z-1$, $c=x+y-1$:

$$(bc-a)^2-(c^2-1)b^2=1-c^2.$$

Now, for arbitrary c, this is a Pell equation in variables $bc-a$ and b, with infinitely many solutions for every $c(\neq 0)$. Solutions may be obtained by familiar procedures, using continued fractions or otherwise.

Solutions may also be obtained based on L. J. Mordell, *On the integer solutions of the equation* $x^2+y^2+z^2+2xyz=n$, Jour. London Math. Soc., 28 (1953) 500–510.

Filling a Plot with Flowers

5186 [1964, 326]. *Proposed by J. D. Nulton, University of California, Berkeley*

A horticulturist wishes to grow a flower garden on a plot of ground which is large enough for only n flowers. Each of his flower bulbs, however, has a probability $p\neq 1$ of not growing into a flower. How many times (expected value) must he replant his garden before he reaches his goal of n flowers? Naturally he replants only those which failed to grow from the last planting.

I. *Solution by A. J. Bosch, Technical University, Eindhoven, Netherlands.* Let $P(x)$ be the probability that one must replant x times (equivalent to $x+1$ plantings), and let $Q(x)$ be the probability that one must replant more than x times. Then we must calculate: $\mu=Ex=\sum_0^\infty xP(x)$. From $P(x)=Q(x-1)-Q(x)$ we have

$$\mu=\sum_{x=0}^{\infty} xP(x)=\sum_{x=0}^{\infty} Q(x).$$

To find $Q(x)$ we place the bulbs on a row $1, 2, \cdots, n$ and at each replanting we replace all the bulbs, marking those places where a flower occurred. The probability that after x replacements of the garden every place has had at least one flower is clearly $(1-p^{x+1})^n$. Hence

$$Q(x) = 1 - (1 - p^{x+1})^n = 1 - \sum_{i=0}^{n} \binom{n}{i} (-p^{x+1})^i = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} p^{(x+1)i}$$

or

$$\mu = \sum_{x=0}^{\infty} Q(x) = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} \sum_{x=0}^{\infty} p^{(x+1)i} = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} \frac{p^i}{1 - p^i}.$$

II. *Solution by S. H. Greene, The Franklin Institute.* After the first planting the horticulturist can expect $(1-p)n$ of the flowers to grow and pn to fail. Of the pn he expects to replant he can expect $(1-p)pn$ to grow and p^2n to fail, so that after the second planting he can expect $(1-p)n+(1-p)pn$ flowers. Of the p^2n he expects to replant he can expect $(1-p)p^2n$ to grow and p^3n to fail, so that after the third planting he can expect $(1-p)n+(1-p)pn+(1-p)p^2n$ flowers. In general, after the jth planting he can expect $(1-p)n+(1-p)pn + \cdots +(1-p)p^{j-1}n$ flowers and p^jn failures, so that the number of his expected successes, $E_j(p, n)$ after j plantings is

$$E_j(p, n) = (1 - p)n \sum_{\nu=0}^{j-1} p^\nu = (1 - p)n \frac{1 - p^j}{1 - p} = (1 - p^j)n.$$

When $E_j(p, n)$ is greater than $n-\frac{1}{2}$, we assume that the total number, n, of flowers can be expected to be growing. Hence

$$E_j(p, n) = (1 - p^j)n > n - \tfrac{1}{2} \Rightarrow p^jn < \tfrac{1}{2},$$

$$j \log p < - \log n - \log 2, \qquad j > (\log n + \log 2)/\log p^{-1}.$$

Thus the expected number, j, of replantings is $[(\log n+\log 2)/\log p^{-1}]$.

Also solved by R. H. Anderson, F. P. Callahan, R. E. Greenwood, G. A. Heuer, E. S. Langford, G. C. Pomraning, J. E. Potter, R. C. Read, H. E. Reinhardt, H. M. Rosenblatt, Y. P. Sabharwal (India), L. A. Shepp, Robert Singleton, W. F. Trench, J. Wessels (Netherlands), and the proposer.

Editorial Note. The expected number of replantings is obviously one less than the expected number of plantings, for which several solvers obtained the equivalent formula $\sum_{i=1}^{n}\binom{n}{i}(-1)^{i-1}(1-p^i)^{-1}$. Shepp proves that this expectancy is asymptotic to $\log_{1/p}n$ and, consequently, to solution II.

Tangents to Conics in Involution

5187 [1964, 326]. *Proposed by D. H. Denniston, University of Canterbury, New Zealand*

A, B, C, D, E, F are general points of a plane. AB', AC', AD' are the tangents at A to the conics $ACDEF$, $ABDEF$, $ABCEF$. Prove that AB, AB'; AC, AC'; AD, AD' are in involution.

Solution by G. Laman, Technological University, Eindhoven, Netherlands. Take trilinear coordinates with $A(1, 0, 0)$, $E(0, 1, 0)$, $F(0, 0, 1)$, $B(b_1, b_2, b_3)$, $C(c_1, c_2, c_3)$, $D(d_1, d_2, d_3)$.

Every conic through A, E and F has equation: $\lambda x_2x_3+\mu x_3x_1+\nu x_1x_2=0$, and

its tangent at $A: \nu x_2+\mu x_3=0$. For $ACDEF$ we have $\lambda c_2c_3+\mu c_3c_1+\nu c_1c_2=0$ and $\lambda d_2d_3+\mu d_3d_1+\nu d_1d_2=0$, so that

$$\mu(c_1c_3d_2d_3 - c_2c_3d_1d_3) + \nu(c_1c_2d_2d_3 - c_2c_3d_1d_2) = 0.$$

Hence the pair (AB, AB') has the equation

$$B^*(x_2, x_3) \equiv (b_3x_2 - b_2x_3)\{(c_1c_3d_2d_3 - c_2c_3d_1d_3)x_2 - (c_1c_2d_2d_3 - c_2c_3d_1d_2)x_3\} = 0.$$

By cyclic permutation of the letters b, c, d we obtain the equations of (AC, AC'): $C^*(x_2, x_3)=0$ and (AD, AD'): $D^*(x_2, x_3)=0$. Straightforward calculation shows that

$$b_1B^* + c_1C^* + d_1D^* = 0,$$

which settles the problem.

Also solved by J. Basile (Belgium), D. Pedoe, O. P. Lossers (Netherlands), S. R. Mandan (India), J. M. Quoniam (France), Harold Simpson (England), and the proposer.

Separation Axioms between T_0 and T_1

5188 [1964, 326]. *Proposed by J. R. Baugh, S. P. Franklin and T. G. McLaughlin, University of California, Los Angeles*

Place the following axioms on a topological space X in the hierarchy of separation axioms and determine what implications exist between them:

A. Every point of X is the intersection of an open set and a closed set.
B. The Boolean algebra generated by the topology of X is 2^X.
C. Every point of X is either open or closed.
D. X is a door space, i.e. every set is either open or closed.

Solution by W. C. Waterhouse, Harvard University. The implications are

$$\begin{array}{ccccccc} & & & & D & & \\ & & & & \Downarrow & & \\ & & T_1 & & B & & \\ & & \Downarrow & & \Downarrow & & \\ D & \Rightarrow & C & \Rightarrow & A & \Rightarrow & T_0. \end{array}$$

Clearly $D \Rightarrow C \Rightarrow A$, $D \Rightarrow B$, $T_1 \Rightarrow C$. Call a set locally closed if it is the intersection of an open set and a closed set; the intersection of finitely many locally closed sets is locally closed, so every set in the Boolean algebra generated by the topology is a finite union of locally closed sets; thus $B \Rightarrow A$.

Let p, q be two distinct points of X. If $\{p\}=G \cup C$ (G open, C closed) then either G is a neighborhood of p not containing q or $X-C$ is a neighborhood of q not containing p; thus $A \Rightarrow T_0$.

To see that no other implications hold, it suffices to check that neither A nor any T_i implies B (let $X=$the reals), that A does not imply C (let $X=\{a, b, c\}$

with open sets $\varnothing$, $\{a\}$, $\{a, b\}$, X), and that T_0 does not imply A (let $X = S \cup \{p\}$, where S is an infinite set, with closed sets X and the finite subsets of S).

Also solved by Helen F. Cullen, Bruce David and Gerald Heuer, W. M. Lambert, Jr., M. D. Mavinkurve (India), S. M. Robinson, A. M. Vaidya, and the proposers.

Vaidya notes that some of these axioms have been considered by Aull and Thron, *Separation axioms between T_0 and T_1*, Indagationes Mathematicae 24 (1962).

Prime Powers in a Binomial Coefficient

5189 [1964, 327]. *Proposed by L. Carlitz, Duke University*

Let $a \equiv b \equiv c \pmod{p^{r+1}}$, where p is a prime and $0 \leqq c < p^{r+1}$. Then, if $m \leqq \min(p^r - 1, c)$ show that the following quotients are integral (mod p):

$$\frac{a(a-1)\cdots(a-m+1)}{b(b-1)\cdots(b-m+1)}, \quad \frac{b(b-1)\cdots(b-m+1)}{a(a-1)\cdots(a-m+1)}.$$

Solution by the proposer. It will evidently suffice to show that the binomial coefficients $\binom{a}{m}$, $\binom{b}{m}$ are divisible by exactly the same power of p. Put

$$a = a_0 + a_1 p + \cdots + a_k p^k \qquad (0 \leqq a_j < p),$$
$$b = b_0 + b_1 p + \cdots + b_k p^k \qquad (0 \leqq b_j < p),$$

so that $a_j = b_j$ $(j = 0, 1, \cdots, r)$ and $c = a_0 + a_1 p + \cdots + a_r p^r$.

We recall that if $n = n_0 + n_1 p + \cdots + n_t p^t$ $(0 \leqq n_j < p)$, then $n! = Tp^w$, where $(T, p) = 1$ and

$$w = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^t}\right] = \frac{n - S(n)}{p-1},$$
$$S(n) = n_0 + n_1 + \cdots + n_t.$$

If $\binom{a}{m}$ is divisible by exactly p^α, we have therefore

$$\alpha = \frac{a - S(a)}{p-1} - \frac{m - S(m)}{p-1} - \frac{a - m - S(a-m)}{p-1} = \frac{S(m) - S(a) + S(a-m)}{p-1}.$$

Similarly if $\binom{c}{m}$ is divisible by exactly p^γ we have

$$\gamma = \frac{S(m) - S(c) + S(c-m)}{p-1},$$

so that

$$\gamma - \alpha = \frac{S(a) - S(a-m)}{p-1} - \frac{S(c) - S(c-m)}{p-1}.$$

Now if we put

$$m = m_0 + m_1 p + \cdots + m_{r-1} p^{r-1} \qquad (0 \leqq m_j < p),$$
$$c - m = m_0' + m_1' p + \cdots + m_{r-1}' p^{r-1} + m_r' p^r \qquad (0 \leqq m_j' < p),$$

then $a-m=(a-c)+(c-m)=m_0'+m_1'p+\cdots+m_r'p^r+a_{r+1}p^{r+1}+\cdots+a_kp^k$, so that

$$\begin{aligned} S(a)-S(a-m) &= (a_0+a_1+\cdots+a_k)-(m_0'+\cdots+m_r'+a_{r+1}+\cdots+a_k) \\ &= (a_0+\cdots+a_r)-(m_0'+\cdots+m_r') \\ &= S(c)-S(c-m). \end{aligned}$$

It follows that $\gamma-\alpha=0$.

In exactly the same way, if $\binom{b}{m}$ is divisible by exactly p^β, we get $\beta=\gamma$. Therefore $\alpha=\beta$, so that $\binom{a}{m}$ and $\binom{b}{m}$ are divisible by exactly the same power of p.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, University of California, Berkeley, and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. Programmed Materials: K. O. May, University of California, Berkeley, Calif. 94704. Films: E. P. Vance, Oberlin College, Oberlin, Ohio 44074

Linear Programming and Extensions. By G. B. Dantzig. Princeton University Press, Princeton, N. J., 1963. 625 pp. $11.50.

It is easy to predict that this book will remain for years a basic work on the subject of Linear Programming. The fact that the author is one of the foremost contributors to the theory is not the only reason warranting this conclusion. The book presents a picture of the present state of development of the subject so accurate that the experienced reader has little difficulty in identifying those parts of the theory which have achieved a near to final state of development (Simplex Method and Duality Theorem in E^n, for example) and those in which a large amount of research, development, and refinement is still to come (Decomposition Principle, Non-Linear Programming, etc.). The book is carefully written at an intermediate level of mathematical elaboration. As a result there is a wide range of possible uses as a text for students of different interests (economists, engineers, planners, and even mathematicians), the more so since it contains a large number of examples and practical illustrations taken from varied fields, together with many exercises of different degrees of difficulty. In fact the two last chapters of the book are only examples in themselves (Nutrition Problems in Chapter 27, and Allocation of Aircraft under Uncertainty in Chapter 28). The first twelve chapters (275 pages) contain a historical introduction and the basic theory of the Simplex Method and Duality, Variants of the Simplex Algorithm, and the interpretation of the Primal Dual Relationship in

Variational Methods for Boundary Value Problems for Systems of Elliptic Equations. By M. A. Lavrent'ev. Noordhoff, Groningen, The Netherlands, 1963. 155 pp. Dfl. 16.25.

The purpose of this monograph is to present a unified approach to the study of elliptic systems consisting of two equations, two dependent and two independent variables. This approach is based upon first conceiving the solution of a problem as a minimum in some set-up, then taking a minimizing sequence belonging to a compact set and concluding, finally, that a convergent subsequence converges to a minimum because otherwise the limit could be modified to yield a smaller value. The modification is performed with the aid of comparison arguments based upon the maximum principle. One of the interesting theorems proved is an extension of Riemann's conformal mapping theorem to nonlinear elliptic systems. The book is somewhat loosely written; the proofs are often sketchy or even partly omitted, although the theorems are stated precisely. A fair knowledge of conformal mappings and elliptic equations seems essential to the understanding of the book.

A. FRIEDMAN, Northwestern University

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Rufus Oldenburger, Director of the Automatic Control Center, School of Mechanical Engineering, Purdue University, has received the $500 Excellence in Documentation Award of the Instrument Society of America for his prize paper on optimal control.

Dean E. L. Canfield, Drake University, represented the Association at the inauguration of Howard R. Bowen as President of the University of Iowa on December 5, 1964.

Professor A. B. Cunningham, West Virginia University, represented the Association at the inauguration of Gordon E. Hermanson as President of Davis and Elkins College on October 21, 1964.

Professor J. H. Curtiss, University of Miami, represented the Association at the inauguration of Kenneth P. Williams as President of Florida Atlantic University on November 12, 1964.

Professor H. J. Ettlinger, University of Texas, represented the Association at the inauguration of J. H. McCracklin as President of Southwest Texas State College on November 20, 1964.

Professor J. J. L. Hinrichsen, Iowa State University, represented the Association at the inauguration of John W. Bachman as President of Wartburg College on November 11, 1964.

Professor James C. Smith, San Fernando Valley State College, represented the Association at the Special Convocation commemorating the Golden Jubilee of Jesuit Higher Education in the City of Los Angeles held at Loyola University on October 22, 1964.

Professor G. C. Webber, University of Delaware, represented the Association at the dedication of the Delaware State College Science Center on October 10, 1964.

University of Alabama: Visiting Professor R. L. Plunkett, University of Alabama, Huntsville, has been appointed Professor; Professor R. M. Fiterre, Athens College, and Dr. R. L. Causey, Lockheed Missiles Systems Division, Palo Alto, California, have been appointed Associate Professors.

Florida State University: Professor O. G. Harrold, University of Tennessee, has been appointed Professor and Chairman of the Mathematics Department; Associate Professor C. W. McArthur has been promoted to Professor.

City College (City University of New York): Professor Bernard Vinograde, Iowa State University, has been appointed Visiting Professor; Mr. Michael Sentlowitz, E. Errett Smith, New York, New York, has been appointed Lecturer; Dr. Sandra O. Jaffe has been promoted to Assistant Professor.

Rice University: Professor M. L. Curtis, Florida State University, and Associate Professor Eldon Dyer, University of Chicago, have been appointed Professors; Dr. E. H. Connell, Brandeis University, has been appointed Associate Professor; Dr. L. C. Glaser, University of Wisconsin, has been appointed Assistant Professor.

Brooklyn College: Assistant Professor D. M. Bloom, University of Massachusetts, has been appointed Assistant Professor; Assistant Professor Meyer Jordan has been promoted to Associate Professor.

St. Louis University: Assistant Professors E. G. Eigel, Jr., and R. W. Freese have been promoted to Associate Professors; Rev. J. F. Daly, S.J., has been promoted from Assistant Professor to Associate Professor.

Colorado State University: Dr. D. P. Squier, California Research Corporation, La Habra, California, has been appointed Associate Professor; Dr. A. P. Baartz, University of Oregon, and Dr. P. W. Mielke, Jr., University of Minnesota, have been appointed Assistant Professors; Assistant Professor Kenzo Seo has been promoted to Associate Professor; Mr. M. L. Kovacic has been promoted to Assistant Professor.

University of Kentucky: Drs J. B. Fugate, State University of Iowa, Lynn Kurtz, University of Utah, and Donald Prullage, Purdue University, have been appointed Assistant Professors.

Ohio State University: Dr. Hans Zassenhaus, Visiting Professor at Ohio State University, has been appointed Research Professor; Assistant Professor Herbert Walum, Harvey Mudd College, and Dr. W. J. Davis, Case Institute of Technology, have been appointed Assistant Professors; Associate Professor R. C. Fisher has been promoted to Professor.

University of Massachusetts: Professor W. L. Strother, University of Florida, has been appointed Head of the Mathematics Department; Assistant Professor R. A. Melter, University of Rhode Island, and Dr. Douglas Crabtree, University of North Carolina, have been appointed Assistant Professors; Assistant Professor W. S. Martindale, Smith College, has been appointed Associate Professor.

Dr. George Adomian, Hughes Aircraft Company, Culver City, California, has been appointed Professor of Engineering Research at Pennsylvania State University.

Dr. Craig Comstock, Harvard University, has been appointed Assistant Professor at Pennsylvania State University.

Dean J. B. Davis, Amarillo College, has been appointed Dean Emeritus and Professor of Mathematics.

Dr. P. O. Frederickson, University of Nebraska, has been appointed Assistant Professor at Case Institute of Technology.

Mr. C. R. McAllister, Booz, Allen Applied Research, Inc., Los Angeles, California, has been promoted to Research Director in the Western Operations Office.

Assistant Professor J. R. McCarthy, College of the Holy Cross, is on leave during 1964–65 for special studies in statistics at the Catholic University of America.

Professor C. E. Miller, University of Saskatchewan, has been appointed Head of the Department of Mathematics.

Professor J. M. Perry, Clarkson College of Technology, is on leave during 1964–65 as a Visiting Member of the Institute for Fluid Dynamics and Applied Mathematics at the University of Maryland.

Associate Professor F. B. Taylor, Manhattan College, has been appointed Head of the Department of Mathematics.

Professor Emeritus B. A. Bernstein, University of California at Berkeley, died on September 25, 1964. He was a charter member of the Association.

Dr. D. F. Gunder, Loveland, Colorado, died on October 21, 1964. He was a member of the Association for 30 years.

Professor Emeritus R. G. Putnam, New York University, died on July 14, 1964. He was a member of the Association for 40 years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE INDIANA SECTION

The Indiana Section met on Saturday, October 31, 1964, at Ball State Teachers College, Muncie. 125 persons attended of whom 60 were members of the Association. Chairman R. E. Dowds of Butler University presided. The morning was devoted to short papers and the afternoon to a business meeting and an invited hour address entitled "Experimental Methods in Number Theory" by Professor Hans Zassenhaus of the Ohio State University.

Papers presented at the morning session were:

1. *Two applications of stochastic processes to actuarial science*, by John A. Beekman, Ball State Teachers College.

Collective risk theory is discussed in terms of stochastic processes. The double Laplace transforms for the two major distribution functions are expressed as iterated integrals involving the characteristic function of the claim distribution. In non-insurance terms, the distribution functions are for the maximum of sums of random numbers of independent, identically distributed random variables, and for the probability that the random sums stay below a line. In insurance terms, the first distribution is for total claims, and the second is for the probability of ruin, for specified claim distribution, initial reserve, premiums and security loadings.

2. *Free topological groups*, by Robert L. Cooley, Wabash College.

The free topological groups $FM(X)$ and $FG(X)$ of Markov and Graev, respectively, are defined and shown to be different for a given completely regular space X. Results include the follow-

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 4

CONTENTS

APRIL 1965

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamilar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00. For nonmembers the subscription price is \$8.00 (\$10.00 effective August–September 1965).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

PRINTED IN THE UNITED STATES OF AMERICA

BEST POSSIBLE INEQUALITIES FOR THE PROBABILITY OF A LOGICAL FUNCTION OF EVENTS

THEODORE HAILPERIN, Sandia Corporation and Lehigh University

1. Introduction. If E denotes the event "A_1 or A_2 or $\cdots$ or A_n", where $A_1, \cdots, A_n$ are n events with respective probabilities $a_1, \cdots, a_n$ and if $\mathsf{P}(E)$ denotes the probability of E, then the following inequality holds:

$$\max [a_1, \cdots, a_n] \leqq \mathsf{P}(E) \leqq \min [1, a_1 + \cdots + a_n]. \tag{1.1}$$

Similarly, if F denotes the event "A_1 and A_2 and $\cdots$ and A_n," then

$$\max [0, a_1 + \cdots + a_n - (n-1)] \leqq \mathsf{P}(F) \leqq \min [a_1, \cdots, a_n]. \tag{1.2}$$

These inequalities, which are frequently used and go back at least to Boole ([1], Chapter XIX, [2]), were shown by Fréchet [3] to be the best possible obtainable if nothing is known about the events $A_1, \cdots, A_n$ except that their probabilities are $a_1, \cdots, a_n$, respectively. Fréchet's proof that these inequalities are the best possible consists in showing, for example in the case of the upper bound in (1.1), that $\mathsf{P}(E) \leqq \min [1, a_1 + \cdots + a_n]$ and, further, that for any n numbers lying between 0 and 1 there are specific events $E_1, \cdots, E_n$ (in some probability space) having probabilities $a_1, \cdots, a_n$ and such that

$$\mathsf{P}(E_1 \text{ or } E_2 \text{ or } \cdots \text{ or } E_n) = \min [1, a_1 + a_2 + \cdots + a_n]. \tag{1.3}$$

Thus if f is a function such that

$$\mathsf{P}(E) \leqq f(a_1, \cdots, a_n) \leqq \min [1, a_1 + a_2 + \cdots + a_n], \tag{1.4}$$

independently of (the probability space and) the choice of events $A_1, \cdots, A_n$ therein, then by choosing for $A_1, \cdots, A_n$ the above described events $E_1, \cdots, E_n$, we would have

$$f(a_1, \cdots, a_n) = \min [1, a_1 + \cdots + a_n];$$

and since this is true for all values of $a_1, \cdots, a_n$ between 0 and 1, the functions must be identical over this range.

In view of the fact that not all events in elementary probability can be expressed as a disjunction or conjunction of simple events, an extension of Fréchet's result to a wider class of compound events should be of interest. Such an extension is presented in this paper: we shall derive best possible inequalities for the probability of a compound event $\phi(A_1, \cdots, A_n)$, where $A_1, \cdots, A_n$ are arbitrary events of which nothing is known except that their respective probabilities are $a_1, \cdots, a_n$. Here $\phi(A_1, \cdots, A_n)$ is any combination of these events put together with the logical connectives 'and', 'or', and 'not'. We shall call ϕ a *Boolean function*; this is the most general type of event which enters into elementary probability if conditioned events are ignored. More exactly, we shall describe an *effective procedure* by which, given any $\phi(A_1, \cdots, A_n)$, one

can obtain the best possible upper and lower bound functions for this ϕ. We emphasize the fact that our effective procedure produces the *functions* (of $a_1, \cdots, a_n$) which are the upper and lower bounds, and not merely a number for any given set of values for $a_1, \cdots, a_n$. The result just mentioned is readily extended to cover the more general problem of finding best possible bounds for $\mathsf{P}(\phi(A_1, \cdots, A_n))$ knowing only that $a_i \leqq \mathsf{P}(\psi_i(A_1, \cdots, A_n)) \leqq b_i$, where the ψ_i are given Boolean functions having bounds as indicated, and $i=1, \cdots, m$.

We have chosen to express our results in terms of the language of probability, but it will be apparent that other interpretations are possible; e.g. "event" can be taken to be "measurable subset of the unit square," or again "$\mathsf{P}(E)$" can be specialized to stand for "ratio of the number of elements of a subset of a fixed finite set to the total number of elements in the fixed set." It is in terms of this latter interpretation that much of Boole's work was framed.

The problem of finding such best possible inequalities was indeed first treated by Boole. In solving certain problems involving the probability of logical combinations of events he was often led to a situation in which the answer was given by one of the roots of an equation of higher degree than the first. Being uncertain of which root to select, he resolved the difficulty by finding between what limits the probability sought must be found. Thus arose the general question of finding the narrowest limits for the probability of a compound event in terms of the probabilities of its simplest components. A brief account of Boole's results pertaining to this question will be given in the last section of our paper.

2. Some preliminaries. By virtue of the well-known association of events with sets in a probability space we may use the language either of propositions or of sets. We assume that the reader is familiar with the rudiments of either truth-functional logic or set algebra. Capital latin letters (with or without subscripts) will be used for variable events (sets), juxtaposition will indicate conjunction (intersection), $\vee$ will be used for disjunction (union), and the application of an accent will indicate negation (complementation). A basic conjunction on $A_1, \cdots, A_n$, represented by $K_j(A_1, \cdots, A_n)$, $(j=1, 2, \cdots, 2^n)$ is a conjunction containing exactly one occurrence of each of the variables $A_1, \cdots, A_n$, and having in this conjunction the variable A_{n-i+1} $(i=1, \cdots, n)$ accented or unaccented according as the binary expansion of the integer 2^n-j has 0 or 1 as the coefficient of 2^{i-1}.

As is well known ([11], Chapter XI, Section 5), each Boolean function $\phi(A_1, \cdots, A_n)$ is equal to a disjunction of basic conjunctions on $A_1, \cdots, A_n$. We shall write

$$\phi(A_1, \cdots, A_n) = \vee^{(\phi)} K_j(A_1, \cdots, A_n), \tag{2.1}$$

where the index ϕ applied to '$\vee$' indicates that in the disjunction there are present those, and only those, basic conjunctions which logically imply ϕ. By '$K_j(A_1, \cdots, A_n)$ logically implies $\phi(A_1, \cdots, A_n)$' we mean that for the assignment of truth-values 'true' or 'false' to the variables in $K_j(A_1, \cdots, A_n)$ which

results in $K_j(A_1, \cdots, A_n)$ having the value 'true', the corresponding assignment to the variables in $\phi(A_1, \cdots, A_n)$ results in $\phi(A_1, \cdots, A_n)$ also having the value 'true'. We admit the "empty" disjunction for logically false events and, of course, the full disjunction of all the 2^n basic conjunctions corresponds to a logically true event. All told there are 2^{2^n} different disjunctions which can be formed from the 2^n basic conjunctions on n variables. Some of these will be degenerate in the sense of being equal to functions of fewer numbers of variables. We make no attempt to single out these degenerate functions for, as far as our problem is concerned, they will be recognizable as those Boolean functions whose probability has bounds which depend on fewer than n variables. These degenerate functions, incidentally, constitute a "negligible" percentage of the total number of functions: of the 2^{2^n} functions of n variables (disjunctions of basic conjunctions) there are $2^{2^{n-1}}$ degeneratively equal to functions of $n-1$ variables. Hence there are

$$2^{2^n} - 2^{2^{n-1}} = 2^{2^n}\left(1 - \frac{1}{2^{2^{n-1}}}\right)$$

functions which are not degenerate.

On the other hand there are many functions which, although different, do not lead, as far as our problem is concerned, to an essentially different solution. Two Boolean functions so related that one can be obtained from the other by a succession of applications of the operations (1) interchanging letters A_i and A_j and (2) replacing A_i by A_i' will be said to be of the same *symmetry type.* It is easy to see that Boolean functions so related will have as bounds functions which can be obtained from each other by making the corresponding interchanges of a_i by a_j and by replacements of a_i by $1-a_i$. A study of the symmetry types of Boolean functions has been made by Pólya [4] and by Slepian [5]. Table I at the end of this paper gives a summary of the computed results. A specific listing of one function for each of the 22 types for three variables, and one for each of the 402 types for four variables will be found in [6], pp. 26–27 and pp. 259–278.

It is easy to see that if BUB_ϕ is the best upper bound of $\mathsf{P}(\phi)$ then $1-\mathrm{BUB}_\phi$ is the best lower bound of $1-\mathsf{P}(\phi)=\mathsf{P}(\phi')$. In what follows we shall therefore limit our attention to finding best upper bounds, since our method will enable us to find best upper bounds for all Boolean functions, and a best lower bound for $\mathsf{P}(\phi)$ will be given by $1-\mathrm{BUB}_{\phi'}$. In this connection the arithmetic fact that $1-\min[a, b, c, \cdots] = \max[1-a, 1-b, 1-c, \cdots]$ is useful.

3. Best possible upper bound. In this section we show that the best possible upper bound is given by the solution of a linear programming problem.

Consider n events $A_1, \cdots, A_n$ with probabilities $a_1, \cdots, a_n$ in some given probability space. As an abbreviation put $k_j = \mathsf{P}(K_j(A_1, \cdots, A_n))$. If now the event $\phi(A_1, \cdots, A_n)$ is the value of a truth-function ϕ of the events $A_1, \cdots,$ A_n, then by (2.1) and the fact that the K_j are mutually exclusive,

(3.1) $$\mathsf{P}(\phi(A_1, \cdots, A_n)) = \sum_{j=1}^{2^n} \delta_j^{(\phi)} k_j = \boldsymbol{\delta}^{(\phi)}\mathbf{k},$$

where

$$\delta_j^{(\phi)} = \begin{cases} 1 & \text{if } K_j \text{ implies } \phi, \\ 0 & \text{if } K_j \text{ implies } \phi', \end{cases}$$

and where $\boldsymbol{\delta}^{(\phi)}\mathbf{k}$ is the product of the row vector

$$\boldsymbol{\delta}^{(\phi)} = (\delta_1^{(\phi)}, \cdots, \delta_{2^n}^{(\phi)})$$

and the column vector $\mathbf{k} = [k_1, \cdots, k_{2^n}]$. We point out that the vector $\boldsymbol{\delta}^{(\phi)}$ is independent of the particular $A_1, \cdots, A_n$ and depends solely on the nature of the function ϕ. On the other hand the vector $\mathbf{k}$ could change with the A_i even if the a_i are held fixed. We wish to examine the variation in $\mathsf{P}(\phi(A_1, \cdots, A_n))$ as the events and the probability space are altered.

Under the circumstances being considered the k_j are subject to certain constraints. For instance, since A_1 is equal to the disjunction of all the basic conjunctions on $A_1, \cdots, A_n$ which have A_1 unnegated, the corresponding probabilities of these conjunctions must add up to a_1. And, in general, for each $i (i = 1, \cdots, n)$

(3.2) $$\sum_{j=1}^{2^n} a_{ij}k_j = a_i, \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if } K_j \text{ implies } A_i \\ 0 & \text{if } K_j \text{ implies } A_i'. \end{cases}$$

To this we must further add

(3.3) $$\sum_{j=1}^{2^n} k_j = 1, \qquad \text{and} \qquad (3.4) \quad k_j \geqq 0 \qquad (j = 1, \cdots, 2^n).$$

Conditions (3.2) and (3.3) may be written in combined form as a matrix equation

(3.5) $$\mathbf{A}\mathbf{k} = \mathbf{a}^+,$$

where $\mathbf{k}$ is the column vector introduced above, where $\mathbf{a}^+$ is the $(n+1)$-component column vector $[a_1, \cdots, a_n, 1]$ and where $\mathbf{A}$, defined in Display 1,

(Display 1) $$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots\cdots & a_{12^n} \\ a_{21} & \cdots\cdots & a_{22^n} \\ \vdots & & \vdots \\ a_{n1} & \cdots\cdots & a_{n2^n} \\ 1 & \cdots\cdots & 1 \end{bmatrix} \qquad a_{ij} = \begin{cases} 1 & \text{if } K_j \text{ implies } A_i \\ 0 & \text{if } K_j \text{ implies } A'_i \end{cases}$$

is an $(n+1) \times 2^n$ matrix composed of the $n \times 2^n$ matrix $[a_{ij}]$ bordered by an

$(n+1)$-st row of ones. Since we have adopted a standard method of arranging the basic conjunctions on n variables the matrix $\mathbf{A}$ is completely fixed, for a_{ij} is 1 or 0 depending on whether a certain K_j has in it A_i or A_i'. Note that in each row of $[a_{ij}]$ there are exactly half ones and half zeros and that the columns of $[a_{ij}]$ consist of all the 2^n possible arrangements of ones and zeros which exactly parallels the adopted order of arrangement of the basic conjunctions. For example, if $n=3$, the adopted arrangement is

$$A_1A_2A_3,\ A_1A_2A_3,\ A_1A_2'A_3,\ A_1A_2'A_3',\ A_1'A_2A_3,\ A_1A_2A_3',\ A_1'A_2'A_3,\ A_1'A_2'A_3'$$

and the matrix $[a_{ij}]$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \tag{3.6}$$

in which the columns correspond in an obvious way with the listed basic conjunctions.

We return now to the question of finding the best upper bound for $\mathsf{P}(\phi)$. Given $\mathbf{a}=(a_1, \cdots, a_n)$ and ϕ, let α be the set of numbers $\mathsf{P}(\phi(A_1, \cdots, A_n))$ for any events A_i in any probability space $\mathcal{P}$ for which $\mathsf{P}(A_i)=a_i$, that is we put

$$\alpha = \{p \mid \text{For some probability space } \mathcal{P} \text{ and events } A_i \text{ in } \mathcal{P} \text{ for which } \mathsf{P}(A_i) = a_i,\ p = \mathsf{P}(\phi(A_1, \cdots, A_n))\}.$$

For any set of real numbers $(a_1, \cdots, a_n)$, all between 0 and 1, the set α is nonempty since for $\mathcal{P}$ we can take the ordinary measurable subsets of the unit square. We now define the best upper bound function BUB_ϕ as the least upper bound of α:

(3.7) DEFINITION. $\mathrm{BUB}_\phi(a_1, \cdots, a_n) = \operatorname{lub} \alpha$.

Clearly, for any events $A_1, \cdots, A_n$ such that $\mathsf{P}(A_i)=a_i$,

$$\mathsf{P}(\phi(A_1, \cdots, A_n)) \leqq \mathrm{BUB}_\phi(a_1, \cdots, a_n),$$

so that BUB_ϕ is an upper bound. Definition (3.7) provides no effective means for calculating $\mathrm{BUB}_\phi(a_1, \cdots, a_n)$ for arbitrary $a_1, \cdots, a_n$. To remedy this and to show that BUB_ϕ is indeed "best possible," it will be helpful to express it in another form.

Let β be the set of numbers $\boldsymbol{\delta}^{(\phi)}\mathbf{k}$ for any $\mathbf{k}$ satisfying conditions (3.4) and (3.5), that is let

$$\beta = \{q \mid \text{For some } \mathbf{k} \text{ such that } \mathbf{k} \geqq 0 \text{ and } \mathbf{Ak} = \mathbf{a}^+,\ q = \boldsymbol{\delta}^{(\phi)}\mathbf{k}\}.$$

THEOREM 3.1. $\alpha=\beta$.

Proof. Let $p\in\alpha$. Then by the discussion connected with (3.1), (3.4) and (3.5), there is a $\mathbf{k}$ such that $p=\boldsymbol{\delta}^{(\phi)}\mathbf{k}$, $\mathbf{Ak}=\mathbf{a}^+$, and $\mathbf{k}\geqq 0$. Hence $p\in\beta$. Suppose that $q\in\beta$. Then for some $\mathbf{k}$ satisfying $\mathbf{k}\geqq 0$ and $\mathbf{Ak}=\mathbf{a}^+$, $q = \boldsymbol{\delta}^{(\phi)}\mathbf{k}$.

Divide the entire unit square into 2^n disjoint subregions $S_1, \cdots, S_{2^n}$ of area $k_1, \cdots, k_{2^n}$, respectively, where $k_1, \cdots, k_{2^n}$ are the components of the vector $\mathbf{k}$ just mentioned. The vector $\mathbf{k}$ also satisfies $\mathbf{Ak}=\mathbf{a}^+$ whose "ith line" is

$$\sum_{j=1}^{2^n} a_{ij}k_j = a_i.$$

We use this equation to define A_i, namely as the union of the S_j for those j's for which $a_{ij}=1$. This shows that there is a probability space (the unit square) with events A_i (as just defined) having $\mathsf{P}(A_i)=a_i$ in the space and such that

$$q = \boldsymbol{\delta}^{(\phi)}\mathbf{k} = \mathsf{P}(\phi(A_1, \cdots, A_n)),$$

i.e. that $q \in \alpha$.

THEOREM 3.2. *The bounded set β is closed, so that* lub $\alpha=$ lub $\beta=\max \beta$.

Proof. The set of points $\mathbf{k}$ in E^{2^n} (Euclidean 2^n-space) satisfying the conditions $\mathbf{k}\geqq 0$ and $\mathbf{Ak}=\mathbf{a}^+$ consists of points lying in the unit hyper-cube $(1\geqq\mathbf{k}\geqq 0)$ and on the (closed) intersection of the $n+1$ hyperplanes $(\mathbf{Ak}=\mathbf{a}^+)$ and hence is a bounded and closed set. Since $q=\boldsymbol{\delta}^{(\phi)}\mathbf{k}$ is a continuous function of $\mathbf{k}$, the corresponding set of values of q, as the continuous image of a compact set, is also closed. Hence lub β is in β and is its maximum.

We now see from Theorem 3.2 that the problem of finding the best possible upper bound for $\mathsf{P}(\phi)$ requires the solution of a

(3.8) LINEAR PROGRAMMING PROBLEM: *Find the maximum of $q=\boldsymbol{\delta}^{(\phi)}\mathbf{k}$ subject to the constraints* $\mathbf{Ak}=\mathbf{a}^+$, $\mathbf{k}\geqq 0$.

It is well known (and easy to see) that the region specified by the constraints of a linear programming problem is the intersection of a finite number of half-spaces and hence, if nonempty, is a convex (hyper-) polyhedral set. Such a region has a finite number of extreme or "corner" points and if a linear function has a maximum on the set it is taken on at one of these extreme points. As we have seen (Theorem 3.2), our function q does have such a maximum. Hence for any given ϕ and $a_1, \cdots, a_n$ there is a convex region and a corner of this region at which $q=\boldsymbol{\delta}^{(\phi)}\mathbf{k}$ assumes its maximum; by using, for example, the simplex method one can calculate this maximum value. We thus have an effective procedure for obtaining the value of $\mathrm{BUB}_\phi(a_1, \cdots, a_n)$ for any given $a_1, \cdots, a_n$. But since the convex region depends upon $a_1, \cdots, a_n$ (being defined via $\mathbf{a}^+=[a_1, \cdots, a_n, 1]$), it would seem that the determination of the *function* BUB_ϕ requires the solution of infinitely many linear programming problems, one for each set $(a_1, \cdots, a_n)$. In the next section, however, we show that by going over to the dual linear programming problem it is necessary only to solve just one linear programming problem to obtain the function BUB_ϕ.

4. The dual problem. As just mentioned, a better grasp of our problem is obtained by going over to the dual linear programming problem (confer, e.g. [7] p. 54, [8] p. 222). The superscript T indicates matrix transpose.

DUALITY THEOREM OF LINEAR PROGRAMMING: *To each (primal) linear programming problem:*

(I) Maximize $\mathbf{cx}$, *subject to the constraints* $\mathbf{Dx} \leqq \mathbf{d}$, $\mathbf{x} \geqq 0$,
there is an equivalent (dual) linear programming problem:

(II) Minimize $\mathbf{d}^T\mathbf{w}$, *subject to the constraints* $\mathbf{D}^T\mathbf{w} \geqq \mathbf{c}^T$, $\mathbf{w} \geqq 0$.

Problem (I) has an optimal vector $\mathbf{x}$ *if and only if (II) has an optimal vector* $\mathbf{w}$ *and, if the vectors exist, the optimal values of* $\mathbf{cx}$ *and* $\mathbf{d}^T\mathbf{w}$ *coincide.*

To cast our probability-logic problem (3.8) into the form of Problem (I) we write the condition '$\mathbf{Ak} = \mathbf{a}^+$' as '$\mathbf{Ak} \leqq \mathbf{a}^+$ and $\mathbf{Ak} \geqq \mathbf{a}^+$' or, equivalently, as '$\mathbf{Ak} \leqq \mathbf{a}^+$ and $-\mathbf{Ak} \leqq -\mathbf{a}^+$.' These $2(n+1)$ equations can then be written in matrix form as

$$\begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix} \mathbf{k} \leqq \begin{bmatrix} \mathbf{a}^+ \\ -\mathbf{a}^+ \end{bmatrix}, \tag{4.1}$$

where the coefficient of $\mathbf{k}$ is the $2(n+1) \times 2^n$ matrix whose upper half is the original $(n+1) \times 2^n$ matrix $\mathbf{A}$ and whose lower half is $-\mathbf{A}$. Similarly for the column matrix on the right of the inequality. Thus problem (3.8) may be restated

$$\text{Maximize } \boldsymbol{\delta}^{(\phi)}\mathbf{k}, \text{ subject to } \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix} \mathbf{k} \leqq \begin{bmatrix} \mathbf{a}^+ \\ -\mathbf{a}^+ \end{bmatrix}, \qquad \mathbf{k} \geqq 0. \tag{4.2}$$

According to the quoted duality theorem, the dual problem equivalent to (4.2) is

$$\text{Minimize } \begin{bmatrix} \mathbf{a}^+ \\ -\mathbf{a}^+ \end{bmatrix}^T \mathbf{w}, \text{ subject to } \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix}^T \mathbf{w} \geqq \boldsymbol{\delta}^{(\phi)T}, \qquad \mathbf{w} \geqq 0. \tag{4.3}$$

The statement of (4.3) can be simplified. The column vector $\mathbf{w}$ has $2(n+1)$ elements w_i $(i = 1, 2, \cdots, 2(n+1))$. Let us rename the first $n+1$ as u_i $(i = 1, \cdots, n+1)$ and the remainder as v_i $(i = 1, \cdots, n+1)$ and write $\mathbf{w} = \left[\begin{smallmatrix} \mathbf{u} \\ \mathbf{v} \end{smallmatrix}\right]$. Also let $\mathbf{x} = \mathbf{u} - \mathbf{v}$ (i.e. let $x_i = u_i - v_i$, $i = 1, \cdots, n+1$). Then

$$\begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix}^T \mathbf{w} = \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix}^T \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \mathbf{A}^T(\mathbf{u} - \mathbf{v}) = \mathbf{A}^T\mathbf{x}; \tag{4.4}$$

and likewise

$$\begin{bmatrix} \mathbf{a}^+ \\ -\mathbf{a}^+ \end{bmatrix}^T \mathbf{w} = \mathbf{a}^{+T}\mathbf{x}. \tag{4.5}$$

Let us now reconceive $\mathbf{a}^+$ as a row vector (i.e. let $\mathbf{a}^{+T}$ be called $\mathbf{a}^+$), let likewise $\boldsymbol{\delta}^{(\phi)}$ be the column vector $\boldsymbol{\delta}^{(\phi)T}$, and also write $\mathbf{B}$ for $\mathbf{A}^T$. With these notational agreements and the use of (4.4) and (4.5), the dual of our problem (4.3) becomes

$$\text{Minimize } \mathbf{a}^+\mathbf{x}, \text{ subject to } \mathbf{Bx} \geqq \boldsymbol{\delta}^{(\phi)}. \tag{4.6}$$

Note that in (4.6) we no longer need the nonnegativity condition since any vector $\mathbf{x}=\mathbf{u}-\mathbf{v}$ is also equal to a $\mathbf{u}^*-\mathbf{v}^*$ with $\mathbf{u}^*\geqq 0$ and $\mathbf{v}^*\geqq 0$. We point out the important fact that in (4.6) the constraints do not depend on the $a_1, \cdots, a_n$ but solely on the function ϕ—thus the convex polyhedral region determined by the constraints of (4.6) is the same for each set $a_1, \cdots, a_n$. Let

$$\mathbf{x}^{(c)} = (x_1^{(c)}, \cdots, x_{n+1}^{(c)}) \qquad c = 1, \cdots, r$$

be the r extreme points of this convex polyhedral region (r depends only on ϕ). Then the minimum value of $\mathbf{a}^+\mathbf{x}$ is the minimum value of the set of r values $\mathbf{a}^+\mathbf{x}^{(c)}$ for $c=1, \cdots, r$. Hence, at last,

$$\mathrm{BUB}_\phi(a_1, \cdots, a_n) = \min\left[\mathbf{a}^+\mathbf{x}^{(1)}, \cdots, \mathbf{a}^+\mathbf{x}^{(r)}\right] \tag{4.7}$$

is the effective description of the BUB_ϕ function.

We illustrate our theoretical discussion with a specific example.

EXAMPLE. *Find the best possible upper bound for the probability of ϕ, where*

$$\phi(A_1, A_2, A_3) = A_1A_2A_3' \vee A_1A_2'A_3 \vee A_1'A_2A_3.$$

Here in row form $\boldsymbol{\delta}^{(\phi)}=(0, 1, 1, 0, 1, 0, 0, 0)$, since the three basic conjunctions of ϕ are the second, third, and fifth in the standard arrangement (see Section 3). In the primal formulation we wish to

$$\text{maximize } \mathbf{k}\boldsymbol{\delta}^{(\phi)} = k_2 + k_3 + k_5$$

subject to the constraints

$$\begin{aligned} &k_1 + k_2 + k_3 + k_4 &&= a_1 \\ &k_1 + k_2 \qquad\qquad + k_5 + k_6 &&= a_2 \\ &k_1 \qquad + k_3 \qquad + k_5 \qquad + k_7 &&= a_3 \\ &k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 &&= 1 \end{aligned} \tag{4.8}$$

$$k_i \geqq 0, \ (i = 1, \cdots, 8).$$

An intuitive picture which one may associate with this problem is that of finding a division of a rectangle of unit area into eight mutually disjoint subregions (let their areas be $k_1, \cdots, k_8$) which will maximize $k_2+k_3+k_5$ subject to the condition that the sum of the first four areas shall equal a_1, the sum of the first, second, fifth and sixth shall equal a_2, and the sum of the first, third, fifth and seventh shall equal a_3. In terms of a Venn diagram (see Fig. 1) the shaded

area is to be maximized while maintaining the areas of A_1, A_2 and A_3 to be a_1, a_2 and a_3 respectively.

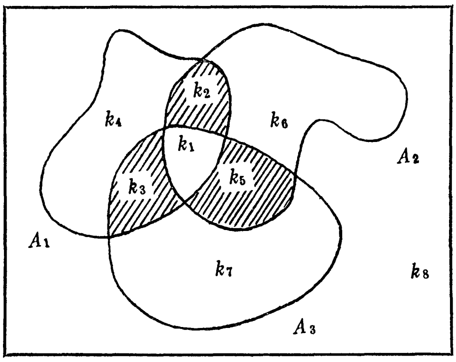


FIG. 1

Turning to the dual formulation of this example, we wish to

$$\text{minimize } \mathbf{a}^+\mathbf{x} = a_1x_1 + a_2x_2 + a_3x_3 + x_4$$

subject to the constraints

$$(4.9)\qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 &\geqq 0 \\ x_1 + x_2 + x_4 &\geqq 1 \\ x_1 + + x_3 + x_4 &\geqq 1 \\ x_1 + x_4 &\geqq 0 \\ x_2 + x_3 + x_4 &\geqq 1 \\ x_2 + x_4 &\geqq 0 \\ x_3 + x_4 &\geqq 0 \\ x_4 &\geqq 0. \end{aligned}$$

To give an intuitive picture here requires first a few preliminary explanations. Consider the unit square and three subregions A_1, A_2, A_3 of areas a_1, a_2, a_3. Any region formed from A_1, A_2, A_3 by the set operations (the unit square being the universe) can be thought of as a region whose area, call it $\mathsf{P}(\phi)$, is a certain portion u_1a_1 of the area a_1 of A_1 plus a portion $v_1(1-a_1)$ of that of A_1', plus a portion u_2a_2 of that of A_2, and so on. That is, any such region has an area expressible in the form

$$\begin{aligned} \mathsf{P}(\phi) &= u_1a_1 + v_1(1 - a_1) + u_2a_2 + v_2(1 - a_2) + u_3a_3 + v_3(1 - a_3) \\ &= a_1x_1 + a_2x_2 + a_3x_3 + x_4. \end{aligned}$$

Since (still using P for area)

$$\begin{aligned}
a_1 = \mathsf{P}(A_1) &= \mathsf{P}(A_1A_2A_3 \vee A_1A_2A_3' \vee A_1A_2'A_3 \vee A_1A_2'A_3') \\
&= \mathsf{P}(A_1A_2A_3) + \mathsf{P}(A_1A_2A_3') + \mathsf{P}(A_1A_2'A_3) + \mathsf{P}(A_1A_2\ A_3\ ) \\
a_2 = \mathsf{P}(A_2) &= \mathsf{P}(A_1A_2A_3) + \mathsf{P}(A_1A_2A_3') + \mathsf{P}(A_1'A_2A_3) + \mathsf{P}(A_1'A_2A_3') \\
a_3 = \mathsf{P}(A_3) &= \mathsf{P}(A_1A_2A_3) + \mathsf{P}(A_1A_2'A_3) + \mathsf{P}(A_1'A_2A_3) + \mathsf{P}(A_1'A_2'A_3)
\end{aligned}$$

and the sum of the probabilities of all the basic conjunctions adds up to 1 we have, by a little algebra,

$$\begin{array}{rl}
a_1x_1 + a_2x_2 + a_3x_3 + x_4 = & (x_1 + x_2 + x_3 + x_4)\,\mathsf{P}(A_1A_2A_3) \\
& + (x_1 + x_2 \phantom{{}+x_3} + x_4)\,\mathsf{P}(A_1A_2A_3') \\
& + (x_1 \phantom{{}+x_2} + x_3 + x_4)\,\mathsf{P}(A_1A_2'A_3) \\
& + (x_1 \phantom{{}+x_2+x_3} + x_4)\,\mathsf{P}(A_1A_2'A_3') \\
& + (\phantom{x_1+{}} x_2 + x_3 + x_4)\,\mathsf{P}(A_1'A_2A_3) \\
& + (\phantom{x_1+{}} x_2 \phantom{{}+x_3} + x_4)\,\mathsf{P}(A_1'A_2A_3') \\
& + (\phantom{x_1+x_2+{}} x_3 + x_4)\,\mathsf{P}(A_1'A_2'A_3) \\
& + (\phantom{x_1+x_2+x_3+{}} x_4)\,\mathsf{P}(A_1'A_2'A_3')
\end{array}$$

We thus see that the inequations (4.9) correspond to the requirements that the portion of area assigned to any basic conjunction present in ϕ be not less than the whole of it, and that all others have nonnegative contributions—subject to these conditions the problem is to minimize $a_1x_1+a_2x_2+a_3x_3+x_4$, the area of the set-theoretic compound defined by $\phi(A_1, A_2, A_3)$, the three regions A_1, A_2, A_3 being specified merely to have areas a_1, a_2, a_3. To conclude this discussion we mention that for this ϕ, namely for

$$\phi(A_1, A_2, A_3) = A_1'A_2A_3 \vee A_1A_2'A_3 \vee A_1A_2A_3',$$

it was found (see the next section) that

$$\mathrm{BUB}_\phi = \min\left[\frac{a_1 + a_2 + a_3}{2},\ a_1 + a_2,\ a_1 + a_3,\ a_2 + a_3,\ \bar{a}_1 + \bar{a}_2 + \bar{a}_3\right],$$

where $\bar{a} = 1 - a$.

5. Computation of best possible bounds. In the preceding section we have seen that obtaining the function BUB_ϕ requires finding all extreme points of a bounded, convex (hyper-)polyhedral region. In the case of a ϕ with two variables this region is one that is bounded by four planes in 3-space. These four planes have up to $C(4, 3) = 4$ possible intersections which one can readily find; one then determines which of these intersections are in the convex region (i.e. satisfy the inequalities). For two-variable ϕ's there are four nontrivial symmetry types (see Table I); all the computations necessary for finding the extreme points for these four types was done by hand computation on a couple of sheets

of paper. (One conveniently uses the pivot exchange, or Jordan elimination, method.) The four bounds thus obtained, and their companion lower bounds, are recorded in Table II. For three-variable ϕ's the convex region is bounded by eight hyperplanes in 4-space, so that there are up to $C(8, 4)=70$ possible intersections of these hyperplanes. An electronic computer was programmed to find these intersections (actually there were only 58 nonsingular cases) and to select from them the extreme points of the convex region for each of the 20 nontrivial symmetry types; the results, which were computed in about one minute, are summarized in Table III. In the case of four-variable ϕ's the convex region is bounded by 16 hyperplanes in 5-space so that there would be $C(16, 5)=4{,}168$ possible intersections. At this stage an exhaustive examination of all intersections to find the extreme points of the convex region seems counter-indicated. A computationally feasible method for finding all vertices of a convex polyhedral region in n-space, which does not exhaustively examine all possible intersections of the hyperplanes taken n at a time, has been presented by M. L. Balinski in [9]. Some method such as this will be required to do a systematic investigation of four-variable ϕ's. It is to be emphasized, however, that for a given $\phi(A_1, \cdots, A_n)$ and a specific set of real numbers $a_1, \cdots, a_n$, the simplex method can be used to compute the optimal value of the linear function in the corresponding linear programming problem without one's having to find all vertices of the convex region.

6. A generalization. Our main result is easily extended to the more general situation in which the given information is not merely that $\mathsf{P}(A_i)=a_i$ $(i=1, \cdots, n)$, but that

$$a_i \leqq \mathsf{P}(\psi_i(A_1, \cdots, A_n)) \leqq b_i, \qquad (i = 1, \cdots, m),$$

where the ψ_i are any m given Boolean functions and the a_i, b_i are given numbers between 0 and 1. From this one obtains as a special case our previous problem by taking $\psi_i\ (A_1, \cdots, A_n)=A_i$, $a_i=b_i$, and $m=n$. We outline how the discussion of sections 3 and 4 is to be modified to cover the more general situation.

The matrix $[a_{ij}]$ is now to be defined as one whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } K_j \text{ implies } \psi_i \\ 0 & \text{if } K_j \text{ implies } \psi_i' \end{cases}$$

so that $\mathbf{A}$ is now an $(m+1)\times 2^n$ matrix, with last row still a row of ones; however, the pattern of ones and zeros in the first m rows now reflects the Boolean expansions of the ψ_i's rather than of the A_i's. Corresponding to the matrix equation $\mathbf{Ak}=\mathbf{a}^+$ we now have

$$\mathbf{a}^+ \leqq \mathbf{Ak} \leqq \mathbf{b}^+,$$

which we convert to

$$\begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix} \mathbf{k} \leqq \begin{bmatrix} \mathbf{b}^+ \\ -\mathbf{a}^+ \end{bmatrix}$$

by replacing $\mathbf{a}^+ \leqq \mathbf{A}\mathbf{k}$ by $-\mathbf{A}\mathbf{k} \leqq -\mathbf{a}^+$. The definition of BUB_ϕ is formally the same, but with α now defined by

$\alpha = \{p \mid$ For some probability space $\mathcal{P}$ and events A_i in $\mathcal{P}$ for which

$$a_i \leqq \mathsf{P}(\psi_i(A_1, \cdots, A_n)) \leqq b_i,\ p = \mathsf{P}(\phi(A_1, \cdots, A_n))\}.$$

This time we cannot assert that α is nonempty since cases can arise in which the m conditions $a_i \leqq \mathsf{P}(\psi_i) \leqq b_i$ are inconsistent. The definition of the set β is now

$$\beta = \left\{ q \,\middle|\, \text{For some } \mathbf{k} \text{ such that } \mathbf{k} \geqq 0 \text{ and } \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \end{bmatrix} \mathbf{k} \leqq \begin{bmatrix} \mathbf{b}^+ \\ -\mathbf{a}^+ \end{bmatrix},\ q = \boldsymbol{\delta}^{(\phi)}\mathbf{k} \right\}$$

with, of course, the new meaning of $\mathbf{A}$. The proof of Theorem 3.1 carries through with only slight changes so that we again have

THEOREM 5.1. $\alpha = \beta$.

However, the theorem corresponding to Theorem 3.2 now becomes

THEOREM 5.2. *The bounded set β is closed and, if β is nonempty, then* lub $\alpha =$ lub $\beta =$ max β.

The proof of this theorem is essentially the same as that for Theorem 3.2. Thus we are again led to a linear programming problem and its equivalent dual formulation, but this time there need not be a solution. If one has specific numbers a_i and b_i then the simplex method can be applied to find the optimal value of $\boldsymbol{\delta}^{(\phi)}\mathbf{k}$ or, in the dual formulation, of

$$\begin{bmatrix} \mathbf{b}^+ \\ -\mathbf{a}^+ \end{bmatrix}^T \mathbf{x}$$

so as to obtain the numerical value of the best upper bound. The simplex method, obligingly enough, supplies one with the information as to whether there is a solution and also which, if any, of the constraint equations are redundant ([8], Chapter 3).

The following example illustrates the method for finding the best possible upper bound function. For typographical convenience (to keep the size of the matrix within reason) we choose as an example one in which the conditions on the ψ_i's are equalities rather than two-sided inequalities.

EXAMPLE (after Boole [10], p. 282). *Given*

$$\begin{aligned} \mathsf{P}(A_1) &= c_1 \\ \mathsf{P}(A_2) &= c_2 \\ \mathsf{P}(A_1A_3) &= c_3 \\ \mathsf{P}(A_2A_3) &= c_4 \\ \mathsf{P}(A_1'A_2'A_3) &= c_5, \end{aligned} \tag{6.1}$$

find the best possible upper bound for $\mathsf{P}(A_3)$.

For this problem we write down the matrix equation $\mathbf{Ak}=\mathbf{c}^+$ and alongside on the left the Boolean functions which generate the corresponding rows of ones and zeros:

$$\begin{array}{l} A_1 \\ A_2 \\ A_1A_3 \\ A_2A_3 \\ A_1'A_2'A_3 \\ \\ \end{array} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \\ k_7 \\ k_8 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ 1 \end{bmatrix}$$

Going over to the equivalent dual problem, we wish to find the

$$\text{minimum of } c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + x_6$$

subject to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \geqq \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{6.2}$$

where the column on the right side of the inequality sign is the $\boldsymbol{\delta}^{(\phi)}$ corresponding to the function $\phi(A_1, A_2, A_3)=A_3$. The coordinates of the extreme points for the convex region specified by (6.2) were found to be (computer time 21 secs.):

$$(0, 0, 1, 1, 1, 0)$$
$$(-1, 0, 1, 0, 0, 1)$$
$$(0, -1, 0, 1, 0, 1)$$

giving a best upper bound for $\mathsf{P}(A_3)$ of $\min[c_3+c_4+c_5,\ \bar{c}_1+c_3,\ \bar{c}_2+c_4]$. At first glance the absence of 1 as one of the arguments inside the brackets of $\min[\quad]$ might seem to indicate a mistake; however, for the given conditions (6.1) to be consistent we must have $c_3 \leqq c_1$ so that $1-c_1+c_3 \leqq 1$, and hence the 1 is indeed not needed.

7. Boole's lost method. As mentioned in the introduction to this paper the problem we have treated here was first considered by George Boole both in the first form we have put it (sections 3–5) as well as in a somewhat more general form in which the compound event $\phi(A_1, \cdots, A_n)$ could be "conditioned" by other compound events, that is, in a form in which the side conditions are $\mathsf{P}(\psi_i) = b_i$ (a special case of the $a_i \leqq \mathsf{P}(\psi_i) \leqq b_i$ just treated in Section 6). To simplify matters in discussing Boole's work we shall ignore this more general form.

In the beginning of Chapter XIX of his *Laws of Thought* [1] there is an initial treatment of the subject, based not on the probability of an event but on the number of elements in a class. Boole uses '$n(x)$' for the number of elements in a class x and '$n(1)$' for the number of elements in the "universe" of which his classes are subclasses. The problem is very clearly stated on p. 300 (his term "constituent" corresponds to our "basic conjunction"):

"PROPOSITION II:

6. To determine the major numerical limit of a class expressed by a series of constituents of the symbols x, y, z, etc., the values of $n(x)$, $n(y)$, $n(z)$, etc., and $n(1)$, being given."

He summarizes his results in the following rule (p. 301):

"RULE.—Take one factor from each constituent, and prefix to it the symbol n, add the several terms or results thus formed together, rejecting all repetitions of the same term; the sum thus obtained will be a major limit of the expression, and the least of all such sums will be the major limit to be employed."

It is not difficult to see that Boole's Rule does give *an* upper bound; however it does not in general give the best possible upper bound. Examples to show this may be obtained from Table III. Boole's Rule can be trivially strengthened by adding: reject those sums which contain (as expressions) any other sum. In this form the sums obtained correspond to the vertices of the convex polyhedra for all cases of two and three variable ϕ's with but the two exceptions mentioned in Table III. In terms of the linear programming formulation these sums are obtainable by the addition of further hyperplanes (readily suggested by the logic) which reduces the convex region to having just one of the original vertices. (Problem: find a simple addition to Boole's Rule so that for an arbitrary $\phi(A_1, \cdots, A_n)$ one obtains all vertices by such a rule.)

One gets the impression that Boole originally thought his Rule gave the best possible upper bound, subsequently discovered that it did not, but then did not thoroughly rewrite his Chapter. Thus he states the problem in Proposition II as that of determining "the [*sic*] major numerical limit" but in his Rule he describes it as "the major limit to be employed." Later in the Chapter (p. 310) Boole acknowledges:

"13. It is to be observed, that the method developed above does not always assign the narrowest limits which it is possible to determine. But in all cases, I believe, it sufficiently limits the solutions of questions in the theory of probabilities.

"The problem of the determination of the *narrowest* limits of numerical extension of a class is, however, always reducible to a purely algebraical form*.

Boole's footnote: "*The author regrets the loss of a manuscript, written about four years ago, in which this method, he believes, was developed at considerable length. His recollection of the contents is almost entirely confined to the *impression* that the principle of the method was the same as above described, and that its sufficiency was proved. The prior methods of this chapter are, it is almost needless to say, easier, though certainly less general."

It is my conjecture that the "lost manuscript" Boole refers to was the one published posthumously by DeMorgan in the *Transactions of the Cambridge Philosophical Society* for 1868 (see [10] pp. 167–186). This paper is described in the table of contents of [10] as "probably written about 1850" and thus accords with the date Boole mentions; moreover it does give a solution to the problem given in Boole's Proposition II quoted above, but in the form of finding the lower bound. The result Boole arrives at is equivalent to the "prior method" of Chapter XIX and hence can be shown to be wrong by the same examples. The "sufficiency" was therefore not proved.

Returning now to Chapter XIX of his *Laws of Thought*, we find Boole now stating his "purely algebraical form" for finding the "highest inferior limit" ([1] pp. 310–311). It is, in essence, our dual form of the linear programming problem, naturally without modern standards of justification nor with modern techniques for handling such problems. Interestingly enough, Boole also discovered subsequently the primal form of the linear programming problem. This he gave in a short paper appearing in *The Philosophical Magazine*, Series 4, Vol. viii, August 1854 (see [10], pp. 280–288). Of this method he states ([10] p. 280): "I propose in this paper to develop an easy and general method of determining such conditions. This object has been attempted in Chapter XIX of my *Laws of Thought*. But the method there developed is somewhat difficult of application, and I am not sure that it is equally general with the one which I am now about to explain." From our present vantage point we now see that the equivalence of Boole's two methods is but an instance of the duality theorem of linear programming. The "easy and general" method which Boole uses to solve his linear programming problem is now known as the Fourier-Motzkin elimination method (see [12], pp. 84–85) and is practical only for small problems.

This work was performed under the auspices of the United States Atomic Energy commission. The author gratefully acknowledges his indebtedness to J. A. Schatz, D. L. Hanson, M. L. Slater and S. Bell for important suggestions in connection with this work.

TABLE I

(From Polya [4], p. 103)

$N_n^{(s)}$ = number of symmetry types for a function of n variables having s basic conjunctions in its Boolean expansion

$N_n^{(s)}$	$s=0$	1	2	3	4	5	6	7	8	$\sum_s N_n^{(s)}$
$n=1$	1	1	1							3
2	1	1	2	1	1					6
3	1	1	3	3	6	3	3	1	1	22
4	1	1	4	6	19	27	50	56	74	402

TABLE II

Best possible inequalities on $\mathrm{P}(\phi(A_1, A_2))$ for all four nontrivial symmetry types ($\bar{a} = 1-a$)

BLB_ϕ = maximum of	$\phi(A_1, A_2)$	BUB_ϕ = Minimum of
$0,\ a_1+a_2-1$	A_1A_2	a_1, a_2
$\bar{a}_1+\bar{a}_2-1, a_1+a_2-1$	$A_1A_2 \vee A_1'A_2'$	$a_1+\bar{a}_2, \bar{a}_1+a_2$
a_1	A_1	a_1
$a_1,\ a_2$	$A_1 \vee A_2$	$1, a_1+a_2$

TABLE III

Best possible inequalities on $\mathrm{P}(\phi(A_1, A_2, A_3))$ for all 20 nontrivial symmetry types; in all cases except as shown the best possible upper bound is obtained by using Boole's Rule (Section 7). For best lower bound use $\mathrm{BLB}_\phi = 1 - \mathrm{BUB}_{\phi'}$.

	$\phi(A_1, A_2, A_3)$	BUB_ϕ = minimum of
1.	$A_1A_2A_3$	
$\vdots$	$\vdots$	$\vdots$
5.	$A_1'A_2A_3 \vee A_1A_2'A_3 \vee A_1A_2A_3'$	$\dfrac{a_1+a_2+a_3}{2}$, $a_1+a_2, a_1+a_3, a_2+a_3,$ $\bar{a}_1+\bar{a}_2+\bar{a}_3$
$\vdots$	$\vdots$	$\vdots$
9.	$\phi_5 \vee A_1A_2A_3$	$\dfrac{a_1+a_2+a_3}{2}$, $a_1+a_2, a_1+a_3, a_2+a_3, 1$
$\vdots$	$\vdots$	$\vdots$
20.	$(A_1A_2A_3)'$	

References

1. G. Boole, Laws of Thought, Amer. reprint of 1854 ed. Dover, New York.
2. ———, Of propositions numerically definite. Trans. of the Cambridge Philos. Soc., Part II, XI (1868). Reprinted as Study IV in [10].
3. M. Fréchet, Généralizations du théorème des probabilités totales, Fund. Math., 25 (1935) 379–387.
4. G. Pólya, Sur les types des propositions composées, J. Symb. Logic, 5 (1940) 98–103.
5. D. Slepian, On the number of symmetry types of Boolean functions of n variables, Canad. J. Math., 5 (1953) 185–193.
6. Harvard Comp. Laboratory Staff, Synthesis of electronic and control circuits, Cambridge, Mass. (1951).
7. A. J. Goldman and A. W. Tucker, Theory of linear programming, Paper 4 in *Linear Inequalities and Related Systems*, ed. H. W. Kuhn and A. W. Tucker, Ann. Math. Studies No. 38. Princeton Univ. Press, Princeton (1956).
8. G. Hadley, Linear Programming, Addison-Wesley, Reading, Mass. (1962).
9. M. L. Balinski, An algorithm for finding all vertices of convex polyhedral sets, J. Soc. Indust. Appl. Math., 9 (1961) 72–88.
10. George Boole, Collected Logical Works, vol. I, Studies in Logic and Probability. ed. R. Rhees. Open Court Publ. Co. LaSalle, Illinois (1952).
11. G. Birkhoff and S. Maclane, A Survey of Modern Algebra, Rev. ed. Macmillan, New York (1953).
12. G. B. Dantzig, Linear Programming and Extensions, Princeton Univ. Press, Princeton, N. J. (1963).

A SUPPLEMENT TO THE STURM SEPARATION THEOREM, WITH APPLICATIONS

LEE LORCH, University of Alberta, Edmonton, Canada, and
DONALD J. NEWMAN, Yeshiva University, New York

To S. Bergman, K. Loewner, D. J. Struik and G. Szegö on their respective birthdays.

1. Introduction. The Sturm Separation Theorem [2, p. 224] asserts that the interior zeros of two linearly independent solutions of a homogeneous linear differential equation of the second order are interlaced. This raises the problem: Given two such solutions, with isolated zeros, determine which has the larger interior zero of specified rank. Such criteria are provided by Theorems 1, 2, and 4 below.

The coefficients of our differential equations are real-valued, and we are interested only in real-valued solutions. By an "interior zero" of a solution we mean a zero inside an interval consisting wholly of ordinary points of the equation. Only interior zeros will be assigned ranks, and these are counted in ascending order. For Bessel functions, e.g., the (strictly) positive zeros are all interior zeros, but 0 is not.

A related question now asks whether there exists, under the same restriction, a "zero-maximal" solution, i.e., a solution having the extremal property that,

among all nontrivial solutions, with isolated zeros, its interior zero of specified rank is the largest. An affirmative answer is contained in Theorem 3.

The corresponding question for the *least* zero of each rank has a negative answer, since there is always a nontrivial solution of such a differential equation which vanishes at a prescribed point arbitrarily close to the lower end-point of an open interval of ordinary points [2, p. 73].

Some applications to special functions are made, particularly of the result concerning zero-maximal solutions.

Our first result is formulated so as to incorporate the Sturm Separation Theorem, in a fashion explained in Remark (i) following Theorem 1.

2. Theorem 1. The first theorem is rather preliminary in character, supplying common ground for the proofs of Theorems 2 and 4. It provides a refinement of the Sturm Separation Theorem and contains the latter theorem.

THEOREM 1. *Let $y_1(x)$, $y_2(x)$ be linearly independent solutions of the differential equation*

$$p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0, \qquad a < x < b \leqq +\infty, \tag{1}$$

where for $a<x<b$, $p_0(x)$, $p_1(x)$, $p_2(x)$ are continuous and $p_0(x)\neq 0$.

Given x_0, $a\leqq x_0<b$. Suppose (i) *that $y_1(x)$ has a first zero in $x_0<x<b$, say x_1;* (ii) *that $y_1(x)>0$, $x_0<x<x_1$, and $y_2(x)>0$ for some x arbitrarily close to x_0, $x>x_0$; and that* (iii) *the Wronskian $W(y_1, y_2; x)<0$, $x_0<x<b$.*

Then $y_2(x)$ has at least as many zeros in (x_0, b) as does $y_1(x)$ and may have precisely one more. Moreover, the k-th zero of $y_2(x)$ in (x_0, b), say ξ_k, lies between the $(k-1)$-st and k-th zeros of $y_1(x)$ in (x_0, b), say x_{k-1} and x_k; i.e., $x_{k-1}<\xi_k<x_k$, $k=1, \cdots$, where, however, x_0 need not be a zero of $y_1(x)$.

Proof. The main point of the proof is to establish the existence of precisely one zero, ξ_1, of $y_2(x)$ with $x_0<\xi_1<x_1$.

To this end, we note that

$$\begin{aligned} 0 > W(y_1, y_2; x_1) &= y_1(x_1)y_2'(x_1) - y_2(x_1)y_1'(x_1) \\ &= - y_2(x_1)y_1'(x_1), \end{aligned}$$

whence $y_2(x_1)<0$, since $y_1'(x_1)<0$.

Therefore, there exists ξ_1, $x_0<\xi_1<x_1$, where $y_2(\xi_1)=0$, since $y_2(x)>0$ for some x near x_0, $x>x_0$. Denoting by ξ any zero of $y_2(x)$ in (x_0, x_1), we have $0>W(y_1, y_2; \xi) =y_1(\xi)y_2'(\xi)$, so that $y_2'(\xi)<0$. Clearly, therefore, there cannot be more than one such ξ in (x_0, x_1), namely ξ_1.

The proof of the remaining assertions can be completed with only obvious modifications of the above reasoning, replacing x_0 by ξ_1, x_1, $\cdots$, in succession.

REMARKS. (i) The Sturm Separation Theorem is obtained by taking $x_0>a$, $y_1(x_0)=0$ and normalizing the solutions so that $y_1(x)>0$, $x_0<x<x_1$, $y_2(x_0)>0$. This can be done simply by changing the signs of either or both solutions, if need be, since sign changes do not affect the zeros, and since $y_2(x_0)\neq 0$, because

linearly independent solutions of (1) cannot have a common interior zero. Moreover, the zeros of $y_1(x)$ and $y_2(x)$ in (x_0, b), $x_0>a$, are isolated (whence $y_1(x)$ has a first zero $x_1>x_0$). Otherwise y_1, y_2 would have infinitely many zeros in a bounded subinterval of (x_0, b), $x_0>a$, and so would vanish identically [**2**, pp. 223–224]. With this in mind, we see that $W(y_1, y_2; x_0)<0$. However, the Wronskian is of constant sign, $a<x<b$, as is clear from the Abel identity [**2**, p. 119], and so $W(y_1, y_2; x)<0$, $a<x_0\leqq x<b$. The Sturm Separation Theorem is now immediate.

We note that, as is well known, this yields the result that the zeros of all nontrivial solutions of (1) are isolated if the zeros of one such solution are.

(ii) Theorem 1 permits the zeros of specified rank of any pair of linearly independent solutions of (1), with isolated zeros, to be ordered according to magnitude. Hypothesis (ii) is merely a notational convenience which can always be realized. Moreover, if the Wronskian is not negative, it can be made so by interchanging $y_1(x)$ and $y_2(x)$.

3. Theorem 2. Here we present another criterion for deciding the order in which the zeros of specified rank occur for a pair of linearly independent solutions of (1).

THEOREM 2. *Let $y_1(x)$, $y_2(x)$ be linearly independent solutions of the differential equation* (1). *Suppose, given x_0, $a\leqq x_0<b$, that the zeros of $y_1(x)$ and $y_2(x)$ in (x_0, b) are isolated, and that, for all x sufficiently near to x_0, $x>x_0$,*

$$\left|\frac{y_1(x)}{y_2(x)}\right| > \liminf_{x\to x_0+}\left|\frac{y_1(x)}{y_2(x)}\right| = A \geqq 0. \tag{2}$$

Then the conclusion of Theorem 1 holds.

Proof. We assume (as can be done without loss of generality) that $y_1(x)$ and $y_2(x)$ are positive for all x sufficiently close to x_0, $x>x_0$. Our result will follow from Theorem 1, once it is shown that (2) implies that $W(y_1, y_2; x)<0$, $x_0<x<b$.

If the constant sign of the Wronskian were not negative, then we would have

$$y_1(x)y_2'(x) \geqq y_1'(x)y_2(x), \qquad x_0 < x < b.$$

Then, fixing $\delta>0$ so that $y_1(x)$ and $y_2(x)$ are positive, $x_0<x<x_0+\delta$, we select ζ, $x_0<\zeta<x_0+\delta$, close enough to x_0 that (2) is satisfied. Then

$$\frac{y_2'(x)}{y_2(x)} \geqq \frac{y_1'(x)}{y_1(x)}, \qquad x_0 < x < \zeta,$$

since the denominators are both positive in the specified interval. Hence, for any ρ, $x_0<\rho<\zeta$,

$$\int_\rho^\zeta \frac{y_2'(x)}{y_2(x)}\,dx \geqq \int_\rho^\zeta \frac{y_1'(x)}{y_1(x)}\,dx,$$

whence

$$\log \frac{y_2(\zeta)}{y_2(\rho)} \geqq \log \frac{y_1(\zeta)}{y_1(\rho)},$$

so that

$$\frac{y_1(\rho)}{y_2(\rho)} \geqq \frac{y_1(\zeta)}{y_2(\zeta)} > A, \qquad x_0 < \rho < \zeta < x_0 + \delta < x_1.$$

Now let $\rho \to x_0+$ through appropriate values. In view of (2), this yields the contradiction

$$A \geqq \frac{y_1(\zeta)}{y_2(\zeta)} > A.$$

Thus, $W(y_1, y_2; x) < 0$, $x_0 < x < b$, and Theorem 1 applies. This completes the proof of Theorem 2.

REMARKS. (i) The Sturm Separation Theorem can be obtained from Theorem 2, with $A = 0$, by an abbreviated form of the reasoning by which the Sturm Theorem was derived from Theorem 1.

(ii) By means of Theorem 2, as well as by Theorem 1, any pair whatever of linearly independent solutions, $y_1(x)$, $y_2(x)$, of (1) can be ordered in accordance with the magnitude of their zeros (if any), assumed isolated, of specified rank. To establish this, we show that $y_1(x)$, $y_2(x)$ must satisfy condition (2) either as they stand or interchanged if need be.

This amounts to observing that $y_1(x)/y_2(x) \neq A < +\infty$ for all x sufficiently close to x_0, $x > x_0$, where we take $y_1(x)$ and $y_2(x)$ both positive for such x. For, suppose that $y_1(t_n)/y_2(t_n) = A$, $t_n \to x_0+$; then $y_1(x) - Ay_2(x)$, a nontrivial solution of (1), would have its zeros cluster at x_0, although those of $y_1(x)$ do not. This is well known to be impossible (cf. Remark (i) of Section 2). If $A = +\infty$, it is sufficient to interchange $y_1(x)$ and $y_2(x)$ to see that the kth zero of $y_2(x)$ exceeds the kth zero of $y_1(x)$ in (x_0, b).

(iii) The above argument proves also that the limit in (2) exists, not merely the "lim inf." If $A = +\infty$, this is obvious. For $0 \leqq A < +\infty$, taking $y_1(x)$ and $y_2(x)$ to be positive for all x near x_0, $x > x_0$, we note that the limit could fail to exist only if one of the nontrivial solutions $y_1(x) - (A \pm \epsilon)y_2(x)$ had zeros arbitrarily close to x_0, $x > x_0$, for any $\epsilon > 0$, and this is impossible.

The use of Theorem 2 is illustrated by the following result.

COROLLARY. *The k-th positive zero of the Bessel function* $\mathcal{C}_\nu(x, \theta) = J_\nu(x) \cos \theta - Y_\nu(x) \sin \theta$ *exceeds that of* $\mathcal{C}_\nu(x, \psi)$ *for* $\nu \geqq 0$, *when* $0 \leqq \theta < \psi < \pi$.

Proof. Under the stated assumptions,

$$\liminf_{x \to 0+} \left| \frac{\mathcal{C}_\nu(x, \theta)}{\mathcal{C}_\nu(x, \psi)} \right| = \frac{\sin \theta}{\sin \psi} < \frac{\mathcal{C}_\nu(x, \theta)}{\mathcal{C}_\nu(x, \psi)}$$

for all sufficiently small positive x.

4. Zero-maximal solutions. The most important case of Theorem 2 occurs when $A=0$. It includes the Sturm Separation Theorem, as observed in Remark (i) of Section 3, and gives substance to the concept of "zero-maximal solutions."

A solution $w_1(x)$ will be called *maximal with respect to the zeros in* (x_0, b) or, simply, *zero-maximal in* (x_0, b), if it has the largest zero of specified rank interior to (x_0, b) among all solutions with zeros isolated in (x_0, b) of the differential equation (1), and if all other nontrivial solutions have as many zeros as $w_1(x)$ interior to (x_0, b). (We note that such another solution could have precisely one more interior zero than $w_1(x)$.)

THEOREM 3. *Let $w_1(x)$ be a solution of the differential equation* (1) *and have isolated zeros in (x_0, b), $a \leqq x_0 < b$. Suppose that there exists a linearly independent solution $w_2(x)$ such that*

$$\liminf_{x \to x_0+} \left| \frac{w_1(x)}{w_2(x)} \right| = 0. \tag{3}$$

Then $w_1(x)$ is zero-maximal in (x_0, b). Conversely, such a zero-maximal $w_1(x)$ always exists, if there is any solution with isolated zeros in (x_0, b).

REMARKS. (i) Clearly, a zero-maximal function is unique except for a constant multiplier.

(ii) The main point of Theorem 3 would appear to be found in the case $x_0 = a$, where the differential equation (1) may have a singular point, as for the Bessel equation with $a=0$. This case shows that, at a finite singular point of the differential equation, there is always a nontrivial solution of smallest order and this solution is zero-maximal in (a, b).

Proof. From the Sturm Separation Theorem we learn that the zeros of $w_2(x)$ are also isolated. From Theorem 2, with $A=0$, it is clear that the zeros of $w_1(x)$ and $w_2(x)$ are related as are the zeros of $y_1(x)$ and $y_2(x)$ in Theorem 1. Furthermore, replacing $w_2(x)$ in (3) by any other solution, $w_3(x)$, linearly independent of $w_1(x)$, will leave (3) valid, since $w_3(x)$ can be expressed as $a_1w_1(x) + a_2w_2(x)$. Hence $w_1(x)$ is zero-maximal in (x_0, b).

On the other hand, let $y_1(x)$, $y_2(x)$ be linearly independent solutions of (1) with isolated zeros in (x_0, b). These solutions can be taken positive for all x sufficiently close to x_0, $x > x_0$, without restricting generality. Now let

$$\liminf_{x \to x_0+} \frac{y_1(x)}{y_2(x)} = A, \qquad 0 \leqq A \leqq +\infty.$$

If $A=0$, then we can define $w_1(x) = y_1(x)$. If $A = +\infty$, then we take $w_1(x) = y_2(x)$. If $0 < A < +\infty$, then we put $w_1(x) = y_1(x) - Ay_2(x)$, $w_2(x) = y_2(x)$. Then

$$\liminf_{x \to x_0+} \left| \frac{w_1(x)}{w_2(x)} \right| = \liminf_{x \to x_0+} \left| \frac{y_1(x)}{y_2(x)} - A \right| = 0,$$

since $y_1(x)>0$, $y_2(x)>0$ for x near x_0, $x>x_0$, and the theorem is proved.

5. Applications of zero-maximal solutions. A canvass of special functions which are solutions of differential equations of the type (1) would produce many easy applications of Theorem 3. The Bateman manuscript project [1] would be a good source. We enumerate some.

(i) *Bessel functions.* The kth positive zero of $J_\nu(x)$ is greater than the kth positive zero of any other solution $\mathcal{C}_\nu(x)\not\equiv J_\nu(x)$ of the Bessel equation, when $\nu\geqq 0$, since $J_\nu(x)/\mathcal{C}_\nu(x)\to 0$ as $x\to 0+$.

(ii) *Coulomb wave functions.* These are solutions of the differential equation

$$y''+\{1-2\eta x^{-1}-L(L+1)x^{-2}\}y=0,\ x>0,\ L\geqq -\tfrac{1}{2};$$

a list of their applications is found in [5]. This equation has one solution which approaches 0 as $x\to 0+$ (the one commonly used in applications). According to Theorem 3, then, this solution is maximal with respect to positive zeros. Other properties of the zeros of solutions of the Coulomb equation are pointed out in [3, p. 457].

(iii) *Hypergeometric functions.* When γ is not an integer, two linearly independent solutions of the hypergeometric equation [4, p. 99] are

$$y_1=F(\alpha,\beta,\gamma,x)\quad\text{and}\quad y_2=x^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,x).$$

Here, as in (i) and (ii), $a=0$ in the differential equation (1), and $F(\alpha,\beta,\gamma,0)=1$. Thus, if $\gamma>1$, $\gamma\neq 2, 3,\cdots$, and if $F(\alpha,\beta,\gamma,x)$ has a kth positive zero, then $F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma,x)$ also has a kth positive zero, one which is smaller than that of $F(\alpha,\beta,\gamma,x)$.

If $\gamma=2, 3,\cdots$, the solution y_2 has to be replaced by one which involves the logarithm function, and similar remarks can be made.

(iv) *Legendre polynomials and functions.* Here two natural questions arise: the zeros interior to $(-1, 1)$ and the positive zeros.

(a) Positive zeros: At $x=0$ the solution of smallest order is [4, p. 103]

$$y_1=x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5-+\cdots.$$

Thus the positive zeros of y_1 are extremal. It is worth noting that for n a nonnegative integer (Legendre polynomials), this solution reduces to a polynomial only when n is odd. Hence the Legendre polynomials of even degree are not maximal in this sense, although those of odd degree are.

(b) Zeros in $(-1, 1)$: Here the Legendre polynomial $P_n(x)$ is of lower order than $Q_n(x)$ [4, p. 103] at $x=-1$, so that it does possess this extremal property for all degrees.

Similar remarks can be made for $P_n(\cos\theta)$, $Q_n(\cos\theta)$.

(v) *Associated spherical functions.* [4, p. 106]. The situation is similar to that in (iv).

(vi) *Confluent hypergeometric function.* As in the case of the hypergeometric function, there are two standard solutions [4, p. 110] $y_1 = F(\alpha, \gamma, x)$, $y_2 = x^{1-\gamma}F(\alpha-\gamma+1, 2-\gamma, x)$ and the remarks made in (iii) can be repeated here with only the obvious changes. The zeros (existence and number) are discussed in [6, Section 3.5].

The following examples are among the prominent special cases of (vi).

(vii) *Laguerre polynomials and functions.* The Laguerre polynomials $L_n^{(\alpha)}(x)$, $\alpha > -1$, have the maximal property for positive zeros, being bounded in the neighborhood of the singular point $a=0$ of equation (1), although $L_n^{(\alpha)}(0) \neq 0$, since the other solutions are unbounded.

(viii) *Hermite polynomials and functions.* The differential equation has no finite singular points. As in (iv), the polynomials of odd degree are maximal with respect to positive zeros, the others not.

6. Theorem 4. With the concept and existence of a zero-maximal solution now available, another theorem like Theorems 1 and 2 can be established.

THEOREM 4. *Let the differential equation* (1) *possess a solution,* $w_1(x)$, *zero-maximal in* (x_0, b) *and let* $w_2(x)$ *be a solution linearly independent of* $w_1(x)$. *Further, let the linearly independent solutions* $y_1(x)$, $y_2(x)$ *be written as*

$$y_1(x) = \alpha w_1(x) + \beta w_2(x), \qquad y_2(x) = \gamma w_1(x) + \delta w_2(x), \tag{4}$$

where the notation is arranged so that $\alpha\delta - \beta\gamma > 0$ *and that* $w_1(x)$, $w_2(x)$, $y_1(x)$, $y_2(x)$ *are each positive for all* x *sufficiently close to* x_0, $x > x_0$. *Then the conclusion of Theorem 1 holds.*

REMARK. The differential equation will possess the desired solution $w_1(x)$ if it possesses any solution with isolated zeros. This was proved in Theorem 3.

Proof. We note first that it is indeed possible, as a matter of notation, to arrange the functions $w_1(x)$, $w_2(x)$, $y_1(x)$, $y_2(x)$ as described after (4). For $w_1(x)$, $w_2(x)$ this is obvious, since their zeros are isolated. Moreover, the zeros of $y_1(x)$, $y_2(x)$ are also isolated, being interlaced with those of, say, $w_1(x)$. Hence we can define all these functions to be positive (without affecting their zeros) for all x sufficiently close to x_0, $x > x_0$.

This done, we note that

$$W(y_1, y_2; x) = (\alpha\delta - \beta\gamma)W(w_1, w_2; x) < 0,$$

where the negativity of $W(w_1, w_2; x)$ is established in the proof of Theorem 2. Thus, Theorem 1 applies and the present proof is complete.

We illustrate this theorem by deriving from it the Corollary of Theorem 2. To do so, put $w_1(x) = J_\nu(x)$, $w_2(x) = -Y_\nu(x)$. Then all the conditions of Theorem 4 are satisfied with $y_1(x) = \mathcal{C}_\nu(x, \theta)$, $y_2(x) = \mathcal{C}_\nu(x, \psi)$, $x_0 = a = 0$, since $\alpha\delta - \beta\gamma = \sin(\psi - \theta) > 0$, for $\nu \geqq 0$.

Another application of this theorem to Bessel functions is:

COROLLARY. *For $\nu > 0$, $\nu \neq 1, 2, \cdots$, let $j_{-\nu,k}$ and $y_{\nu k}$ be the respective k-th positive zeros of $J_{-\nu}(x)$ and $Y_\nu(x)$. Then $j_{\nu k} > j_{-\nu,k} > y_{\nu k} > \nu$, when $\nu\pi$ is in quadrant* I *or* III, *and $j_{-\nu,k} < y_{\nu k} < j_{\nu k}$, when $\nu\pi$ is in quadrant* II *or* IV, $k = 1, 2, \cdots$.

Proof. In Section 5 (i) we have shown that $J_\nu(x)$ is maximal with respect to positive zeros. This establishes the upper bound throughout, since $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent solutions of the Bessel differential equation when ν is not an integer. To verify the remaining inequalities we need also the identity [1, vol. 2, p. 4(4)]

$$J_{-\nu}(x) = J_\nu(x) \cos \nu\pi - Y_\nu(x) \sin \nu\pi.$$

Here $x_0 = 0$, $w_1(x) = J_\nu(x)$ and $w_2(x) = -Y_\nu(x)$.

In quadrant I, we take $y_1(x) = J_{-\nu}(x)$, $y_2(x) = -Y_\nu(x)$. Then $\alpha\delta - \beta\gamma = \cos \nu\pi > 0$ and the middle inequality is established for this case (that $y_{\nu 1} > \nu$, for $\nu > 0$, is standard).

In quadrant III, we put $y_1(x) = -J_{-\nu}(x)$ so that $y_1(x)$ is positive for x near $x_0 = 0$ (since $J_\nu(0) = 0$ while $-Y_\nu(0) = +\infty$), and retain $y_2(x) = -Y_\nu(x)$. Then $\alpha\delta - \beta\gamma = -\cos \nu\pi > 0$, and this case is complete.

In quadrant II, we put $y_1(x) = -Y_\nu(x)$, $y_2(x) = J_{-\nu}(x)$; in quadrant IV, $y_1(x) = -Y_\nu(x)$, $y_2(x) = -J_{-\nu}(x)$.

This work was supported by the U. S. National Science Foundation through grants NSF-GP-98 to the Courant Institute of Mathematical Sciences, New York University (where the first-named author was in residence while this work was done) and NSF-GP-62 to Yeshiva University.

References

1. A. Erdélyi *et al.*, Higher transcendental functions, 3 vols., McGraw-Hill, New York, 1953.
2. E. L. Ince, Ordinary differential equations, Dover reprint, New York, 1944.
3. Lee Lorch and Peter Szegö, Higher monotonicity properties of certain Sturm-Liouville functions, II, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys., 11 (1963) 455–457.
4. Erwin Madelung, Die mathematischen Hilfsmittel des Physikers, 6th ed., Springer, Berlin, 1957. (We are indebted to Dr. J. Burlak for this reference.)
5. Tables of Coulomb wave functions, vol. 1, Nat. Bur. Standards. Appl. Math. Ser. 17, 1952.
6. F. G. Tricomi, Funzioni ipergeometriche confluenti, Cremonese, Rome, 1954.

AN ELEMENTARY PROOF OF THE STATIONARY DISTRIBUTION FOR AN IRREDUCIBLE MARKOV CHAIN

NORMAN LEVINSON, Massachusetts Institute of Technology

A finite or infinite Markov chain with stationary transition probabilities from state j to state k given by p_{jk}; $j, k = 1, 2, \cdots$, will be considered. No use will be made of a Tauberian theorem such as that of Littlewood or that of Erdös, Pollard and Feller [1, 2] in proving the existence of a stationary distribution in the irreducible case.

Let state j be called E_j, $j=1, 2, \cdots$. Let the n step transition probability from E_j to E_k be denoted by $p_{jk}(n)$. Let the probability of a first passage from E_j to E_k in n steps be denoted by $f_{jk}(n)$. The generating functions

$$F_{jk}(s) = \sum_{1}^{\infty} f_{jk}(n)s^n,$$

$$G_{jj}(s) = 1 + \sum_{1}^{\infty} p_{jj}(n)s^n,$$

$$G_{jk}(s) = \sum_{1}^{\infty} p_{jk}(n)s^n, \qquad j \neq k,$$

are then related as follows for $0 \leqq s < 1$, $G_{jj}(s) - 1 = F_{jj}(s)G_{jj}(s)$, or

$$G_{jj}(s) = 1/(1 - F_{jj}(s)) \tag{1}$$

and

$$G_{ij}(s) = F_{ij}(s)G_{jj}(s) \qquad i \neq j. \tag{2}$$

From the definition of $f_{jk}(n)$ follows $F_{jk}(1) \leqq 1$.

E_j is said to be a *transient* state if $F_{jj}(1) < 1$. From (1) follows that the transient case is characterized by

$$G_{jj}(1) < \infty.$$

If $F_{jj}(1) = 1$ then E_j is said to be *recurrent*. From (1) follows that the recurrent case is characterized by $G_{jj}(1) = \infty$. In the recurrent case the mean time of first return to E_j from E_j is given by

$$\mu_j = \sum_{1}^{\infty} nf_{jj}(n) = F'_{jj}(1).$$

If $\mu_j < \infty$, E_j is called *positive recurrent* while if $\mu_j = \infty$ then E_j is called *null recurrent*. Let $u_j = 1/\mu_j$. From (1) and the definition of derivative follows

$$\lim_{s \to 1} (1 - s)G_{jj}(s) = 1/\mu_j = u_j. \tag{3}$$

Thus the positive recurrent case is characterized by $u_j > 0$ and the null recurrent case by $u_j = 0$. Note that in the transient case the notation

$$\lim_{s \to 1} (1 - s)G_{jj}(s) = u_j = 0$$

can also be used.

The chain may be said to be irreducible if every state can be reached from every state, that is, if $F_{jk}(1) > 0$ for all j, k. Let N be the shortest path from E_j to E_k with positive probability and let M be the shortest path from E_k to E_j

with positive probability. Then $p_{jk}(N)=\alpha>0$ and $p_{kj}(M)=\beta>0$ and for any $n>0$

$$p_{jj}(n+M+N) \geqq p_{jk}(N)p_{kk}(n)p_{kj}(M) = \alpha\beta p_{kk}(n)$$
$$p_{kk}(n+M+N) \geqq p_{kj}(M)p_{jj}(n)p_{jk}(N) = \alpha\beta p_{jj}(n).$$

Thus $G_{jj}(1)$ and $G_{kk}(1)$ are both finite or are both infinite. Hence all states of an irreducible chain are either transient or all are recurrent. It also follows that

$$\frac{1}{\alpha\beta}\lim_{s\to 1}(1-s)G_{kk}(s) \geqq \lim_{s\to 1}(1-s)G_{jj}(s) \geqq \alpha\beta\lim_{s\to 1}(1-s)G_{kk}(s).$$

Hence by (3) all recurrent states are either null recurrent states or all are positive recurrent states.

Clearly $F_{kk}(1) \leqq F_{kj}(1)F_{jk}(1)+(1-F_{kj}(1))$. In the irreducible recurrent case $F_{kk}(1)=1$ and $F_{kj}(1)>0$. Hence,

$$F_{kj}(1) \leqq F_{kj}(1)F_{jk}(1)$$

or $F_{jk}(1)\geqq 1$. Hence,

(4) $$F_{jk}(1) = 1.$$

A Markov chain is said to have a stationary distribution $\{v_j\}, j\geqq 1, \sum v_j=1, v_j\geqq 0$ if

(5) $$v_j = \sum v_i p_{ij}.$$

THEOREM. *An irreducible Markov chain has a stationary distribution if and only if the chain is positive recurrent. Then the stationary distribution is* $\{u_j\}$ *and is unique.*

Proof. Let a stationary distribution $\{v_j\}$ exist. From iterations of (5) it follows that

$$v_j = \sum_i v_i p_{ij}(n).$$

Multiplying by s^n and adding $v_j\sum_1^\infty s^n = \sum_i v_i G_{ij}(s) - v_j$ for $0\leqq s<1$. Using (2) and the equation above (1) we obtain

$$sv_j = (1-s)G_{jj}(s)\sum v_i F_{ij}(s).$$

Letting $s\to 1$ and using (3)

(6) $$v_j = u_j\sum v_i F_{ij}(1).$$

Since $F_{ij}(1)\leqq 1$ and $\sum v_i=1$ this gives $v_j\leqq u_j$. In the transient or null recurrent cases $u_j=0$. Hence $v_j=0$ which is impossible. Thus $u_j>0$ for at least one j and hence the positive recurrent case occurs so that $u_j>0$ for all j. In the positive recurrent case it follows from (4) and (6) that

$$v_j = u_j, \qquad j \geqq 1.$$

This proves the "only if" statement and that $v_j = u_j$.

Next let the chain be positive recurrent so that $u_j > 0$. From $\sum_k p_{jk}(n) = 1$ it follows that

$$\sum_1^\infty s^n \sum_k p_{jk}(n) = s/(1-s)$$

or $(1-s)\sum_k G_{jk}(s) - (1-s) = s$. Using (2) we see that

$$(1-s)\sum_k G_{kk}(s) F_{jk}(s) = s.$$

Letting $s \to 1$ and taking first a finite sum we obtain $\sum_k F_{jk}(1) u_k \leqq 1$, and hence the same inequality holds for the infinite sum. By (4)

$$\sum u_k \leqq 1. \tag{7}$$

Multiplying $p_{jk}(n+1) = \sum_i p_{ji}(n) p_{ik}$ by s^n and summing on n, we get

$$(1/s)[G_{jk}(s) - p_{jk}s] = \sum_i G_{ji}(s) p_{ik} - p_{jk}. \qquad j \neq k.$$

Using (2) and the equation above (1), we obtain

$$(1/s)[F_{jk}(s) G_{kk}(s) - p_{jk}s] = \sum_i F_{ji}(s) G_{ii}(s) p_{ik}.$$

Multiplying by $1-s$, letting $s \to 1$ and using (3) and (4), we get

$$u_k \geqq \sum_i u_i p_{ik} \tag{8}$$

first for a finite sum and hence for an infinite sum. $\sum u_k$ is finite by (7). Hence, summing (8) on k and using $\sum_k p_{ik} = 1$ shows that equality must hold. Hence,

$$u_k = \sum_i u_i p_{ik}.$$

Recall that $u_j > 0$. Let $v_k = u_k / \sum u_j$. Then $\{v_k\}$ is a stationary distribution. But then it has been proved that $v_k = u_k$. Hence $\sum u_k = 1$.

The preparation of this paper was supported in part by the Office of Naval Research and in part by the National Science Foundation Grant No. GP-149.

References

1. William Feller, An introduction to probability theory and its applications I, 2nd ed., Wiley, New York, 1957.

2. M. Rosenblatt, Random processes, Oxford University Press, New York, 1962.

FUNCTIONS CONTINUOUS AT THE IRRATIONALS AND DISCONTINUOUS AT THE RATIONALS

G. A. HEUER and Undergraduate Research Participation Group, Concordia College

1. Introduction and summary. Functions which are continuous on an everywhere dense set and discontinuous on another everywhere dense set seldom fail to provide for the student a new appreciation for one of the rigorous characterizations of continuity. This may partly explain the popularity these functions have enjoyed in certain quarters of the mathematical literature.

Probably the most famous example is the one sometimes called the ruler function: $f(x)=0$ for irrational x; $f(p/q)=|1/q|$ when p and q are relatively prime integers; $f(0)=1$. For decades a fairly standard exercise in analysis textbooks has been to show that f is continuous at the irrationals and discontinuous at the rationals. Some authors have expanded on this example; Apostol ([1], p. 248) gives a class of functions having the same property; the ruler function is a member. Hardy ([3], p. 190) provides an example not in the class of Apostol. In 1950 the problem section of this MONTHLY [4] asked for an example of a function which was discontinuous on an everywhere dense set and also differentiable on an everywhere dense set. This prompted a note by M. K. Fort, Jr. [2] to the effect that such a function must be continuous and nondifferentiable on a residual set (i.e., on the complement of the union of a sequence of nowhere dense sets and, thus, on a dense set). More recently Porter [6] has shown that the ruler function is nowhere differentiable.

The present paper is concerned primarily with the ruler function and its powers, and investigates both differentiability and Lipschitz conditions. Recall that a function g is *Lipschitzian* at x if and only if there exists a neighborhood $N(x)$ of x and a number $M>0$ such that $|g(y)-g(x)|\leqq M|y-x|$ for all y in $N(x)$. "Dense" will always mean "everywhere dense in the reals." Two generalizations of the theorem of Porter are obtained. Examples are obtained of functions for which there are three dense sets, on one of which the function is discontinuous, on another continuous but not Lipschitzian, and on the third differentiable. In addition, a function is given for which there are three dense sets, on one of which the function is discontinuous, on another continuous but not Lipschitzian, and on the third Lipschitzian but not differentiable. Finally, for every function discontinuous on the rationals, there exists an uncountable dense subset of the reals on which the function fails to satisfy a Lipschitz condition.

2. The ruler function and its powers. In this section f always denotes the ruler function.

THEOREM 1. *If $0<\alpha<2$, then the function f^α is nowhere Lipschitzian. Moreover, f^2 is nowhere differentiable.*

Proof. Let x be an irrational number. By a well-known theorem ([5], Theorem 4.1), every neighborhood of x contains a rational number p/q such that

$|x-p/q|<1/q^2$, and therefore $|f^\alpha(x)-f^\alpha(p/q)|>|q^{2-\alpha}|\ |x-p/q|$. But every neighborhood also contains $y\neq x$ for which $f^2(x)-f^2(y)=0$. Both statements in the theorem follow immediately.

Note that each statement in Theorem 1 implies that f is nowhere differentiable.

One expects, however, that raising f to higher powers will have smoothing effects, and that this is the case is seen in the next three theorems. For this purpose we shall make use of several concepts from the theory of rational approximation of real numbers.

A real number x is *algebraic of degree n* if there is a polynomial P of degree n, but none of lower degree, with rational coefficients, such that $P(x)=0$. A real number x is *approximable by rationals to order n* if there is a positive number c such that the inequality $|x-p/q|<c/q^n$ is satisfied for infinitely many rational numbers p/q. A *Liouville number* is a real number x such that for every positive integer m there exists a rational number p_m/q_m, with $q_m>1$, satisfying the inequality $|x-p_m/q_m|<1/q_m^m$. Every Liouville number is transcendental ([5] Theorem 7.9).

THEOREM 2. *The function f^α is*: (A) *discontinuous at the rationals for every* $\alpha>0$; (B) *continuous but not Lipschitzian at the Liouville numbers, for every* $\alpha>0$; (C) *differentiable at every irrational algebraic number of degree* $\leqq\alpha-1$, *if* $\alpha>3$.

Proof. (A) is obvious. (B) Continuity is immediate because the Liouville numbers are irrational. Now if x is a Liouville number, let M and δ be arbitrary positive numbers. Choose the integer m so that $m>\alpha$, $2^{m-\alpha}>M$ and $2^{-m}<\delta$. Since x is a Liouville number there is a rational number p_m/q_m, with $q_m>1$, such that

$$|\ x - p_m/q_m < q_m^{-m} \leqq 2^{-m} < \delta,$$

while

$$|f^\alpha(x) - f^\alpha(p_m/q_m)| = q_m^{m-\alpha}\ q_m^{-m} \geqq 2^{m-\alpha}\ q_m^{-m} > M\,|x - p_m/q_m|\,.$$

Thus f^α is not Lipschitzian at x.

(C) Assume $\alpha>3$, and x is an irrational algebraic number of degree $n\leqq\alpha-1$. Then ([5] Theorem 7.8) x is not approximable by rationals to degree $n+1$; hence, given $\epsilon>0$, the set of rationals $\{p/q: |x-p/q|\leqq 1/(\epsilon\cdot q^{n+1})\}$ is finite. When y lies in a neighborhood of x excluding this finite set,

$$\left|\frac{f^\alpha(y) - f^\alpha(x)}{y - x}\right| < \epsilon,$$

so $f'(x)=0$.

The result (C) may be sharpened by the use of the celebrated Thue-Siegel-Roth theorem [7]. According to this theorem, if $\beta>2$, no algebraic number is

approximable by rationals to order β. Our reason for separating Theorems 2 and 3 is that the former is based upon readily accessible results, while the Thue-Siegel-Roth theorem lies somewhat deeper.

THEOREM 3. *The function f^α is differentiable at every algebraic irrational number, if $\alpha>2$ (and by Theorem 1, at none if $\alpha\leqq 2$).*

Proof. Let x be an algebraic irrational number, and let $\alpha>2$. Choose β such that $\alpha>\beta>2$. The Thue-Siegel-Roth result implies that $|x-p/q|<q^{-\beta}$ is satisfied by only a finite number of rationals p/q. The remainder of the proof is like that of Theorem 2, part (C).

Since the algebraic irrational numbers (in fact, those of each degree) are dense in the reals, and the Liouville numbers form an uncountable subset of the transcendentals also dense, we see that when $\alpha>2$ there are sets D, E, and F, each dense in the reals, such that f^α is discontinuous on D, continuous but not Lipschitzian on E, and differentiable on F. It would be interesting to produce a fourth dense set on which f^α is Lipschitzian but not differentiable. We have not been able to do so for $\alpha>2$, but can show the existence of such a set for f^2.

LEMMA 1. *If m and n are integers such that $a=m^2-4n$ is positive but not a perfect square, $x=(m-\sqrt{a})/2$, u is the distance of x from the integer nearest to x, and t is the smaller of u and $1/(2\sqrt{a})$, then no rational number p/q satisfies the inequality $|x-p/q|<t/q^2$.*

Proof. Suppose that such a rational p/q exists, with p and q relatively prime. Let v be defined by $(m-\sqrt{a})/2-p/q=v/q^2$. Then $0<|v|<t$, and $v/q+q\sqrt{a}/2=mq/2-p$. Thus $v^2/q^2+v\sqrt{a}=p^2-pmq+q^2n$. The right-hand member of the last equation is an integer, so the left member must be. But

$$|v^2/q^2+v\sqrt{a}| \leqq |v^2/q^2| + |v\sqrt{a}| < 1/4aq^2+1/2<1,$$

so the left member is zero. It follows that $p^2=q(pm-qn)$, and from this that each prime factor of q divides p. This contradicts the fact that p and q are relatively prime, unless $|q|=1$. But $|q|=1$ is precluded by the fact that $t\leqq u$. Therefore, no such rational exists.

LEMMA 2. *Let S be the set of numbers of the form $(m-\sqrt{a})/2$, with a and m as in Lemma 1. Then S is dense in the real numbers.*

Proof. Let x and y be real numbers; $x<y$. The proof is complete if we find integers m and n such that $x<(m-\sqrt{a})/2<y$, with $a=m^2-4n$ positive and not a square.

Let $z=(y+x)/2$, and $\epsilon=(y-x)/2$. We seek integers m and n such that $z-\epsilon<(m-\sqrt{m^2-4n})/2<z+\epsilon$, or

$$z-m/2-\epsilon<\sqrt{(m/2)^2-n}<z-m/2+\epsilon. \tag{1}$$

If we impose the condition $m/2>z+\epsilon$, then all three members of (1) are

negative, and that inequality is satisfied if

$$(2)\qquad (z-m/2)^2+2\epsilon(z-m/2)+\epsilon^2<(m/2)^2-n<(z-m/2)^2-2\epsilon(z-m/2)+\epsilon^2.$$

The right member of (2) exceeds the left by $4\epsilon(m/2-z)$, which is greater than 2 when m is sufficiently large. For such m there exists an integer n such that both $(m/2)^2-n$ and $(m/2)^2-(n+1)$ lie between the left and right members of (2). Since m^2-4n and $m^2-4(n+1)$ are not both perfect squares, the desired m and n exist.

THEOREM 4. *The function f^2 is Lipschitzian but not differentiable at the points of the set S in Lemma 2.*

Proof. That f^2 is nowhere differentiable has been shown.

If x is in S and t is chosen as in Lemma 1, then

$$|f^2(y)-f^2(x)| \leqq \frac{1}{t}|y-x|$$

for all y; i.e., f^2 is Lipschitzian at x.

Thus we have dense sets D, E, and S such that f^2 is discontinuous on D, continuous but not Lipschitzian on E, and Lipschitzian but not differentiable on S.

3. A related result. Fort has shown in [2] that a function discontinuous on a dense set is differentiable on at most a set of first category. Thus a function discontinuous at the rationals must be nondifferentiable on a residual set. In this connection the following is of some interest.

THEOREM 5. *If g is a function discontinuous at the rationals and continuous at the irrationals, then there is a dense uncountable subset of the reals at each point of which g fails to satisfy a Lipschitz condition.*

We omit the proof, because it is rather lengthy, and one would hope to generalize the theorem by replacing the rationals by an arbitrary dense set, and possibly to show that the set of points at which g fails to be Lipschitzian is a residual set.

The research reported in this paper was supported by the NSF Undergraduate Science Education program under Grant No. GE-1082. Undergraduates contributing to this work were Lynn Erbe, Richard G. Lee, and Ronald Rietz.

References

1. T. M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1957.

2. M. K. Fort, Jr., A theorem concerning functions discontinuous on a dense set, this MONTHLY, 58 (1951) 408–410.

3. G. H. Hardy, A Course of Pure Mathematics, Macmillan, New York, 1943.

4. Problem E935, this MONTHLY, 57 (1950) 557.

5. Ivan Niven, Irrational Numbers, MAA, Carus Monograph no. 11, 1956.

6. G. J. Porter, On the differentiability of a certain well-known function, this MONTHLY, 69 (1962) 142.

7. K. F. Roth, Rational approximations to algebraic numbers, Mathematika, 2 (1955) 1–20; corrigendum p. 168.

ON THE FUNCTIONAL EQUATION

$f(x+y)+f(x-y)=2f(x)f(y)$

PL. KANNAPPAN, University of Washington

The purpose of this note is to describe all of the complex-valued, continuous solutions of the functional equation

$$f(x+y)+f(x-y)=2\,f(x)f(y), \tag{A}$$

where the function f is defined on R^n, $n \geqq 1$.

Solutions of (A) when: 1) f is a real-valued, continuous function of a real variable, 2) f is a complex-valued, measurable function of a real variable, 3) f is a complex-valued, continuous function of a complex variable are given in [1], [3] and [2] respectively. Our method is an extension of that used by Flett in [2], for the case $n=2$. (While Flett considers the complex numbers K as his domain of definition, only additive properties of K are needed.)

We will prove the following

THEOREM. *Let n be a positive integer ($\geqq 1$) and consider the additive group R^n, with elements $x=(x_1, x_2, \cdots, x_n)$ etc. Let f be a complex-valued function which is continuous at some point and is a solution of* (A). *Then if $f \not\equiv 0$, we have*

$$f(x)=\cosh(\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n), \tag{1}$$

where α_i $(i=1, 2, \cdots, n)$ are complex numbers.

The proof depends on induction on n.

First, it is easy to verify that if f satisfies (A) and is continuous at some point, then it is continuous everywhere [2].

The case $n=1$ has been treated by various authors; we can shorten somewhat the usual discussion. First, f has derivatives of all orders (see [3], p. 126). Now differentiating both sides of (A) with respect to y twice, we have

$$f''(x+y)+f''(x-y)=2f(x)f''(y). \tag{2}$$

Put $y=0$ in (2). Then we have

$$f''(x)=f''(0)f(x). \tag{3}$$

It is elementary to show from (3) that $f(x)=\cosh \alpha x$, for all real x and some complex number α. This completes the proof for $n=1$.

To prove the theorem by induction for all n, suppose that f is of the form (1) for all $m \leqq n$. We will prove the theorem for $(n+1)$. Write $x=(x_1, x_2, \cdots, x_{n+1}) \in R^{n+1}$. Then the function $(x_1, x_2, \cdots, x_n) \rightarrow f(x_1, x_2, \cdots, x_n, 0)$ satisfies (A) on R^n, and so $f(x_1, x_2, \cdots, x_n, 0)=\cosh(\alpha_1 x_1+\alpha_2 x_2+\cdots+\alpha_n x_n)$ for some complex numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$. The function $x_{n+1} \rightarrow f(0, 0, \cdots, 0, x_{n+1})$ also satisfies (A) on R, and so

$$f(0, 0, \cdots, 0, x_{n+1}) = \cosh(\alpha_{n+1}x_{n+1})$$

for a complex number α_{n+1}. Take $x=(x_1, x_2, \cdots, x_n, 0)$ and $y=(0, 0, \cdots, 0, x_{n+1})$ in (A). Then we have

$$\begin{aligned} &f(x_1, x_2, \cdots, x_n, x_{n+1}) + f(x_1, x_2, \cdots, x_n, -x_{n+1}) \\ &\quad = 2f(x_1, x_2, \cdots, x_n, 0)f(0, 0, \cdots, 0, x_{n+1}) \\ &\quad = 2\cosh(\alpha_1x_1 + \cdots + \alpha_nx_n)\cosh\alpha_{n+1}x_{n+1}. \end{aligned} \tag{4}$$

It can be easily shown, as is done in [2], that

$$[f(x+y) - f(x-y)]^2 = 4[f^2(x) - 1][f^2(y) - 1]. \tag{5}$$

In (5), set $x=(x_1, x_2, \cdots, x_n, 0)$ and $y=(0, 0, \cdots, 0, x_{n+1})$. Then we have

$$\begin{aligned} &f(x_1, x_2, \cdots, x_n, x_{n+1}) - f(x_1, x_2, \cdots, x_n, -x_{n+1}) \\ &\quad = 2\epsilon(x_1, x_2, \cdots, x_{n+1})\sinh(\alpha_1x_1 + \cdots + \alpha_nx_n)\sinh\alpha_{n+1}x_{n+1}, \end{aligned} \tag{6}$$

where $\epsilon(x_1, x_2, \cdots, x_{n+1})$ assumes only the values ± 1. Adding (4) and (6) we obtain

$$f(x_1, x_2, \cdots, x_n, x_{n+1}) = \cosh(\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n + \epsilon\alpha_{n+1}x_{n+1}). \tag{7}$$

We will show that for every $x \in R^{n+1}$, the choice of the sign $\pm$ is the same, i.e. ϵ is a constant on R^{n+1}. If $\alpha_{n+1}=0$, the proof is complete.

We suppose henceforward that $\alpha_{n+1} \neq 0$. Let $e_i=(0, 0, \cdots, 0, 1_{(i)}, 0, \cdots, 0)$, $(i=1, 2, \cdots, n+1)$, let $x \in R^{n+1}$, and let $y=te_{n+1}$, for $t \in R$. From (A) we have

$$2f(x)f(te_{n+1}) = f(x+te_{n+1}) + f(x-te_{n+1}). \tag{8}$$

Now integrate the identity (8) with respect to t between 0 to v, where v is real and positive. This yields

$$\begin{aligned} 2f(x)\int_0^v f(te_{n+1})dt &= \int_0^v f(x+te_{n+1})dt + \int_0^v f(x-te_{n+1})dt \\ &= \int_{-v}^v f(x+te_{n+1})dt. \end{aligned} \tag{9}$$

For every positive number h, (9) shows that

$$\begin{aligned} &2/h[f(x+he_{n+1}) - f(x)]\int_0^v f(te_{n+1})\,dt \\ &\quad = 1/h\int_{-v}^v f(x+(t+h)e_{n+1})dt - 1/h\int_{-v}^v f(x+te_{n+1})\,dt \\ &\quad = 1/h\int_v^{v+h} f(x+te_{n+1})dt - 1/h\int_{-v}^{-v+h} f(x+te_{n+1})\,dt. \end{aligned} \tag{10}$$

Since f is continuous, the limit as $h \downarrow 0$ of the right side of (10) exists and is equal to $f(x+ve_{n+1})-f(x-ve_{n+1})$.

We infer that

$$\frac{\partial f}{\partial x_{n+1}}(x_1, x_2, \cdots, x_{n+1}) = \lim_{h \downarrow 0} 1/h[f(x+he_{n+1})-f(x)]$$

exists, because $f(0, 0, \cdots, 0)=1$ and, for small v, $\int_0^v f(te_{n+1})dt \neq 0$. Hence we have

$$\begin{aligned} f(x+ve_{n+1})-f(x-ve_{n+1}) &= 2\frac{\partial f}{\partial x_{n+1}}(x_1, x_2, \cdots, x_n, x_{n+1})\int_0^v f(te_{n+1})\,dt \\ &= 2\frac{\partial f}{\partial x_{n+1}}(x_1, x_2, \cdots, x_n, x_{n+1})\int_0^v \cosh \alpha_{n+1}t\,dt \qquad (11) \\ &= 2\frac{\partial f}{\partial x_{n+1}}(x_1, x_2, \cdots, x_n, x_{n+1})\overset{-1}{\alpha_{n+1}} \sinh \alpha_{n+1}v. \end{aligned}$$

Putting $x_{n+1}=0$ in x, we infer from (6) and (11) that

$$\begin{aligned} (12) \quad &\frac{\partial f}{\partial x_{n+1}}(x_1, x_2, \cdots, x_n, 0)\overset{-1}{\alpha_{n+1}} \sinh \alpha_{n+1}v \\ &= \epsilon(x_1, x_2, \cdots, x_n, v) \sinh(\alpha_1 x_1 + \cdots + \alpha_n x_n) \sinh \alpha_{n+1}v. \end{aligned}$$

The equality (12) shows that ϵ is constant on the set

$$D = \{(x_1, x_2, \cdots, x_{n+1}) : \sinh(\alpha_1 x_1 + \cdots + \alpha_n x_n) \sinh \alpha_{n+1}x_{n+1} \neq 0\}.$$

Let δ be the value of ϵ on D. On D, we have

$$\begin{aligned} (13) \quad &f(x_1, x_2, \cdots, x_n, x_{n+1}) + f(x_1, x_2, \cdots, x_n, -x_{n+1}) \\ &= 2\cosh(\alpha_1 x_1 + \cdots + \alpha_n x_n)\cosh \alpha_{n+1}x_{n+1} \end{aligned}$$

$$\begin{aligned} (14) \quad &f(x_1, x_2, \cdots, x_n, x_{n+1}) - f(x_1, x_2, \cdots, x_n, -x_{n+1}) \\ &= 2\delta \sinh(\alpha_1 x_1 + \cdots + \alpha_n x_n)\sinh \alpha_{n+1}x_{n+1}. \end{aligned}$$

For $(x_1, x_2, \cdots, x_n, x_{n+1}) \notin D$, we have $\sinh(\alpha_1 x_1 + \cdots + \alpha_n x_n) \sinh \alpha_{n+1}x_{n+1} = 0$, and so (14) holds with a fixed value of δ, either ± 1, on the complement of D. If $\delta = 1$, (13) and (14) show that

$$f(x_1, x_2, \cdots, x_n, x_{n+1}) = \cosh(\alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_{n+1}x_{n+1}).$$

If $\delta = -1$, (13) and (14) show that

$$f(x_1, x_2, \cdots, x_n, x_{n+1}) = \cosh(\alpha_1 x_1 + \cdots + \alpha_n x_n - \alpha_{n+1}x_{n+1}).$$

This completes the proof.

I wish to acknowledge my deep sense of gratitude to Prof. Edwin Hewitt for his guidance during the preparation of this paper.

References

1. A. L. Cauchy, Cours d'Analyse, Paris, 1821; Oeuvres Complètes (2), 3 (1897) 106–113.

2. T. M. Flett, Continuous solutions of the functional equation $f(x+y)+f(x-y)=2f(x)f(y)$, this MONTHLY, 70 (1963) 392–397.

3. S. Kaczmarz, Sur l'équation fonctionnelle $f(x)+f(x+y)=\phi(y)f(x+y/2)$, Fund. Math., 6 (1924) 122–129.

PROFESSOR RICHARD COURANT'S ACCEPTANCE SPEECH FOR THE ASSOCIATION'S DISTINGUISHED SERVICE AWARD

JANUARY 29, 1965

With deep gratitude I accept the award as a recognition for efforts on behalf of education in the mathematical sciences. What makes me particularly happy is the fact that my old friends, Warren Weaver and Mina Rees, played a part in this occasion. I was fortunate indeed to be associated with both of them during the critical war years in the Mathematics Panel of the Office of Research and Development where Warren Weaver's ingenious and effective leadership made a profound impression on me; ever since those days, I have also admired Mina Rees for her many-sided activities.

Permit me a few personal remarks about a specific subject mentioned in the Citation just read. Since my early days in Göttingen I have been concerned with "Applied Mathematics." Yet until today I have not been able to define what the word means or should mean. On the contrary, I have become more and more wary of semantic definitions; they seem futile, and even dangerously restrictive.

In fact, until the early years of this century there did not exist a serious separation between "pure" and "applied" mathematicians. The great mathematicians of the 19th century neither acknowledged nor practiced discrimination. The greatest of all, C. F. Gauss, initiated the era of pure mathematical existentialism by proving in his thesis the existence of roots of algebraic equations. Immediately afterwards, he earned world fame by his gigantic and sensationally successful work of numerical analysis in computing the orbit of the small planet Ceres which had been observed for only a few days, was then lost, and was eventually rediscovered, after many months, at the place computed by Gauss and far removed from the position where the astronomical experts expected it. This major achievement of a definitely applied character was almost immediately succeeded by his youthful masterpiece, the "disquisitiones arithmeticae," a volume in which he established modern algebraic number theory. In Gauss' long scientific life we always find great achievements of a highly applied and concrete character alternating and sometimes interacting with profound purely theoretical investigations—Riemann's Theory of Func-

tions, one of the greatest and most consequential creations of theoretical mathematics in the 19th century reflects inspirations and motivations from the intuitive theory of incompressible fluids.

Only later, when specialization narrowed the angle of vision for everybody, did emphatic purism emerge. High priests of beauty and purity displayed an attitude of cool disinterest in mathematics which was not directed towards "the glory of the human mind."

To balance such puristic attitudes Felix Klein tried to create in Göttingen and other German universities special chairs for applied mathematics. Unfortunately, however, the charter of this new academic field was too narrow, embracing mostly numerical analysis detached from live scientific substance. The experiment failed, and gradually the chairs of applied mathematics in Germany fell into the hands of incumbents who sometimes did not even give lip service to applications. At the same time truly great applied scientists and mathematicians arose in vital and vigorously active fields such as mechanics, aerodynamics, branches of engineering and others. I only mention Prandtl, von Karman, and G. I. Taylor—who must be rated among the leading scientists of the whole past era, and were highly inventive in mathematical constructive thinking.

Certainly their work and success shows that applied mathematics is fascinating and immensely fruitful if imbedded in the live structure of active science and not artificially dissociated from it.

Altogether, I want to emphasize that in my opinion and experience, applied mathematics is not a definable scientific field but a human attitude. The attitude of the applied scientist is directed towards finding clear cut answers which can stand the test of empirical observation. To obtain the answers to theoretically often insuperably difficult problems, he must be willing to make compromises regarding rigorous mathematical completeness; he must supplement theoretical reasoning by numerical work, plausibility considerations and so on.

In contrast, the purist must strive for theoretical perfection and logical completeness of unbroken chains of rigorous reasoning. And still, the purist also is compelled to make compromises, sometimes not openly. He sometimes adjusts and modifies or greatly changes his problem, or the meaning of the concept of solution, to bring it within the power of his ability. Without malice one might sometimes be reminded of the man desperately searching for his house key at night under a bright street lamp and admits that he lost the key somewhere else; but "there it is dark and here it is light."

Today, in connection with the enormous sociological changes which are transforming our whole society, the pursuit of sciences and, in particular, mathematics has radically changed; the separation between the purist attitude and the applied attitude has become threatening indeed, the more so as the spectrum of mathematical activities is ever broadening, far beyond the reach of any individual.

I believe that it is vital to counteract these dangerous tendencies by fighting

over-specialization and fragmentation of mathematics and by a vigorous effort at building bridges between the diverging mathematical fields as well as between mathematics and other sciences and human intellectual activities. There exist great obstacles. But the challenge is overwhelming, and I am sure that it will be met and the strong forces for unity of science will assert themselves more and more.

The Mathematical Association of America can play an important role in this inevitable development.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

A REMARK ON EXTREME DOUBLY STOCHASTIC MEASURES

JORAM LINDENSTRAUSS, Yale University

Let I be the unit interval with the Lebesgue measure m on it. A positive regular measure μ on $I\times I$ is called doubly stochastic if $\mu(A\times I)=\mu(I\times A)=m(A)$ for every Borel subset A of I. The set of all doubly stochastic measures on $I\times I$ is clearly a convex set. We are here concerned with the extreme points of this set. R. R. Phelps has conjectured that every such extreme measure is singular with respect to the Lebesgue measure on $I\times I$ (that is $m\times m$). We shall give here a simple characterization of the extreme doubly stochastic measures and use it to verify the conjecture of Phelps. The paper ends with simple remarks on related questions.

I am indebted to Professor S. Kakutani for helpful conversations concerning the subject of this note.

Our characterization of extreme doubly stochastic measure is

THEOREM 1. *Let μ be a doubly stochastic measure on $I\times I$. μ is an extreme point of the set of all doubly stochastic measures on $I\times I$ if and only if the subspace of $L_1(\mu)$ consisting of all the functions of the form $f(x)+g(y)$ with $f, g\in L_1(m)$ is norm dense in $L_1(\mu)$.*

Proof. μ is not extreme if and only if there is a measure $\nu\neq 0$ on $I\times I$ satisfying $|\nu(S)|\leqq\mu(S)$ for every Borel set $S\subset I\times I$ and $\nu(A\times I)=\nu(I\times A)=0$ for every Borel set $A\subset I$. By the Radon Nikodym Theorem these conditions on ν can be written as $d\nu=F(x, y)d\mu$ with $|F(x, y)|\leqq 1$ and

$$\int_{I\times I} f(x)d\nu = \int_{I\times I} f(y)d\nu = 0,$$

for every $f \in L_1(m)$. In other words μ is not extreme if and only if there exists a nonzero element $F(x, y)$ of $L_\infty(\mu)$ such that

$$\int_{I\times I} (f(x) + g(y))F(x, y)d\mu = 0$$

for every $f, g \in L_1(m)$. The theorem now follows from the fact that $L_1(\mu)^* = L_\infty(\mu)$.

PROPOSITION 1. *Every extreme doubly stochastic measure μ on $I \times I$ is singular with respect to $m \times m$.*

Proof. Let μ be a doubly stochastic measure on $I \times I$ and let $d\mu = F(x, y)d(m\times m) + d\mu_0$, with $F(x, y) \geqq 0$ and μ_0 a nonnegative singular measure on $I\times I$. If $F(x, y)$ is not almost everywhere 0 then there exists a $\delta > 0$ and a square S inside $I \times I$ such that

$$(m \times m)\{(x, y); (x, y) \in S, F(x, y) \geqq 8\delta\} \geqq 31(m \times m)(S)/32.$$

(This follows from the density theorem of Lebesgue.) Let (x_0, y_0) be the center of S and let $2h$ be the length of each of its sides. We denote by S_1 the set $\{(x, y); x_0 \leqq x \leqq x_0 + h, y_0 \leqq y \leqq y_0 + h\}$ and by S_i, $i = 2, 3, 4$, the other three quarters of S numbered in the counter-clockwise direction. Let $h(x, y) = 0$ for $(x, y) \in S_1$ and $= 1$ for $(x, y) \in I\times I \sim S_1$, and let f and g be any elements of $L_1(m)$. Then

$$\int_{I\times I} | h(x, y) - f(x) - g(y) | d\mu \geqq \int_{S_1} | f(x) + g(y) | F(x, y)d(m \times m)$$
$$+ \sum_{i=2}^{4} \int_{S_i} | f(x) + g(y) - 1 | F(x, y)d(m \times m).$$

Assume now that $\int_{S_1} |f(x) + g(y)| F(x, y)d(m\times m)$ is less than some given $\epsilon > 0$. From our choice of S it follows that $|f(x) + g(y)| \leqq \epsilon/\delta h^2$ for a set of points (x, y) in S_1 whose measure is not less than $3h^2/4$. Let $x_1 \in (x_0, x_0 + h)$ be such that $|f(x_1) + g(y)| \leqq \epsilon/\delta h^2$ for a subset B_1 of $(y_0, y_0 + h)$ with $m(B_1) \geqq 3h/4$. Let $y_1 \in B_1$ be such that $|f(x) + g(y_1)| \leqq \epsilon/\delta h^2$ for a subset A_1 of $(x_0, x_0 + h)$ with $m(A_1) \geqq 2h/3$. Putting $f(x_1) = a$ we thus get subsets $A_1 \subset (x_0, x_0 + h)$ and $B_1 \subset (y_0, y_0 + h)$ both of measure $\geqq 2h/3$, such that $x \in A_1$ implies $|f(x) - a| \leqq 2\epsilon/\delta h^2$ and $y \in B_1$ implies $|g(y) + a| \leqq 2\epsilon/\delta h^2$. Using the same argument for S_2 we see that if $\int_{S_2} |f(x) - 1 + g(y)| F(x, y)d(m\times m)$ is less than ϵ then there are $A_2 \subset (x_0 - h, x_0)$, $B_2 \subset (y_0, y_0 + h)$ and a number b such that $m(A_2), m(B_2) \geqq 2h/3$ and $|f(x) - 1 - b| \leqq 2\epsilon/\delta h^2$ for $x \in A_2$, $|g(y) + b| \leqq 2\epsilon/\delta h^2$ for $y \in B_2$. Since $B_1 \cap B_2$ cannot be empty we get that $|a - b| \leqq 4\epsilon/\delta h^2$. Similarly by requiring that

$$\int_{S_i} | f(x) - 1 + g(y) | F(x, y)d(m \times m) \leqq \epsilon \qquad i = 3, 4$$

we will eventually get a number d with $|d - a| \leqq 12\epsilon/\delta h^2$ and a subset A_4 of

(x_0, x_0+h) such that $m(A_4)\geqq 2h/3$ and $|f(x)-1-d|\leqq 2\epsilon/\delta h^2$ for $x\in A_4$. But since $A_1\cap A_4\neq\varnothing$ it follows that $16\epsilon/\delta h^2\geqq 1$. Thus for every f and g in $L_1(m)$ we get

$$\int_{I\times I} | h(x, y) - f(x) - g(y) | \, d\mu \geqq \delta h^2/16.$$

By Theorem 1 μ is not an extreme measure. This concludes the proof of Proposition 1.

Theorem 1 cannot be strengthened by asserting that for every extreme doubly stochastic measure μ every $h\in L_1(\mu)$ can be written as $h(x, y)=f(x)+g(y)$ μ almost everywhere with f and g in $L_1(m)$. To see this we shall consider the following example. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a sequence of positive numbers with $\sum\lambda_i=1$. Let

$$x_0 = 0, \quad x_1 = \lambda_1, \qquad x_2 = \lambda_1+\lambda_2+\lambda_3, \cdots, x_n = \lambda_1+\lambda_2+\cdots+\lambda_{2n-1}, \cdots$$

$$y_0 = 0, \quad y_1 = \lambda_1+\lambda_2, \quad y_2 = \lambda_1+\lambda_2+\lambda_3+\lambda_4, \cdots,$$

$$y_n = \lambda_1+\lambda_2+\cdots+\lambda_{2n}, \cdots.$$

Let μ be a measure distributed uniformly on each of the segments joining (x_n, y_n) with (x_{n+1}, y_{n+1}) with total mass λ_{2n+1} and on each of the segments joining (x_{n+1}, y_n) with (x_{n+2}, y_{n+1}) with total mass λ_{2n+2}. It is easily checked that for every choice of the λ_i we get a doubly stochastic measure. By Theorem 1 it also follows easily that such a μ is always an extreme measure. Every $h\in L_1(\mu)$ is of the form $h(x, y)=f(x)+g(y)$ with f, $g\in L_1(m)$ if and only if $\sup_n(r_n/\lambda_n)<\infty$, where $r_n=\sum_{i=n}^{\infty}\lambda_i$. Indeed, let $h(x, y)=a_n$ on the segment to which we gave the mass λ_n, where $\sum\lambda_n|a_n|<\infty$, and suppose that $h(x, y)=f(x)+g(y)$ μ almost everywhere with f, $g\in L_1(m)$. Put $a_0=\int_{x_0}^{x_1}f(x)dm/\lambda_1$; then it follows easily that

$$(\lambda_1+\lambda_2)(a_1-a_0) = \int_{y_0}^{y_1} g(y)dm, \qquad (\lambda_2+\lambda_3)(a_2-a_1+a_0) = \int_{x_1}^{x_2} f(x)dm,$$

and in general

$$(\lambda_{2n-1}+\lambda_{2n})(a_{2n-1}-a_{2n-2}+-\cdots-a_0) = \int_{y_{n-1}}^{y_n} g(y)dy,$$

$$(\lambda_{2n}+\lambda_{2n+1})(a_{2n}-a_{2n-1}+-\cdots+a_0) = \int_{x_n}^{x_{n+1}} f(x)dx.$$

Hence, since $g\in L_1(m)$, we must have

$$\sum_n (\lambda_{2n-1}+\lambda_{2n}) \, | a_{2n-1}-a_{2n-2}+-\cdots-a_0 | < \infty.$$

Putting $a_i=(-1)^i b_i$ for $i\geqq 1$ with $b_i\geqq 0$ we deduce that, for every sequence of positive b_i such that $\sum\lambda_i b_i<\infty$, also $\sum r_i b_i<\infty$. Therefore $\sup_n(r_n/\lambda_n)<\infty$.

Conversely, if $\sup_n(r_n/\lambda_n) < \infty$ it can be shown by essentially the same argument that every $h \in L_1(\mu)$ can be written as $h(x, y) = f(x) + g(y)$ for (x, y) on the support of μ with $f, g \in L_1(m)$.

Theorem 1 can be generalized easily to the following situation. Let I_1 and I_2 be two measure spaces with positive measures m_1 and m_2 respectively. Consider now the measures which are doubly stochastic with respect to (m_1, m_2), that is, positive measures μ on $I_1 \times I_2$ such that $\mu(I_1 \times A_2) = m_2(A_2)$ for measurable $A_2 \subset I_2$, and $\mu(A_1 \times I_2) = m_1(A_1)$ for measurable $A_1 \subset I_1$. In particular, if I_1 and I_2 are finite sets then Theorem 1 gives an easy way to describe the extreme doubly stochastic measures with respect to (m_1, m_2).

PROPOSITION 2. *Let (I_1, m_1) and (I_2, m_2) be two measure spaces, each consisting of a finite number of points. A measure μ on $I_1 \times I_2$ which is doubly stochastic with respect to (m_1, m_2) is extreme if and only if for every integer k and for every two subsets A_1 of I_1 and A_2 of I_2, each consisting of k elements, the number of points of $A_1 \times A_2$ whose μ measure is positive is less than $2k$.*

REMARK. The values of k for which the condition above is meaningful are of course $1 < k \leqq \min(\text{card } I_1, \text{card } I_2)$.

Proof. The generalization of Theorem 1 to the present situation states that μ is extreme if and only if for every choice of numbers $a_{i,j}$, $i \in I_1$, $j \in I_2$, there exist b_i, $i \in I_1$, and c_j, $j \in I_2$, such that $a_{i,j} = b_i + c_j$ for all i and j for which $\mu(i, j) \neq 0$. Suppose now that there are $A_1 \subset I_1$, $A_2 \subset I_2$ with card $A_1 =$ card $A_2 = k$ such that $\mu(i, j) > 0$ for at least $2k$ points (i, j) in $A_1 \times A_2$. Then the equations $a_{i,j} = b_i + c_j$ with the $2k$ unknowns b_i, $i \in A_1$, and c_j, $j \in A_2$, cannot have a solution for every given $a_{i,j}$ (observe that the homogeneous equations $b_i + c_j = 0$ always have the nontrivial solution $b_i = 1$, $c_j = -1$). Conversely, if the condition on μ in Proposition 2 is satisfied then μ is extreme. We shall prove this by induction on $p+q$ where $p =$ card I_1 and $q =$ card I_2. For $p+q=2$ the assertion is obvious. Suppose now that $p \geqq q$ and let $a_{i,j}$ be given. By our assumption on μ there exists at least one $i_0 \in I_1$ such that $\mu(i_0, j) \neq 0$ for exactly one $j \in I_2$, say j_0. By the induction hypothesis there exist b_i, $i \in I_1 \sim i_0$, and c_j, $j \in I_2$, such that $a_{i,j} = b_i + c_j$ if $\mu(i, j) > 0$ and $i \neq i_0$. Choose now $b_{i_0} = a_{i_0, j_0} - c_{j_0}$ then $a_{i,j} = b_i + c_j$ for all (i, j) such that $\mu(i, j) > 0$ and this concludes the proof.

We remark that if $I_1 = I_2$ and if $m_1 = m_2$ give equal measure to every point we get immediately from Proposition 2 the well-known result of G. Birkhoff stating that the extreme doubly stochastic matrices are the permutation matrices.

Research supported in part by NSF Grant 25222.

Added in proof: After this note was accepted for publication (in January 1964) there appeared a paper of R. G. Douglas (Mich. Math. J. 11 (1964) 243–247) which contains Theorem 1 of this note in a more general setting. The proof is the same as ours.

A SEQUENCE WITHOUT REPEATS ON x, x^{-1}, y, y^{-1}.

RICHARD A. DEAN, California Institute of Technology

A problem of continued interest is to construct infinite sequences $a_1, a_2, \cdots$ on n distinct symbols so that for every positive integer k no two consecutive blocks of k terms are identical, i.e. for no k and i is it possible that

$$a_i, a_{i+1}, \cdots, a_{i+k-1} = a_{i+k}, a_{i+k-1}, \cdots, a_{i+2k-1}.$$

Arshon [1] gives a general procedure for obtaining solutions to this problem for all $n \geqq 3$. Independently, Morse and Hedlund [6] find different solutions for the cases $n=3$ and 4. The Morse-Hedlund solutions come from a sequence, the Morse symbolic trajectory [5, 6] on two symbols. In [6] it is proved that this sequence has no block repeats of the form EEe, where e is the first symbol of E. As a matter of historical interest, Arshon [1] constructs this sequence but he proves only that it has no triple block repeats of the form EEE. Other discussions of this problem are found in [3, 4, 7].

Professor Marshall Hall, Jr. has asked for a sequence on the symbols x, x^{-1}, y, y^{-1} without a 2 block of the form $a^{\epsilon}a^{-\epsilon}(a=x$ or y, $\epsilon=\pm 1)$ and with no repeated k blocks. The existence of such a sequence is helpful in work on the Burnside problem in groups [2]. We recast the problem by replacing x by 1, y by 2, x^{-1} by 3 and y^{-1} by 4 and we construct a sequence on 1, 2, 3, 4 without k block repeats so that the odd-numbered terms are odd and the even-numbered terms are even. The sequence given by Arshon for the case $n=4$ can be shown to have this property while the Morse-Hedlund sequence does not. In this note we give a new sequence on 1, 2, 3, 4 whose method of construction permits us to give especially short and simple proofs of the desired properties.

DEFINITION. *Let α_0 be the sequence* 1, 2, 3, 4. *Proceeding by induction, let α_n be a sequence of length 2^{n+2}. Quarter α_n into four blocks, A, B, C, D each of length 2^n, so that $\alpha_n = ABCD$. Define $\alpha_{n+1} = ABCDADCB$.*

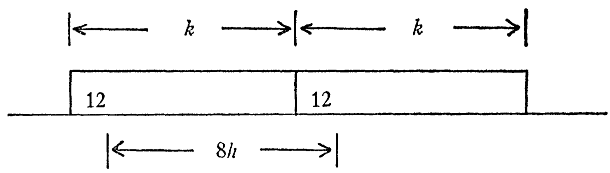
α_4 can be generated as follows:

$$\begin{aligned}
\alpha_0 &= 1\ 2\ 3\ 4\\
\alpha_1 &= \underline{1\ 2\ 3\ 4\ 1\ 4\ 3\ 2}\\
\alpha_2 &= \underline{1\ 2\ 3\ 4\ 1\ 4\ 3\ 2\ 1\ 2\ 3\ 2\ 1\ 4\ 3\ 4}\\
\alpha_3 &= \underline{1\ 2\ 3\ 4\ 1\ 4\ 3\ 2\ 1\ 2\ 3\ 2\ 1\ 4\ 3\ 4}\\
&\ \underline{1\ 2\ 3\ 4\ 1\ 4\ 3\ 4\ 1\ 2\ 3\ 2\ 1\ 4\ 3\ 2}\\
\alpha_4 &= 1\ 2\ 3\ 4\ 1\ 4\ 3\ 2\ 1\ 2\ 3\ 2\ 1\ 4\ 3\ 4\\
&\ 1\ 2\ 3\ 4\ 1\ 4\ 3\ 4\ 1\ 2\ 3\ 2\ 1\ 4\ 3\ 2\\
&\ 1\ 2\ 3\ 4\ 1\ 4\ 3\ 2\cdot 1\ 2\ 3\ 2\ 1\ 4\ 3\ 2\\
&\ 1\ 2\ 3\ 4\ 1\ 4\ 3\ 4\ 1\ 2\ 3\ 2\ 1\ 4\ 3\ 4.
\end{aligned}$$

THEOREM. *For each $n \geqq 0$, α_n is an initial segment of α_{n+1} and the infinite sequence having α_n as its initial segment for all $n \geqq 0$ has consecutive terms of opposite parity and has no k block repeats for all $k \geqq 1$.*

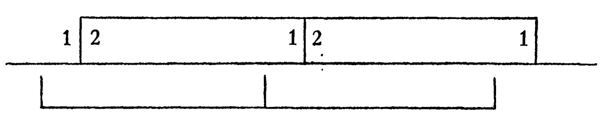
Proof. It is an immediate consequence of the definition that α_n is an initial segment of α_{n+1} and it is easy to prove that α_n and so the infinite sequence has the form 1 2 3_ 1 4 3_ 1 2 3_ 1 4 3_ $\cdots$ with the blanks being filled with either 2 or 4. From this it is clear that there are no k block repeats for $k=1$ or 2.

Let us suppose that we have a repeated block EE; we denote the length of E by k and the index of the first element in the first block E by i. Our next observation is that 8 divides k. This is so because, since $k \geqq 2$, 1 must appear in EE, hence in E. Consider its first occurrence. Its successor is an even number x and the pairs $1x$ occur periodically with period 8. Thus it follows that $8 \mid k$. The diagram below is a visualization in the case $x=2$



$k=8h.$

Now we order in a lexicographic fashion the pairs (i, k) corresponding to repeated blocks and we consider the repeat EE corresponding to the first such pair. If the first element of E is even then the preceding element is odd and since $8 \mid k$ the last term of E will be that same odd number. Hence we should have had a repeat with the pair $(i-1, k)$, a contradiction. The diagram below shows the case when E begins with 2 preceded by 1.



Thus we may assume for minimal (i, k) the repeat E starts with an odd number. But now we define $1^*=12$, $2^*=34$, $3^*=14$, $4^*=32$. It is clear that each initial segment α_n for $n \geqq 1$ may be written as an initial sequence α_{n-1}^* on $1^*, 2^*, 3^*, 4^*$ with the same rule of formation. Moreover, since $8 \mid k$, we should obtain from the repeat EE a repeat E^*E^* in the starred sequence, the length of E^* being $k/2$. By removing the stars we obtain a repeat in the original sequence whose associated pair is $(j, k/2)$ where $j \leqq i$, a contradiction.

This research was partially supported by NSF Grant GP212.

References

1. S. Arshon, Démonstration de l'éxistence des suites asymétrique infinies, Mat. Sb. (Moscow), 44 (1937) 769–777 (Russian) 777–779 (French Summary).

2. Marshall Hall, Jr. Lectures on Modern Mathematics, vol. 2 (T. L. Saaty, Editor), Wiley, New York. (To appear.)

3. D. Hawkins and W. E. Mientka, On sequences which contain no repetitions, Math. Student, 24 (1956) 185–187.

4. John Leech, A problem on strings of beads, Math. Gazette, 41 (1957) 277–278.

5. Marston Morse, A solution of the problem of infinite play in chess, Abstract 360, Bull. Am. Math. Soc., 44 (1938) 632.

6. Marston Morse and Gustav A. Hedlund, Unending chess, symbolic dynamics and a problem in semi-groups, Duke Math. J., 11 (1944) 1–7.

7. Problem 5030 [1962, 439], solution, this MONTHLY, 70 (1963) 675–676.

Added in proof: It has recently been noted that many of the results contained in 5 and 6 were anticipated by the fundamental work of Axel Thue, Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen, Norske Videnskaps Akademi (Oslo), Matematisk Naturvidenskapelig Klasse, Skifter No. 1 (1912) 67. (Formerly Skr. Norske Vid.-Akad., Kristiania I Mat. Naturv. Klasse.) A characterization of these sequences on two symbols is given by G. A. Hedlund and W. H. Gottschalk, A characterization of the Morse minimal set, Proc. Am. Math. Soc., 15 (1964) 70–74.

A SHORT PROOF OF WEDDERBURN'S THEOREM

D. W. HENDERSON, University of Wisconsin

Wedderburn's theorem is an important theorem which gives an ideal-theoretic characterization of complete matrix rings over division rings. The theorem has long been known and a proof is included in most treatises on ring theory (e.g., [1]). In this paper a complete and elementary proof of Wedderburn's theorem is given which appears significantly simpler than any other published proof. The author's contribution lies primarily in the simple proof of the lemma below (a more complicated proof appears in [2], Theorem A.2(g)). In the interest of completeness, however, we show how the lemma can be used to give a short proof of Wedderburn's theorem.

Notation and terminology. M_R will indicate that M is to be considered a right R-module; $\mathrm{Hom}(M_R, M_R)$ will designate the ring of R-homomorphisms of M into M; a ring with identity will be called simple if it contains no proper two-sided ideals; and if A and B are subsets of a ring, AB will denote the set of all finite sums of elements of the form ab, $a\in A$ and $b\in B$.

LEMMA. *If A is a ring with identity 1 and with an idempotent e such that $AeA=A$, then $A\cong\mathrm{Hom}(M_D, M_D)$, where $D=eAe$ and $M=Ae$.*

Proof. 1) Define $f\colon A\to\mathrm{Hom}(M_D, M_D)$ by $[f(a)](m)=am$, for $m\in M$ and $a\in A$. It is easy to show that $f(a)\in\mathrm{Hom}(M_D, M_D)$, for all $a\in A$, and that f is a ring homomorphism.

2) If $aM=aAe=0$, then $aAeA=aA=0$ and hence $a=a1=0$. Therefore the kernel of f is 0.

3) Since $A=AeA$, we have $1=\sum a_ieb_i$, for some a_i, $b_i\in A$, $i=1, 2, \cdots, k$. Let $\delta\in\mathrm{Hom}(M_D, M_D)$ and $m=me\in M$. Then, since $M=Ae$ and $e^2=e$, we have

$$\begin{aligned}\delta(m) &= \delta(1m) = \delta([\textstyle\sum a_ieb_i]m) = \sum \delta(a_ieb_im) = \sum \delta[a_ie(eb_i)(me)]\\ &= \textstyle\sum [\delta(a_ie)eb_ime] \text{ (since } eb_ime\in D) = \{\sum [\delta(a_ie)eb_i]\}m.\end{aligned}$$

Because $\sum[\delta(a_ie)eb_i]$ does not depend on m, we have $\delta=f(\sum[\delta(a_ie)eb_i])$ and hence f is onto. Thus we have shown that $A\cong\mathrm{Hom}(M_D, M_D)$.

WEDDERBURN'S THEOREM. *If A is a simple ring with identity* 1 *and with a minimal left ideal $M \neq 0$, then A is isomorphic to the ring of all $n \times n$ matrices over a division ring.*

Remark. The converse of this theorem is true and is easily proved. In addition there is a uniqueness theorem which is often proved with Wedderburn's theorem (see [1]).

Proof. 1) For any $b \in A$, $b \neq 0$, it is true that $AbA = A$, since $1b1 = b \neq 0$ and A is a simple ring.

2) If $M^2 = 0$, then for $0 \neq m \in M$ we would have $0 = (Am)^2 = (AmA)m = Am$. This is impossible because $0 \neq m = 1m \in Am$. Therefore, since M is minimal, $M^2 = M$. For $m \in M$, Mm is a left ideal contained in M; therefore, for some $0 \neq x \in M$, $Mx = M$ and thus there is an $e \in M$ such that $ex = x$ and $(e^2 - e)\,x = 0$. The left annihilator in M of x is a left ideal contained in M and $ex = x \neq 0$, therefore since M is minimal, the annihilator in M of x is 0. Thus $e^2 - e = 0$. Since $0 \neq e = e^2 \in Ae$ and M is minimal, we have $M = Ae$.

3) We can now apply the Lemma to get $A \cong \mathrm{Hom}(M_D, M_D)$, where $D = eAe$.

4) If $0 \neq eae \in eAe$, then $eae \in Aeeae = Ae$ since Ae is minimal. Therefore there exists an $a' \in A$ such that $a'eeae = e$. Thus $(ea'e)(eae) = e$ and every nonzero element in eAe has a left inverse with respect to the identity e. Thus there is an $s \in eAe$ such that $e = s(eae)(ea'e) = s(eae)e(ea'e) = s(eae)\,[(ea'e)(eae)\,](ea'e) = (eae)(ea'e)$. We conclude that all left inverses are right inverses and therefore eAe is a division ring.

5) If M were infinite dimensional over D, then the transformations of finite dimensional range in $\mathrm{Hom}(H_D, M_D)$ would form a proper two-sided ideal; therefore M must be finite dimensional over D.

6) It is well known that 4) and 5) imply that $\mathrm{Hom}(M_D, M_D)$ is isomorphic to the ring of $n \times n$ matrices over D. (See [3], page 212.)

The author was a National Science Foundation Graduate Fellow and is now at the Institute for Advanced Study.

References

1. C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York, 1962.

2. M. Auslander and O. Goldman, Maximal Orders, Trans. Amer. Math. Soc., 97 (1960) 1–24.

3. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, rev. ed., Macmillan, New York, 1953.

ORDER IN LOGIC AND INTEGRAL DOMAINS

ARNOLD A. JOHNSON, The University of Toledo

1. Introduction. There is an analogy between order in integral domains [1] and order in the propositional calculus [2]. The propositional calculus is ordered by the material implication connective, $\supset$, where "$p \supset q$" is definable as

"$\neg p \vee q$ is true" (i.e. "either not p or q is true") while an ordered integral domain is ordered by the relation $<$ where "$a<b$" is definable as "$-a+b$ is positive." There are many theorems that the two realms share. The one that started the author thinking along the lines of this paper is the theorem "*if* $a<b$ *then* $-b<-a$" which in logic is the law of contraposition: "if p implies q then not q implies not p." The examples suggest that there is a common theory.

It is well known that a Boolean algebra of statements can be made into a ring [**3**] by interpreting "$p+q$" as "either p and not q or q and not p." However, "(not p)$+q$" then becomes "either not p and not q or p and q," which means "p is equivalent to q" and not "if p then q." Consequently we will look in another direction for the common theory.

Integral domains and Boolean algebras both have two closed binary operations which we will denote by $+$ and $\cdot$. The operations are commutative and associative and the operation $\cdot$ distributes over the operation $+$. Such a system we will call a *commutative semiring* [**4**]. In both propositional calculus and ordered integral domains there is a negative operator, which we denote by $-$, such that $-(-a)=a$ and such that $-[a+(-a)]$ is a zero element 0, obeying the laws $x+0=x$ and $x\cdot 0=0$ for all elements x. Moreover there is a set of negative elements (called false sentences in the propositional calculus) such that if a is negative then $-a$ is nonnegative.

In order to generate a common theory of ordered semirings, therefore, we need only lay down axioms of order similar to but not identical to those of an ordered integral domain. There are, of course, some striking differences. For example, in the propositional calculus it is false that $-0=0$ since 0 may be interpreted as "both a and not a" and -0 may be interpreted as "either a or not a." In fact in the propositional calculus 0 is a negative element (i.e. false) and -0 is a nonnegative element (i.e. true) while among integers 0 is nonnegative and equal to -0. Hence the theory cannot depend on whether 0 is negative or nonnegative.

Since the group structure under addition cannot be assumed, the number of axioms for an ordered semiring will be more numerous than the number usually given for an integral domain. We will begin with a simple ordered structure called an *ordered commutative semigroup*. Here we need four axioms instead of the two or three needed for an ordered Abelian group. Most of the theorems of interest can be proved using this system. The extension to ordered semigroups with a zero element requires the axiom that $-[a+(-a)]=0$. Finally, using two more axioms, we extend the theory of order to semirings.

2. Ordered semigroups. A *semigroup* is a set A together with an associative binary operation $A\times A\to A$. Let S be a commutative semigroup (whose operation will be denoted by $+$) having a negation operation denoted by $-$ such that $-(-a)=a$. Suppose further that S contains a subset of elements called the *negative* elements of S. If S satisfies the following axioms then S will be said to be an *ordered semigroup*. (An ordered semigroup is not to be confused with a lattice ordered semigroup [**5**].)

I. *a is negative yields $-a$ is nonnegative*, (law of the excluded middle).

II. *$-a+b$ is nonnegative and a is nonnegative yield b is nonnegative*, (modus ponens or law of detachment).

III. *$(-a+b)+(-b+a)$ is nonnegative*, (law of dichotomy).

IV. *$-(-a+b)+[-(a+c)]+(b+c)$ is nonnegative*, (additivity).

Axiom IV is primitive proposition 1.6 of *Principia Mathematica* [2]. We will first deduce some useful and elementary consequences. By a double application of axiom II to axiom III we deduce the result: (A) *if a and b are nonnegative then $a+b$ is nonnegative* (closure). From axioms I and II and the law $-(-a)=a$ we obtain: (B) *if $a+b$ is nonnegative then a is nonnegative or b is nonnegative.* The contrapositive of (B) is: (C) *if a and b are negative then $a+b$ is negative.* From axiom III and (B) we obtain the formal counterpart of axiom I (law of the excluded middle): (D) *$a+(-a)$ is nonnegative.* It follows from III and (B) that: (E) *if $a+b$ is negative then $-a+(-b)$ is nonnegative.*

We give the usual definition of the order relation $<$ (weakened so as to allow $a<b$ when $a=b$).

DEFINITION: *"$a<b$" for "$-a+b$ is nonnegative".*

We will prove that the above definition leads to the usual properties of weak order. The following are simple consequences of the definition and the four axioms:

1. $a<a$ (reflexive law).
2. $a<b$ *or* $b<a$ (law of dichotomy).
3. *If $a<b$ then $a+c<b+c$* (additivity).
4. *If $a<b$ and $b<c$ then $a<c$* (transitive law).
5. *If a is negative then $a<-a$ and if a is nonnegative then $-a<a$.*
6. *If $a<b$ then $-b<-a$*, (law of contraposition).

In the following simple proofs of these consequences we shall abbreviate nonnegative to n.n. and negative to neg. The reflexive law (1) $a<a$ follows from the definition of $<$ together with the law saying that $a+(-a)$ is n.n.

The law of dichotomy (2) ($a<b$ or $b<a$) is axiom III translated by the use of (B) and the definition $<$. In any order relation $<$ it is an important question whether $<$ is antisymmetric, i.e. whether from $a<b$ and $b<a$ it follows that $a=b$. If in the propositional calculus one interprets "$p=q$" as "p is equivalent to q" then obviously $p\supset q$ and $q\supset p$ yield $p=q$. But if "$p=q$" is interpreted as "p has the same meaning as q" then it is not necessarily the case that $p=q$ follows from $p\supset q$ and $q\supset p$. Since the system under consideration admits the latter interpretation of "$=$" it does not follow without an additional hypothesis that $<$ is antisymmetric.

Applying the definition of $<$ and modus ponens (ax. II) to axiom IV we obtain additivity (3).

From modus ponens (ax. II) together with the definition of $<$ we obtain: if $a<b$ then (a is n.n. yields b is n.n.). Consequently by additivity we have:

if $a<b$ then ($a+x$ is n.n. yields $b+x$ is n.n.) for all x. We next show that if for all x ($a+x$ is n.n. yields $b+x$ is n.n.) then $b+(-a)$ is n.n. The proof is immediate upon setting $x=-a$ and observing that $-a+a$ is n.n. Consequently we establish (F): *$a<b$ if and only if for all x, $(a+x)$ is n.n. yields $(b+x)$ is n.n.* The transitivity (4) of $<$ follows from (F.)

Property (5) results from the definition of $<$, (B), and $-(-a)=a$. Finally the law of contraposition (6) follows from $-a+b=-(-b)+(-a)$.

As an additional simple consequence we have:

(7) *If e is a neutral element* (such that $e+a=a$ for all a) *and if a is negative then $a<e$.*

We will say that an ordered semigroup S is an *ordered semigroup with a zero element* if $-(-a+a)$ is a neutral element and we will write $-(-a+a)=0$. It follows in this case that:

(8) *If a is nonnegative then $-a<0<a$.*

Property (8) holds because -0 is nonnegative. The uniqueness of the neutral element 0 follows by the familiar argument: if 0 and $0'$ are neutral elements then $0=0+0'=0'$.

3. Ordered semirings. We now extend these results to a commutative semiring S. S has two operations $+$ and $\cdot$ such that S is a commutative semigroup under both operations and satisfies the distributive law $a\cdot(b+c)=a\cdot b+a\cdot c$. We will assume that S is an ordered semigroup with a zero element under addition. (Thus $-(-a+a)=0$ and $0+x=x$ for all x.) We will not assume the law $a\cdot 0=0$ since it will be provable. S is an *ordered semiring* provided it satisfies the axioms:

V. *a and b are nonnegative yields $a\cdot b$ is nonnegative*, (closure).

VI. *a is nonnegative yields $a\cdot(-b)=-(a\cdot b)$*, (rule of unlike signs).

Grassman used an axiom similar to VI in his axiomatization of arithmetic [6].

THEOREM. $a\cdot 0=0$.

Proof. If a is n.n. then the rule of unlike signs (axiom VI), the distributive law and the definition of 0 yield:

$$a\cdot[-(-b+b)]=-[a\cdot(-b+b)]=-[a\cdot(-b)+a\cdot b]=-[-(a\cdot b)+a\cdot b]=0.$$

If a is neg. then for the same reasons

$$a\cdot(-b+b)=-[(-a)\cdot(-b+b)]=(-a)\cdot[-(-b+b)]=0.$$

Consequently, if a is neg.,

$$a\cdot[-(-b+b)]=a\cdot[-(-b+b)]+a\cdot(-b+b)=a\cdot[-(-b+b)+(-b+b)]=0.$$

In the course of the above proof we have established the

LEMMA. *If a is negative then $a\cdot(-b+b)=0$.*

It follows from axiom V and VI that: (G) *if a is negative and b is nonnegative then $-(a\cdot b)$ is nonnegative.*

Proofs of the following are left to the reader:

(9) *If a is nonnegative and $b<c$ then $a\cdot b<a\cdot c$.*

(10) *If a is negative and $b<c$ then $a\cdot c<a\cdot b$.*

4. Concluding remarks. For an ordered semiring to be an ordered integral domain it is only necessary that the following two conditions be true: (1) $0=-0$ and (2) *both a and $-a$ are nonnegative iff $a=0$.* The nonzero nonnegative elements are called positive and we have a trichotomy instead of a dichotomy.

In order to obtain the propositional calculus from an ordered semi-group or semiring we need only the following three conditions:

(1) $a+a=a$,

(2) $-a+(a+b)$ *is nonnegative,*

(3) $a\cdot b=-[(-a)+(-b)]$.

Here "a is nonnegative" may be read "a is true" and written "$\vdash a$." Condition (3) is usually introduced as a definition.

References

1. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, 2nd ed. ,Macmillan, New York, 1953.
2. A. Whitehead and B. Russell, Principia Mathematica, vol. I., Cambridge Univ. Press, 1910, 2nd ed., 1925.
3. M. H. Stone, The theory of representation for Boolean algebra, Trans. A.M.S., 40 (1936) 37–111.
4. H. S. Vandiver, On the imbedding of one semi-group in another, with application to semirings, Amer. J. Math., 62 (1940) 72–78.
5. G. Birkhoff, Lattice Theory, New York, 1948.
6. H. Wang, The axiomatization of arithmetic, J. Symb. Logic, 22 (1957) 145–157.

CONGRUENCES FOR THE COEFFICIENTS OF THE k-th POWER OF A POWER SERIES (II)

JOSEPH ARKIN, Spring Valley, New York

1. Introduction. In a previous paper, [1] I proved the following: Put

$$\left(\sum_{n=0}^{\infty} C_n x^n\right)^k = \sum_{n=0}^{\infty} C_n^{(k)} x^n, \tag{1.1}$$

where the C_n are integers and k is an integer $\geqq 1$. We define $C_n^{(k)}$ by means of (1.1) for all integral k. (The series considered in this paper are purely formal power series, and the question of their convergence is irrelevant.)

We then have:

$$C_n^{(k)} \equiv 0 \pmod{k/(n,k)} \qquad n = 1, 2, 3, \cdots. \tag{1.2}$$

and when $C_0=1$,

$$C_n^{(-k)} \equiv 0 \pmod{k/(n, k)}. \tag{1.3}$$

2. Supplementary results.

THEOREM I. *If for a prime p and for some integers t and $r(r\geqq 1)$, the congruence*

$$C_{mp+t}^{(d)} \equiv 0 \pmod{p^{r-s}}$$

holds for $m=0, 1, 2, \cdots$ and for $0\leqq s\leqq r$, then

$$C_{mp+t}^{(p^r a+d)} \equiv 0 \pmod{p^{r-s}}$$

for any integer a.

Before giving a proof of Theorem I, we state two lemmas.

LEMMA I.

$$\sum_{n=0}^{\infty} C_n^{(p^r a)} x^n \equiv \sum_{n=0}^{\infty} C_{np}^{(p^r a)} x^{np} \pmod{p^r}. \tag{2.1}$$

Proof. Lemma I is an immediate consequence of (1.2). Since p is a prime, all the coefficients in $\sum_{n=0}^{\infty} C_n^{(p^r a)} x^n$ are divisible by p^r, except those of $1, x^p, x^{2p}, x^{3p}, \cdots$, and Lemma I is proved.

LEMMA II. *When $C_0=1$,*

$$\sum_{n=0}^{\infty} C_n^{(-p^r a)} x^n \equiv \sum_{n=0}^{\infty} C_{np}^{(-p\ a)} x^{np} \pmod{p^r}. \tag{2.2}$$

Proof. Using (1.3), we prove Lemma II just as we proved Lemma I. We restate Lemmas I and II as follows:

LEMMA I.

$$\sum_{n=0}^{\infty} C_n^{(p^r a)} x^n \equiv \sum_{n=0}^{\infty} C_{np}^{(-p^r a)} x^{np} \pmod{p^{r-s}}. \tag{2.11}$$

LEMMA II. *When $C_0=1$,*

$$\sum_{n=0}^{\infty} C_n^{(-p^r a)} x^n \equiv \sum_{n=0}^{\infty} C_{np}^{(-p^r a)} x^{np} \pmod{p^{r-s}}. \tag{2.21}$$

We are now in a position to prove Theorem I. Multiplying the congruence of Lemma I (2.11) through by $\sum_{n=0}^{\infty} C_n^{(d)} x^n$, we get

$$\sum_{n=0}^{\infty} C_n^{(p^r a+d)} x^n \equiv \left(\sum_{n=0}^{\infty} C_{np}^{(p^r a)} x^{np}\right)\left(\sum_{n=0}^{\infty} C_n^{(d)} x^n\right) \pmod{p^{r-s}}.$$

By comparing coefficients we have

$$C_{mp+t}^{(p^r a+d)} \equiv \sum_{v=0}^{m} C_{vp}^{(p^r a)} C_{(m-v)p+t}^{(d)} \pmod{p^{r-s}}, \qquad m=0, 1, 2, \cdots. \tag{2.3}$$

Since it is evident that $mp+t$ runs through the same values as $(m-v)p+t$, Theorem I is proved.

The following corollaries are easily proved.

COROLLARY I. *If* $C_0=1$, *and if* $C_{mp+t}^{(-d)} \equiv 0 \pmod{p^{r-s}}$, *$p$ being a prime, and t is any integer for which the congruence holds, then*

$$C_{mp+t}^{(p^r a-d)} \equiv 0 \pmod{p^{r-s}}.$$

COROLLARY II. *If* $C_0=1$, *and if* $C_{mp+t}^{(d)} \equiv 0 \pmod{p^{r-s}}$, *$p$ being a prime, and t is any integer for which the congruence holds, then*

$$C_{mp+t}^{(-p^r a+d)} \equiv 0 \pmod{p^{r-s}}.$$

COROLLARY III. *If* $C_0=1$, *and if* $C_{mp+t}^{(-d)} \equiv 0 \pmod{p^{r-s}}$, *$p$ being a prime, and t is any integer for which the congruence holds, then*

$$C_{mp+t}^{(-p^r a-d)} \equiv 0 \pmod{p^{r-s}}.$$

3. In this section we discuss an application of our results. Let

$$\left(\sum_{n=0}^{\infty} \sigma(2n+1)x^n\right)^{-1} = \sum_{n=0}^{\infty} B_n x^n, \tag{3.1}$$

where $\sigma(2n+1)$ equals the sum of the divisors of $2n+1$.

THEOREM II.

$$B_{5m+2} \equiv 0 \quad \text{and} \quad B_{5m+4} \equiv 0 \pmod 5, \qquad m = 0, 1, 2, \cdots. \tag{3.2}$$

Proof. By substituting the last digit endings of v into $v(v+1)$, it is seen that $v(v+1)$ ends in 0, 2, and 6, but never in 4 or 8. Hence $v(v+1) \neq 10m+4$ or $10m+8$ and we have

$$v(v+1)/2 \neq 5m+2 \quad \text{or} \quad 5m+4. \tag{3.3}$$

Put

$$\sum_{v=0}^{\infty} x^{v(v+1)/2} = \sum_{n=0}^{\infty} C_n x^n; \tag{3.4}$$

here $C_0=1$. By (3.3), $C_{5m+2}=0$ and $C_{5m+4}=0$, so that we have

$$C_{5m+2} \equiv 0 \quad \text{and} \quad C_{5m+4} \equiv 0 \pmod 5. \tag{3.5}$$

Since $C_0=1$, we apply (1.3) to the identity in (3.4), and get

$$C_{5m+2}^{(-5)} \equiv 0 \quad \text{and} \quad C_{5m+4}^{(-5)} \equiv 0 \pmod 5. \tag{3.6}$$

Applying the results of (3.5) and (3.6) to Corollary II, (where since $d=1$, $r=1$, $s=0$, $p=5$, and $t=2$ and 4), we have the following congruences

$$C_{5m+2}^{(-5+1)} \equiv 0 \quad \text{and} \quad C_{5m+4}^{(-5+1)} \equiv 0 \pmod 5. \tag{3.7}$$

The coefficients $C_{5m+2}^{(-4)}$ and $C_{5m+4}^{(-4)}$ are defined by

$$\sum_{n=0}^{\infty} C_n^{(-4)} x^n = \left(\sum_{v=0}^{\infty} x^{v(v+1)/2} \right)^{-4}. \tag{3.8}$$

A well-known identity of Jacobi (see [2]) is,

$$\sum_{n=0}^{\infty} C_n^{(-4)} x^n = \left(\sum_{v=0}^{\infty} x^{v(v+1)/2} \right)^{-4} = \left(\sum_{k=0}^{\infty} \sigma(2k+1)x^k \right)^{-1}, \tag{3.9}$$

where $\sigma(2k+1)$ equals the sum of the divisors of $2k+1$. Since (3.8) and (3.9) are identical, we see that, in

$$\left(\sum_{k=0}^{\infty} \sigma(2k+1)x^k \right)^{-1} = \sum_{n=0}^{\infty} C_n^{(-4)} x^n,$$

the congruences of (3.7) hold, and Theorem II is proved.

The author wishes to thank the referee for his suggestions.

References

1. J. Arkin, Congruences for the coefficients of the kth power of a power series (I), this MONTHLY, 71 (1964) 899–900.

2. L. E. Dickson, History of the Theory of Numbers, 3 vols., Washington, 1919–27, (vol. I, p. 381).

THE INVERSION OF A CONVOLUTION TRANSFORM WHOSE KERNEL IS A BESSEL FUNCTION

P. L. BHARATIYA, St. Aloysius' College, Jabalpur, India

1. In the case of some functions it is possible to invert a certain convolution transform by a similar convolution transform. Ta Li [4] has proved this for the kernel which is a Tchebycheff polynomial and R. G. Buschman [1] obtained a similar result where the kernel is a Legendre polynomial.

Recently D. V. Widder [5] has given new proofs for these inversions using Laplace transforms and he applied the method to invert the convolution transform which involves a Laguerre polynomial as its kernel.

The object of this paper is to obtain similar results for the convolution transform whose kernel is a Bessel function. The method adopted is that of D. V. Widder.

2. A Bessel function $J_n(z)$ is defined [see **3**, p. 109] for n which is not a negative integer, by

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+n}}{2^{2k+n} k! \Gamma(1+n+k)}, \tag{2.1}$$

and if n is a negative integer, by

$$J_n(z) = (-1)^n J_{-n}(z). \tag{2.2}$$

We shall represent the Laplace transform $F(p) = p\int_0^\infty e^{-pt} f(t)dt$, Re $(p) > 0$, by

$$F(p) \doteq f(t). \tag{2.3}$$

3. Results required in the proof. We have [2]

$$\frac{\Gamma(n+\frac{1}{2})(2a)^n p}{\sqrt{\pi}(p^2+a^2)^{n+1/2}} \doteq t^n J_n(at) \qquad \operatorname{Re} n > -\tfrac{1}{2};\ \operatorname{Re} p > |\operatorname{Im} a| \tag{3.1}$$

$$a^{n/2} p^{-n} e^{-a/p} \doteq t^{n/2} J_n(2\sqrt{at}) \qquad \operatorname{Re} n > -1 \tag{3.2}$$

$$a^{n/2} p^{-n} e^{a/p} \doteq t^{n/2} I_n(2\sqrt{at}) \qquad \operatorname{Re} n > -1 \tag{3.3}$$

$$ap/(p^2+a^2) \doteq \sin(at) \qquad \operatorname{Re} p > |\operatorname{Im} a| \tag{3.4}$$

4. THEOREM 1. *If $g(x) \in C^1$ for $0 \leqq x < \infty$, $g(0) = 0$, and*

$$f(x) = \int_0^x J_0(x-t) g(t)\, dt \tag{4.1}$$

then $f(x) \in C^2$ for $0 \leqq x < \infty$, $f(0) = f'(0) = 0$, and

$$g(x) = \int_0^x J_0(x-t)[D^2+1] f(t)\, dt. \tag{4.2}$$

Proof. Since $g(x) \in C^1$ for $0 \leqq x < \infty$ and $g(0) = 0$, we clearly have $f(x) \in C^2$ for $0 \leqq x < \infty$ and $f(0) = f'(0) = 0$.

Also if $F(p) \doteq f(t)$ and $G(p) \doteq g(t)$, $(p^2+1)F(p) \doteq (D^2+1)f(t)$ and equation (4.1) transforms into

$$F(p) = \frac{1}{p}\left[\frac{p}{\sqrt{(p^2+1)}}\right] \cdot G(p)$$

so that

$$G(p) = \frac{1}{p}\left[\frac{p}{\sqrt{(p^2+1)}}\right] \cdot (p^2+1) F(p).$$

Inverting this result we get

$$g(x) = \int_0^x J_0(x-t)(D^2+1) f(t)\, dt.$$

The theorem is thus proved.

5. Alternative proof. The second part of the theorem can also be proved as follows. We have the identity,

$$\sqrt{(p^2+1)} = (p^2+1)/\sqrt{(p^2+1)}.$$

Hence

$$\frac{1}{p}\cdot\frac{p}{\sqrt{(p^2+1)}}\cdot\frac{p}{\sqrt{(p^2+1)}} = \frac{p}{p^2+1}.$$

Inverting this result, we get

$$\int_0^x J_0(x-t)J_0(t)\,dt = \sin(x). \tag{5.1}$$

Now let

$$h(x) = \int_0^x J_0(x-t)f(t)\,dt \tag{5.2}$$

$$= \int_0^x J_0(t)f(x-t)\,dt. \tag{5.3}$$

As $f(0)=f'(0)=0$, direct differentiation of (5.3) yields $(D^2+1)h(x)=$ the right member of (4.2).

Now substituting the value of $f(t)$ from (4.1) in (5.2), we obtain

$$\begin{aligned} h(x) &= \int_0^x J_0(x-t)\left[\int_0^t J_0(t-y)g(y)\,dy\right]dt \\ &= \int_0^x g(y)\left[\int_y^x J_0(x-t)J_0(t-y)\,dt\right]dy \\ &= \int_0^x g(y)\left[\int_0^{(x-y)} J_0(x-y-t)J_0(t)\,dt\right]dy. \end{aligned}$$

It follows from (5.1) for the inner integral that

$$h(x) = \int_0^x \sin(x-y)g(y)dy,$$

and we easily get $(D^2+1)h(x)=g(x)$. The proof is thus complete.

6. The following general form of Theorem 1 can be proved similarly.

If $g(x)\in C^{2r+1}$ for $0\leqq x<\infty$, and $g_i(0)=0$, for $i=0, 1, 2, \cdots, 2r$, (i.e. if $g(x)$ and all its $(2r+1)$ differential coefficients are continuous for $0\leqq x<\infty$, and it and its $2r$ differential coefficients vanish at $x=0$), and $f(x)=\int_x^0 J_0(x-t)$ $g(t)dt$, then

$$f(x) \in C^{2r+2} \quad \text{for} \quad 0 \leqq x < \infty,$$
$$f_s(0) = 0 \qquad \text{for} \quad s = 0, 1, 2, \cdots, (2r+1),$$

and $g(x) = \sqrt{\pi}/[\Gamma(r+\frac{1}{2})2^r]\int_0^x (x-t)^r J_r(x-t)(D^2+1)^{r+1}f(t)dt$, where r is any nonnegative integer.

The proof depends upon the simple identity

$$\sqrt{(p^2 + 1)} = (p^2 + 1)^{r+1}/(p^2 + 1)^{r+1/2}.$$

The Theorem can be easily generalized further and we have the following.

THEOREM. *If $2n$ is a nonnegative integer,*

$$g(x) \in C^{s+1} \quad \textit{for} \quad 0 \leqq x < \infty,$$
$$g_i(0) = 0 \qquad \textit{for} \quad i = 0, 1, 2, \cdots, s,$$

where k and s are the greatest integers in $(n+3/2)$ and $2(r+k-n-\frac{1}{2})$ respectively, r being any nonnegative integer, and $f(x) = \int_0^x (x-t)^n J_n(x-t)g(t)dt$ then

$$f(x) \in C^{2k+2r} \quad \textit{for} \quad 0 \leqq x < \infty,$$
$$f_j(0) = 0 \qquad \textit{for} \quad j = 0, 1, 2, \cdots, (2k+2r-1),$$

and

$$g(x) = \frac{\pi}{\Gamma(n+\frac{1}{2})\Gamma(r+k-n-\frac{1}{2})(2)^{r+k-1}} \int_0^x (x-t)^{r+k-n-1} J_{r+k-n-1}(x-t)\cdot(D^2+1)^{k+r}f(t)\,dt.$$

The proof of this theorem depends on the simple identity

$$(p^2 + 1)^{n+1/2} = \frac{(p^2+1)^{r+k}}{(p^2+1)^{r+k-n-1/2}}\cdot$$

7. THEOREM 2. *If $g(x) \in C^1$ for $0 \leqq x < \infty$, $g(0) = 0$, and*

$$f(x) = \int_0^x J_0(2\sqrt{(x-t)})g(t)\,dt,$$

then $f(x) \in C^2$ for $0 \leqq x < \infty$, $f(0) = f'(0) = 0$ and

$$g(x) = \int_0^x I_0(2\sqrt{(x-t)})\,D^2 f(t)\,dt,$$

where $I_0(x)$ is a modified Bessel function.

The proof of this theorem is similar to that of Theorem 1. It depends on the simple identity $e^{-1/p}\cdot e^{1/p} = 1$.

The theorem can be generalized easily as follows: *If n is a nonnegative integer*

and $g(x) \in C^{n+1}$ *for* $0 \leqq x < \infty$, *and* $g_i(0) = 0$ *for* $i = 0, 1, 2, \cdots, n$,

$$f(x) = \int_0^x (x-t)^{n/2} J_n(2\sqrt{(x-t)})g(t)\,dt$$

then $f(x) \in C^{2n+2}$ *for* $0 \leqq x < \infty$, $f_j(0) = 0$ *for* $j = 0, 1, 2, \cdots, (2n+1)$, *and*

$$g(x) = \int_0^x (x-t)^{n/2} I_n(2\sqrt{(x-t)})\, D^{2n+2} f(t)\,dt,$$

where $I_n(x)$ *is a modified Bessel function.*

My sincere thanks are due to Dr. B. R. Bhonsle for suggesting the problem and for guidance in the preparation of this paper. I am also thankful to the Editor and the referee for their kind advice.

References

1. R. G. Buschman, An inversion integral for the Legendre transformation, this MONTHLY, 69 (1962) 288–289.
2. Mc Lachlan, Modern Operational Calculus, Macmillan, London, 1948.
3. E. D. Rainville, Special Functions, Macmillan, New York, 1960.
4. Ta Li, A new class of integral transforms, Proc. Amer. Math. Soc., 11 (1960) 290–298.
5. D. V. Widder, The inversion of a convolution transform whose kernel is a Laguerre polynomial, this MONTHLY, 70 (1963) 291–295.

A THEOREM ON POWER SETS

E. S. WOLK, University of Connecticut

Let $P(S)$ denote the power set of the set S. Problem 5127 (this MONTHLY, August–September, 1963, p. 275) asks whether $P(S)$ contains an uncountable chain if S is countable. To see that the answer is affirmative, map S in a 1:1 manner onto the set R of all rational numbers in the interval $[0, 1]$ on the real axis. For any irrational x in $[0, 1]$, let $E_x = \{r \in R: r < x\}$. Then the family of sets $\{E_x\}$ is an uncountable chain.

The purpose of this note is to show that, assuming the Generalized Continuum Hypothesis, a much more general result is true. Let $|S|$ denote the cardinal number of the set S. We shall prove the following:

THEOREM. *Let k be any infinite cardinal number and S a set with $|S| = k$. Assuming the Generalized Continuum Hypothesis, there exists a chain $\mathcal{C}$ in $P(S)$ with $|\mathcal{C}| = 2^k$.*

If S is a subset of a linearly ordered set L, let us say that S is *dense in L* if and only if, whenever a and b are elements of L with $a < b$, there exists $c \in S$ with $a < c < b$. The proof of the theorem is based on the following lemma (in which the Generalized Continuum Hypothesis is assumed).

LEMMA. *If k is any infinite cardinal number, then there exists a linearly ordered set L of cardinal 2^k which contains a dense subset Z of cardinal k.*

Proof. Let Ω_k denote the first ordinal number whose cardinal is k. Let F be the set of all functions ("transfinite sequences") with domain $\{\alpha:\alpha$ is an ordinal $<\Omega_k\}$ and range $\{0, 1\}$. If $f\in F$ and there exists α such that $f(\beta)=1$ for all $\beta\geqq\alpha$, we say that f is *eventually* 1. Remove from F all those functions which are eventually 1, and call the remaining set of functions L.

If $f, g\in L, f\neq g$, let $\delta(f, g)$ denote the least element of the set $\{\alpha: f(\alpha)\neq g(\alpha)\}$. We linearly order L "lexicographically" by defining $f<g$ if and only if $f(\delta)<g(\delta)$, where $\delta=\delta(f, g)$. Let $Z=\{f\in L: f$ is eventually $0\}$. We assert that $|Z|=k$. For let α be any ordinal $<\Omega_k$, and define $Z_\alpha=\{f\in Z: f(\beta)=0$ for all $\beta\geqq\alpha\}$. Then $|Z_\alpha|=2^{|\alpha|}$. But $|\alpha|<k$ implies $2^{|\alpha|}\leqq k$, by the Generalized Continuum Hypothesis. Thus $|Z_\alpha|\leqq k$ for all α, and hence

$$|Z| = |\cup\{Z_\alpha: \alpha < \Omega_k\}| \leqq k\cdot k = k.$$

Since we obviously also have $|Z|\geqq k$, our assertion follows.

In the same way, it follows that $|\{f\in F: f$ is eventually $1\}|=k$. Since $|F|=2^k$, we then also have $|L|=2^k$.

Furthermore, Z is dense in L. For suppose that $f, g\in L$ and $f<g$. Then $f(\delta)=0$ and $g(\delta)=1$, where $\delta=\delta(f, g)$. Let γ be the first ordinal $>\delta$ such that $f(\gamma)=0$. Such a γ exists since f is not eventually 1. Define $h\in Z$ by $h(\alpha)=f(\alpha)$, for $\alpha<\gamma$, $h(\gamma)=1$, $h(\alpha)=0$, for $\alpha>\gamma$. Then clearly $f<h<g$.

Now to prove the theorem, let S be a given set of cardinal k, and let L and Z be the sets of the lemma. For each $h\in L$, let $E_h=\{f\in Z: f<h\}$. If $\mathfrak{C}=\{E_h: h\in L\}$, then $\mathfrak{C}$ is a chain in $P(Z)$, and $|\mathfrak{C}|=2^k$. Since S is equivalent to Z, the theorem is proved.

Editorial Note. In the paper, "Some Remarks on Orbits in Invertible Spaces" by A. J. Umen, this MONTHLY, 71 (1964) pp. 643–646, the following remark is made on p. 645: "It is not difficult to show that if S and T are invertible spaces then their topological product $S\times T$ is an invertible space." However apparently it would be rather difficult to show this because the statement is false, as P. H. Doyle and S. A. Naimpally have separately pointed out. Professor Doyle cited the circle squared as a counter example. Professor Naimpally wrote as follows to the Editor of this Department: "A counterexample is the torus which is the product $S\times S$ of the circle S with itself. The circle is invertible but if U is a nonempty proper open subset of S then although $U\times U$ is open in $S\times S$ there is no self-homeomorphism of $S\times S$ which carries $S\times S-U\times U$ into $U\times U$. However, such an inverting homeomorphism exists for each open set of the form $U\times S$ or $S\times U$. Let a topological space S be called *subinvertible* if S has a subbasis B such that corresponding to each UB there exists a self-homeomorphism h of S such that $h(S-U)\subset U$. We can say that if S and T are invertible then $S\times T$ is *subinvertible*."

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Md. 20740.

MATRIX NUMBER THEORY: AN EXAMPLE OF NONUNIQUE FACTORIZATION

BERNARD JACOBSON, Franklin and Marshall College and CUPM; ROBERT J. WISNER, CUPM

1. Introduction. One of the most difficult tasks faced by an instructor of elementary undergraduate mathematics is that of properly motivating the need for proofs of theorems that appear to be "obvious" to the students. Nowhere is this need more striking than in a discussion of the Fundamental Theorem of Arithmetic. Most textbooks in elementary number theory ignore this problem.

Several examples of multiplicative semigroups which do not possess unique factorization are given in [1], [2] and [3]. None of these sets is closed under addition, however, and thus these examples do not provide the advantages of the example given below.

The set T of two by two matrices with equal integral entries provides an excellent example of a commutative ring with nonunique factorization. Furthermore, all of the "primes" can be given explicitly by formula, ideals can be introduced into the discussion, and proofs of a "Goldbach conjecture" and a "prime number theorem" are trivial. It should be noted that T is "almost" an integral domain, lacking only a multiplicative identity.

2. Definitions. Notation. Statement of Theorems. Let S be the set of all two by two matrices with equal positive integral entries, i.e.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S, \text{ if and only if, } a = b = c = d.$$

The matrix

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

will be denoted by (x). Since there exist no units in S, (x) will be called composite in S if, and only if, $(x)=(y)(z)$ where $(y), (z)\in S$. Otherwise (x) will be called prime in S. We will not consider two factorizations of (x) different if they differ only in the order in which the factors are written. Let $P(x)$ be the number of prime matrices (y) in S for which $y \leqq x$. The following theorems will be proved:

THEOREM 2.1. *The matrix (x) is prime in S if, and only if, x is odd.*

COROLLARY 2.1.

$$\operatorname*{limit}_{x\to\infty}\frac{2P(x)}{x} = 1.$$

COROLLARY 2.2. *Every composite in S can be written as the sum of two primes.*

THEOREM 2.2. *If (x) is composite in S, then (x) is uniquely factorable in S if, and only if, $x=2^n$ or $x=2^np$, where $n\geqq 1$ and p is an odd prime.*

3. Proof of the Theorems. If (x) is composite in S, then $(x)=(y)(z)=(2yz)$ and x is even. Conversely, if $x=2a$, then $(x)=(2a)=(1)(a)$. Thus the proof of Theorem 2.1 is complete. Theorem 2.2 follows as a result of the following lemmas.

LEMMA 3.1. *If $x=2^n$, where $n\geqq 1$, then (x) is uniquely factorable in S as $(1)^{n+1}$.*

Proof. The proof is by induction. Clearly Lemma 3.1 is true when $n=1$. We assume its validity for $n\leqq k$ and consider $(2^{k+1})=(a)(b)=(2ab)$. It follows that $a=2^r$ and $b=2^{k-r}$ where $0\leqq r\leqq k$. Applying the inductive hypothesis, we obtain $(2^{k+1})=(2^r)(2^{k-r})=(1)^{r+1}(1)^{k-r+1}=(1)^{k+2}$ and the proof is complete.

LEMMA 3.2. *If $x=2^np$, where $n\geqq 1$ and p is an odd prime, then (x) is uniquely factorable in S as $(1)^n(p)$.*

Proof. The proof is again by induction, and follows along the lines of the proof of Lemma 3.1, because $(2^{k+1}p)=(a)(b)=(2ab)$ implies that $a=2^rp$ and $b=2^{k-r}$ with $0\leqq r\leqq k$, the order of the factors $(a)(b)$ being immaterial because of the commutativity.

LEMMA 3.3. *If $x=2^npQ$ where $n\geqq 1$, p is an odd prime and Q is an odd positive integer greater than one, then (x) is not uniquely factorable in S.*

Proof. We merely exhibit two factorizations of (x) as a product of primes in S. If $n>1$, then $(x)=(1)^{n-1}(p)(Q)=(1)^n(pQ)$ and if $n=1$, then $(x)=(p)(Q)=(1)(pQ)$.

Lemmas 3.1, 3.2, and 3.3 exhaust all possible cases, and thus the proof of Theorem 2.2 is complete.

4. Two by two matrices with equal integral entries. In this section we will drop the restriction that the entries be positive. Let T be the set of all two by two matrices with equal integral entries. We adopt definitions of composite and prime which are analogous to the definitions given in 2. (x) is called composite in T if, and only if, $(x)=(y)(z)$ where (y), $(z)\in T$. Otherwise (x) is called prime in T.

THEOREM 4.1. *The matrix (x) is prime in T if, and only if, x is odd.*

The proof is identical to the proof of Theorem 2.1.

THEOREM 4.2. *If* (x) *is composite in* T, *then* (x) *is uniquely factorable in* T *if, and only if,* $x=-2$.

Proof. We exhibit two factorizations for all matrices (x), where $x\neq-2$. In this proof $n\geqq 1$ and Q represents an odd positive integer. If $x=2^nQ$ and $n=1$, then $(x)=(1)(Q)=(-1)(-Q)$. If $x=2^nQ$ and $n>1$, then $(x)=(1)^n(Q)=(1)^{n-1}(-1)(-Q)$. If $x=-2^nQ$ and $n=1$ then $(x)=(-1)(Q)=(1)(-Q)$. If $x=-2^nQ$ and $n>1$, then $(x)=(1)^{n-1}(-1)(Q)=(1)^n(-Q)$. If $x=2$, then $(x)=(1)^2=(-1)^2$. If $x=2^n$ and $n>1$, then $(x)=(1)^{n+1}=(-1)^2(1)^{n-1}$. If $x=-2^n$ and $n>2$, then $(x)=(-1)(1)^n=(-1)^3(1)^{n-2}$. If $x=-4$, then $(x)=(-1)(1)^2=(-1)^3$. Finally, if $x=-2$, then $(x)=(-1)(1)$ is uniquely factorable.

The use of the negatives to display two factorizations seems like cheating, and this is remedied in the next section where the ideals of T are considered.

5. Ideals in T. In this section we will first show that every ideal is principal and then show that the irreducible ideals are those generated by the primes of S. Furthermore, a reducible ideal can be uniquely factored as a product of irreducible ideals if, and only if, its generating element can be uniquely factored as a product of primes in S. We will call (x) the least positive element of an ideal α if, and only if, $y>0$ and $(y)\in\alpha$ implies $0<x\leqq y$. In this section we will adopt the definition of irreducible ideal given in [4]: An ideal α is called irreducible if there exist no ideals β and γ such that $\alpha=\beta\cdot\gamma$.

THEOREM 5.1. *Every ideal in* T *is principal.*

Proof. Let (x) be the least positive element of the ideal α.

$$\sum_{i=1}^{q}(x)=q(x)=(qx)$$

is an element of α for every positive integer q. $(-qx)$ is also an element of α. Thus $(qx)=q(x)$ is an element of α for every integer q. If (y) is any element of α, then $(y)-(qx)=(y-qx)\in\alpha$. By the Euclid Algorithm there exist integers q and r such that $y-qx=r$ and $0\leqq r<x$. Since (x) is the least positive element of α, $r=0$, $y=qx$ and $(y)=(qx)=q(x)$. Therefore, α is a principal ideal.

THEOREM 5.2. *The ideal* α *is a reducible ideal of* T *if, and only if, the least positive element of* α *is composite in* S.

Proof. Let (x) be the least positive element of α. If $\alpha=\beta\gamma$ is reducible, then $(x)=(y)(z)=(-y)(-z)$, where $(y)\in\beta$ and $(z)\in\gamma$. Since $x>0$, y and z are both positive or both negative, either (y) and (z) are elements of S or $(-y)$ and $(-z)$ are elements of S, and thus (x) is composite in S. Conversely, suppose $(x)=(y)(z)$ is composite in S, where $\alpha=\{(x)\}$, $\beta=\{(y)\}$, and $\gamma=\{(z)\}$. If (w) is any element of α, then $(w)=q(x)=q(y)(z)=(qy)(z)$, where $(qy)\in\beta$ and $z\in\gamma$. Also, if $(t)\in\beta$ and $(v)\in\gamma$ then $t=q_1(y)$ and $v=q_2(z)$. Therefore, $(t)(v)=q_1q_2(y)(z)=q(x)\in\alpha$. Therefore $\alpha=\beta\cdot\gamma$ is reducible.

COROLLARY 5.2. *The irreducible ideals of T are exactly those ideals generated by the primes of S.*

THEOREM 5.3. *A reducible ideal can be factored uniquely as a product of irreducible ideals if, and only if, its generating element can be uniquely factored as a product of primes in S.*

Proof. If $(x) = (y_1)(y_2) \cdots (y_j) = (z_1)(z_2) \cdots (z_k)$ where each (y_s), $s=1, 2, \cdots, j$ and each (z_i), $i=1, 2, \cdots, k$, is a prime in S and $(y_l) \neq (z_r)$ for some l and $1 \leqq r \leqq k$, then $\{(x)\} = \{(y_1)\} \cdots \{(y_j)\} = \{(z_1)\} \cdots \{(z_k)\}$, where $\{(a)\}$ denotes the ideal generated by (a). It follows from Theorem 5.2 that each $\{(y_s)\}$ and $\{(z_i)\}$ is a prime ideal. Conversely, suppose that $\{(x)\} = \{(y_1)\} \cdots \{(y_j)\} = \{(z_1)\} \cdots \{(z_k)\}$, where each $\{(y_i)\}$ and $\{(z_i)\}$ is a prime ideal and $\{(y_l)\} \neq \{(z_r)\}$ for some l and $1 \leqq r \leqq k$. Since $\{(a)\} = \{(-a)\}$ we may choose $0 < y_s$ and $0 < z_i$, for $s=1, 2, \cdots, j$ and $i=1, 2, \cdots, k$. Thus $(x) = (y_1) \cdots (y_j) = (z_1) \cdots (z_k)$, and it follows from Theorem 5.2 that y_s and z_i are primes in S.

EXAMPLE. In S, $(2)=(1)^2$ is uniquely factorable, while in T $(2)=(1)^2 =(-1)^2$; but

$$\begin{aligned}\{(2)\} &= \{\cdots, (-4), (-2), (0), (2), (4), \cdots\} \\ &= \{(-3), (-2), (-1), (0), (1), (2), \cdots\}^2 = \{(1)\}^2\end{aligned}$$

is uniquely factorable.

6. The set T reinterpreted. It will be shown in this section that the ring T is merely the ring E of even rational integers in disguised form. Consider the following one-to-one correspondence between the elements of S and the elements of E: $(x) \rightarrow 2x$.

$$(x) + (y) = (x + y) \rightarrow 2[x + y] = 2x + 2y \quad \text{and} \quad (x) + (y) \rightarrow 2x + 2y.$$

$$(x)(y) = (2xy) \rightarrow 4xy \quad \text{and} \quad (x)(y) \rightarrow 2x \cdot 2y = 4xy.$$

Thus the rings S and E are isomorphic.

7. Conclusion. The illustration by means of two by two matrices should provide greater impact than could be provided by the even integers, even after the isomorphism is pointed out to the students. The use of a subset of the integers to provide an illustration of a property not possessed by the integers themselves seems somewhat contrived to students. This example has the further advantage that generalization to the $n \times n$ case can be left as an exercise for any bright freshman.

References

1. Newcomb Greenleaf and Robert J. Wisner, The Unique Factorization Theorem, Math. Teacher, 52 (1959) 600–603.

2. Bernard Jacobson and Robert J. Wisner, Matrix Number Theory I: Factorization of 2×2 unimodular matrices, AMS Notices, 10 (1963) 264 (Submitted for publication).

3. ———, Matrix Number Theory II: Factorization of 2×2 singular matrices, *ibid.*

4. Harry Pollard, Theory of Algebraic Numbers, Carus Monograph no. 9, 1950.

A FUNCTIONAL DEFINITION OF THE DETERMINANT

C. G. CULLEN AND K. J. GALE, University of Pittsburgh

The classical definition of the determinant is a rather awkward and cumbersome way of introducing this important function on square matrices (at least for students of linear algebra). The idea of giving a functional definition of this function is by no means a new idea; it was first investigated by Weierstrass about 80 years ago. Several recent textbooks on linear algebra [2], [3] utilize functional definitions of the determinant. None of these definitions employ, as an essential ingredient, the relation $|AB|=|A|\ |B|$ which is clearly the most significant property of the determinant function.

In this paper we provide a functional definition of the determinant, using the above multiplicative property as the essential ingredient, which is believed to be as efficient as possible.

Let C_n be the algebra of n by n matrices with elements from the complex field C. We begin by defining elementary matrices on C_n in the usual way: $E_{ij}, i\neq j$, is obtained from I, the identity matrix, by interchanging rows i and j, $E_j(z)$ is obtained from I by multiplying row j by the scalar z, and $E_{ij}(k)$ is obtained from I by replacing row i by itself plus k times row j. Except for $E_j(0)$, these elementary matrices are all nonsingular: $E_j(z)^{-1}=E_j(1/z)$, $E_{ij}^{-1}=E_{ij}$, and $E_{ij}(k)^{-1}=E_{ij}(-k)$. It is well known that every nonsingular A in C_n is a product of nonsingular elementary matrices [3], and it is immediate that, by appropriately using the matrices $E_j(0)$, we can write any singular matrix of C_n as a product of our elementary matrices.

DEFINITION 1. *Let d be a function from C_n to C such that $d(E_1(z))=z$ and $d(AB)=d(A)d(B)$ for all z in C and for all A, B from C_n.*

It is clear from our definition and our first theorem that d is the determinant.

THEOREM 1. $d(E_j(k))=k$, $d(E_{ij}(k))=1$ *for all k in C, and* $d(E_{ij})=-1$.

Proof. $E_1(1)=I$, the identity matrix, so it is immediate that $d(I)=1$.

Since $E_j(k)=E_{j1}E_1(k)E_{j1}$ we have $d(E_j(k))=d(E_{j1}E_{j1})d(E_1(k))=k$.

From the easily checked identity, $E_{ij}(k)=E_j(1/k)E_{ij}(1)E_j(k)$, $(k\neq 0)$, we have

$$\begin{aligned} d(E_{ij}(k)) &= d(E_j(1/k))d(E_j(k))d(E_{ij}(1)) = d(E_j(1/k)E_j(k))d(E_{ij}(1)) \\ &= d(I)d(E_{ij}(1)), \end{aligned}$$

and hence

$$d(E_{ij}(k)) = d(E_{ij}(1)) \quad \text{for} \quad k \neq 0. \tag{1}$$

Since $E_{ij}(1)^2=E_{ij}(2)$, we see from (1) that

$$d(E_{ij}(1))^2 = d(E_{ij}(1)^2) = d(E_{ij}(2)) = d(E_{ij}(1)),$$

and hence that $d(E_{ij}(1))=1$ or $d(E_{ij}(1))=0$. From $I=E_{ij}(1)E_{ij}(-1)$ it is clear

that $d(E_{ij}(1))=0$ is inconsistent with $d(I)=1$. Thus $d(E_{ij}(k))=d(E_{ij}(1))=1$ and the proof of the first two assertions is completed.

A direct calculation establishes the identity.

$$E_{ij} = E_i(-1)E_{ji}(1)E_{ij}(-1)E_{ji}(1) \tag{2}$$

and thus we have

$$d(E_{ij}) = d(E_i(-1))d(E_{ji}(1))d(E_{ij}(-1))d(E_{ji}(1)) = (-1)(1)^3 = -1$$

which completes the proof.

The identities (1) and (2) are displayed since they are of some interest in their own right. This theorem makes it clear that $d(A)$ is actually the determinant of A and also yields the usual results about how elementary row and column operations on A effect the value of $d(A)=|A|$.

All of the usual theorems about $d(A)$ can now be proven easily. As an example, $d(A)=0$ if and only if A is singular since $d(A)=0$ if and only if, for some j, $E_j(0)$ is a term in the representation of A as a product of elementary matrices.

Essentially the same proof as is given on pages 89 and 90 of [3] can now be used to establish

THEOREM 2. *d is a linear function of its columns (rows), i.e., if A_j represents the j-th column of A and $A_i=\alpha+\beta$ then $d(A_1, \cdots, A_{i-1}, \alpha+\beta, A_{i+1}, \cdots, A_n)=d(\cdots, \alpha, \cdots)+d(\cdots, \beta, \cdots)$.*

For A in C_n let $M_{ij}(A)$ in C_{n-1} denote the $n-1$ by $n-1$ submatrix of A obtained by deleting the ith row and the jth column of A. We will now indicate how our development yields the Laplace expansion for $d(A)$. We first prove a preliminary result.

LEMMA. *If $B=(b_{ij})$ in C_n has the form*

$$B = \begin{bmatrix} 1 & 0 \\ B_{21} & M_{11}(B) \end{bmatrix},$$

with $B_{21}=(b_{21}, \cdots, b_{n1})^T$, then $d(B)=d(M_{11}(B))$.

Proof. Since

$$B = \left[\prod_{j=2}^{n} E_{j1}(b_{j1})\right]\begin{bmatrix} 1 & 0 \\ 0 & M_{11}(B) \end{bmatrix}$$

we see, from Theorem 1, that

$$d(B) = d\left(\begin{bmatrix} 1 & 0 \\ 0 & M_{11}(B) \end{bmatrix}\right).$$

If $E_1E_2 \cdots E_t=M_{11}(B)$ is a representation of $M_{11}(B)$ as a product of elementary matrices, then

$$\begin{bmatrix} 1 & 0 \\ 0 & E_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & E_2 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & E_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & M_{11}(B) \end{bmatrix}.$$

Since

$$E_i' = \begin{bmatrix} 1 & 0 \\ 0 & E_i \end{bmatrix}$$

and E_i are elementary matrices of the same type it is clear that $d(E_i) = d(E_i')$ and hence that

$$d(B) = d\left(\begin{bmatrix} 1 & 0 \\ 0 & M_{11}(B) \end{bmatrix}\right) = \prod_{i=1}^{t} d(E_i') = \prod_{i=1}^{t} d(E_i) = d(M_{11}(B)).$$

THEOREM 3. *If $A = (a_{ij})$ is in C_n, then*

$$d(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} d(M_{ij}(A)) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} d(M_{ij}(A)).$$

The proof of this theorem follows easily from the lemma and our first two theorems and will not be given in detail.

It should also be clear that Theorem 3, used recursively, will yield the classical formula for $d(A)$ in terms of the elements of A.

We have found that it is sufficient to specify d at all of the matrices $E_1(k)$. If we assume that our function is continuous it is only necessary to specify the value at one of the matrices $E_1(k)$.

To see this let f be a nonconstant continuous multiplicative function from C_n to C such that $f(E_1(\gamma)) = \gamma$ for $\gamma = 1+i$. It is clear that $f(E_1(k))$ is a function of k only, say $f(E_1(k)) = g(k)$. We see immediately that $g(k)$ must be a nonconstant continuous multiplicative function from C to C such that $g(\gamma) = \gamma$.

Now $g(16) = g(\gamma^4) = g(\gamma)^4 = \gamma^4 = 16$ and it follows that $g(16^p) = g(16)^p = 16^p$ for any positive integer p. Moreover, if q is another positive integer, we have $g(16^{p/q})^q = g(16^p) = 16^p$, which yields $g(16^r) = 16^r$ for any positive rational r. Since the set of numbers of the form 16^r, r a positive rational, is dense on the positive real axis we must have, as a consequence of the continuity of g, that $g(x) = x$ for x any positive real number.

Now consider $\omega = (1/\sqrt{2})\gamma$. From our previous discussion we have $g(\omega) = g(1/\sqrt{2})g(\gamma) = (1/\sqrt{2})\gamma = \omega$. For q a positive integer let $\omega^{1/q}$ denote the principal qth root of ω so that $\omega^{p/q} = (\omega^{1/q})^p$ is well defined for all positive integers p and q. Now

$$g(\omega^{p/q})^q = g(\omega^p) = \omega^p$$

which yields $g(\omega^r) = \omega^r$ for r a positive rational.

The set $\Gamma = \{\rho\omega^r : \rho \text{ is a positive real, } r \text{ is a positive rational}\}$ is dense in C. For $z = \rho\omega^r$ in Γ we have $g(z) = g(\rho\omega^r) = g(\rho)g(\omega^r) = \rho\omega^r = z$. From the continuity of g it follows that $g(z) = z$ for all z in C and hence that $f(E_i(z)) = z$ for all z in C.

We summarize the foregoing discussion in

THEOREM 4. *A nonconstant continuous function d, from C_n to C, such that $d(E_1(1+i))=1+i$ must be the determinant.*

We could just as well have specified d at some other matrix of the same type. However, using $d(E_1(\sigma))=\sigma$ for σ either real or pure imaginary will not yield the desired results.

The same type of argument which was used to prove Theorem 4 will also produce Cater's result [2] that any continuous multiplicative mapping from C_n to C must be a power of the determinant.

References

1. S. Cater, Scalar valued mappings of square matrices, this MONTHLY, 70 (1963) 163–169.
2. K. Hoffman and R. Kunze, Linear Algebra, Prentice Hall, Englewood Cliffs, N. J., 1961.
3. Robert R. Stoll, Linear Algebra and Matrix Theory, McGraw-Hill, New York, 1952.

THE TWO FUNDAMENTAL THEOREMS OF CALCULUS

F. CUNNINGHAM, JR., Bryn Mawr College

This note is purely polemical, rather than offering anything new. The Fundamental Theorems of calculus are those which state the quasi-inverse relationship between differentiation and integration. There are really two of them, and they are independent of each other.

THEOREM 1. *If F is differentiable on $[a, b]$ and F' is (Riemann) integrable there, then $\int_a^b F' = F(b) - F(a)$.*

Proof. For any partition $a=x_0<x_1< \cdots <x_n=b$ we have

$$F(b) - F(a) = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$$

$$= \sum_{i=1}^{n} F'(c_i)(x_i - x_{i-1}) \qquad c_i \in (x_{i-1}, x_i)$$

by the Mean Value Theorem for derivatives. As the partition gets finer, the Riemann sums in the last line converge by definition to $\int_a^b F'$, because by hypothesis this limit exists.

THEOREM 2. *If f is continuous on $[a, b]$ and $F(x)=\int_a^x f$ for $x\in[a, b]$ then $F'(x)=f(x)$.*

The well-known proof consists in estimating

$$[F(x + \Delta x) - F(x)]/\Delta x = \int_x^{x+\Delta x} f\Delta x,$$

and then using the continuity of f at x to get $f(x)$ for the limit as $\Delta x\to 0$.

The procedure followed by so many current calculus texts, that it has to be regarded as standard, is to prove Theorem 2 first, and then base the proof of Theorem 1 on it. (One compares F with G defined by $G(x)=\int_a^x F'$. Since F and G have the same derivative by Theorem 2, they differ by a constant. Hence $F(b)-F(a)=G(b)-G(a)=\int_a^b F'$.) I submit that this procedure suffers from the following defects:

(a) It proves Theorem 1 only with the additional and unnecessary hypothesis that F' be continuous.

(b) Making Theorem 1 depend on Theorem 2 obscures the fact that the two theorems are saying different things, having different applications, and may give the student the impression that Theorem 2 is *the* fundamental theorem. This also requires that Theorem 2 be presented first, whereas actually Theorem 1 most naturally precedes Theorem 2, both in motivation (i.e. usefulness) and in ease of understanding. (To understand Theorem 2, you have to envisage the indefinite integral defined by replacing one limit of integration by x.)

The need for Theorem 1 is immediate. No sooner has one motivated the definite integral by applications such as area and work and then produced the definition itself, than one wants to evaluate definite integrals. Theorem 1 tells how to do this in those numerous cases where the integrand can be recognized as the derivative of some function F. The proof given above is simple, direct and rigorous, and requires no machinery which is not available at this point. I feel further that it connects with the motivating applications of the definite integral in a way which encourages understanding. One annoying detail which the student will of course not understand is that to apply Theorem 1, you have to know that the integral exists. How much to insist on this will depend on taste and circumstances.

The more sophisticated problem to which Theorem 2 is applicable is the creation of transcendental functions, for instance the logarithm, having a given derivative. You need first an existence theorem to make $F(x)=\int_a^x f$ defined (f being continuous), and then Theorem 2 to conclude that $F'=f$ as desired.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

ACTIVITY AND MOTIVATION IN MATHEMATICS

EDWIN E. MOISE, Harvard University

It often happens that when a committee meets to discuss a question of educational policy, the mere fact that the committee exists and is holding a meeting tends to suggest a commitment on the question under discussion. For example,

at a meeting of a Committee on the Teaching of Calculus to Engineers, it is hard to avoid the suggestion that the teaching of calculus to engineers is a special problem, requiring a special solution. This in itself is a decision. (In my own opinion, it is a wrong decision.)

It seems to me that we need to be on guard against such suggestions. And I believe that one such suggestion is present at any conference devoted to curricular planning in the large. If we look at a four-year college program, as a whole, it is very natural to judge it by the quantity of mathematical knowledge that it appears to contain, and by the speed with which it reaches the concepts and methods of contemporary mathematics. Surely these are important criteria. I believe, however, that there are other criteria which it is easy to neglect, because they are harder to describe and to apply.

In the first place, I do not believe the effect of an educational program can safely be judged merely by the quantity of knowledge which the student can demonstrate on an examination. It seems to me that this sort of simple bookkeeping is not just slightly inaccurate but grossly so. An extreme example may make this point plain.

I worked for the doctorate in mathematics under Professor R. L. Moore, of the University of Texas. In three and a half years of graduate work, under his direction, I believe that I heard him lecture (in a style that a conventional professor would recognize as lecturing) for about two hours at most, and it may well have been less. All of his graduate courses worked in the same way: he would present postulates, definitions, and propositions; and the student's job was to find out which of the propositions were true, and present proofs or counter-examples. The theorems were not mere exercises; they were the substance of the sort of set-theoretic topology that Moore was teaching. Six months might elapse, between the time a problem was proposed and the time it was solved. At first, a very large number of the students' proofs were wrong. When a student had gone astray, Moore did not correct him; he would merely say that he didn't understand; and it took us quite a while to realize that anything that he "didn't understand" was surely wrong.

Under Moore's regime, all use of the literature is forbidden. Most of the time, therefore, a student of Moore doesn't even know whether the problems that he is working on are research problems. Eventually, they turn out to be, and the result is a thesis. Thus, at the time when I wrote my first research paper, I had never read one. And I had never discussed topology with fellow-students outside of class.

If this scheme of teaching seems misguided, we should remember that the list of Moore's students includes the names of R. L. Wilder, G. T. Whyburn, R. H. Bing, and many other research mathematicians of distinguished achievements. (I could give a long list, but would prefer to avoid the problem of deciding where to stop.) And Moore's record as a teacher is even more impressive in the light of the circumstances under which he has taught. There are some universities in the United States where a professor can take for granted that the

best and most ambitious graduate students in the country will arrive, every year, already well trained and already fully committed as mathematicians. The University of Texas is not one of these, and so Moore has had the task of recruitment as well as the task of teaching. When he first encountered them, R. L. Wilder intended to be an actuary, and G. T. Whyburn intended to be a chemist. Bing made his first appearance as a summer student, on vacation from his regular job as a high-school teacher.

Thus Moore's teaching has been effective, to say the least. I believe that all teaching would profit if the basis of his method were better understood. By this I do not mean to suggest that his method can simply be copied, in the extreme form in which he practices it. It seems rather likely that the success of the method, in its extreme form, depends on special conditions.

First, the method may require that the teacher be a genius. Teaching in this style is not merely a negative matter of not lecturing; rather, it is a striking example of the art that conceals art, and it makes great demands on both mathematical and psychological depth.

Second, it may require that the teacher dominate the environment that his students live in. Surely Moore has done this at Texas, but I doubt that he or anybody else could do it, at Harvard, Princeton, or Chicago, well enough to persuade students not to talk to each other, or well enough to persuade other professors to avoid telling them things that would help in their problems.

Third, the method may require that the subject-matter be logically primitive, and fairly isolated from that of other courses.

In spite of all these reservations, however, I believe that Moore's work proves something of broad significance. It proves that sheer knowledge does not play the crucial role in mathematical development that most people suppose. The amount of knowledge that a small class can acquire, struggling at every stage to produce its own proofs, is quite small. The resulting ignorance ought to be a hopeless handicap, but in fact it isn't; and the only way that I can see to resolve this paradox is to conclude that mathematics is capable of being learned as an *activity*, and that knowledge which is acquired in this way has a power which is out of all proportion to its quantity.

For this reason, it seems to me quite unrealistic to judge a curriculum by its general outline, or to judge a course by its syllabus. We can "cover" very impressive material, if we are willing to turn the student into a spectator. But if you cast the student in a passive role, then saying that he has "studied" your course may mean no more than saying of a cat that he has looked at a king. Mathematics is something that one *does*. In elementary courses, this idea is well understood; but later it often gets lost, only to reappear, as a brutal surprise, when the time comes to write a dissertation. When this happens, a high price has been paid for erudition: we seem to be relying on the idea that the creative spirit is both self-discovered and indestructible. This is like the idea that "murder will out": if it were true, how would anybody know it?

It is, I believe, a fundamental task of the teacher to introduce students to

the intellectual life that he, the teacher, really leads. Under this standard, the cram-school conception of learning is a falsification of the subject. And the subject can be falsified in other ways, by presenting ideas in advance of their motivations. This can be explained best by a few examples.

1. A distinguished algebraist once served as an examiner in a final oral for the doctorate, based on a dissertation on Banach algebras. Toward the end of the examination, the algebraist asked the student to describe some examples of Banach algebras. The student was able, at length, to name one example, but his one example was trivial.

The director of the dissertation could have given a serious answer to the question; he knew of good reasons for investigating Banach algebras. But apparently his teaching had taken a shortcut, past "classical mathematics" to "modern mathematics," and the result was that the student was working on Banach algebras for no good reason known to himself. Thus the student's training omitted an essential part of the teacher's intellectual equipment.

2. In a course in analysis, the students were furnished with a very elegant definition of a Riemann surface. One of them showed the definition to a personal friend who was an analyst and a member of the National Academy of Science. It took the academician quite a while to figure out how the definition worked.

To the teacher in this course, the "right" definition of a Riemann surface was the solution of a difficult problem. But it is hard to understand the answer if you don't know what the question was.

3. One of the veterans of Bourbaki once told me that he had been annoyed by people who approached him at meetings and told him about trivial syntactical tricks which they had devised. These people expected Bourbaki to admire their tricks, but had the bad luck to try them on a man who was distinguished (even among the Bourbakistes) for his candor. It seems that the source of this misunderstanding is that while the Bourbaki series is self-contained in a logical sense, it is not psychologically self-contained; the first few volumes, taken at face value by a naive reader, give a misleading impression of the mentalities of the authors. The authors are learned and tough-minded; and some of the simplest-looking things, in the first few volumes, are the solutions of difficult problems. (Where do you begin? and how? Above all, what do you omit?)

This is not, in itself, an example of the sort of misguided teaching that I am talking about. (The Bourbaki books are not courses, and I hardly think that courses taught by Bourbakistes have such an effect.) With this reservation, the preceding three examples have something in common. Each of them is an example of first-rate mathematics presented apart from the context which makes it significant; in each case, the solution was presented before the problem.

The point, I believe, is that mathematics is arranged not only in a logical order but also in a psychological order; and at many stages, the orders are different. Thus in algebra, the abstract and general logically precede the con-

crete and special, but most of the time the abstractions are intended to be codifications, and may easily seem pointless to a student who doesn't know what is being codified.

These remarks are *not* intended as a defense of "traditional" teaching. Mathematics has no traditions comparable to those of literature and art: some of the best of its past is dead, at least in a stylistic sense. Thus the task of a calculus student is to understand calculus, and for this it is neither necessary nor sufficient that he understand Newton (let alone Leibniz). The problems that I have been discussing would not even arise if we were not reformulating the curriculum; and they are important precisely because such reformulations are necessary.

The problem of arranging a curriculum in an order which is logically sensible and psychologically well-motivated appears to be highly empirical. As far as I know, there are no general principles which can be relied upon. Lately, some authors have proposed the so-called *genetic principle*, under which mathematics should be presented in the order in which it was discovered. But this involves difficulties. Manipulative algebra, and negative real numbers, were developed during the Renaissance. Therefore, under the genetic principle, you should teach the geometry of Euclid (including the Eudoxian theory of proportionality, for incommensurables) before you teach the student to write or solve the equation $2x+6=0$. In fact, you should teach Euclid before you teach the representation of positive integers in decimal form. No one seriously proposes to do this. But a rule which admits massive exceptions is not a very reliable rule. Indeed, to rescue the genetic principle from absurdity, we must construe it so loosely that it can hardly be used at all. It is not even safe, as far as it goes. If you are teaching, say, Eighteenth Century mathematics, in a given year, then you cannot assume that the student knows all the mathematics of earlier centuries, because in fact you have taught him only a small portion of it. You must make sure, in each case, that what you are teaching is not based (logically or psychologically) on an appeal to earlier mathematics which has not been taught.

For these reasons, I believe that the people who have proposed the genetic principle were really thinking about the problem of motivation. In practice, most unmotivated treatments are violations of it. But in a serious attack on the problems of course design, the genetic principle ought to be regarded merely as a fund of suggestions, and not as a standard of judgment.

We also hear the maxim that the particular should precede the general. Usually this gives reasonable results, but here again there are exceptions. The easiest way that I know of, to see that the number 5 has a cube root, is to observe (somehow) that the cube function is continuous, and so must take on every value between 0 and 8. I see no advantage in specializing the mean-value theorem to which this argument appeals; its natural habitat, both logically and psychologically, is the set of functions continuous on intervals.

For these reasons, I believe that there are no simple answers to the questions that I have been raising. The fundamental task of a mathematician who teaches

is to convey to his students not only what mathematicians know, but also what they do, and how, and why. This is a problem in full and honest self-revelation. And (as Raymond Chandler has remarked in another connection) ***honesty is an art.***

A revised version of an address delivered at a conference on the teaching of college mathematics, in Katada, Japan, Sept. 6, 1964.

SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN

MICHAEL N. BLEICHER, The University of Wisconsin

Several years ago, Professor L. C. Young conceived the idea that The University of Wisconsin might conduct a mathematical problem competition similar to those which have met with great success in other countries, most notably, Hungary. Through Professor Young's efforts and enthusiasm the National Science Foundation became interested and agreed to support a pilot program. The task of designing and organizing the specifics of the program fell to a three-man committee consisting of Professor L. Fejes-Toth from Budapest, who was visiting at Madison for a year, Professor A. Beck, and the author, as chairman. The program consisted of sending four problem sets of five problems each to every student in the Racine high schools, every student taking a mathematics course in the Madison and Milwaukee schools, and ten copies to every other high school in the state. The purpose of the wide distribution in Racine was to see if we could interest the student who through lack of previous motivation had shown no inclination to excel in or study mathematics. Unfortunately, none of the students who actually submitted good papers fell into this category.

We attempted to pick problems which were solvable without any specific knowledge of course work, if the student was sufficiently clever. Of course, the students who had taken advanced mathematics courses might be able to use methods which they had learned in class or analogous methods, and thus find solutions more easily. The number of sophomores and juniors who did well, however, indicates some success in our problem choice. Among the top 46 students were 16 seniors, 20 juniors, 9 sophomores, and 1 freshman. Of these only four were girls.

In the Appendix is a copy of the letter which we sent with the first problem set to introduce the students to the competition. Also listed are the problems themselves, along with the number of papers received and the number of students who solved each problem completely. Every problem was solved by at least one student, most were solved by many students. Several students got all but one or two of the problems.

We were pleasantly surprised at the large response to the problems. In addition, letters from both students and faculty of the high schools indicate that there were many more students who actively worked and discussed the problems, but for any one of a number of reasons did not submit their results.

On the basis of the first three problem sets, the top forty-six students and

Problem Set 4—50 Responses

1. In Problem No. 3, Set No. 3 it was shown that $(a+b)/2 \leqq \sqrt{[(a^2+b^2)/2]}$ and

$$\frac{a+b+c+d}{4} \leqq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}},$$

where a, b, c, and d are arbitrary numbers. Show that

$$\frac{a_1+a_2+a_3+\cdots+a_k}{k} \leqq \sqrt{\frac{a_1^2+a_2^2+\cdots+a_k^2}{k}},$$

where $a_1, a_2, \cdots, a_k$ are arbitrary numbers, and k is a positive integer. Hint: Consider the case $k=2^n$ first. (14)

2. If x and y are chosen to be integers, then $2x+6y$ is divisible by 19 if and only if $5x-4y$ is divisible by 19. (19)

3. A brick of dimensions $a\times b\times c$ is inscribed in a unit sphere. Let $\widehat{PQ}$ be the shortest distance on the surface of the sphere between the points P and Q. Let S be the sum of all the distances $\widehat{PQ}$ where the sum is taken over all possible pairs P, Q of vertices of the brick. For what values of a, b, and c is S largest? (19)

4. In a group of n boys and n girls certain marriages are allowable. For each girl or boy there may be several allowable marriage partners. Show that it is possible to marry all of them to allowable partners only when for every integer $k\leqq n$, every group of k boys has a total of at least k possible partners between them. (1)

5. An unending sequence $a_0, a_1, a_2, \cdots,$ of positive integers is given. This sequence satisfies the conditions $a_0<a_1\leqq a_0+20$, $a_1<a_2\leqq a_1+20$ $a_2<a_3\leqq a_2+20, \cdots$. Show that the unending decimal $b=.a_0a_1a_2\cdots$ obtaining by writing the digits of $a_0, a_1, a_2, \cdots$ successively in order is not a repeating decimal; i.e., b is an irrational number. (14)

MODERN COORDINATE GEOMETRY A WESLEYAN EXPERIMENTAL CURRICULAR STUDY

HARRY SITOMER, Queens College, Flushing, N. Y.

During the summer of 1964, at Wesleyan University, Middletown, Connecticut, under a National Science Foundation grant, a team directed by Professor R. A. Rosenbaum, wrote a modern coordinate geometry text for high school students and a commentary for teachers.

The writing project developed directly out of a pilot experiment sponsored by the School Mathematics Study Group in 1961–62. In the experiment, teachers worked from a text based on Howard Levi's "Foundations of Geometry and Trigonometry" (Prentice-Hall, 1956). All six teachers were unanimous in stating that the results were sufficiently encouraging to warrant the writing of a suitable text for this course.

The text written at Wesleyan University is now the course of study in five experimental centers throughout the country. Each center has a mathematics consultant and six teachers, one of whom acts as group chairman. Written reports are submitted regularly for each chapter, and comments are encouraged on text difficulty, suitability, and so on. This information is being collated, and will guide a rewrite committee during the summer of 1965.

"Modern Coordinate Geometry" uses only five axioms pertaining to geo-

metric entities, and since it is an analytic course, it assumes the properties of real numbers. The five axioms are:

Axioms for the line l.

L 1. If A and B are distinct points of l, there is exactly one of the members of the set C (coordinate systems) which assigns the number 0 to A and the number 1 to B.

L 2. Let F and G be any one-to-one correspondences in the set C. There exist numbers $a \neq 0$ and b such that if x and x' denote the numbers assigned by F and G respectively to each point, X, of l, then $x' = ax + b$.

Axioms for the Plane $\mathcal{A}$.

P 1. If A and B are distinct points of $\mathcal{A}$, there is one and only one line of $\mathcal{A}$ which contains them.

P 2. If l is a line of $\mathcal{A}$ and if P is a point of $\mathcal{A}$, then there is one and only one line of $\mathcal{A}$ which contains P and is parallel to l.

P 3. If l and l' are lines of $\mathcal{A}$, then any parallel projection from l to l' is an affinity.

The concept of a function plays an important role in the course. Starting with the mathematics of a linear function, the various kinds of functions that are treated are coordinate functions, affinities from one line to another, parallel projections, similitudes, similarities, isometries, and congruences. The vocabulary of functions and the symbolism to denote functions are an integral part of the course.

An early study of triangles and parallelograms in an affine plane, restricted to division points and parallelism, makes possible a simplified approach via equations which can be handled by tenth year students. The techniques of algebra are in use through the year, thus avoiding the discontinuity in the development of mathematics that usually occurs in the tenth year. Moreover, proofs using algebra are often short and elegant.

To date (early December, 1964) judging from reports so far received, several observations seem indicated:

1. The course content is proving to be quite interesting to both teachers and students.

2. Students are thinking more deeply, working harder, and asking better questions.

3. The function concept makes for a more profound understanding of relations among geometric objects.

A PROJECT FOR THE IMPROVED USE OF NEWER EDUCATIONAL MEDIA IN ELEMENTARY SCHOOL MATHEMATICS

JAMES H. ZANT, Project Director, Oklahoma State University

This project is sponsored by the National Council of Teachers of Mathematics and supported by a grant from the U. S. Office of Education. The purpose of the project is two-fold:

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before August 31, 1965.

E 1776. *Proposed by Carolyn C. Styles, San Diego Mesa College*

For any even integer m, $m>2$, there is an infinity of odd integers not the sum of a prime and a positive power (>1) of m.

E 1777. *Proposed by Horváth Sándor, University of Technology, Budapest, Hungary*

For which positive values of a and c is $a^n n! > cn^n$ true for every positive integer n?

E 1778. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If R, r, r_1, r_2, r_3 are the circumradius, inradius and exradii of a triangle, prove that

$$\frac{1}{r^3} - \frac{1}{r_1^3} - \frac{1}{r_2^3} - \frac{1}{r_3^3} = \frac{12R}{r \cdot r_1 \cdot r_2 \cdot r_3}.$$

E 1779. *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey*

If h_i and r_i are the altitudes and exradii of a triangle prove that

$$\frac{r_1}{h_1} + \frac{r_2}{h_2} + \frac{r_3}{h_3} \geqq 3.$$

E 1780. *Proposed by Norman Brenner, Harvard University*

Find all pairs (α, β) of real numbers which satisfy the equation

$$\frac{\sin 2\alpha}{\sin (2\alpha + \beta)} = \frac{\sin 2\beta}{\sin (2\beta + \alpha)}.$$

E 1781. *Proposed by Ray Redheffer, University of California, Los Angeles*

On an interval (a, b) let f and g be continuous, and f or g monotone. Suppose that some sequence $\{x_n\}$ satisfying $f(x_n)=g(x_{n-1})$ has a limit point on (a, b). Prove that then $f(x)=g(x)$ has a solution on (a, b).

E 1782. *Proposed by V. V. Menon, Indian Statistical Institute, Calcutta*

Given an $n\times n$ chessboard, a coloring of the board with k colors is defined as an assignment of k colors to the cells of the board such that (a) each cell is assigned a color, and (b) no two cells which are assigned the same color lie in the same row, same column, or same diagonal. What is the minimum number of colors required for a coloring?

E 1783. *Proposed by V. V. Menon, Indian Statistical Institute, Calcutta*

A figure is a compact convex set with nonempty interior in the Euclidean plane. A bisector of a figure is defined to be a straight line which divides the figure into two portions of equal area. Given $n (n\geqq 3)$, does there exist a figure such that through some point in it there pass exactly n (and no more) bisectors of the figure?

E 1784. *Proposed by A. W. Hales, Institute for Defense Analyses, Princeton, N. J.*

Consider the first order differential equation $dy/dx=f(y)$, where f is continuous in a neighborhood of y_0. Assume that the solutions through y_0 are not unique, that is, assume that there are two solutions ϕ_1 and ϕ_2 such that $\phi_1(0)=\phi_2(0)=y_0$ and that ϕ_1 and ϕ_2 differ in every neighborhood of 0. Prove that $f(y_0)=0$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Representation of a Power of a Prime

E 1684 [1964, 430] (Corrected). *Proposed by E. A. Lee, Albuquerque, New Mexico*

If p and q are positive integers, $p>1$, q is a prime, and $(p, q)=(p-1, q)=1$, show that there is at most one positive power of q which is representable as a sum of consecutive nonnegative integer powers of p.

Solution (Patterned on that of the proposer.)

Since $(p, q)=1$, it is clear that if $q^m=\sum_{i=k}^{l} p^i$, for nonnegative integers k and l and a positive integer m, then $k=0$. Hence, we must show that there is at most one pair (m, ν) of positive integers such that $q^m=\sum_{i=0}^{\nu} p^i$. Put another way, letting J^+ denote the set of positive integers and

$$S_{pq} = \left\{m \in J^+ \mid q^m = \sum_{i=0}^{\nu(m)} p^i \text{ for some } \nu(m) \in J^+\right\},$$

we must show that S_{pq} is at most a singleton.

We will use consistently the following notation:

$$\mu_{pq} = \min S_{pq} \text{ (when } S_{pq} \text{ is nonempty)}, \qquad \mu = \mu_{pq} \quad \text{and} \quad n = \nu(\mu_{pq}).$$

We use the assumption that q is a prime only at the last step.

LEMMA 1. *If $(p, q)=1$, then $1+\nu(\mu_{pq})$ divides $1+\nu(s)$ for each $s\in S_{pq}$.*

Proof. With $r=\nu(s)$, suppose that $r=(n+1)w+\rho$ with $0\leqq\rho<n+1$. Then

$$\begin{aligned} q^s &= \frac{p^{r+1}-1}{p-1} = \sum_{i=0}^{r} p^i = \sum_{i=0}^{(n+1)w+\rho} p^i \\ &= \sum_{i=0}^{(n+1)w-1} p^i + \sum_{i=(n+1)w}^{(n+1)w+\rho} p^i = \sum_{j=0}^{w-1}\sum_{i=0}^{n} p^{j(n+1)+i} + p^{(n+1)w}\sum_{i=0}^{\rho} p^i \\ &= \left(\sum_{i=0}^{n} p^i\right)\sum_{j=0}^{w-1} p^{j(n+1)} + p^{(n+1)w}\frac{p^{\rho+1}-1}{p-1} \\ &= \frac{p^{n+1}-1}{p-1}\sum_{j=0}^{w-1} p^{j(n+1)} + p^{(n+1)w}\frac{p^{\rho+1}-1}{p-1} \\ &= q^\mu \sum_{j=0}^{w-1} p^{j(n+1)} + p^{(n+1)w}\frac{p^{\rho+1}-1}{p-1}. \end{aligned}$$

Now, q^μ divides q^s $(s\geqq\mu)$, so q^μ divides $p^{(n+1)w}(p^{\rho+1}-1)/(p-1)$. Since $(p, q)=1$, q^μ divides $(p^{\rho+1}-1)/(p-1)$. But then $\rho+1\geqq n+1$, (i.e., $\rho=n$), and $r+1=(n+1)(w+1)$.

LEMMA 2. *If $k, l\in J^+$, $s\in J^+-\{1\}$, and $t\in J^+-\{1\}$, then*

$$\frac{s^{kt}-1}{(s^k-1)^2} = \sum_{j=0}^{t-2}(t-1-j)s^{jk} + \frac{t}{s^k-1}.$$

Proof. If $x\neq 1$, then

$$\frac{x^t-1}{(x-1)^2} = \sum_{j=0}^{t-1}(t-1-j)x^j + \frac{t}{x-1}.$$

LEMMA 3. *If $(p, q)=1$, then $q^{\mu_{pq}}(1+\nu(\mu_{pq}))$ divides $1+\nu(s)$ for each $s\in S_{pq}-\{\mu_{pq}\}$.*

Proof. By Lemma 1, $\nu(s)+1=r+1=(n+1)w$ for some $w\in J^+$. By Lemma 2 with $s=p$, $k=n+1$, and $t=w$,

$$\begin{aligned} q^{s-2\mu} &= (p-1)\frac{p^{(n+1)w}-1}{(p^{n+1}-1)^2} = (p-1)\sum_{j=0}^{w-2}(w-1-j)p^{(n+1)j} + \frac{(p-1)w}{p^{n+1}-1} \\ &= (p-1)\sum_{j=0}^{w-2}(w-1-j)p^{(n+1)j} + \frac{w}{q^\mu}. \end{aligned}$$

Since w/q^μ must be an integer, $q^\mu | w$; i.e., $\nu(s)+1=(n+1)Dq^\mu=q^\mu(1+\nu(\mu))D$ for some $D\in J^+$.

THEOREM. *If* $(p, q)=1=(p-1, q)$ *and* q *is a prime, then* S_{pq} *is at most a singleton.*

Proof. If $s\in S_{pq}-\{\mu_{pq}\}$, Lemma 3 tells us that $\nu(s)+1=q^\mu(n+1)D$ for some $D\in J^+$. But then

$$q^s = \frac{p^{q(n+1)q^{\mu-1}D}-1}{p-1} = \frac{p^q-1}{p-1}\cdot\frac{p^{\nu(s)+1}-1}{p^{(n+1)q^{\mu-1}D}-1}.$$

Since q is a prime and $(p^q-1)/(p-1)$ divides q^s, $(p^q-1)/(p-1)\equiv 0 \pmod q$. But $(p^q-1)/(p-1)\equiv(p-1)/(p-1)=1 \pmod q$, a contradiction.

Also solved (in the original version, to the extent of providing counterexamples) by J. C. Abad, A. N. Aheart, Shair Ahmad, Joseph Bechely, B. A. Hausmann, Hajna Janos, E. S. Langford, W. I. Nissen, Jr., R. T. Hood, Fritz Herzog, D. C. B. Marsh, and Simon Vatriquant.

An Essentially Known Limit

E 1686 [1964, 430]. *Proposed by Richard Sinkhorn, University of Houston*

If p is a nonnegative integer, evaluate

$$\lim_{x\to 1-}(1-x)^{p+1}\sum_{n=1}^{\infty} n^p x^n.$$

I. *Solution by Sylvan H. Greene, Franklin Institute of Pennsylvania.* We have

$$f_p(x) = (1-x)^{p+1}\sum_{n=1}^{\infty} n^p x^n = \left(\sum_{n=1}^{\infty} n^p x^n\right)\Big/(1-x)^{-p-1}.$$

Since the series diverges for $x=1$, l'Hospital's Rule is suggested. We find then that

$$\lim_{x\to 1-}(p+1)f_p(x) = \lim_{x\to 1-} f_{p+1}(x)/x = \lim_{x\to 1-} f_{p+1}(x).$$

We observe that $f_0(x)\to 1$ as $x\to 1-$; an induction argument then shows that $f_p(x)\to p!$ as $x\to 1-$.

II. *Solution by Miltiades S. Demos, Drexel Institute of Technology.* It is of course known that $\sum_{n=0}^{\infty} x^n=1/(1-x)$ for $|x|<1$. We differentiate and multiply by x, p times. By easy induction, we find that

$$\sum_{n=1}^{\infty} n^p x^n = p!x^p(1-x)^{-p-1}+f(x)(1-x)^{-p} \qquad (|x|<1),$$

where $f(x)$ is a polynomial. The required limit is therefore $p!$.

III. *Solution by Stanton Philipp, Seal Beach, California.* We have

$$(1 - x)^{-\alpha-1} = \sum_{n=0}^{\infty} \binom{\alpha + n}{n} x^n \quad \text{for} \quad |x| < 1$$

and α any complex number. Observing that

$$n^\alpha \Big/ \binom{\alpha + n}{n} \to \Gamma(\alpha + 1) \quad \text{as} \quad n \to \infty$$

and that the series above diverges at $x=1$ if $\operatorname{Re}(\alpha) > -1$, we find that

$$\lim_{x\to 1-} (1 - x)^{\alpha+1} \sum_{n=1}^{\infty} n^\alpha x^n = \Gamma(\alpha + 1)$$

for $\alpha > -1$ because of the well-known

THEOREM. *If $\lim_{n\to\infty} a_n/b_n = C$, b_n has one sign for all large n, and $\sum_{n=0}^{\infty} b_n x^n \to \infty$ as $x \to 1-$, then*

$$\lim_{x\to 1-} \left(\sum_{n=0}^{\infty} a_n x^n\right) \Big/ \sum_{n=0}^{\infty} b_n x^n = C.$$

Also solved by J. D. Anderson and J. B. Langworthy (jointly), P. M. Anselone (2 solutions), Raymond Balbes, Randy Barron, J. A. Burslem, M. S. Demos, G. E. Engebretsen, N. J. Fine, W. T. French, Ernest Gore, S. H. Greene, Arthur Greenspoon, Eldon Hansen, Ray Hines, Stephen Hoffman, J. M. Horner, R. F. Jackson, J. P. Jordan, Kenneth Kramer, E. S. Langford, J. Lehner, J. N. McDonald, M. A. Malik, D. C. B. Marsh, H. F. Mattson, W. I. Nissen, Jr., Horvath Sandor, P. A. Scheinok, Lawrence Schulman, C. Seguin, K. Soni and R. P. Soni (jointly), Andy Vince, J. E. Wilkins, Jr., Max Wyman, K. L. Yocom, David Zeitlin, an unknown contributor, and the proposer.

Anderson and Langworthy and Zeitlin base their solutions on the known identity

$$f_p(x) = (1 - x)^{p+1} \sum_{n=1}^{\infty} n^p x^n = \sum_{r=1}^{p} \left\{ \sum_{m=1}^{r} (-1)^{m+1} \binom{p+1}{m-1} (r - m + 1)^p \right\} x^r.$$

[R. Stalley, this MONTHLY 56 (1949) 325, or D. Zeitlin, *Two methods for the evaluation of*

$$\sum_{n=1}^{\infty} n^p x^n,$$

this MONTHLY 68 (1961) 986–989]. Hansen cites the formulas $T_p(p) = (-1)^p p!$ and

$$f_p(x)/(1 - x)^{p+1} = \sum_{k=1}^{p} (-x)^k T_p(k)/(1 - x)^{k+1},$$

where

$$T_p(k) = \sum_{m=1}^{k} (-1)^m \binom{k}{m} m^p.$$

[I. J. Schwatt, *An Introduction to the Operations with Series*, 2nd ed., Chelsea (1962), pp. 33 and 38.] The unsigned solution called attention to H. Ory, *Sommation de séries numériques de la forme* $\sum n^{k+1} r^{n\lambda}$, Rend. Circ. Mat. Palermo, 54 (1930) 366–370.

An Application of Menelaus' Theorem

E 1687 [1964, 430]. *Proposed by Daniel Pedoe, Purdue University*

UVW is an equilateral triangle; A, B, C are the respective midpoints of the sides VW, WU, UV; A' is any point on line VW, B' any point on line WU, and C' any point on line UV. If P is the intersection of BC and $B'C'$, Q of CA and $C'A'$, R of AB and $A'B'$, prove that (1) the lines $A'P$, $B'Q$, $C'R$ are concurrent, (2) the areal coordinates of the point of concurrency with respect to triangle ABC are, with a suitable sign convention, $(AA')^{-1}:(BB')^{-1}:(CC')^{-1}$.

Generalize both (1) and (2) by means of an affine projection, and generalize (1) by a general projection.

I. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.* (1) We first set $BC=CA=AB=1$ and consider VW, WU, UV; $AA'=a'$, $BB'=b'$, $CC'=c'$ as directed segments. Let λ, μ, ν be the ratios in which P, Q, R divide the sides of $A'B'C'$. Applying the Menelaus theorem to the pair $UC'B'$, CB we get $(PB'/PC')\cdot(CC'/CU)\cdot(BU/BB')=1$ or $\lambda(-c')(1/b')=1$; i.e., $\lambda=-b'/c'$. Considering also two other pairs we get $\mu=-c'/a'$ and $\nu=-a'/b'$ which give $\lambda\cdot\mu\cdot\nu=-1$ proving the concurrency at a point T.

(2) We denote the areal coordinates of T by the matrices $(l'm'n')$ and $(l\ m\ n)$ in the triangles $A'B'C'$ and ABC respectively, and from $l':n'=-\mu$, $m':l'=-\nu$ we obtain

$$(\alpha)\qquad l':m':n' = l':\frac{a'}{b'}\,l':\frac{a'}{c'}\,l' = \frac{1}{a'}:\frac{1}{b'}:\frac{1}{c'}\,.$$

Now to find $l:m:n=(l\ m\ n)$, let us first introduce the following symbol

$$LMN/XYZ = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

where the columns are the areal coordinates of L, M, N in the triangle XYZ. We have

$$(\beta)\quad A'B'C'/UVW = \begin{bmatrix} 0 & 1-a' & 1+a' \\ 1+b' & 0 & 1-b' \\ 1-c' & 1+c' & 0 \end{bmatrix},\quad UVW/ABC = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

and $T/A'B'C'=(1/a'\ 1/b'\ 1/c')$. It is not difficult to see the general identity

$$(\gamma)\qquad T/ABC = (T/A'B'C')\cdot(A'B'C'/UVW)\cdot(UVW/ABC).$$

Substituting (α) and (β) in (γ),

$$(l\ m\ n) = (l'\ m'\ n')\cdot\begin{bmatrix} 0 & 1-a' & 1-a' \\ 1-b' & 0 & 1-b' \\ 1-c' & 1-c' & 0 \end{bmatrix}\cdot\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= (1/a'\ 1/b'\ 1/c')\begin{bmatrix} 2 & 2a' & -2a' \\ -2b' & 2 & 2b' \\ 2c' & -2c' & 2 \end{bmatrix} = (2/a'\ 2/b'\ 2/c')$$

$$(\delta) \qquad T/ABC = l{:}m{:}n = 1/a'{:}1/b'{:}1/c' = AA'^{-1}{:}BB'^{-1}{:}CC'^{-1}.$$

(3) An affine projection transforms UVW into an arbitrary triangle and ABC into its medial triangle in which the concurrency holds.

In an affine transformation the ratio of segments and the ratio of areas being preserved, replacing $AA'/1$, $BB'/1$, $CC'/1$ by AA'/VW, BB'/WU, CC'/UV, (δ) becomes

$$T/ABC = \left(\frac{AA'}{VW}\right)^{-1} : \left(\frac{BB'}{WU}\right)^{-1} : \left(\frac{CC'}{UV}\right)^{-1}.$$

By a general projection the perspective triangles UVW and ABC are transformed into such triangles. So the generalization is obtained for any UVW and for any inscribable triangles ABC and $A'B'C'$, such that AU, BV, CW are concurrent. Furthermore, the point P has the same areal coordinates in ABC and $A'B'C'$.

Also solved by the proposer who points out that the generalization of (1) to general projections occurs as a problem in a 1909 Mathematical Tripos, Part I.

An Equation for a Regular $2n$-gon

E 1688 [1964, 430]. *Proposed by R. A. Jacobson, South Dakota State College*

The graph of the relation $|x+y|+|x-y|=2$ is a square with sides of length 2. Find a relation of the form $\sum_i |a_ix+b_iy+c_i|=A$ such that its graph is a regular octagon with sides of length 2.

Solution by Harold Gabow, Martin Van Buren High School. If $a_i^2+b_i^2=1$, then $|a_ix_1+b_iy_1+c_i|$ is equal to the distance between the point (x_1, y_1) and the line $a_ix+b_iy+c_i=0$. Hence the relation $\sum_i |a_ix+b_iy+c_i|=A$ represents the locus of points whose distances from the lines $a_ix+b_iy+c_i=0$ sum to A. In a regular $2n$-gon, the distances of any point to the diagonals passing through the center sum to a constant. This is readily proved by using the result that the points on the base of an isosceles triangle have distances to the sides which sum to a constant. Based on these remarks, one obtains the following equation of a regular $2n$-gon of side k:

$$\sum_{i=1}^{n}\left|\left[\sin(2i-1)\frac{\pi}{2n}\right]y + \left[\cos(2i-1)\frac{\pi}{2n}\right]x\right| = \frac{k}{2}\cot\frac{\pi}{2n}\sum_{j=1}^{n}\sin(2j-1)\frac{\pi}{2n}\cdot$$

For $2n=8$, $k=2$, this relation simplifies to:

$$|y+(\sqrt{2}-1)x| + |y-(\sqrt{2}-1)x| + |(\sqrt{2}-1)y+x| + |(\sqrt{2}-1)y-x| = 4+2\sqrt{2}.$$

Also solved by J. C. Abad, J. A. Burslem, M. L. Chachere, M. S. Demos, E. S. Eby, G. E. Engebretsen, Charles Franti, G. A. Heuer, J. E. Homer, Jr., R. F. Jackson, E. S. Langford, D. C. B. Marsh, Gus Maurigian, Norman Miller, F. D. Parker, Stanton Philipp, P. A. Scheinok, K. Soni, Stephen Wilson, and the proposer.

Simon Vatriquant calls attention to G. Tachella, *Fondamenti di Geometria Analitica a Due ed a Tre Dimensioni dei Luoghi Lineari Composti*, Premio Savoia Brabante, Genova (1933), wherein are discussed figures defined by $\sum_i |a_ix+b_iy+c_i| = m$ and other similar equations.

An Integer-Valued Arithmetic Function with no Infinite Linear Subset

E 1689 [1964, 430]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

Let f be an integral-valued function defined for all nonnegative integers x, such that $0 \leqq f(x) \leqq x$. Must the graph of f have an infinite subset whose points are collinear?

Solution by E. S. Langford, Autonetics Division, North American Aviation. Let f be any increasing function from the nonnegative integers into themselves which satisfies

(1) $\quad f(x) \leqq x$ and $\qquad$ (2) $\quad f(x) = o(x)$ but not $O(1)$.

For example, $f(x)$ might be $[\sqrt{x}]$. Since f is not bounded but is increasing, the graph of f contains no infinite collinear subset of points parallel to the x-axis. But there can be no infinite collinear subset of the graph with slope $\epsilon > 0$, since for all x', $f(x) < f(x') + \epsilon(x - x')$ ultimately. This is a consequence of assumption (2), since $f(x) = o(x)$ implies

$$\frac{f(x) - f(x')}{x - x'} \to 0 \quad \text{as} \quad x \to \infty \text{ for any } x'.$$

This answers the question in the negative.

Also solved by Shair Ahmad, Raymond Balbes and Thelma Balbes (jointly), E. D. Bender, T. J. Burke, Leonard Carlitz, R. E. Chandler, Allan Chuck and Peter Goldstein (jointly), Frank Dapkus, F. M. Eccles, N. J. Fine, David Forthoffer, Fred Galvin, W. E. Gould, J. O. Herzog, R. A. Jacobson, A. J. Keeping, Joel Kugelmass, C. R. MacCluer, D. C. B. Marsh, Bill Mitchell, D. A. Moran, W. O. Moser, J. B. Muskat, C. B. A. Peck, Horvath Sandor, Lawrence Schulman, C. Seguin, T. R. Smith, and the proposers.

A Divisibility Problem in $J_p[x]$

E 1690 [1964, 430]. *Proposed by D. E. Daykin, University of Reading, England*

Let n be a positive integer, p a prime, $q = p^n$, and $E = (q^p - 1)/(q - 1)$. Prove that $f(x) = x^q - x - 1$ divides $x^E - 1$ modulo p if and only if $n = 1$.

Solution by Leonard Carlitz, Duke University. We have, modulo p and $f(x)$,

$$x^q \equiv x + 1, \qquad x^{q^2} \equiv (x + 1)^q \equiv x^q + 1 \equiv x + 2, \cdots, x^{q^r} \equiv x + r,$$

so that

$$x^E = x^{1+q+q^2+\cdots+q^{p-1}} \equiv x(x+1)\cdots(x+p-1) \equiv x^p - x.$$

Then $x^E - 1 \equiv x^p - x - 1 \equiv 0 \pmod{x^q - x - 1}$ if and only if $n = 1$.

Also solved by the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before October 31, 1965, to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903.

5280. *Proposed by T. J. Guglielmo, Chaminade College, Hawaii*

Does the following limit exist and, if so, what is it?

$$\prod_{n=1}^{\infty} \begin{pmatrix} \dfrac{1}{2n} & \dfrac{1}{3n} \\ \dfrac{1}{4n} & \dfrac{1}{5n} \end{pmatrix}.$$

5281. *Proposed by D. J. Newman, Yeshiva University*

Prove that the general solution of the $N \times N$ algebraic system

$$x_n = \sum_{\substack{i=1 \\ i \neq n}}^{N} \frac{1}{x_n - x_i}, \qquad n = 1, 2, \cdots, N$$

is given by the N zeros of $(d/dx)^N e^{-x^2}$ in some order.

5282. *Proposed by G. Di Antonio, Fresno State College, California*

Given $T = (az+b)/(cz+d)$, $ad \neq bc$, and let I: $|cz+d| = 1$, I': $|cz-a| = 1$, be the isometric circles of T and T^{-1}, respectively. It is known (Ford, *Automorphic Functions*, Ch. I) that T is an inversion in I plus a reflection on L, the radical axis of I and I', for the nonloxodromic case. With this information show that, for the hyperbolic case, I and I' each contain a fixed point, and that each fixed point is a limit point of a nested sequence of successively transformed circles starting with I in one case and with I' in the other.

5283. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove the inequality

$$\frac{b-a}{(a^2+r^2)^{1/2}(b^2+r^2)^{1/2}} < \frac{1}{r}\left(\arctan\frac{b}{r} - \arctan\frac{a}{r}\right),$$

where $a < b$ and $r \neq 0$ (arc tan x designates the principal value).

5284. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Determine all functions X_k $(k=1, 2, \cdots, n)$ each of which depends on a single variable x_k in such a manner that

$$\left(\sum_{k=1}^{n} X_k\right)\left(\sum_{k=1}^{n} X_k''\right) = \sum_{k=1}^{n} (X_k')^2.$$

with $X_k' = dX_k/dx_k$, $X_k'' = d^2X_k/dx_k^2$.

5285. *Proposed by A. J. Goldman, National Bureau of Standards*

Which (total) functions of one variable can be computed by a simple Turing Machine with only one internal configuration? (We refer to the definition of simple Turing Machine given in Davis, *Computability and Unsolvability*.)

5286. *Proposed by Ralph Greenberg, University of Pennsylvania*

Let k be an integer. Prove the existence of infinitely many primes p for which $p+k$ is square-free. Find an asymptotic formula for the number of such primes under a given limit.

5287. *Proposed by Michael Gemignani, University of Notre Dame*

Prove or disprove: Let X be a topological space. Then any covering of X by open sets contains a subcovering C' which is minimal in the sense that if any member of C' is removed, then C' no longer covers X.

5288. *Proposed by R. F. Pavley and S. I. Mack, Radio Corporation of America, Moorestown, N. J.*

Does there exist a continuous real-valued function f and a real number x such that no subsequence of partial sums of the Fourier series of f, for this x, converges to $f(x)$?

5289. *Proposed by S. E. Payne, Florida State University*

Let the odd primes be denoted by $q_1, q_2, \cdots$. Let $a_1 = 1/\phi(q_1)$; $a_n = \{1/\phi(q_n)\}(1-a_{n-1})$ for n greater than 1 (ϕ is the Euler totient). Then show that $\sum_{n=1}^{\infty} a_n = 1$.

SOLUTIONS OF ADVANCED PROBLEMS

Mappings of groups of matrices

5190 [1964, 327]. *Proposed by J. L. Brenner, Stanford Research Institute, Menlo Park, California*

Let ϕ be a mapping of the full linear group of $n \times n$ matrices (over the field of complex numbers, say) into the underlying field. It is known that if $\phi(AB) = \phi(A)\phi(B)$, then $\phi(A)$ is a multiplicative function of the determinant of A. In this MONTHLY [1963, 163], S. Cater proved that the same conclusion follows if $\phi(ABC) = \phi(CBA)$ for all triples of matrices, but that the hypothesis $\phi(AB)$

$=\phi(BA)$, which is satisfied by the trace function, is not enough. (1) Show that, even if $n=2$, there are infinitely many maps ϕ satisfying this last hypothesis. (2) If both hypotheses $\phi(AB)=\phi(BA)$, $\phi(A^{-1})=[\phi(A)]^{-1}$ are satisfied, how is the map restricted?

Solution by the proposer. That $\phi(AB)=\phi(BA)$ gives very little restriction is seen from a result of H. Flanders (Proc. Amer. Math. Soc., 2 (1951) 871–874) that may be paraphrased as follows in case the dimension of the matrices is 2. Let S and T be similar matrices (invertible or not). Then A, B can be found such that $S=AB$, $T=BA$. Conversely, AB and BA are always similar (in the 2×2 case) with the single exception that the class representatives

$$O=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\qquad H=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

can be respectively written in the forms AB, BA. Thus the requirement $\phi(AB)=\phi(BA)$ merely specifies that ϕ is constant on a similarity class, and that $\phi(O)=\phi(H)$. Since there are infinitely many similarity classes, and ϕ is otherwise arbitrary, there are 2^c possible choices for ϕ. The restriction $\phi(A^{-1})=[\phi(A)]^{-1}$ does not alter the situation in any essential way. To be sure of obtaining the determinant function and no other, it would be necessary to specify that the matrices

$$\begin{pmatrix}x & 0\\ 0 & y\end{pmatrix},\qquad \begin{pmatrix}x & 1\\ 0 & x\end{pmatrix}$$

are mapped into their determinants for all x, y: these sets of matrices exhaust the similarity classes.

Elliptic functions with prescribed invariants

5192 [1964, 440]. *Proposed by J. M. Horner, University of Alabama*

Let $\wp(z)$ be a Weierstrass elliptic function whose invariants are of the form $g_2=-3r^2$, $g_3=2s^3$, where r and s are rational, $s\neq 0$, $r^2-s^2\neq 0$. Prove that $\wp(\omega+\omega')$ is real, $\operatorname{sgn}(\wp(\omega+\omega'))=\operatorname{sgn}(s)$, and $\wp(\omega+\omega')$ is irrational.

Solution by Oswald Wyler, University of New Mexico. The equation

$$f(w) = 4w^3 - g_2w - g_3 = 4w^3 + 3r^2w - 2s^3 = 0$$

has exactly one real solution since $f'(w)>0$ for all real w. As s and $f(0)$ have opposite signs, the real solution and s have the same sign. The three solutions of $f(w)=0$ are $\wp(\omega)$, $\wp(\omega')$, $\wp(\omega+\omega')$ for primary half-periods ω, ω', and by a suitable choice of ω and ω' (e.g. ω, ω' conjugate complex), one can make sure that the real solution of $f(w)=0$ is indeed $\wp(\omega+\omega')$.

Let now $u=w/s$, $p=r/s$. The equation $f(w)=0$ is equivalent to

$$(u+\tfrac{1}{2}p)^3+(u-\tfrac{1}{2}p)^3 = 1.$$

Since the only rational solutions of the equation $x^3+y^3=1$ are the pairs $x=1$, $y=0$ and $x=0$, $y=1$, we must have $u=\pm\frac{1}{2}p$, $p=\pm 1$ if p and u are rational. But $p=\pm 1$ means $r^2=s^2$, and this was excluded. Thus $\wp(\omega+\omega')$ is not rational.

Also solved by L. Carlitz and by the proposer.

Zeros of a Power Series

5193. *Proposed by Lewis Batson, University of Southeastern Louisiana*

Find the zeros of the function

$$f(x) = 1 + \sum_{n=1}^{\infty}\left[(2x)^n \Big/ \prod_{k=1}^{n}(2^k - 1)\right].$$

Solution by N. J. Fine, Pennsylvania State University. More generally, for $|r|>1$, consider the entire function

$$f(x) = \sum_{n=0}^{\infty} A_n x^n,$$

where $A_0=1$, $A_{n+1}=rA_n/(r^{n+1}-1)$, $(n\geqq 0)$. Multiply both sides by $x^{n+1}(r^{n+1}-1)$ and sum for $n\geqq 0$, to get

$$\begin{aligned}
(1)\qquad f(rx) - f(x) &= rxf(x),\\
f(rx) &= (1+rx)f(x),\\
f(x) &= (1+x)f(x/r),\\
&= (1+x)(1+x/r)f(x/r^2),\cdots,\\
&= \lim_{N\to\infty}\prod_{n=0}^{N}(1+x/r^n)f(x/r^{N+1}),\\
(2)\qquad f(x) &= \prod_{n=0}^{\infty}(1+x/r^n).
\end{aligned}$$

Thus, the zeros of $f(x)$ are at $x=-r^n$ $(n\geqq 0)$.

Also solved by I. N. Baker (England), L. Carlitz, John Doles, P. G. Engstrom, Sylvan Greene, Hajna Jânos (Hungary), R. P. Kelisky, J. Koekoek (Netherlands), Viktors Linis, Stanton Philipp, D. Suryanarayana (India), J. H. van Lint (Netherlands), W. C. Waterhouse, Oswald Wyler, and the proposer.

Several solvers obtain the zeros of $f(x)$ directly from the functional equation (1) above. Philipp and Suryanarayana refer to Barnard and Childs, *Higher Algebra*, p. 490, for the identity (2).

The Airy Equation: $u''+tu=0$

5194 [1964, 440]. *Proposed by John V. Ryff, Harvard University*

If u is a solution of the Airy equation $u''+tu=0$ which satisfies $u(0)=0$, show that ξ, the first positive zero of u, is such that $\pi^{2/3}<\xi<2^{1/3}\pi^{2/3}$.

I. *Solution by L. E. Ward, Sr., Escondido, California.* The desired results are obtained by Sturm's comparison methods. We use the comparison function $v(t)=\sin \pi t/\xi$, which satisfies the differential equation $v''+(\pi^2/\xi^2)v=0$. On multiplying this differential equation by $-u$, the given differential equation by v, adding and integrating, it is found that

$$u'v - v'u\Big]_0^\xi + \int_0^\xi (t - \pi^2/\xi^2)uvdt = 0$$

or

$$\int_0^\xi (t - \pi^2/\xi^2)uvdt = 0, \tag{1}$$

since u and v both vanish at $t=0$ and at $t=\xi$. The vanishing of this integral requires that the linear factor $t-\pi^2/\xi^2$ be positive on part of the interval $(0, \xi)$ and negative on part of the interval since neither u nor v changes sign. This requires that the linear factor be positive at $t=\xi$. Hence $\xi>\pi^2/\xi^2$, which is equivalent to one of the desired results.

The other result comes from looking at the same integral in a different way. Let this integral be written

$$\int_0^{\xi/2} (t - \pi^2/\xi^2)uvdt + \int_{\xi/2}^{\xi} (t - \pi^2/\xi^2)uvdt = 0.$$

In the second of these integrals change the integration variable to t' by the relation $t=\xi-t'$. We are then able to combine the integrals and obtain

$$\int_0^{\xi/2} [(t - \pi^2/\xi^2)u(t) + (\xi - t - \pi^2/\xi^2)u(\xi - t)]v(t)dt = 0. \tag{2}$$

It is convenient to assume that $u(t)$ is positive on $0<t<\xi$. This is not a restriction of the generality of the solution.

It will now be shown that

$$u(\xi - t) > u(t), \qquad 0 < t < \xi/2. \tag{3}$$

For convenience let $w(t)=u(\xi-t)$, and note that $w''+(\xi-t)w=0$. Multiply this differential equation by $-u$, the differential equation of the problem by w, add, and integrate, getting

$$u'w - w'u\Big]_0^t - \int_0^t (\xi - 2t)uwdt = 0,$$

or, since both u and w vanish at $t=0$,

$$u'(t)w(t) - w'(t)u(t) - \int_0^t (\xi - 2t)uwdt = 0$$

In this relation let t be restricted to $0<t\leqq\xi/2$. Then, since u and w are both positive, the integral is positive. Hence $u'(t)w(t)-w'(t)u(t)>0$ and

$$\int_t^{\xi/2}\frac{u'(t)}{u(t)}\,dt \geqq \int_t^{\xi/2}\frac{w'(t)}{w(t)}\,dt,$$

equality occurring at $t=\xi/2$ and only there.

It follows that $\log u(\xi/2)-\log u(t)\geqq\log w(\xi/2)-\log w(t)$, or, since $u(\xi/2)=w(\xi/2)$, $u(t)\leqq u(\xi-t)$, $0\leqq t\leqq\xi/2$, equality occurring only at $t=0$ and $t=\xi/2$.

Returning to (2), suppose it were possible that $\xi/2\geqq\pi^2/\xi^2$. Then, using (3) it is seen that

$$(\pi^2/\xi^2-t)u(t)<(\xi-t-\pi^2/\xi^2)u(\xi-t), \qquad 0<t<\xi/2.$$

But in this case (2) could not be true. Hence the present supposition fails, and it follows that $\xi/2<\pi^2/\xi^2$. This is equivalent to the second of the desired relations.

II. *Solution by J. Koekoek, Technological University, Eindhoven, Netherlands.* The nontrivial solution of the Airy equation $u''+tu=0$, which satisfies $u(0)=0$, is

$$u(t)=at^{1/2}J_{1/3}(\tfrac{2}{3}t^{3/2}) \qquad a\neq 0.$$

The first positive zero ξ of this function satisfies $2.90<\frac{2}{3}\xi^{3/2}<2.91$ and leads to $\xi=2.666$. (Cf. *Tables of Bessel functions of fractional order*, The Computation Laboratory of the National Bureau of Standards, vol. I, Columbia University Press, 1948.)

Also solved by J. J. A. Brands (Netherlands), Robert Breusch, Harley Flanders, Hajna János (Hungary), A. E. Livingston, G. H. Ryder, Sidney Spital, J. Ernest Wilkins, Jr., and the proposer.

The equation $y'''-p(x)y=0$

5195 [1964, 440]. *Proposed by A. C. Lazer, Carnegie Institute of Technology.*

Let $p(x)$ be any function positive and continuous for x in an interval I. (a) Show that no nontrivial integral of the differential equation $y'''-p(x)y=0$ can have more than one point of inflection in I, nor (b) can it be tangent to the line $y=0$ more than once in I, nor (c) can it have both an inflection point and a point of tangency at the line $y=0$ in I.

(a) *Solution by J. H. van Lint, Technological University, Eindhoven, Netherlands.* The example $p(x)=1$, $y=e^{-x/2}\cos\frac{1}{2}x\sqrt{3}$, shows that it is possible to have any number of points of inflection, thereby refuting the assertion of the problem.

(b) *Solution by the proposer.* If $u'''=p(x)u(x)$, $v(x)=u''u-[u'(x)]^2/2$, then $v'(x)=p(x)[u(x)]^2$. If $p(x)$ is positive and $u(x)$ does not vanish identically on some subinterval of I, then $v(x)$ is strictly increasing and vanishes at most once in I. Thus $u(x)$ and $u'(x)$ can become zero simultaneously at most once.

(c) *Solution by Oswald Wyler, University of New Mexico.* The example $p(x)=60(x^3+10)^{-1}$, $y(x)=x^5+10x^2$, $x\geqq -2$, exhibits a tangent to $y=0$ at (0, 0) and an inflection point at $x=-1$, thereby contradicting the statement of the problem.

Also solved by Sidney Spital.

The curve $r(\theta)=r(\theta+\alpha)$

5197 [1964, 441]. *Proposed by H. Guggenheimer, University of Minnesota*

Let $r=r(\theta)$ be the polar equation of a smooth, convex, simple closed curve, the origin being placed at the area centroid of the curve. Show that for any constant α the equation $r(\theta)=r(\theta+\alpha)$ has at least four distinct solutions.

Solution by K. A. Post, Technological University, Eindhoven, Netherlands. The conditions about convexity and smoothness are not necessary. The fact that the area centroid is at the origin leads at once to the equations

$$\int_0^{2\pi} r^3(\theta)\cos\theta d\theta = \int_0^{2\pi} r^3(\theta)\sin\theta d\theta = 0,$$

$$\int_0^{2\pi} r^3(\theta+\alpha)\cos\theta d\theta = \int_0^{2\pi} r^3(\theta+\alpha)\sin\theta d\theta = 0.$$

The function $f(\theta)=r^3(\theta+\alpha)-r^3(\theta)$ now satisfies the following conditions:

$$\int_0^{2\pi} f(\theta)d\theta = 0, \qquad \int_0^{2\pi} f(\theta)\cos\theta d\theta = 0, \qquad \int_0^{2\pi} f(\theta)\sin\theta d\theta = 0.$$

By exercise 141 in Polya u. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, I, $f(\theta)=0$ has at least four solutions mod 2π.

Also solved by Robert Breusch, Harley Flanders, Tudor Zamfirescu (Rumania), and the proposer.

Coefficients of the characteristic polynomial

5198 [1964, 441]. *Proposed by H. S. Shapiro, New York University and the University of Michigan*

Let $\|a_{ij}\|$ be a square matrix of complex numbers, and $\Delta(\lambda)=\lambda^n+c_1\lambda^{n-1}+\cdots+c_n$ its characteristic polynomial. Show that

$$|c_k| \leqq n^{k/2}\binom{n}{k}M^k,$$

where $M=\text{Max }|a_{ij}|$.

Solution by Sidney Spital, California State Polytechnic College. Let the k-rowed principal minors of $\|a_{ij}\|$ be labelled D_i, $i=1, 2, \cdots, \binom{n}{k}$. Then $|c_k|=|\sum_{i=1}^p D_i|\leqq\sum_{i=1}^p |D_i|$, where $p=\binom{n}{k}$. From Hadamard's inequality

it follows that all the minors satisfy $|D_i|^2 \leqq k^k M^{2k}$. Thus

$$|c_k| \leqq k^{k/2}\binom{n}{k}M^k$$

which implies the required bound.

Also solved by Robert Breusch, L. Carlitz, M. M. Chawla (India), John H. E. Cohn (England), J. L. Ercolano, Harley Flanders, Yasuhiko Ikebe, A. E. Livingston, Marvin Marcus, Leo Moser and M. G. Murdeshwar, P. J. Nikolai, Joseph Reitberg, John V. Ryff, R. C. Thompson, W. C. Waterhouse, and the proposer.

Subspaces of $L^2[0, 1]$ in L^p, $p>2$

5199 [1964, 441]. *Proposed by H. S. Shapiro, New York University and the University of Michigan*

Prove or disprove the following conjecture: A closed subspace of $L^2[0, 1]$ which is a subset of L^p for some $p>2$ is finite dimensional.

Solution by James Williams, University of Michigan. The conjecture is true if $p=\infty$ and false otherwise.

It is well known that if $\sum(a_k^2+b_k^2)=\gamma^2<\infty$, then $\sum(a_k \cos n_k t+b_k \sin n_k t)$ is the Fourier series of an L_2 function f. If the series is *lacunary*, i.e., if $n_{k+1}/n_k \geqq q>1$ for all k, much more is true. Indeed Zygmund (*Trigonometric Series*, vol. 1, 1959, p. 215) proves that there is a constant $B_{p,q}$ depending only on p and q such that for every p, $1<p<\infty$,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p dt\right)^{1/p} \leqq B_{p,q}\cdot\gamma.$$

It clearly follows from this that the conjecture is false for $1<p<\infty$. For example, the closed subspace of $L_2(0, 2\pi)$ spanned by the functions $\{\cos 2^k x, \sin 2^k x; k=1, 2, \cdots\}$ is a subspace of $L_p(0, 2\pi)$ if $1<p<\infty$.

Considering now the case $p=\infty$, let M be a closed subspace of $L_2(0, 1)$ with the property that each function in M is essentially bounded. In view of the obvious inequality $\|f\|_2 \leqq \|f\|_\infty$ it follows that M is closed in $L_\infty(0, 1)$. In other words, the vector space M is a Banach space with respect to these two (comparable) norms. It follows from the Closed Graph Theorem that these norms are actually equivalent on M, i.e., for some constant $K>0$ we have

$$\|f\|_\infty \leqq K\|f\|_2 \qquad \text{all } f \in M.$$

To complete the proof we show that if $\phi_1, \phi_2, \cdots, \phi_n$ is any orthonormal set in M, then $n \leqq K^2$.

Let $\alpha_k=(\alpha_1^k, \alpha_2^k, \cdots, \alpha_n^k)$, $k=1, 2, \cdots$ be a dense set in the Hilbert space C^n of complex n-tuples and let

$$E_k = \{t \in [0, 1]: |\langle \alpha_k, \phi(t)\rangle| \leqq K\|\alpha_k\|\} \qquad k = 1, 2, \cdots,$$

where

$$\langle \alpha_k, \phi(t)\rangle = \sum_{j=1}^{n} \bar{\alpha}_j^k \phi_j(t), \qquad \|\alpha_k\|^2 = \sum_{j=1}^{n} |\alpha_j^k|^2.$$

If $E = \bigcap_{k=1}^{\infty} E_k$, then E consists of those $t \in [0, 1]$ for which

$$|\langle \alpha_k, \phi(t)\rangle| \leqq K\|\alpha_k\|, \qquad k = 1, 2, \cdots.$$

Using the density of the α_k this means that $t \in E$ if and only if

$$(*) \qquad \|\phi(t)\|^2 = \sum_{j=1}^{n} |\phi_j(t)|^2 \leqq K^2.$$

Now for each fixed k the vector $\sum_{j=1}^{n} \bar{\alpha}_j^k \phi_j$ belongs to M and hence for almost all $t \in [0, 1]$ we have

$$\left| \sum_{j=1}^{n} \bar{\alpha}_j^k \phi_j(t) \right|^2 \leqq K^2 \left\| \sum_{j=1}^{n} \bar{\alpha}_j^k \phi_j \right\|_2^2 = K^2 \cdot \sum_{j=1}^{n} |\alpha_j^k|^2.$$

Thus the sets E_k have measure 1 and consequently $m(E) = 1$, i.e., (*) holds for almost all t. Hence

$$n = \sum_{k=1}^{n} \|\phi_k\|_2^2 = \sum_{k=1}^{n} \int_0^1 |\phi_k(t)|^2 dt \leqq K^2 \cdot 1.$$

Also solved by Harry Furstenberg and Charles McCarthy, M. A. Malik, M. Rajagopalan and A. Wilansky, and the proposer.

Notes. See also S. Banach, *Theorie des Opérations Linéaires*, pp. 203–204 for the application of lacunary trigonometric series above.

The proposer settles the case $p < \infty$ with the closed subspaces generated by the Rademacher functions; see Zygmund, *Trigonometric Series*, 1st ed., pp. 122, ff.

Furstenberg shows that M (in L^∞) is contained in the eigenspace of a Hilbert-Schmidt operator on L^2 (0, 1) corresponding to the eigenvalue 1 and is, therefore, finite dimensional. He observes, too, that a closed subspace of l^2 consisting of sequences in l^1 is finite dimensional.

A New Method of Catching a Lion

In this note a definitive procedure will be provided for catching a lion in a desert (see [1]).

Let Q be the operator that encloses a word in quotation marks. Its square Q^2 encloses a word in double quotes. The operator clearly satisfies the law of indices, $Q^m Q^n = Q^{m+n}$. Write down the word 'lion,' without quotation marks. Apply to it the operator Q^{-1}. Then a lion will appear on the page. It is advisable to enclose the page in a cage before applying the operator.

I. J. Good

1. H. Petard, A contribution to the mathematical theory of big game hunting, this Monthly 45 (1938) 446–447.

curves, rational parametrization of some plane curves; and, as examples of higher dimensional objects, some algebraic groups, the Segre variety, and the Plücker variety of lines in three-space are offered.

My chief criticism is that there are only four or five theorems in the book, the diagonalization of a quadratic form and the existence of Plücker coordinates being the deepest. My own prejudice is that mathematics books—no matter how small or how modestly conceived—must, like mathematics itself, aim at proving theorems. An intellectual diet composed largely of definitions and examples is as unsustaining as cocktail party chatter.

On the technical side, the author gets into trouble early by trying to identify a variety with its point set, a procedure which is not possible if the ground field k is finite and the space is composed only of the k-rational points. For instance, every point set in the plane must then be called a curve, but "when everybody's somebody, then no one's anybody." The definition of vector space given in the preliminary chapter is incorrect. As compensation, there are full page pictures of curves like $y^2=x^3$; the author confesses that he likes pictures, a refreshing admission indeed for a 20-th century geometer.

ARTHUR MATTUCK, Massachusetts Institute of Technology

Introduction to Abstract Algebra. By Wilfred E. Barnes. Heath, Boston, 1963. 215 pp. $6.00.

This is a short book of 210 pages treating basic material on groups, rings, and fields. The text is meant to acquaint advanced undergraduate or beginning graduate students with terminology and techniques of algebra.

The book begins with a brief discussion of sets and functions. This chapter is too brief for the uninitiated, however, and is inserted only to establish terminology and notation. The author actually assumes that the student has a fair degree of maturity. Specifically, he assumes that the reader has already completed at least one course in linear algebra and is familiar with the algebra and arithmetic of the integers.

This text contains a number of topics not usually encountered in a text designed strictly for undergraduates. Among the topics covered are: Groups with operators, composition series, the Jordan-Hölder theorem, the Sylow theorems, the decomposition theorems for abelian groups, finiteness conditions for ideals, Noetherian rings, field extensions, the Wedderburn theorem for finite division algebras, Galois theory, and ordered fields. Most of the proofs given are standard and well known. This book abounds with problems, many quite challenging.

The number of examples in the text is perhaps adequate for a mature student, but less experienced students may feel the need of further examples. There is essentially no discussion of the role of algebra in other parts of mathematics. While linear algebra is assumed, there is little discussion of the interrelation between it and the topics under consideration.

D. J. LEWIS, University of Michigan

Numerical Analysis. By Nathaniel Macon. Wiley, New York, 1963. 161 pp. $5.50.

A first course in numerical analysis *and* high-speed computers often tends to be a difficult compromise between mathematical theory and practice in programming, especially when the course is offered at the sophomore level. Professor Macon's book includes enough mathematics to be reasonably rigorous, and at the same time stresses the applications of the theory to high-speed computers. Each mathematical topic is well documented with easily understandable examples, and, whenever possible, flow charts for programming use are included. Also, each chapter contains many good exercises, with most answers being supplied.

As a first course in numerical analysis, this book covers the standard topics: approximation of functions by polynomials (Ch. 2), iterative methods, such as Newton's method, for solving equations (Ch. 3), computational methods for matrix equations, and the eigenvalue-eigenvector problem (Ch. 4, 5, 6), interpolation (Ch. 7), differentiation and quadrature (Ch. 8), and ordinary differential equations (Ch. 10). It is rather clear that the author leans somewhat more toward mathematics than programming. For example, he does not hesitate to prove the Weierstrass theorem, as a foundation for polynomial approximation, by means of Bernstein polynomials. Similarly, the convergence of the Cauchy polygon method and the Cauchy-Lipschitz method are presented in detail. It is thus the reviewer's opinion that the scope of the mathematics is generally beyond that of most sophomores. More advanced students should have no difficulty with the text.

The exposition is at all times very clear, and only a few easily corrected errors were found.

R. S. VARGA, Case Institute of Technology

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. Smith Park, Eastern Kentucky State College, represented the Association at the Centennial Year Celebration of Founders Day at the University of Kentucky on February 22, 1965.

Professor R. G. Selfridge, University of Florida, represented the Association at the inauguration of Robert H. Spiro as President of Jacksonville University on November 20, 1964.

Arlington State College: Dr. E. C. Kennedy, General Dynamics Corporation, Daingerfield, Texas, has been appointed Professor; Dr. Douglas Stocks, University of Texas, and Mr. J. R. Harvey, University of Dallas, have been appointed Assistant Professors.

University of Colorado: Dr. G. M. Wing, University of California, Berkeley, has been appointed Professor; Dr. W. B. Jones, National Bureau of Standards, has been appointed Assistant Professor.

Eastern Michigan University: Dr. D. E. Ryan, University of Texas, and Mr. James Walter, Valparaiso University, have been appointed Assistant Professors; Assistant Professors G. D. Anderson and James Northey have been promoted to Associate Professors; Mrs. Nelly Ullman has been promoted to Assistant Professor.

University of Missouri at Rolla: Professor Charles Hatfield, University of North Dakota, has been appointed Professor and Chairman of the Mathematics Department; Mr. A. J. Garver has been promoted to Assistant Professor.

Mr. D. R. Baker, I.T.T., Paramus, New Jersey, has accepted a position as Senior Mathematician at the Advanced Computer Laboratories of Melpar, Inc., Falls Church, Virginia.

Dr. W. O. Buschman, California State Polytechnic College, has been appointed Director of the Computer Center.

Professor B. G. Clark, Vanderbilt University, retired on June 30, 1964, with the title of Professor Emeritus. (This is a correction of an item in the January 1965 issue.)

Dr. R. M. Fesq, University of California, Berkeley, has been appointed Assistant Professor at Kenyon College.

Associate Professor B. M. Seelbinder, Wake Forest College, has been promoted to Professor.

Assistant Professor C. J. Stuth, East Texas State College, has been promoted to Associate Professor.

Professor J. W. T. Youngs, Indiana University, has been appointed Professor at the University of California, Santa Cruz.

Professor Emeritus W. C. Brenke, University of Nebraska, died on August 20, 1964. He was a charter member of the Association.

Professor T. C. Carson, retired from East Tennessee State College, died on October 25, 1964. He was a member of the Association for 30 years.

Professor Evelyn W. Casner, Randolph-Macon Woman's College, died on November 5, 1964. She was a member of the Association for 35 years.

Associate Professor R. S. Christian, Georgia State College of Business Administration, died on February 19, 1965. He was a member of the Association for 15 years.

Dean Emeritus J. W. Clawson, Ursinus College, died on October 26, 1964. He was a charter member of the Association.

Associate Professor Olga Larson, retired from State College for Women, Tallahassee, Florida, died on July 16, 1964. She was a member of the Association for 40 years.

Mr. Lionel London, Aerojet-General Corporation, Sacramento, California, died on August 29, 1964. He was a member of the Association for 20 years.

Associate Professor Imanuel Marx, Purdue University, died on November 13, 1964. He was a member of the Association for 7 years.

Associate Professor Emeritus E. J. Miles, Yale University, died on November 3, 1964. He was a charter member of the Association.

Professor Emeritus Mary W. Newson, Eureka College, died in December 1959. She was a charter member of the Association.

UNIVERSITY OF MONTREAL—SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES

Under the sponsorship of the North Atlantic Treaty Organization (NATO) and the Canadian Mathematical Congress, the fourth session of the University of Montreal "SÉMINAIRE DE MATHÉMATIQUES SUPÉRIEURES" will be held next summer, from June 28 to August 6.

This year, the Seminar will be on Partial Differential Equations. The programme will consist of six main courses: Shmuel Agmon, Université de Jérusalem, "Unicité et propriétés de convexité dans les problèmes différentiels"; Marcel Brelot, Université de Paris, "Théorie axiomatique du potentiel s'appliquant aux équations aux dérivées partielles du second ordre"; Felix Browder, University of Chicago, "Problèmes non-linéaires"; Guido Stampacchia, Université de Pise, "Équations elliptiques du second ordre à coefficients discontinus"; José Barros-Neto, Université de Montréal, "Problèmes aux limites non-homogènes"; Samuel Zaidman, Université de Montréal, "Équations différentielles abstraites."

Apart from these courses, the programme will include a certain number of lectures given by guest speakers. Registrants may make application for financial assistance to cover travelling and living expenses. To obtain further information and registration forms, please write to: Département de Mathématiques, Université de Montréal, Case postale 6128, Montréal 3, Québec (Canada).

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-EIGHTH ANNUAL MEETING OF THE ASSOCIATION

The Forty-eighth Annual Meeting of the Mathematical Association of America was held at the Denver-Hilton Hotel, Denver, Colorado, from Thursday to Saturday, January 28 to 30, 1965, in conjunction with the Annual Meeting of the American Mathematical Society, and meetings of the National Council of Teachers of Mathematics and the Association for Symbolic Logic. The session of the Association on Saturday was a joint session with the National Council of Teachers of Mathematics. There were registered 2095 persons, including 1292 members of the Association.

Sessions of the Association were held on Thursday morning, Friday morning, and Saturday morning and afternoon in the Grand Ballroom of the Denver-Hilton Hotel. Presiding officers were Professor C. E. Burgess for the Panel and General Discussion on the Role of Topology in the Undergraduate Program on Thursday morning, Professor W. T. Reid for the hour address on Thursday morning, Professor George Springer for the first hour address on Friday morning and Professor W. J. Thron for the second one, Professor F. B. Allen for the first two parts of the Saturday program, Professor R. A. Rosenbaum for part III, and Professor Ruth I. Hoffman for part IV. The Program Committee for the meeting consisted of W. J. Thron, Chairman; C. E. Burgess, Ruth I. Hoffman, R. D. Mayer, W. T. Reid and George Springer.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 5

CONTENTS

MAY 1965

THE AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00. For nonmembers the subscription price is \$8.00 (\$10.00 effective August–September 1965).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

GAMES, AN INFORMAL TALK

H. STEINHAUS, Wrocław

In the first decade of our century I had the occasion as a student in Göttingen to hear many comments about Felix Klein. He was praised as a great mathematician and, at the same time, blamed because of his proof for the existence of solutions of the potential equation. He considered the distribution of electricity on a metallic conductor as a convincing argument for such existence. Most mathematicians avoid such statements; nevertheless, they have played and will play a role in the evolution of our science. The short story of my encounter with the theory of games does not take into account exact principles (which I consider to be laudable as a rule) and is not to be considered as a lecture for professionals.

1. There are two definitions of continuity: one known as Cauchy's definition, another attributed to Heine. Let us call them C-continuity and H-continuity respectively. A function $f(x)$ defined in $(-1, +1)$, $f(0)=0$, is C-continuous at 0 if and only if to any natural n there exists a natural $m(n)$ such that $|x| < 1/m$ implies $|f(x)| < 1/n$. It is H-continuous at 0 if and only if $|x_m| < 1$ for all natural m and

$$\lim_{m\to\infty} x_m = 0 \quad \text{implies} \quad \lim_{m\to\infty} f(x_m) = 0.$$

C means that f is C-continuous at 0, H means that it is H-continuous at 0. The proof of $C\Rightarrow H$ is quite easy. The converse theorem $H\Rightarrow C$ can be established by an argument based on the Axiom of Choice (Zermelo's Axiom); we call this axiom AZ. There are mathematicians who don't like AZ but they apply it in class when proving that $H\Rightarrow C$.

Let us deal with this question as with a claim brought before a law court. The usher asks if there is anybody who does not believe that $H\Rightarrow C$. If nobody acts in reply to this challenge, the court assumes that everybody admits $H\Rightarrow C$ as being true, and proclaims $H\Rightarrow C$ as proved by this *consensus ingeniorum*, in spite of no formal proof being submitted to the court. It may happen that things are different: Mr. A who has brought the question to the court believes $H\Rightarrow C$ to be true but a Mr. B appears as his opponent. "I know," says Mr. B, "that $H\Rightarrow C$ can be proved if the Axiom of Choice is admitted but I decline to accept proofs based on principles rejected by some mathematicians of high rank." "I share your opinion," answers A, "as to AZ, nevertheless I believe $H\Rightarrow C$, and if nobody manages to define a $f(x)$ H-continuous at 0 without being C-continuous at 0, the High Court will issue the verdict $H\Rightarrow C$." The judge summons Mr. B to define a function $f(x)$ necessary to convince Mr. A. Let us suppose that Mr. B claims to have such an example and calls it $g(x)$. Its C-discontinuity at 0 involves the existence of a natural number N such that every interval $I_m=(-1/m, 1/m)$ contains a x_m such that $|g(x_m)| \geqq 1/N$ whatever the natural m. Mr. B is obliged to propose such an N, and he does it. Mr. A risks now the con-

jecture that $|g(x)| < 1/N$ for all x in I_1. Were this conjecture true Mr. B's situation would be disagreeable. It is not the case: he puts his finger on a point x_1 in I_1 and proclaims triumphantly: "$g(x_1) \geqq 1/N$." Mr. A does not know the definition of $g(x)$, but he accepts this statement as correct. Now the *onus probandi* is on him and he is compelled to try his luck with I_2. Mr. B answers promptly, choosing x_2 in I_2 and pretending $|g(x_2)| \geqq 1/N$; Mr. A has no means to contest it. Now it is the judge who stops the game by the question: "Mr. B, are you sure you will always succeed with your fixed N ('always' means here 'in every I_m')?" Before Mr. B answers, Mr. A interrupts the proceedings: "Sir, whatever B answers his cause is lost !: 1^0 if he says 'no' he disavows his example $g(x)$; 2^0 if he says 'yes' his answer will be equivalent to the existence of an infinite sequence $\{x_m\}$, $x_m \in I_m$, such that $g(x_m) \geqq 1/N$ for every m, which is incompatible with the H-continuity of $g(x)$ at $x=0$; consequently, $g(x)$ being H-discontinuous at 0, it cannot be a counter-example for $H \Rightarrow C$."

The trick applied by Mr. A is as follows: Mr. B disavows proofs for $H \Rightarrow C$ if Mr. A would try to prove it with the aid of AZ, i.e., with the Axiom of Choice; Mr. A asks from B a *Gegenbeispiel* against $H \Rightarrow C$, without even being interested in the definition of such an example. Mr. A shows that if B has a well-defined $g(x)$ with H-continuity and without C-continuity he can be entangled in contradictions. In 1923 I met O. Toeplitz in Stein near Kiel and told him that I did not feel convinced by proofs based on Zermelo's axiom; he answered: "Would you try to construct counterexamples against theorems proved with this axiom?" My answer was: "No!"

The idea of appealing to a law court in such questions as the equivalence of two definitions of continuity is perhaps extravagant for most mathematicians but not for people interested in the theory of games; legal proceedings are often similar to games with fixed rules. Some years ago I told the case of A *versus* B to E. Čech, a universally known geometer who was visiting Poland. His opinion was a disapproval. After a couple of years I met him again—I am sorry to say that it was his last visit—and he asked me about legal mathematics. He assured me that he accepts them as correct . . . "Too late!" I answered, "because in the meantime your objections have destroyed my belief."

Dr. Jan Mycielski, having read the lines above, told me that he considers the legal action A *versus* B as an argument for Zermelo's Axiom of Choice in general. His are the following lines:

Mr. A believes Zermelo to be right. Mr. B brings a counterexample. He pretends to know a family F of sets X such that no set X belonging to F is void and such that there is no function $f(X)$ defined for every X belonging to F and verifying $f(X) \in X$ for every $X \in F$. As B declares that no X belonging to F is void, he can find x in X for every X in F, if he cannot he has lost his case already. If he says that he can do it, he has a function f described above; this is a contradiction against what he has told about the family F, which means his defeat too.

2. The story of two adversaries who bring their opposite views on a mathematical problem before a law court is not a bad joke as it could be called by the reader of Section 1; there are in applied mathematics questions which can be presented as games and their solutions are the best strategies of fictitious (sometimes of real) partners; a controversy in a court conducted in conformity with the rules of legal procedure being a game too, the mathematical world is not separated from the legal one by an impermeable wall. (For an application of games to pure mathematics cf. [6].) When speaking about games we mean their modern theory.

The first important result of this theory is Zermelo's lecture presented to the Fifth International Congress of Mathematicians in 1912 [1]. His result is the determinateness of such games as chess. To explain it briefly, let us consider N chessboards and N pairs playing; we assume that every partner is perfect: he makes always those moves which are absolutely best. Then all N results will be the same, which means that either White will win on N chessboards, or Black on N chessboards, or all N games will end in a draw. Zermelo's theorem is valid for all games sharing with chess the following properties: (a) alternate moves, (b) finite number of moves, (c) complete information. (a) means that White moves first, then Black, then White, etc. (b) means that there is a number M such that no game with more moves is possible, i.e., that all games yield a definite result after n moves, n being always $\leqq M$. (c) means that there are no circumstances having an influence on the result, except the alternate moves and the initial position known to both partners; all moves are made openly.

World War I is the reason why Zermelo's paper failed to be appreciated according to its merit. The second step is due to Émile Borel [2]. He found the minimax concept after World War I, but his papers were not duly appreciated in France and failed to reach Polish mathematicians. I was interested in the theory of pursuit. My paper of 1925 was translated in 1960 [3]. My problem was to define the best strategies in a game older than chess and perhaps older than human civilization: the game of chase and escape. To define this particular game let us call A the partner who is trying to catch B in the shortest time possible, whereas B is trying to enjoy his freedom as long as possible.

Let us treat the simplified version of this game in which A and B move always with velocities a, b respectively, $a > b$. The positions of the partners are functions of time, $A(t)$ and $B(t)$, t being the time elapsed since the initial time zero. The initial distance, i.e., the segment $A(0)\ B(0)$, has length d. The problem for A is to choose a function $\alpha(A, B)$, α being the angle between East and the course of A. This function is called the strategy of partner A. An analogous definition gives β, the angle between East and the course of B; B has to choose a strategy $\beta(A, B)$. It is easily seen that the best strategy for A is given by $\alpha =$ the angle between the direction East and the vector $A(t)\ B(t)$; also for B, by $\beta =$ angle between East and vector $A(t)\ B(t)$. This statement is to read as follows: If A chooses his course in conformity with the rule above, i.e., if he keeps B always

ahead, he will reduce the time T of the chase to $d/(a-b)$ or less; if B always chooses his course so as to see A astern, he will augment T to $d/(a-b)$ or more. This is the minimax principle for the special game of "chase and escape" in its simplest form. It can be written as follows:

$$\min_{\alpha} \max_{\beta} T(\alpha(A, B), \beta(A, B)) \geqq \max_{\beta} \min_{\alpha} T(\alpha(A, B), \beta(A, B)). \tag{1}$$

The proof of (1) is easy: It is evident that the real time T resulting from arbitrary choice of strategies α, β lies between the left and right side of inequality (1). The replacing of $\geqq$ in (1) by the equality sign would be an improvement which I felt to be true but I was unable to establish the general problem of pursuit in 1925—"general" means here a theory of pursuit in a limited plane or on arbitrary surfaces such as an ellipsoid or a torus. This inability was a consequence of the ignorance of Zermelo's paper [1] in spite of its having been published in 1913. Two proofs of (1), with the equality sign, valid for the general theory of pursuit and evasion, are to appear in [**10**] and [**11**]. J. von Neumann was aware of the importance of the minimax principle (cf. [**4**]); it is, however, difficult to understand the absence of a quotation of Zermelo's lecture in his publications (cf. [**13**]). Let us explain Zermelo's result.

Partner A has n purses $A_1, A_2, \cdots, A_n$ containing counters of different kinds, each counter carrying a symbol; partner B has n purses $B_1, B_2, \cdots, B_n$ with counters marked with symbols; partner A knows the content of all $2n$ purses, B knows it too. Partner A chooses one counter from A_1 and places it on a table; B takes notice of the symbol a_1 visible on the table and chooses from B_1 a counter which he places besides a_1; A reads its symbol b_1 and chooses a counter from A_2; and so on. These alternating choices stop when B has placed a counter from B_n; the sequence $a_1b_2a_1b_2 \cdots a_nb_n$ resulting therefrom is a "word." Rules, known to A and B, define a dichotomy of the "vocabulary" S, i.e., of the set of all words: $S = \mathfrak{A} + \mathfrak{B}$, $\mathfrak{A}\mathfrak{B} = 0$. If the word resulting from the finished game belongs to $\mathfrak{A}$ then A is the winner, if it belongs to $\mathfrak{B}$, B is the winner. Zermelo's theorem applies to games of the kind specified above. It asserts the existence of a strategy for A, assuring him of victory whatever B does, or the existence of such a strategy for B, i.e., it proclaims the intrinsic nullity of such games when played by two partners endowed with complete knowledge relative to the game in question—grown up people playing "wolf-and-sheep" [**14**] are an example: they know that the sheep will win if led according to a certain strategy with which both of them are familiar; the behavior of the wolf cannot change his destiny. There are no such partners in the world of chess. Even the champions are very far from perfect knowledge, they play chess like children playing "wolf-and-sheep." Nevertheless Zermelo's theorem applies to chess in spite of our present ignorance of the result; we do not know which it is: White wins, Black wins, or they draw, but we know that the different results obtained by chessplayers of high rank are a proof that chess is a complicated puzzle.

Dr. Jan Mycielski has found a few years ago the following formulation of Zermelo's theorem, which is at the same time its proof:

$$
\begin{aligned}
&\sim \left\{ \bigvee_{a_1\in A_1} \bigwedge_{b_1\in B_1} \cdots \bigvee_{a_n\in A_n} \bigwedge_{b_n\in B_n} (a_1b_1a_2b_2 \cdots a_nb_n) \in \mathfrak{A} \right\} \\
(2) \qquad &\Rightarrow \left\{ \bigwedge_{a_1\in A_1} \bigvee_{b_1\in B_1} \cdots \bigwedge_{a_n\in A_n} \bigvee_{b_n\in B_n} (a_1b_1a_2b_2 \cdots a_nb_n) \notin \mathfrak{A} \right\}.
\end{aligned}
$$

Formula (2) is one of several connected with the name of Augustus de Morgan [5] and belongs to formal logic. Its wording does not contain letters $\mathfrak{B}$ and S—instead of writing that a word belongs to $\mathfrak{B}$ the author writes that it does not belong to $\mathfrak{A}$.

3. As yet there is a gap between the formulae (1) and (2). To fill it let us consider ordinary chess problems published in newspapers; they show a chessboard with white and black chessmen artificially distributed; "artificially" means here that they display a situation which could have been observed during a real game but in most cases has been invented by the proposer. Most of them challenge the reader: "White starts and wins in k moves" k being specified as 2, 3, 4 moves; some of them give to k greater values. Let us assume $k=4$. The problem is then to be read as follows: There is a strategy for White which guarantees to him a victory in four moves, and there is a strategy for Black which guarantees to him that his defeat will not happen during the first 3 moves W B W B W B. In such cases we define "4" to be the value of the game. Now, whatever the distribution of the chessmen, there is a number k (k is finite) with the following property:

"There is a strategy for White which guarantees him a victory at his kth move at latest, and there is such strategy for Black which makes impossible to White a victory earlier than at White's kth move." The sentence: k is the value of the game defined by the distribution of the chessmen means exactly the property between quotation marks. The analogue of k in the game of pursuit and evasion is the minimum time T of the chase, i.e., the time which the best strategy guarantees to A.

To have a perfect analogy between chess and pursuit, we must consider arbitrary initial positions in both games. Naval pursuit applies obviously to arbitrary initial positions, whereas classic chess requires an initial formation fixed once for all by international rules. This is the reason why chess problems, admitting arbitrary starting formations, are the discontinuous counterpart of pursuit. Let us emphasize that in both games there are so-called "solutions"—this name is given to the course of events resulting from the application of the minimax strategies by both partners; it is interesting that the usual solutions of chess problems, printed in newspapers, are minimax solutions; they show only the "point-counter-point" battle—usually no proof is given for the "principal variant" to be minimax.

There is still a difference: formula (2) cannot be improved, whereas (1) suggests an improvement by replacing the sign $\geqq$ by $=$. When writing my paper [3] I did not know Zermelo's lecture [1]; I have called games with $=$ in (1) "closed" and with $>$ "open," and irrational reasons led me to the conviction

that the "chase and escape" game is closed. The paper of C. Ryll-Nardzewski [10] quoted in Section 2 is based on Zermelo's theorem (2); its author considers the continuous game of pursuit to be a limit of discontinuous games which are closed according to (2). He makes use of the circumstance that the minimax procedure as defined in Section 3 becomes simpler when partner A, instead of considering his antagonist B as a human being thinking in such terms as "strategy," considers all possible functions $B=f(t)$ representing the itinerary of B, t being the time.

If $\alpha(A, B)$ are all strategies of A defined in Section 2, then the time of capture T becomes now a functional $T(\alpha(A, B), f(t))$ to which A applies the minimax operation:

$$\min_{\alpha(A,B)} \max_{f(t)} T(\alpha(A, B), f(t))$$

instead of the operation $\min_\alpha \max_\beta$ of (1). This leads to the best strategy for A. Of course, B can apply an analogous method. The struggle yields the same solution as (1) but is more satisfactory from the logical point of view: the notion of strategy is more sophisticated than that of a trajectory; the new formula shows that it is not necessary to exclude the case of a partner choosing his course at random.

To explain simply the difference between open and closed games, let us imagine a contest of a dog and a rabbit. There is a rectangular garden and the rabbit can choose any point in it. The dog can choose any point along the fence separating the garden from the street but he cannot leave the street. Dog D is interested in reducing his distance DR from rabbit R as much as possible while the rabbit would like to increase DR so as to be as far from the dog as circumstances permit. The game resulting from these contradictory tendencies has two different mathematical models.

MODEL 1. D and R have maps so that they can choose simultaneously the spots of their respective preferences; then an arbiter transfers the chosen points on his map and shows their respective places on it—D and R are obliged to occupy them on the real battle-field. This model defines a game without perfect information. It is obvious (cf. Fig. 1) that the dog can reduce the distance DR so that $DR \leqq 5$ by choosing the midpoint M of the sidewalk. It is also obvious that any choice $D \neq M$ implies the possibility that $DR > 5$. Thus 5 is the value of the game for the dog. Things are different for the rabbit: he risks always being faced by the dog staring at him from 4 yards distance. The inequality $5 > 4$ shows the game to be open. The choice $D = M$ is evidently the best solution for the dog. What is our advice for the rabbit? Our theory gives him no means to decide—whatever he does he risks $DR = 4$ but never $DR < 4$. There exists, however, a reason for R to choose one of the corners $Q_{1,2}$. The minimax approach is based on the pessimistic hypothesis that the opponent chooses a move which proves to be best for him. This hypothesis does not give any advice. The new principle is

as follows: we assume that the partner applies the minimax principle. Our help for the rabbit is based on this principle, which involves that D will choose M, and, consequently R has to occupy Q_1 or Q_2. Such a trick closes the game in a new sense, $5=QM$ becomes the value of the game for both partners.

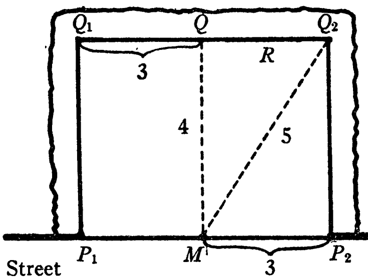


FIG. 1.

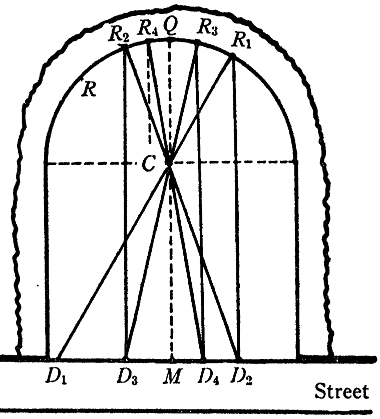


FIG. 2.

MODEL 2. What happens if the animals try to play an alternating game with perfect information, the garden of Fig. 1 being still their battle-field? Such a game would degenerate in perpetual hunting, D would run along the street and R along the wall Q_1Q_2. If the initial situation has been given as $D=P_1$, $R=Q_2$, the course would be as follows:

1. DP_1P_2; RQ_2Q_1.
2. DP_2P_1; RQ_1Q_2, the 3rd move $=1$, the 4th move $=2$, etc.

Things are quite different (Fig. 2) if the garden has a semicircular background. In this case Model 1 admits a minimax solution which places R in the midpoint Q of the circular wall and D in M. The distance $MP=7$ is the value of the game, the same for D and R. Model 2: Let D_1 be D's first position; obviously R_1 is the best reply of R, D_2 is D's answer. It is easy to prove that the game is infinite. Nevertheless the successive positions D_n, R_n yield convergent sequences:

$$\lim D_n = M, \qquad \lim R_n = Q.$$

Although $n \to \infty$, the limits M, Q are reached after a finite time if both D and R have a constant velocity.

One of J. von Neumann's important achievements in the theory of games is his theorem about closing open games [4]. Dr. Ryll-Nardzewski's version of this theorem assumes that there are teams of players led by captains who are responsible for the decisions of their teams. The simplest example is matching

pennies. It is obvious that this game, played by two adversaries A, B, is open: no strategy excludes the loss of a penny by A which is the worst result; the situation of B is the same. Now, if A has a team of two subordinates, A_1, A_2, playing simultaneously against B, he can order A_1 to bet on heads and A_2 to bet on tails—whatever B does the result $A:B$ will be $0:0$, thus 0 is the value of the game for A and for B. This example has the drawback that matching pennies is a game lacking perfect information. Nevertheless it explains von Neumann's idea of closing an open game by variable strategies, the distribution of strategies being a superstrategy; then the minimax theory applies to the global result and the supergame is closed.

4. Immediately after the publication of [**3**], S. Banach and St. Mazur resolved to investigate infinite games to see if the minimax rule (2) applies to such alternating games with perfect information; in 1925 they found such games, refuting by their discovery my conjecture that all alternating games with perfect information are closed. The simplest example of a two-person infinite game of A against B is as follows: The interval $[0, 1]$ is split into two sets of points, $\mathcal{A}$ and $\mathcal{B}$. The partners A, B know for any x given in $[0, 1]$ whether it belongs to $\mathcal{A}$ or to $\mathcal{B}$. They have to write alternately on a strip digits $a_1, b_1, a_2, b_2, \cdots$ *ad inf.*, and they see every digit immediately after its appearance on the strip. The symbol $0.\, a_1b_1a_2b_2 \cdots a_nb_n \cdots$ defines a point x in $[0,1]$ when read as a binary development. If $x \in \mathcal{A}$, A is the winner, if $x \in \mathcal{B}$, B is the winner.

How to play an infinite game? It is possible for mortals to play it really by writing their respective strategies on two sheets apart and the umpire who is given both sheets has to determine x which is perfectly defined by this information; he can deduce whether x belongs to $\mathcal{A}$ or to $\mathcal{B}$; consequently, he can decide whether A or B has won the game. In some cases the task of the umpire will be extremely difficult, but theoretically it will, in every particular case, be a well-defined mathematical problem. As to the rules for A and B, they are as follows: Partner A decides first whether $a_1 = 0$ or 1. This he writes on his sheet. Then he must write on it the definition of a function $c(k)$ whose values are 0 or 1 for every natural number k. The umpire reads this definition as an obligation of A to write always the digit $c(k)$ as a_{n+1} if the sequence $a_1 \cdots b_n$ read as a binary expansion, yields k. Partner B must define $d(k)$. His procedure is analogous to that of A, $d(k)$ being used now, instead of $c(k)$, to be written on B's sheet; there is no analogy to the choice of a_1.

The importance of the trivial statements above is their being a proof for the possibility of playing really infinite games in a finite time.

EXAMPLE. $\mathcal{A}$ = set of rational points on $[0, 1]$, $\mathcal{B}$ = set of irrational points on $[0, 1]$; partner B can secure his victory by a strategy the reader will easily guess.

On the other hand it has been shown (cf. [**9**]) that when $\mathcal{A}$ and $\mathcal{B}$ are two totally imperfect sets, neither A nor B can have a victorious strategy.

I have been tempted to extend formula (2) to $n = \infty$ to show that infinite alternating two-person games with perfect information are closed, but J.

Mycielski called my attention to the contradictions such extensions involve. The crucial point in the controversy is the appearance of Zermelo's Axiom of Choice (AZ) in all examples of the Banach-Mazur kind; it is employed to define the sets $\mathfrak{A}$, $\mathfrak{B}$. It is known that AZ produces such consequences as the decomposition of a ball into five parts which can be put together to build up a new ball of twice the volume of the old one, a result considered as paradoxical by many scientists. There is another objection: how are we to speak of perfect information for A and B if it is impossible to verify whether both of them think of the same set when they speak of "$\mathfrak{A}$"? This impossibility is inherent in every set having only AZ as its certificate of birth. In such circumstances it is doubtful whether human beings will ever play really a BM (Banach-Mazur) game.

All these considerations impelled me to place the blame on the Axiom of Choice. Sixty years of the theory of sets have elapsed since this Axiom was proclaimed, and some ideas like those in Section 1 of this talk have convinced me that a purely negative attitude against AZ would be dangerous to propose. Thus I have chosen the idea of replacing AZ by the following "Axiom of Determinacy" (AD):

All two-person alternating games with perfect information are closed.

I have encountered difficulties; the extension of (2) to infinite games yields AZ; on the other hand, AZ generates BM-games, i.e., open infinite games which I will not admit. My collaboration with Jan Mycielski started at this point. It had such episodes as two telegrams (Berkeley-Zakopane) in summer 1961: "The axiom is dead," and a day later: "Axiom still living." A note of both authors [7] presented October 14, 1961 by A. Mostowski contains a formal expression of AD. This formulation generalizes to infinite games Zermelo's theorem on chess: infinite games are closed too. The formula (A) on p. 3 of [7] can be read as a generalization of (2), i.e., as a new formula of A. de Morgan's type. It would be preposterous, however, to do it *verbatim*: Classic formulae apply to general sets, finite or infinite, under the condition that the number of quantifiers be finite, whereas in our case (A) has ω quantifiers. If we admit general sets beneath the quantifiers, we get false statements. For our purpose it would be sufficient to consider the simplest kind of infinite games which we have described in Section 4 and to formulate AD as follows: Either there exists a digit a_1 and a function $c(k)$ such that A wins whatever the function $d(k)$, or there exists a function $d(k)$ such that B wins whatever a_1 and $c(k)$.

Axiom AD and its consequences have been studied by Jan Mycielski [9]. He mentions there the equivalence of Cauchy's definition of continuity with that given by Heine: it results from the usual system of axioms (i.e., the Zermelo-Fraenkel-Skolem system) with AD but without AZ. Let us call this system $ZFSAD$. The nonexistence of a cardinal number $\mathfrak{m}$ such that $\aleph_0 < \mathfrak{m} < 2^{\aleph_0}$ is a consequence of $ZFSAD$. He and S. Świerczkowski [8] have shown that $ZFSAD$ involves the Lebesgue measurability of every subset of the real line. Thus AD, our axiom, destroys many results proved commonly with the help of AZ. One

of them is the Banach-Tarski decomposition of the ball: the parts of the ball resulting from such decomposition have no Lebesgue-volume, which contradicts $ZFSAD$. Instead, AD brings positive results such as the property of Baire and those already quoted. Mycielski [**9**] writes that the conjecture of the consistency of $ZFSAD$ is allowed in the actual state of knowledge; I am quoting here this opinion because our collaboration was primarily based on contradictory attitudes, his being the pessimistic one. Such caution had a serious reason: K. Gödel's result about AZ being compatible with ZFS. This result involves the possibility of AZ turning out in the future to be a true theorem in ZFS and (consequently) AD being false in ZFS. This danger has lost a great part of its imminence: P. J. Cohen's recent result [**12**] proves AZ to be independent of ZFS. As to the question whether AD is an improvement of mathematics, my answer is positive: in the modern theory of probability it is often necessary to prove the measurability of sets and functions; it is a gain if such proofs can be canceled once for all as superfluous.

5. Most persons are interested rather in games of chance than in games with perfect information. Heads and tails is the simplest game of chance, called also "matching pennies." Mr. A puts a dime on the table and covers it with his palm, Mr. B puts his dime on the table, Mr. A lifts his hand; if the dimes are showing the same image, i.e., head and head or tail and tail, B takes both; if they are discordant A puts them in his pocket. The value of the game is -10 cents for A and $+10$ cents for B, consequently the gap is 20 cents: the game is open. Things are different if the game consists of an infinite sequence of such trials as above, g_n being the total gain of A in the n first trials which, however, is not paid to A; the score is recorded in a book and $\lim_{n\to\infty} g_n/n$ is the amount to be paid by B to A; there are strategies for which this limit exists. The best for both partners is to ignore which side of the dime is visible when they put it on the table; the strong law of large numbers guarantees 0 as the limit. Thus the game is closed, 0 being its value for A and for B. This example is the simplest particular case of the general law discovered by J. von Neumann [**4**]: open games become closed by repetition of successive plays. Dr. Ryll Nardzewski remarks that instead of repeating the game endlessly, we could arrange a simultaneous contest of n pairs of players, n being a large integer. There are n players A_k ($k=1, 2, \cdots, n$) led by A, and n players B_k led by B; consequently there are n fields F_k; every pair A_k, B_k has its field F_k and the rules are the same on every field. As to strategy the captain A dictates it to his men A_k but these strategies are not the same for different fields. B, the captain of the team $\{B_k\}$, has analogous rights. Let us assume that the stake s is $1/n$ for every field F_k and that the game is open in every field, the gap being g_n, and that A, B are the respective treasurers of their clubs. Nardzewski's result is that there is a strategy best for A (in the minimax sense) and best for B (in the same sense), such that the gap between the respective values for A and B becomes very small for great n, which makes the game closed as $n\to\infty$ although $g_n>0$ for every finite n.

Games of chance can also be treated from a different point of view. To close them we can consider the expected gain instead of the accidental one. There is a mention in [3] about this point of view. At that time a two-person game called *baccara* was popular in gambling circles. The banker B takes a card from the pack, his opponent too. They have done it blindly and now they look at the cards chosen without showing them to the adversary. Then the player can take a second card if he likes, or resign; this second card is shown to B. Now B takes a second card or not: eventually both of them, B and partner P, show the cards to compare the points. No card has more points than 9; if s is the sum of points on a pair of cards and $s>9$, then the pair in question counts for $s-9$. The stake is fixed before the game starts; the winner is the player exhibiting more points. It is easy to find an approximate best strategy P_1 for the player. Let B_1 be the best strategy for B against a player applying P_1; this relatively best strategy can be computed by the rules of the calculus of probability: we have to maximize the expected gain for B. Assuming the banker's strategy to be B_1 we can find P_2, the relatively best strategy for P. Proceeding to B_2 we find $B_2=B_1$. Then P_3 results as equal to P_2. Consequently P_2, B_2 turn out to be absolutely best, by the general principle that two reciprocally best strategies are minimax strategies. Thus P_2, B_2, represent the minimax solution of the two-person game of cards called baccara. This special computation (which has never been published) is an illustration of a general law discovered by J. von Neumann [4] about the randomization of open games: because of imperfect information, every single play of baccara is open, the repetition of plays makes the game closed. The P_2, B_2 solution shows that the game is not fair; the gain expected by the banker is positive; the repetition of plays gives in the limit a mean game equal to this expectation.

6. To finish this talk about games let us admit that we don't know if AD is consistent with the ZFS theory of sets. Nevertheless there are the following reasons to be put forward for the new axiom:

It simplifies mathematics by cancelling some consequences of AZ which have always shocked many serious scientists; on the other hand AD does not destroy those theorems which the majority of mathematicians consider as essential tools in modern science and which used to be proved with the help of AZ (Zermelo's Axiom); the new axiom AD takes over the job of AZ by rescuing such theorems. To be scrupulous let us admit that we are not sure whether this remark applies to the full list of such theorems.

AZ has the advantage of being immediately comprehensible and intuitively clear. AD is exposed to the objection that the closedness of infinite games is by no means evident. On the other hand AZ involves consequences which are far from being intuitively clear.

Dr. A. Mostowski's objection against AD is as follows: AD is a proposition no logician would find unless he were accustomed to the gambler's fashion of thinking. Let me remark that the calculus of probability has been founded too

on this fashion of thinking: it was considered not to be mathematics and this view prevailed until World War I. Now it is considered to be a chapter of mathematics. This evolution is the reason why AD belongs to mathematics: AD belongs to games, games belong to mathematics, thus AD is mathematics. It is not impossible that the future will bring propositions exactly equivalent to AD and formally as clear as most principles of the ZFS theory of sets.

Dr. Jan Mycielski contributed to this essay about games by his criticisms; I acknowledge his help with gratitude.

References

1. E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels, Proc. of the Fifth International Congress of Mathematicians, Cambridge, 2 (1912) 501–504.

2. E. Borel, La théorie du jeu et les équations intégrales à noyau symétrique gauche, C. R. Acad. Sci. Paris, 173 (1921) 1304–1308. Sur les jeux où interviennent le hasard et l'habileté des joueurs, Théorie des Probabilités, Paris, 1924, pp. 202–224.

3. H. Steinhaus, Definicje potrzebne do teorji gry i pościgu. Myśl Akademicka, Lwów, 1925, nr 1. (This note has been reprinted in English, Naval Res. Logist. Quart., 7 (1960) 105–107.)

4. J. v. Neumann, Zur Theorie der Gesellschaftsspiele, Math. Ann., 100 (1928) 295–320.

5. A. de Morgan, Formal Logic or the Calculus of Inference necessary and probable, 1847.

6. A. Ehrenfeucht, An application of games to the completeness problem for formalized theories, Fund. Math., 49 (1961) 129–141.

7. Jan Mycielski and H. Steinhaus, A mathematical axiom contradicting the axiom of choice, Bull. Acad., Polan. Sci. Sér. Sci. Math., Astr., Phys., 10 (1962) 1–3.

8. Jan Mycielski and S. Świerczkowski, On the Lebesgue measurability and the axiom of determinateness, Fund. Math., 54 (1964) 67–71.

9. Jan Mycielski, On the axiom of determinateness, Fund. Math., 53 (1964) 205–224.

10. C. Ryll-Nardzewski, Theory of pursuit and evasion. Contributions to the Theory of Games, vol. 5, Annals of Math. Stud., to appear.

11. Jan Mycielski, Continuous games with perfect information, *ibid.*, to appear.

12. P. J. Cohen, The independence of the continuum hypothesis, I, Proc. Nat. Acad. Sci., 50 (1963) 1143–1148; II, *ibidem*, 51 (1964) 105–110.

13. H. W. Kuhn and A. W. Tucker, John v. Neumann's work in the theory of games, Bull. Amer. Math. Soc., 64 (1958) 100–122.

14. Maurice Kraitchik, Mathematical Recreations, George Allen and Unwin, London, 1955, pp. 309–310.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION: EARLY HISTORY

G. BIRKHOFF, Harvard University

The basic idea underlying the Putnam Competition was expressed by William Lowell Putnam in a three-page article in the Harvard Graduates' Magazine of December, 1921 (reprinted here as an Appendix) from which the following excerpt is taken: " . . . it is a curious fact that no effort has ever been made to organize contesting teams in regular college studies. All rewards for scholarship are strictly individual and are given in money, or in prizes or in honorable mention. No opportunity is offered a student by diligence and high marks in examinations to win or help in winning honor for his college. All that is offered to him is the chance of personal reward. Little appeal is made to high ideals or to unselfish motives.

Is not this one of the reasons why the effort to interest the great bulk of the undergraduate body in their studies is such an uphill task?"

These words expressed Mr. Putnam's profound conviction in the value of intellectual competition, especially by *teams* whose members took pride in the achievements of their team as a whole and the standing of the institution which it represented. This conviction was shared by Mrs. Putnam, and by her brother Abbott Lawrence Lowell, then President of Harvard. In a letter about the competition some years later, Mr. Lowell referred to the importance of "the promotion of scholarship by contest, as athletics have been," and of "the spirit of intercollegiate emulation in scholarship."

To give substance to the ideas expressed above, Mrs. Putnam established a $125,000 trust in her will, written in 1927. She directed that the trustees "shall have always in mind the purpose of this trust, as set out in my husband's own words," and incorporated a copy of his article in her will.

It was some years after Mrs. Putnam wrote her will, however, before the Putnam Competition assumed its present form. In the meantime, various short-lived experiments were tried.

The first experiment was a competition in 1928 in the field of English between Harvard and Yale. The Yale and Harvard teams each contained ten senior concentrators in English. One member of the Harvard team was Nathan Marsh Pusey, now president of that institution. It was won by Harvard, a prize of $5,000 going to the winning institution. Both Yale and Princeton declined to repeat the contest, and an offer to compete in economics with Cambridge University was also declined, partly because of a fear of undesirable publicity.

The second experiment was a similar mathematical competition in 1933 between Harvard and West Point juniors. The examination stressed ability to solve problems in the calculus, analytic geometry, and elementary differential equations. This close competition was won by the somewhat more mature and (ac-

cording to local tradition) better trained West Point cadets. Again, the competition was not repeated, though all ten West Point participants expressed enthuasiasm in personal letters to Mrs. Putnam.

Mr. Lowell having retired as President of Harvard in 1933, no further contest was held until after the death of Mrs. Putnam in 1935. In her will, she appointed her sons, George Putnam and August Lowell Putnam, as trustees for the William Lowell Putnam Memorial Fund. These two brothers then consulted George David Birkhoff at Harvard as to the best use to be made of the bequest, a question which the latter had often discussed with Mrs. Putnam. It seemed natural to have the competition in mathematics, both because mathematical ability can better be tested by a set examination than ability in most subjects, and because of the strong mathematical traditions of the Putnam family.

The possibility of discovering outstanding mathematical ability at an early age by set examinations had, indeed, been demonstrated conclusively in Hungary, where a competitive examination (named after the physicist Roland Eötvös) had been given annually (since 1894) to young people entering the University. It certainly contributed to the development of Hungarian mathematics; among the winners were such internationally known names as Leopold Fejér, Theodore von Kármán, Dénes König, Alfred Haar, Marcel Riesz, Gabor Szegö, Tibor Radó, and Edward Teller.

Somewhat similarly, the Cambridge University mathematical examinations were used in the nineteenth century, as a means of detecting both mathematical and legal ability, the Senior Wrangler being considered a man "most likely to succeed" in either profession.

The mathematical interests of the Putnam family were unusual, to say the least. Mrs. Putnam's brother, former President Abbott Lawrence Lowell of Harvard, received in 1877 his A.B. *summa cum laude* in Mathematics at Harvard, and his undergraduate thesis was published in volume 13 (1877), of the Proceedings of the American Academy of Arts and Sciences, pages 222–250. Mr. William Lowell Putnam received in 1882 an A.B. *magna cum laude* in Mathematics at Harvard, and was for many years Chairman of the Visiting Committee of the Mathematics Department there. George Putnam, a trustee of the Putnam Fund, had also majored in mathematics and was active in his turn on the same Visiting Committee, while the present form of the Putnam Competition was being planned. Roger Lowell Putnam also majored in mathematics at Harvard, graduating *magna cum laude* in 1915, while William Lowell Putnam's grandson, McGeorge Bundy (later Dean of the Faculty at Harvard), and his brother Harvey majored in mathematics at Yale.

George David Birkhoff saw in these various facts an important opportunity to stimulate interest in mathematics, to establish national standards of undergraduate mathematical achievement, and to discover and encourage mathematical talent at a time when good fellowships for first-year graduate students were few and far between. With these objectives in mind he proposed, in essence,

the principles governing the present Putnam Competition.

The first principle was that the competition should be open to three-man teams selected by the faculty of, *and* to individuals from, *any* American or Canadian college. This was to encourage participation by smaller colleges, who might have only one or two unusually able mathematical concentrators in any given year. The second principle was to have the competition administered by the Mathematical Association of America, which is the professional organization representing college mathematics teachers. The third principle was to distribute prizes and honorable mention to several teams and individuals, so that distinguished performance would be broadly recognized. (To be one of the top 25 or even 50 competitors in any one year, out of 1500 competitors selected from many thousands of mathematics concentrators, is no mean achievement.) The fourth principle was to offer a graduate fellowship at Harvard to *one* of the *five* top competitors. This choice was allowed because Birkhoff recognized that other evidence of talent—especially evidence of creative originality—should be weighed in evaluating men having nearly equal examination scores.

Birkhoff took a very active part in making up the first examination, which was made up by the Harvard Mathematics Department in 1938. So as to avoid favoring the few schools where advanced mathematical courses were then open to able undergraduates, a deliberate effort was made to stress questions covered in standard courses in the calculus, analytic geometry, and elementary mechanics. The emphasis was correspondingly on thoroughness, accuracy, and a clear command of detail.

The first competition was won by the University of Toronto, which was asked to set the next examination and disqualify itself from competing the following year. A similar procedure was followed for about five years, the winning teams being alternately from Toronto and Brooklyn College. These victories were attributed partly to the greater maturity and more intensive concentration program of Canadian mathematical concentrators, partly to the British tradition of problem-solving, and partly to the ability and devotion of Professor Harris MacNeish of Brooklyn in training students how to diagnose and solve set problems.

During the years 1943–5 the Putnam competition was not held because of war conditions. After the war, it was renewed with a few minor modifications. Thoroughness and facility in handling standard course material was given less emphasis, ingenuity in devising and using algorithms and in logical analysis being emphasized in their place. To construct an examination testing these qualities, a special committee was appointed consisting of the distinguished Hungarian-born mathematicians George Pólya and Tibor Radó, and the first Putnam Scholar, Irving Kaplansky. Since that time, membership in the committee making out the examination has rotated, and institutions have been free to win the competition in successive years whenever the ability of their teams made this possible!

APPENDIX

A SUGGESTION FOR INCREASING THE UNDERGRADUATE INTEREST IN STUDIES

WILLIAM LOWELL PUTNAM

[The following article, which appeared in the December, 1921, issue of the *Harvard Graduates' Magazine*, and is reprinted with permission, was the genesis of the William Lowell Putnam Mathematical Competition.]

The idealism of the undergraduate student, his eagerness to achieve something for his college, for his country or for any cause which fills him with enthusiasm is constantly referred to with admiration by those in charge of universities. This unselfish impulse is recognized as one of the strongest forces in a student's life, and great results have been and are being accomplished by appealing to it.

The most telling of these appeals at present are those made in behalf of athletics, where boys are asked and expected to work hard to win laurels for the college on the gridiron or the river or the track. They have to train regularly and monotonously for long periods, to give up smoking, drinking, and late entertainments, to keep off probation, which often requires laborious and irksome study, and they make all these sacrifices willingly and even eagerly for the sake of their college.

In none of these cases is the undergraduate primarily interested in winning honor for himself. He is anxious to have his Alma Mater win, and very glad to play a useful even if an inconspicuous part in the preparation of the team by which her victory is secured. A Harvard man prefers, for example, to be a member of a tennis team to play against Yale and to win a match in that contest rather than to be champion of Harvard, which is only an individual honor. In short, the undergraduate likes to work for the success of his college and particularly likes to work for it as one of a team.

These intercollegiate contests are not confined to athletics. There are competing debating teams which attract some attention from the newspapers and there are chess teams which bring victories that are noticed by those who are interested in this game. But it is a curious fact that no effort has ever been made to organize contesting teams in regular college studies. All rewards for scholarship are strictly individual and are given in money, or in prizes or in honorable mention. No opportunity is offered a student by diligence and high marks in examinations to win or help in winning honor for his college. All that is offered to him is the chance of personal reward. Little appeal is made to high ideals or to unselfish motives.

Is not this one of the reasons why the effort to interest the great bulk of the undergraduate body in their studies is such an uphill task? Some few of them may be wise enough to realize the truth of what they are told often; namely, that if they work hard at their books they will lay the foundation for a future career which will reflect credit on their college; but this is a rather conceited view and the prospect is too remote to appeal to most young men. Those who distin-

guish themselves as students usually do so either from a desire to improve their own chances of obtaining assistance in college or good employment after graduation, or else from a wish to keep up some family tradition of scholarship or to please some inspiring instructor or from some other similar incentive.

Probably one reason why undergraduate teams do not contest in scholarship is the difficulty of fixing the terms and conditions of such a contest and arranging the details, but in these days, when all contests are fought in the presence of umpires and under highly artificial and technical rules, is there any more difficulty in fixing the terms of a contest of scholars than in arranging for the International Olympic Games or other similar events?

I want to lay particular stress on the point that the competition to be valuable should be between teams and not individuals. Undoubtedly, one reason for the surpassing interest in football games is that the victory is the result of an immense amount of cooperative work, much of it done by people who are never seen by the public. Some twenty-five or more men actually take part in the crowning event of the season, but the efficiency of the team is the result of the coordinate efforts of three times as many players and coaches, each of whom contributes an essential part to the final result. It is a very inspiring thing in itself to form part of such a large body working harmoniously to achieve a great end.

The absence of the opportunity for team-work in undergraduate studies is the more noteworthy from the fact that both before entering college and after graduation most boys do their work as members of a competing team or organization. The preparatory schools take a definite pride in entering their boys without conditions and in a creditable manner and a prize is given to the school whose candidates do best. Every boy as he prepares for his examination feels that he is a part of the school team competing for this prize.

After graduation men usually get into the employ of a firm or corporation or similar body and become intensely loyal to its interests and eager to work hard for its success. Even those who continue their studies at the Law School get an opportunity to become editors of the *Law Review*, or to become members of law clubs which have interclub contests, so that students are able to do work directly connected with their studies as part of a team which wins credit for the club or the *Review*.

But the regular work of a man's college years—those four most impressionable years—must be done individually and for his own selfish advantage, and one can but ask whether the effort to induce the bulk of the students and their parents to take more interest in their studies would not accomplish more if an appeal were made to higher motives and an opportunity offered for teamwork and for winning laurels for the college.

It seems probable that the competition which has inspired young men to undertake and undergo so much for the sake of athletic victories might accomplish some result in academic fields.

THE WILLIAM LOWELL PUTNAM COMPETITION: LATER HISTORY AND SUMMARY OF RESULTS

L. E. BUSH, Kent State University

It was most fortunate for undergraduate mathematics in America that the Birkhoffs, father (George David) and son (Garrett), were close friends of the Putnam family. As such, they had the opportunity of formulating the principles of the Putnam Competition in accordance with Mr. Putnam's ideas as expressed in his article. In formulating these principles and in advising the Putnam Trustees on all changes in the regulations governing the Competition, they have always held in mind Mr. Putnam's belief in the value of intellectual competition and his belief that this competition should not be by individuals but by pre-selected teams whose members take pride in the achievement of their team as a whole.

In accordance with Professor Birkhoff's second principle, the competition was placed under the administration of the Mathematical Association of America in the person of its secretary, Professor W. D. Cairns of Oberlin College. Professor Cairns published the general regulations governing the contest and award in this MONTHLY, 45 (1938) 64–66. These regulations have served as a model for all later revisions. The regulations provided that: "The president of the Association will appoint a committee of three (not necessarily members of the Association) which will select two examiners, mathematicians qualified and willing to undertake the preparation of the examination, and a reader qualified to grade the examination books; in view of the special relation of the donor of the Fund to Harvard University, one member of the committee shall be a member of the Division of Mathematics at Harvard University." In fact, the first examination was prepared by the Harvard Department of Mathematics, and Harvard did not enter a team. The examination was held on April 16, 1938. This competition was won by the University of Toronto, which alternated with Brooklyn College in winning the first five competitions.

The second, third, and fourth examinations were each prepared by the department of mathematics of the institution winning the previous competition, and it appears that these institutions disqualified themselves from the competition for which their department had prepared the examination. The examination for the fifth competition (1942) seems, however, to have been prepared by a committee consisting of Professor B. H. Brown of Dartmouth College and Professor Marie Litzinger of Mount Holyoke College.

The original three team prizes were \$500, \$300, and \$200, but these were changed for the fifth competition to four prizes of \$400, \$300, \$200, and \$100, respectively. In 1958 the number of team prizes was increased to five, with values of \$500, \$400, \$300, \$200, and \$100, respectively; in addition, the members of each team received prizes of \$50, \$40, \$30, \$20, and \$10, respectively, while the five highest ranking individuals received \$75 and the next five highest \$35 each.

The William Lowell Putnam Fellowship had a value during the first five years of \$1000. The value has been gradually increased until it is now worth \$2500 plus tuition at Harvard (\$1520 for 1963–64), or a total monetary value of \$4020.

In the October, 1942 issue of the MONTHLY it was announced that the Putnam Competition was being postponed indefinitely by agreement of the Putnam Trustees and the Board of Governors, "mainly because of the preoccupation of both teachers and students with war courses in mathematics." When the Competition was renewed after the war the Association was not able to resume its administration immediately, although the President of the Association appointed a committee of three persons to prepare the examination. Members of the Department of Mathematics of Harvard University were asked to assume the responsibility of administration. The first post-war competition, the sixth, was administered by Professor Garrett Birkhoff, and the examination was held June 1, 1946. The seventh and eighth competitions were held May 24, 1947, and March 20, 1948, under the direction of Professor George Mackey.

Beginning with the first post-war competition, the examination has been prepared by a committee appointed by the President of the Association. The members of this committee are appointed for three-year terms, one member retiring each year. The third-year-member of the committee serves as Chairman. Since the resumption of the Competition after the war, every otherwise eligible institution has been free to compete regardless of whether one of their staff members is on the examination committee or is a paper grader. It has been assumed that persons officially connected with the Competition would not take part in organizing or coaching a team. Some of the most distinguished mathematicians of the United States and Canada have served on the committee which makes up the examination, including Professors Tibor Radó, George Pólya, Mark Kac, Irving Kaplansky, Orrin Frink, Jr., Bancroft H. Brown, Emory P. Starke, Ralph G. Sanger, Andrew M. Gleason, L. M. Kelly, R. E. Greenwood, Leo Moser, W. R. Scott, R. E. Bellman, Ivan Niven, D. E. Richmond, John M. H. Olmsted, Gian-Carlo Rota, H. S. M. Coxeter, and A. M. Garsia.

At the September 6, 1948, meeting of the Board of Governors it was agreed that the Association should resume the direction of the Putnam Competition and that the President should be authorized to appoint a director of the competition for a term of five years. Professor Lester R. Ford, Sr., President of the Association, appointed this writer, then of the College of Saint Thomas, as director. I was reappointed in 1953, 1958 and 1963. In 1964 I asked to be relieved after the completion of the twenty-fifth competition.

My first task was to propose a revision of the regulations of the competition for the approval of the Board of Governors and the Putnam Trustees. This revision was necessitated by the new administrative arrangements. It was patterned after Professor Cairns' original general regulations and diverged from that pattern only where the new structure made it necessary. These regulations

were approved and, except for several amendments to take care of special situations, they are the present governing instrument of the competition.

That there has been a steady increase in interest in the competition is evidenced by the steadily increasing number of inquiries, both written and oral, about such matters as the level of grades on the examination and the mechanics of operation. The increase in interest also is shown by the growth of participation, as indicated in Table I. The figures given for the first eight competitions may not be exactly comparable with those given for the succeeding competitions.

TABLE I

PARTICIPATION IN THE PUTNAM COMPETITION

Competition Number	*Date of Examination*	*Teams Participating*	*Individuals Participating*	*Institutions Registered*
1	4/16/38	42	163	67
2	3/4/39	41	200	69
3	3/2/40	45	208	68
4	3/1/41	28	146	44
5	3/17/42	—	114	31
6	6/1/46	14	—	17
7	5/24/47	32	145	36
8	3/20/48	24	120	29
9	3/26/49	33	155	51
10	3/25/50	41	223	56
11	3/31/51	42	209	56
12	3/22/52	52	295	76
13	3/23/53	49	256	63
14	3/6/54	44	231	67
15	3/5/55	48	256	70
16	3/3/56	54	291	78
17	3/2/57	65	378	103
18	2/8/58	71	430	96
19	11/22/58	86	506	119
20	11/21/59	95	633	141
21	12/3/60	128	867	166
22	12/2/61	165	1094	197
23	12/1/62	157	1187	192
24	12/7/63	170	1260	205

The first twenty-four competitions show twenty-four institutions whose teams have shared in the prizes and sixteen additional ones which have won at least one honorable mention. Table II shows the distribution of the team honors. One year there was a tie for second place, and both teams were counted as second place winners; subsequently, ties for lower positions were treated similarly. Fourth place winners were not named until the fifth competition; fifth place winners were not named until the nineteenth competition.

TABLE II

WINNING TEAMS IN THE FIRST TWENTY-FOUR COMPETITIONS

Name of Institution	*First Place*	*Second Place*	*Third Place*	*Fourth Place*	*Fifth Place*	*Honorable Mention*
Harvard University	7	5	3	2		2
University of Toronto	4	4	2	2		4
Brooklyn College	3	2	1			3
California Institute of Technology	2	1	2	2		6
Cornell University	2	1	1	1	1	5
Michigan State University	2			1		1
Polytechnic Institute of Brooklyn	2				1	3
University of California at Berkeley	1	1		1		7
Queen's University (Kingston)	1		1	1		1
Massachusetts Institute of Technology		3	4	2	1	5
Yale University		3	1			5
Columbia University		2	3			6
University of Pennsylvania		1	1	1		1
College of the City of New York		1		4		5
Dartmouth College		1			1	1
Cooper Union			2			3
Carnegie Institute of Technology			1	1		8
New York University			1			3
Mississippi Women's College			1			
Kenyon College				1		2
McGill University				1		1
University of Manitoba				1		1
University of California at Los Angeles					1	3
Case Institute of Technology					1	1
Swarthmore College, Yeshiva University						3 each
Oberlin College, University of British Columbia						2 each
Brown U., Northwestern U., Princeton U., Reed College, Rice U., U. of Alberta, U. of Colorado, U. of Michigan, U. of Montreal, U. of Notre Dame, U. of Rochester, and Wesleyan U.						1 each
40 Institutions	24	25	24	21	6	99

Like the Eötvös Competition in Hungary, the Putnam Competition lists among its early winners a large number of the most creative mathematicians of the present day. The number of these is even more impressive when it is realized that the winners of the very first competition probably are still well under fifty years old and that the winners of the first post-war competition, the sixth, probably are under forty.

There have been four contestants who have been among the five highest individual contestants three times. They are Edward L. Kaplan, Carnegie Institute of Technology; Andrew M. Gleason, Yale University; D. J. Newman, College of the City of New York; and J. B. Herreshoff, IV, University of California at Berkeley.

There have been sixteen contestants who have been among the five highest individual contestants twice. They are Bernard Sherman, Brooklyn College; Maxwell Rosenlicht, Columbia University; W. F. Stinespring, Harvard University; Eoin L. Whitney, University of Alberta; J. W. Milnor, Princeton University; Kenneth G. Wilson, Harvard University; Trevor Barker, Kenyon College; Everett C. Dade, Harvard University; David Mumford, Harvard University; David Bloom, Columbia University; Joseph Lipman, University of Toronto; Alfred W. Hales, California Institute of Technology; Samuel Klein, College of the City of New York; Edward Bender, California Institute of Technology; John Hathaway Lindsey, California Institute of Technology; and William C. Waterhouse, Harvard University.

The five highest ranking individuals in each of the first twenty-four competitions are listed in Table III by years in alphabetical order. Where there were ties for fifth place, all those tying are listed. In parentheses following the name of the contestant is the institution which he represented in the competition. Next is given his present position and his present address (whenever we were able to obtain this information). Those awarded the William Lowell Putnam Fellowship are marked by an asterisk (*); although awarded the Fellowship, Mr. Gleason (1940) and Mr. Shepp (1958) never actually held it.

TABLE III

WINNERS OF THE PUTNAM COMPETITIONS

Abbreviations

S = State	C = College	I = Institute	T = Technology
U = University	a = Assistant	A = Associate	P = Professor

Academic subject is mathematics unless noted otherwise.

First Competition—1938

Robert W. Gibson (Fort Hayes Kansas S. C.)—aP, Oklahoma S. C.
*Irving Kaplansky (U. of Toronto)—P, U. of Chicago.
George W. Mackey (Rice I.)—P, Harvard U.
Michael J. Norris (C. of Saint Thomas)—Head, Analysis Dept., Bellcomm Inc., 1100 17th St., N. W., Washington, D. C.
Bernard Sherman (Brooklyn C.)—Member, Technical Staff, Applied Math. Dept., Space Technical Laboratories, Redondo Beach, Cal.

Second Competition—1939

Richard P. Feyneman (Massachusetts I. of T.)—Physics, California I. of T.
Abraham Hillman (Brooklyn C.)—AP, U. of Santa Clara.
*Edward L. Kaplan (Carnegie I. of T.)—AP, Oregon S. U.
William Nierenberg (C. of the City of New York)—P, Physics, U. of California at Berkeley.
Bernard Sherman (See First Competition).

Third Competition—1940

W. J. R. Crosby (U. of Toronto)—aP, U. of Toronto.
*Andrew M. Gleason (Yale U.)—P, Harvard U.
Edward L. Kaplan (See Second Competition).
John Cotton Maynard (U. of Toronto)—Assistant Actuary, Cannon Life Assurance Co., 330 University Ave., Toronto, Ontario.
R. M. Snow (George Washington U.)—(Last known address (1960): 2520 Harrison Blvd., Harrison, Utah.)

Fourth Competition—1941

*Richard F. Arens (U. of California at Los Angeles)—AP, U. of California at Los Angeles.
Samuel I. Askovitz (U. of Pennsylvania)—Medicine (Ophthalmology), Albert Einstein Medical Center, 4900 N. Ninth St., Philadelphia, Pa.
Andrew M. Gleason (See Third Competition).
Edward L. Kaplan (See Second Competition).
Paul C. Rosenbloom (U. of Pennsylvania)—P, U. of Minnesota.

Fifth Competition—1942

*Harvey Cohn (C. of City of New York)—P, U. of Arizona.
Andrew M. Gleason (See Third Competition).
Warren S. Loud (Massachusetts I. of T.)—P, U. of Minnesota.
Harold Victor Lyons (U. of Toronto)—Actuary, State Mutual Life Assurance, 340 Main St., Worcester, Mass.
Melvin A. Preston (U. of Toronto)—P, Physics, McMaster U.

Sixth Competition—1946

Felix Browder (Massachusetts I. of T.)—P, U. of Chicago.
Eugenio Calabi (Massachusetts I. of T.)—P, U. of Minnesota; U. of Pisa, Pisa, Italy.
Donald A. Fraser (U. of Toronto)—P, Statistics, U. of Toronto.
J. Arthur Greenwood (Harvard U.)—Statistics, Editor, International Encyclopedia of Social Sciences, 351 Park Ave. S., New York, N. Y.
*Maxwell Rosenlicht (Columbia U.)—P, U. of California at Berkeley.

Seventh Competition—1947

Clarence Wilson Hewlett, Jr. (Harvard U.)—Electrical Engineer, General Electric, Woodland Road, Hampton, N. H.
Maxwell Rosenlicht (See Sixth Competition).
W. Forrest Stinespring (Harvard U.)—aP, U. of Chicago.
*William Turanski (U. of Pennsylvania)—Died in automobile accident.
Eoin L. Whitney (U. of Alberta)—aP, U. of Alberta.

Eighth Competition—1948

George F. D. Duff (U. of Toronto)—AP, U. of Toronto.
Leonard Geller (Brooklyn C.)—Chemistry; Sr. Associate, S. M. Stoller Associates, 22 Irving Place, New York, N. Y.
Harry Gonshor (McGill U.)—aP, Rutgers U.
Robert L. Mills (Columbia U.)—Physics; Sr. Engineer, North American Aviation, Los Angeles, Cal.
D. J. Newman (C. of the City of New York)—AP, Yeshiva U.
*Eoin L. Whitney (See Seventh Competition).

Ninth Competition—1949

J. W. Milnor (Princeton U.)—P, Princeton U.
*D. J. Newman (See Eighth Competition).
W. Forrest Stinespring (See Seventh Competition).
David L. Yarmush (Harvard U.)—Technical Research Group, Inc., 2 Aerial Way, Syosset, N. Y.
Ariel Zemach (Harvard U.)—Physics, U. of California at Berkeley.

Tenth Competition—1950

John P. Mayberry (U. of Toronto)—Operations Analyst GS-15, Hq. U. S. Air Force AFCOA, Washington, D. C.
*Z. Alexander Melzak (U. of British Columbia)—AP, U. of British Columbia.
J. W. Milnor (See Ninth Competition).
D. J. Newman (See Eighth Competition).
Richard J. Semple (U. of Toronto)—AP, Carleton U., Waterloo, Ontario.

Eleventh Competition—1951

A. P. Demster (U. of Toronto)—Statistics, Harvard U.
*James B. Herreshoff, IV (U. of California at Berkeley)—Resuming graduate work at U. of California at Berkeley. Home address: 1862 Arch St., Berkeley 9, Cal.
Herbert G. Kranzer (New York U.)—P, Adelphi C.
P. J. Redmond (Cooper Union)—Physics; 6873 Fortuna Road, Goleta, Cal.
Harold Widom (C. of the City of New York)—AP, Cornell U.

Twelfth Competition—1952

Walter L. Baily, Jr. (Massachusetts I. of T.)—aP, U. of Chicago.
James B. Herreshoff, IV (See Eleventh Competition).
Gerhard Rayna (Harvard U.)—Instructor, Lehigh U.
*Eugene R. Rodemich (Washington U., St. Louis)—Address: 10405 Eastborne Ave., Los Angeles, Cal.
Richard G. Swan (Princeton U.)—Instructor, U. of Chicago.

Thirteenth Competition—1953

Norman Bauman (Harvard U.)—Assistant Resident, Dept. of Medicine, Duke U.
Marshall L. Freimer (Harvard U.)—AP Business Admin., U. of Rochester.
J. B. Herreshoff, IV (See Eleventh Competition).
Samuel Jacob Klein (C. of the City of New York)—620 W. 116th St., Apt. 82, New York, N. Y.
*Tai Tsun Wu (U. of Minnesota)—Physics, Gordon McKay Lab. of Applied Science, Harvard U.

Fourteenth Competition—1954

*James Daniel Bjorken (Massachusetts I. of T.)—AP, Stanford Linear Accelerator Center, Stanford U.
Leonard Evens (Cornell U.)—aP, U. of California at Berkeley.
William P. Hanf (U. of California at Berkeley)—Staff Mathematician, General Products Division Lab., International Bus. Machines Corp., Endicott, N. Y.
Benjamin Muckenhoupt (Harvard U.)—AP, Rutgers U.
Kenneth G. Wilson (Harvard U.)—7 Martha's Point Rd., Concord, Mass.

Fifteenth Competition—1955

*Trevor Barker (Kenyon C.)—Math. and (primarily) programmer for research project in Brain Model Theory, Cornell U.
Everett C. Dade (Harvard U.)—Fellow, California I. of T.
David B. Mumford (Harvard U.)—AP, Harvard U.

Howard C. Rumsey, Jr. (California I. of T.)—Sr. Research Engineer, Jet Propulsion Lab., Pasadena, Cal.
Jack Towber (Brooklyn C.)—Last known address (1956): 1234 Stratford Ave., Bronx 72, N. Y.

Sixteenth Competition—1956

Trevor Barker (See Fifteenth Competition).
David M. Bloom (Columbia U.)—aP, U. of Massachusetts.
*Richard M. Friedberg (Harvard U.)—Physics, I. for Advanced Study, Princeton, N. J.
David B. Mumford (See Fifteenth Competition).
Kenneth G. Wilson (See Fourteenth Competition).

Seventeenth Competition—1957

Richard T. Bumby (Massachusetts I. of T.)—Instructor, Rutgers U.
David M. Bloom (See Sixteenth Competition).
Everett C. Dade (See Fifteenth Competition).
*Rohit J. Parikh (Harvard U.)—Instructor, Stanford U.
J. Ian Richards (U. of Minnesota)—aP, U. of Minnesota.

Eighteenth Competition—1958 (Spring)

David R. Brillinger (U. of Toronto)—Member Tech. Staff, Bell Telephone Labs., Murray Hill, N. J.
Donald J. C. Bures (Queen's U.)—P, U. of British Columbia.
Richard M. Dudley (Harvard U.)—aP, U. of California at Berkeley.
Joseph Lipman (U. of Toronto)—Graduate Student in Math. and Physics, Harvard U. Residence: 9 Vincent Ave., Cambridge, Mass.
*Lawrence A. Shepp (Polytechnic I. of Brooklyn)—Member Tech. Staff, Bell Telephone Labs., Murray Hill, N. J.

Nineteenth Competition—1958 (Fall)

Alfred W. Hales (California I. of T.)—Instructor, Harvard U.
Robert C. Hartshorne (Harvard U.)—Graduate Student, Harvard U.
John Rex Forrester Hewett (U. of Toronto)—Graduate Student, King's C., Cambridge, England. Home Address: Apt. 1407, 500 Avenue Rd., Toronto, Ontario.
Joseph Lipman (See Eighteenth Competition).
*Alan Gaisford Waterman (San Diego S. C.)—Graduate Student, Harvard U.

Twentieth Competition—1959

Stephen L. Adler (Harvard U.)—Home Address: 32 Chambers St., Princeton, N. J.
Donald Gorman (Harvard U.)—Graduate Student, Harvard U.
Alfred W. Hales (See Nineteenth Competition).
Martin Isaacs (Polytechnic I. of Brooklyn)—Graduate Student, Harvard U.
Samuel Klein (C. of the City of New York)—620 W. 116th St., Apt. 82, New York, N. Y.
*Stephen Lichtenbaum (Harvard U.)—Graduate Student, Harvard U.
Donald Passman (Polytechnic I. of Brooklyn)—Student, Harvard U.
Daniel G. Quillen (Harvard U.)—178 Hayes St., Cambridge 40, Mass.

Twenty-First Competition—1960

William R. Emerson (California I. of T.)—U. of California at Berkeley.
Jon H. Folkman (U. of California at Berkeley)—U. of California at Berkeley.
Melvin Hochster (Harvard U.)—Harvard U.
*Louis Jaeckel (U. of California at Los Angeles)—N. S. F. Fellow, Harvard U., 1962–63.
Samuel Klein (See Twentieth Competition).

Twenty-Second Competition—1961

Edward Anton Bender (California I. of T.)—California I. of T.
Elwyn R. Berlekamp (Massachusetts I. of T.)—Electrical Engineering.
*John Hathaway Lindsey (California I. of T.)—Lecturer, U. of Toronto.
William C. Waterhouse (Harvard U.)—Harvard U., Perkins Hall 7.
Barry Wolk (U. of Manitoba)—Math. and Physics. Home Address: 500 Inkster Blvd., Winnipeg 4, Canada.

Twenty-Third Competition—1962

Edward Anton Bender (See Twenty-Second Competition).
John Hathaway Lindsey (See Twenty-Second Competition).
Robert S. Stricharz (Dartmouth C.)
*William C. Waterhouse (See Twenty-Second Competition).
John Wood (Harvard U.).

Twenty-Fourth Competition—1963

Lawrence Corwin (Harvard U.).
*Stephen E. Crick, Jr. (Michigan S. U.).
Robert E. Green (Michigan S. U.).
Joel H. Spencer (Massachusetts I. of T.).
Lawrence Zalcman (Dartmouth C.).

Students and faculty of the competing institutions have expressed great interest in the details of the scoring of the examination papers. Since the ninth competition, the examination has always consisted of twelve questions (six questions in the morning and six in the afternoon), each of which counted ten points. The graders have given partial credit for partially answered questions and for imaginative attempts whenever these seemed to deserve recognition. Thus the maximum score on each examination was 120 points. Information relative to the grading of the first eight examinations is not available, but since the ninth competition no paper has been judged worthy of the full 120 points. The lowest score on every one of these sixteen examinations has been zero, and on six of them more than ten per cent of the papers had a score of zero. Table IV gives for each of these examinations the scores, to the nearest integer, of contestants falling in the tenth, twenty-fifth, fiftieth, seventy-fifth, and ninetieth percentiles. It also shows the highest score on each examination.

Some readers may wish to make deductions about the relative difficulty of the examinations and the relative number of easy and difficult questions in each examination. Since any such deductions must depend also on the grader, the following information is pertinent. There have been four graders involved in the above sixteen competitions, and since they must remain anonymous, I shall call them A, B, C, and D. Where two graders graded a competition, one graded the morning papers and one the afternoon papers. Competitions nine through nineteen were graded by A alone, competition twenty by A and B, competition twenty-one and twenty-two by B and C, and the last two competitions by C and D.

TABLE IV

SCORES OF CONTESTANTS IN THE NINTH THROUGH THE TWENTY-FOURTH COMPETITION

Competition	*Percentile* 10	25	50	75	90	*Highest Score*
9	1	4	9	20	32	74
10	0	2	8	22	42	100
11	6	14	29	49	65	103
12	6	16	31	51	69	117
13	4	11	21	38	53	114
14	0	5	16	31	53	86
15	1	5	16	33	51	91
16	1	6	15	26	45	107
17	0	2	9	22	34	89
18	0	1	5	22	41	97
19	0	3	10	25	43	106
20	3	10	21	36	55	96
21	5	10	18	28	40	87
22	0	6	14	26	44	107
23	2	6	13	25	41	90
24	5	9	14	21	28	62

Many examination supervisors have written me apologetically regarding the showing made by their contestants. They can be encouraged by the fact that on six of the last sixteen competitions at least ten per cent of the scores were zero and that a score of six is the highest tenth percentile rank on any of these sixteen examinations. A contestant does not have to earn a high score to feel that it has been worth his while to spend six hours on the examination. He may derive great benefit from seeing the kind of problems that the ablest undergraduates are interested in. It is my hope that he may thus gain encouragement to delve more deeply into this kind of mathematical activity.

A STUDY OF 60,000 DIGITS OF THE TRANSCENDENTAL "*e*"

R. G. STONEHAM, City College, City University of New York

1. Introduction. In November 1957 (see *Notices AMS*, 5(1958), 64) we presented to the American Mathematical Society some incomplete statistical data on 60,000 digits of "*e*" which had been computed on the Illiac at the University of Illinois in 1952 by Prof. D. Wheeler.(1) We now offer the complete results of the studies initiated in 1956 and, in addition, a number of new statistical studies as well as further information on the "normality" of "*e*" in the sense of Borel [1, p. 94]. We submit the Kendall-Smith tests for randomness, i.e. the frequency,

poker, serial, and gap test (we carried out the study of gaps for all pairs, 00, 11, $\cdots$, 99; usually only the gaps between zeros are chosen), χ^2 measures for all these tests, the χ^2's for the accumulated frequencies of the counts of single digits computed every 500 digits, the χ^2 values for 60 blocks of 1000 digits, a goodness-of-fit test using the χ^2's for 120 blocks of 500 digits, a study of these 120 blocks as Bernoulli trials, plots of the deviation of each separate digit from 0 to 9 from expectation for the accumulated counts of 600 blocks of 100 digits, the values of S_n^i/n and S_n^{ij}/n, where these ratios represent relative frequencies of single and pairs of digits for normality studies, and finally a test of randomness based on a criterion for randomness in a "collective" proposed by von Mises [2, p. 29] wherein one chooses subsets by some choice of place selection.

In general, all the statistical tests support the hypothesis of pure chance selection in the sequence of digits in "e" but with a few anomalies and also the relative frequency of the single and pairs of digits in "e" are approximately 1/10 and $1/10^2$, respectively, thus empirically supporting normality in the sense of Borel.

Since the first study of "e" to 2000 places by John von Neumann *et al.* in 1950, [3], the concern in subsequent studies was largely the hope of establishing the "random" character of the decimal digits in transcendentals like "e" and π and possibly making such transcendentals a computer source of internally programmed pseudo-random numbers for Monte Carlo methods. For example, in 1961, R. K. Pathria [4, 5] offered a study of 2500 digits of "e" and in 1962, a study of 10,000 digits of π. Each of these studies was motivated by the search for random sequences.

H. Geiringer, in 1954, [6] discussed some of the theoretical problems in the statistical investigation of transcendentals. In this study, Geiringer maintains that since numbers like "e," π, or γ (Euler's constant) are formed by mathematical laws, they do not form a "random" sequence or a "collective" in the sense of von Mises [2, p. 29] even though we are not yet able to establish any prevailing regularities. Such sequences can, however, exhibit some sort of "restricted" or "local" randomness.

We would like to emphasize that our point of view in performing these extensive calculations is not necessarily to accumulate more statistics on "e" or to seek sources of random numbers, but rather to search for any "prevailing regularities" in the decimal expansions of irrationals (if they exist!) in order to obtain a deeper understanding of the numerical distributions in a number theoretic sense. For example, in Section 8, figure 2, the plot for the sixes shows a consistent excess above pure chance expectation for 97.7% of the 60,000 place sample. It is interesting to note that in 1938, Fisher and Yates [7, p. 18] observed that there was an "excess of sixes" in their attempts to construct "random" numbers by selecting digits from the 15–19 places of a table of 20 place logarithms to the base 10. In responding to a question proposed by Henri Poincaré in 1899 [8, p. 224] concerning the number of even or odd digits in the ith decimal place of the sequence of logarithms of the natural integers n,

J. Franel in 1917 [9] proved that $P_i(n)/n$ *does not* tend to a limit as n increases without bound if i is fixed and $P_i(n)$ represents the number of even digits in the ith place of the sequence of logarithms of the natural integers. Poincaré had suggested that the intuitive result was $\frac{1}{2}$, yet Franel proved the contrary. Such results would appear to caution us against the firm conviction that "e" is simply normal from this 60,000 place sample even though there is strong empirical evidence. From the numerical data as well as the graphs of the relative frequencies of particular digits, there is the suggestion from the data that the mode of approach to 1/10 may be distinctive, i.e. by means of oscillating sequences which, for some digits, appears to approach 1/10 by sequences of values consistently above or below 1/10 and for others, by what might be called a "damped" oscillation toward 1/10. We shall discuss this in more detail in Section 8.

A normal number was defined by Borel in 1909 [1, p. 94] as a real number x which has all possible blocks of j digits appearing with the relative frequency of $1/10^j$ (or $1/g^j$, where g is the base of representation of x). Also, Borel defined an "absolutely" normal number as one which is normal to every base. He proved [1, p. 115] that "almost all" real numbers are normal to every base, i.e., the set of those real numbers which are not normal to every base is of measure zero.

Several theorems have been established concerning the properties of normal numbers from the point of view of measure theory. In 1960, Schmidt [10] proved that there exist numbers which are normal to one base but not absolutely normal. Unfortunately, many of these theorems based on measure theory describe the properties of a given normal number but do not offer techniques which can decide whether a given real number is normal or not. At present, it is not known whether irrationals like "e," π, or $\sqrt{2}$ are normal to any base [1, p. 112]. Geiringer [6, p. 311] states, "The fact that a set of nonnormal numbers is of measure zero does not help in any way in the extremely difficult problem of deciding whether a given number is normal or not." Explicit formulas for the frequency of a given digit or block of digits in any irrational within n places of decimals are not known at the present time.

Recently [12], we have been able to derive analytic expressions by means of which we can show that the reciprocals of integral powers of primes represented in a base g which is a primitive root mod p^2, where p is an odd prime and n any positive integer will have a relative frequency of all possible blocks of j digits of approximately $1/g^j$ up to a limit on block size given by $j \leqq [n \log_g p]$, where $[x]$ designates the greatest integer $\leqq x$. In a sense, we find that the distribution of digits in the reciprocal of an integral power of a prime is "approximately" normal, i.e., up to a limit on block size, the relative frequency counts can be made arbitrarily close to $1/g^j$ for a sufficiently large prime and (or) integral power of the prime. Is this property of "approximate" normality in the reciprocal of an integral power of a prime related to that which is the apparent normality of "e" when we recall the structure of the factorial series for "e"? Certainly, in the rational approximation of "e" to 60,000 places, we could expect to find a numerically similar phenomenon (as we already will show in Section 8, at least for

$j=1, 2$), i.e., an "approximate" sort of normality with upper bounds on the block size j for which we could expect that the relative frequencies for all possible blocks B_j in the 60,000 places would approximate to $1/10^j$ with some variable difference depending on block size. At present, however, let us continue machine studies which may indicate significant trends in the various aspects of these expansions that may point to theoretical endeavors.

2. Statistics, randomness and irrationals. In deciding whether a given sequence of digits is "random" or not, the usual practice is to set up a number of statistical tests designed to detect any unusual frequencies, patterns, runs, gaps, etc. If the null hypothesis is not rejected at confidence levels of say 1% and 99% on some statistical measure, usually the chi-square test, then we conclude that the sequence is "random." It is more precise, however, to say that, based upon the choice of a *particular* set of tests, the sequence tested possesses "locally" random properties, i.e. the sequence presents various observed frequency groupings which do not differ significantly from similar frequency groupings of a sample of equal size drawn from a normally distributed infinite discrete sample space. Thus, we define "randomness" as that quality of a particular numerical sequence which has deviations from expectancy similar to a sample of equal size formed by pure chance selection. It is to be remembered, however, that the given sequence is "random" *only* with respect to a particular fixed set of probability measures. No doubt, order or a "prevailing regularity" could be found by other "tests." It appears impossible to give a practical definition of randomness which will meet all the philosophical subtleties of the notion itself, i.e., for example, the sequence contains no "prevailing regularities." Essentially, the deeper problem is deciding that a given sequence *does not* possess certain qualities which are to be denied in making the decision, but we cannot deny the existence of those "regularities" for which we have no awareness.

Von Mises [2, p. 29] stated an axiomatic concept of randomness, viz., if a collective (i.e., a discrete sample space) is random then every subset consisting of members chosen from the original collective in such a way as to be independent of place selection should have the same relative frequency for identical members as in the original collective. Here the problem in carrying out this criterion is attaining the "independent of place selection" in some practical sense, which relates to our comments above. In order to test this concept of randomness on "e," however, we chose 50 subsets of 1200 each taken from the following digit positions in the 60,000 place sequence for practical machine considerations; the 1st, 51st, 101st, $\cdots$,; the 2nd, 52nd, 102nd, $\cdots$, out to the 50th, 100th, $\cdots$, 60,000th place.

We present in Table 10, the χ^2's for the 50 subsets, and in Table 9 a sample of the actual frequency counts of the digits in 10 of the 50 subsets. The data shows that the von Mises criterion is firmly supported in the 60,000 place "collective" (finite!) using the indicated place selection. It would be interesting in testing the von Mises criterion further to have followed this same procedure

but to have chosen the 50 subsets of 1200 digits each based on a place selection which used a table of "random" numbers.

Geiringer in [6, p. 315] develops the point of view that perhaps a sounder statistical basis for studying samples to n places drawn from the decimal expansions of irrationals is to compare the probability expectations in the sample with a finite discrete sample space consisting of the 10^n possible arrangements of n decimals chosen from the digits from 0 to 9. Thus, for example, one can say that the accumulated frequency count of the digits from 0 to 9 in the first 2000 places of $e-2$ has the remarkably low value of $\chi^2=1.06$. (The value $\chi^2=1.11$ was reported by von Neumann *et al.* in 1950 [3] for the first 2000 places of "e" and was thought to indicate some possible number theoretic phenomenon, but we reported to the Society in 1957 that this was a singular value which does not occur again (see our Table 1) for the rest of the 60,000 place sample of "e.")

TABLE 1. The accumulated frequency χ^2 for single digits in $e-2$

N	χ^2	N	χ^2	N	χ^2	N	χ^2	N	χ^2	N	χ^2
5	6.72	105	7.81	205	8.38	305	5.98	405	4.49	505	6.86
10	4.86	110	7.83	210	7.77	310	5.82	410	5.61	510	7.19
15	3.56	115	7.53	215	7.44	315	4.89	415	6.15	515	7.43
20	1.06	120	8.42	220	7.14	320	5.62	420	6.52	520	7.14
25	1.96	125	10.05	225	6.30	325	5.05	425	6.52	525	7.21
30	3.05	130	8.78	230	6.93	330	4.72	430	6.62	530	6.77
35	5.00	135	7.79	235	6.72	335	5.74	435	7.47	535	6.83
40	2.89	140	9.19	240	6.84	340	5.50	440	6.72	540	7.17
45	2.73	145	7.31	245	7.48	345	5.47	445	7.26	545	6.83
50	9.86	150	8.35	250	7.73	350	5.41	450	7.76	550	7.03
55	9.28	155	8.17	255	7.39	355	4.74	455	7.48	555	6.98
60	7.68	160	7.85	260	7.28	360	5.38	460	7.17	560	7.42
65	9.56	165	9.09	265	7.88	365	5.81	465	7.62	565	7.96
70	9.83	170	9.01	270	7.22	370	5.25	470	8.25	570	8.78
75	9.49	175	10.22	275	7.50	375	6.09	475	8.78	575	7.97
80	9.74	180	10.08	280	7.51	380	5.96	480	8.42	580	8.73
85	9.83	185	10.12	285	7.68	385	5.74	485	7.95	585	7.81
90	8.90	190	8.44	290	6.46	390	5.27	490	8.76	590	8.16
95	7.68	195	8.78	295	6.56	395	4.70	495	8.15	595	8.27
100	8.61	200	7.87	300	5.83	400	4.82	500	7.25	600	9.03

From these frequencies one would expect that much less than 1% of all the 10^{2000} possible arrangements of 2000 digits chosen from the digits from 0 to 9 would show a χ^2 less than 1.06. Further, it is proved [6, p. 311] that the average of all possible χ^2 values for these 10^n arrangements associated with the frequency counts defined by

$$\chi^2 = 10/n \sum_{i=0}^{i=9} (n_i - n/10)^2,$$

with n_i denoting the count of the ith digit in the first n decimals, is

$$\chi^2 = 1/N \sum_{\nu=1}^{\nu=N} \chi_\nu^2 = 9,$$

where $N = 10^n$ and χ_ν^2 is computed for each of the 10^n combinations of the digits from 0 to 9. Also, the "variance" of these χ_ν^2 values is shown to be

$$\sigma(\chi_\nu^2) = 1/N \sum_{\nu=1}^{N} (\chi_\nu^2 - 9)^2 = 18(1 - 1/n).$$

Since 10^n is large for these applications, the ordinary tables for 9 degrees of freedom can be used [6, p. 315]. We find from our data for "e" that the average χ^2 value over the first 100 blocks of 100 digits is 9.81 which compares favorably with 9. The variance of the χ^2 over the same 100 blocks is 17.97 which compares quite favorably with $18(1-1/100) = 17.82$. For 120 blocks of 500 digits, the average χ^2 value is 8.93 and the variance is 22.52 which is somewhat larger than the expected 17.96. It is to be pointed out that these measures for a sample space presupposes no infinite sequence, no randomness, and no probability concept whatsoever.

TABLE 2. Goodness-of-fit χ^2 for 120 blocks of 500 digits

i	0	1	2	3	4	5	6	7	8	9
χ^2	0–4.17	4.18–5.38	5.39–6.39	6.40–7.36	7.37–8.34	8.35–9.41	9.42–10.70	10.71–12.20	12.21–14.70	$\chi^2 \geqq$ 14.71
f	9	18	16	15	10	10	9	9	13	11

3. Frequency of digits. In Table 1, we find the χ^2 values for the accumulated frequency counts(2) in $e-2$ of the digits from 0 to 9 in blocks of 500 digits where the number of places in each entry is $n = 100N$. We note, after the unusually low value of $\chi^2 = 1.06$ at $n = 2000$, that the χ^2 values are such that $4 < \chi^2 < 11$, approximately, for the rest of the sample. The value of $\chi^2 = 1.11$ reported by von Neumann *et al.* in 1950 [3] appeared unusually low, stimulating their remark that more digits of "e" ought to be studied since there was the possibility of some number-theoretic phenomenon. The value $\chi^2 = 1.11$ is based on the first 2000 digits of "e" starting with the digit 2 to the left of the decimal point. The data for the first 2000 places of $e-2$ indicates an even more improbable value of $\chi^2 = 1.06$.

At confidence levels of 1% and 99%, the acceptable range of χ^2 values is such that $2.08 \leqq \chi^2 \leqq 21.67$. We note that all the tabulated values are acceptable except those for $n = 2000$ and 2500 places. The probability that $\chi^2 < 1.15$ for 9 degrees of freedom is .001 and thus $\chi^2 = 1.06$ is an extremely improbable event. The frequency counts for $n = 2000$ are "too close" to expectation, like hitting a

TABLE 3. Bernoulli trials

k	N_k^0	N_k^1	N_k^2	N_k^3	N_k^4	N_k^5	N_k^6	N_k^7	N_k^8	N_k^9	Total
27	0	0	0	1	0	0	0	0	0	0	1
28	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	1	1
30	0	0	0	0	0	0	0	0	0	0	0
31	0	1	0	0	0	0	0	0	1	0	2
32	0	0	0	1	1	0	0	0	0	0	2
33	0	0	0	0	0	0	0	1	0	0	1
34	0	0	0	0	1	0	0	0	1	1	3
35	0	0	1	0	1	0	0	0	1	1	4
36	1	1	0	0	0	2	1	1	1	0	7
37	1	3	2	0	0	0	1	1	0	2	10
38	1	0	1	1	1	2	2	2	1	2	13
39	3	0	0	4	3	2	0	2	0	3	17
40	2	3	4	3	3	1	1	5	7	1	30
41	5	2	5	2	3	2	1	4	2	3	29
42	6	0	4	3	2	7	6	3	4	7	42
43	4	6	1	4	2	2	3	3	6	5	36
44	2	5	3	5	6	5	1	3	8	3	41
45	7	3	5	5	5	6	9	6	2	7	55
46	10	6	4	6	3	8	6	6	7	8	64
47	10	6	10	9	7	11	7	7	4	5	76
48	7	7	8	3	9	6	6	4	6	5	61
49	5	12	8	4	10	4	4	10	5	5	67
50	7	9	13	9	7	5	7	6	8	8	79
51	6	5	8	4	7	8	9	7	13	7	74
52	4	4	8	5	7	3	6	6	8	5	56
53	3	8	6	10	7	7	5	9	6	12	73
54	8	9	2	4	8	6	7	12	5	1	62
55	4	7	2	9	8	7	9	1	7	3	57
56	6	2	7	8	6	6	6	5	5	10	61
57	2	1	4	4	1	2	6	4	1	4	29
58	4	2	2	3	3	2	2	5	5	2	30
59	4	6	3	4	2	5	3	0	0	2	29
60	2	5	4	1	3	2	2	3	1	3	26
61	1	0	1	2	1	1	2	0	1	2	11
62	1	3	0	0	0	2	3	1	0	1	11
63	0	1	3	2	0	2	2	0	1	0	11
64	1	2	0	0	2	1	0	0	1	0	7
65	0	0	1	1	2	1	0	1	0	1	7
66	2	0	0	0	0	1	0	0	1	0	4
67	0	1	0	0	0	0	1	2	0	0	4
68	0	0	0	1	0	0	0	0	0	0	1
69	1	0	0	0	0	0	1	0	0	0	2
72	0	0	0	2	0	0	0	0	0	0	2
76	0	0	0	0	0	0	0	0	1	0	1
81	0	0	0	0	0	0	1	0	0	0	1
											$\sum = 1200$

bull's eye with almost *every* throw of a dart! We also made a study of the frequency counts of the digits from 0 to 9 in 120 separate blocks of 500 digits and computed a χ^2 for each block. These 120 values of χ^2 provided an empirical fit to the distribution (9 degrees of freedom). A goodness-of-fit chi-square was computed using 10 class intervals each of which was expected to contain 10% of the values. The results are given in Table 2, where we have set up class limits for the χ^2-distribution based on equal probability intervals, and have grouped the 120 χ^2 values accordingly. For convenience, we have designated the ith probability interval $i/10<p<(i+1)/10$ by column headings $i=0, 1, \cdots, 9$ with their corresponding χ^2 values, f is the count of the number of χ^2 in the given range, and $\Sigma f=120$.

Under the expectation that each interval should contain 12 values, we find a goodness-of-fit $\chi^2=8.17$ with $p=.48$ which supports the 50% expectation for local randomness.

Furthermore, we may view these 120 consecutive blocks of 500 digits as Bernoulli trials, i.e., in n places of "e," we may obtain k "successes" of the ith digit from 0 to 9 and $n-k$ "failures." Thus, if S_n is a random variable which is the number of successes in n trials, then $b_i(k, n)=P\{S_n=k\}$, where $b_i(k, n) = C_k^n 9^{n-k}/10^n$ is the probability that we obtain the ith digit k times in n places. Our results for each digit from 0 to 9 are shown in Table 3, Bernoulli Trials, and we find for the whole sample $27 \leqq k \leqq 81$, where N_k^i denotes the number of blocks which contained k successes of the ith digit for some given k. For example, noting the values for $k=50$, we see that 7 blocks out of 120 contained exactly 50 zeros or $N_{50}^0=7$, 9 contained exactly 50 ones or $N_{50}^1=9$, etc. (After $k=69$, we entered only those k up to 81 which had nonzero counts.) First, we tested the hypothesis that the data in Table 3 was drawn from a normal population by comparing, in Table 4, the theoretical percentage expectations for the unit

TABLE 4. Bernoulli trials in "e" as normal variates

k(incl.)	44–56	37–63	46–54	$k \geqq 55$	$k \leqq 45$
Th.%	68.26	95.44	50.00	25.00	25.00
Obs.%	68.83	95.83	51.00	24.50	24.50
$n(k)$	826	1150	612	294	294

standard deviations and quartiles of normal variates with the observed percentages given by N_k^i summed over the ranges of k given by

$$\mu - t\sigma < k \leqq \mu + t\sigma,$$

where $t=1$, 2 and .6745 and $k>\mu+.6745\sigma$ or $k \leqq \mu-.6745\sigma$ with $\mu=50$ and the theoretical deviation $\sigma=3\sqrt{5}=6.7082$. Let $n(k)$ be the total number of counts for all k in a specified range. The results in Table 4 indicate remarkable agreement.

Secondly, we found a mean value of k using the totals as frequency counts and obtained $\bar{k}=49.915$ and standard deviation $\sigma_{ob}=6.5135$ which compares quite favorably with expected values $\mu=50$ and $\sigma_{th}=6.7082$.

Finally, we computed a goodness-of-fit for the graduated frequencies compared with the observed frequencies where we grouped the data into 34 class intervals so that each cell frequency was about 5 or larger near the ends of the data. The class intervals for the k values were: 26.5–32.5; 32.5–35.5, then in unit intervals 35.5–36.5 etc. up to 65.5, 65.5–67.5, and 67.5–81.5. Thus, with 31 degrees of freedom, $\bar{k}=49.915$, and $\sigma_{ob}=6.5135$, we found $\chi^2=30.85$ and $p=.475$.

It is interesting to note the close agreement between the goodness-of-fit for Table 2 with $p=.48$ and $p=.475$ that we have just obtained. All of these results strongly support the pure chance selection or "locally random" character of this sample of "e" from the point of view of frequency studies.

TABLE 5. Poker test

Hands Th. f / Block	*abcde* 302	*aabcd* 504	*aabbc* 108	*aaabc* 72	*aaabb* 9	$\left\{\begin{matrix} aaaab \\ aaaaa \end{matrix}\right\}$ 4.6	χ^2
1	316	506	98	70	5	5	3.420
2	307	499	108	80	6	0	6.620
3	317	503	90	72	13	5	5.530
4	299	511	114	58	12	6	4.610
5	299	498	99	84	18	2	5.320
6	307	503	111	67	8	4	7.050
7	299	509	107	72	9	4	.167
8	295	522	100	73	8	2	2.990
9	313	505	110	56	12	4	5.070
10	304	497	116	70	8	5	.904
11	308	501	95	77	15	4	6.130
12	314	485	109	80	10	2	3.672
Tot. Ob.	3678	6039	1257	859	124	43	$\chi_T^2=$
Tot. Th.	3624	6048	1296	864	108	55.2	7.087

4. The poker test. In the poker test, we considered 12 blocks of 5000 digits, each containing 1000 sets of nonoverlapping "poker hands." In Table 5, we have tabulated the results. For 5 degrees of freedom, the acceptable χ^2's are such that $P(.55<\chi^2<15.1)=.98$. We note one unusually small value $\chi^2=.167$ for which the probability that $\chi^2<.167$ is .001. All the other values do not reject the null hypothesis, i.e. local randomness at the required confidence levels. We have also computed an over-all value of $\chi_T^2=7.06$ from the total observed frequencies and total expected frequencies indicated. This value is such that $.20<P\{\chi^2>7.06\}<.30$. At the top of each column heading, we have given the theoretical frequency for each type of poker hand in 5000 digits and lumped the frequencies for the types *aaaab* and *aaaaa*.

TABLE 6. Serial test

Block	$2\chi^2$	z	p
1	186.16	.265	.3955
2	192.16	.447	.3375
3	189.12	.373	.3546
4	248.64	2.389	.0085
5	244.48	2.257	.0120
6	222.32	1.532	.0628
7	243.76	2.234	.0127
8	200.96	.797	.2128
9	208.96	1.076	.1410
10	181.60	.097	.4614
11	202.32	.845	.1991
12	215.36	1.295	.0976

5. The serial test. In the serial test, we used 12 blocks of 5000 digits and computed a χ^2 for each matrix of pairs. The pairs were overlapping with the last digit and the first digit of each block forming the final pair. In Table 6, we find for each block $2\chi^2$ and for 90 degrees of freedom, the normal deviate $z=\sqrt{2\chi^2}-\sqrt{179}$ yields the indicated probabilities p. For each block of 5000 digits, the expected cell frequency in the matrix for each pair from 00 to 99 was 50, thus $\chi^2=\sum(f_0-50)^2/50$ with $.01<p<.99$. We note that all blocks satisfy the acceptable probabilities for local randomness except block 4, for which $p=.0085$.

TABLE 7. Gap test χ^2

Block	00	11	22	33	44	55	66	77	88	99
1	13.5	14.8	11.7	20.2	15.9	14.1	33.4	29.8	24.3	8.5
2	22.4	14.5	29.9	13.8	18.7	10.9	18.9	12.3	8.4	16.4
3	18.4	16.8	19.6	33.3	29.4	17.7	12.0	17.8	13.2	16.6
4	21.0	24.7	29.0	44.8	25.3	17.2	11.0	20.0	24.0	19.6
5	32.8	27.5	17.8	28.8	10.3	16.5	27.9	28.9	16.5	12.3
6	16.1	29.3	19.1	8.6	18.6	33.8	16.4	19.8	17.3	18.3
7	16.9	27.8	16.1	19.4	19.9	17.4	22.2	35.0	18.1	32.9
8	35.6	22.8	13.5	21.3	19.2	20.4	15.5	31.7	9.5	17.1
9	24.7	12.6	20.2	23.9	22.4	22.7	28.2	20.8	14.9	21.4
10	18.4	20.5	11.1	17.1	20.4	19.9	18.9	21.9	25.9	21.3
11	16.5	27.1	16.4	15.1	18.3	23.8	15.1	20.2	16.0	19.5
12	29.2	12.9	16.6	31.3	5.9	19.9	17.5	20.6	9.9	8.9

6. Gap test. In the gap test, we computed gap sizes 0, 1, $\cdots$, 15, 16–20 incl., 21–25 incl., and 26 and over for the gaps between all zeros, ones, . . . , nines, in each of the 12 blocks of 5000 digits. This provided us with 12 blocks

of 19×10 counts for all possible gaps in the 60,000 place sample. A χ^2 was computed for each of the 12 blocks for $i=0, 1, \cdots, 9$. (The column heading 00, 11, $\cdots$, etc. denotes the χ^2 for the gaps between these digits.) We present this data in Table 7. At the 1% and 99% levels, the acceptable range of χ^2 is $7.015 \leqq \chi^2 \leqq 34.80$ for 18 degrees of freedom. We note 4 unacceptable values, viz. 44.80, 5.95, 36.6 and 35.0 in the 120 values. A frequency distribution of the χ^2 values using 10 equal probability intervals is given in Table 8, following the same format as in Table 2.

TABLE 8. Goodness-of-fit χ^2 for the gap test

i	0	1	2	3	4	5	6	7	8	9
χ^2	0–10.90	10.91–12.90	12.91–14.40	14.41–15.90	15.91–17.30	17.31–18.90	18.91–20.60	20.61–22.80	22.81–26.00	$\chi^2 \geqq$ 26.01
f	8	9	5	6	18	12	19	11	9	23

Here the horizontal sum of the f values is 120. With the expectation of 12 in each interval the χ^2 goodness-of-fit is $\chi^2=27.16$ which is acceptable since $.05 < P(\chi^2 > 27.16) < .10$.

7. Von Mises randomness and subsequences. As described in the introduction, we computed the frequencies of digits in 50 subsets drawn by the already stated place selection. (Von Mises calls these "Bernoulli" sequences.) This provided us with 50 frequency counts of the digits from 0 to 9 in the subsets, each consisting of 1200 digits. Instead of preparing a large table of the 500 values,

TABLE 9. Frequency counts, subsequences

S	0	1	2	3	4	5	6	7	8	9
1	119	111	118	112	115	119	147	118	117	124
5	135	129	103	120	113	125	130	116	116	113
10	127	111	121	129	107	115	139	118	114	119
15	111	112	114	119	124	127	102	129	150	112
20	120	106	114	116	135	135	120	127	104	123
25	120	105	113	121	111	124	139	126	121	120
30	93	134	116	124	113	119	133	107	141	120
35	126	120	134	105	119	134	109	124	117	112
40	126	104	119	132	106	106	117	156	114	120
45	123	116	120	131	132	96	127	133	100	122
50	124	119	112	128	105	103	148	102	134	125

we offer in Table 9 the total exact counts of a given digit (indicated at the top of each column) in the 1st, 5th, 10th, . . . , 50th subsequence, and in Table 10 we present the χ^2 values for all of the 50 subsequences. We note that each sub-

TABLE 10. χ^2 for 50 subsequences in "e"

S	χ^2	S	χ^2	S	χ^2	S	χ^2	S	χ^2
1	7.78	11	14.87	21	2.41	31	9.43	41	13.17
2	6.68	12	19.27	22	9.32	32	5.65	42	12.32
3	11.43	13	10.92	23	5.93	33	15.07	43	13.47
4	6.00	14	5.63	24	10.55	34	10.55	44	9.75
5	7.08	15	13.47	25	6.42	35	7.20	45	12.40
6	6.12	16	5.83	26	5.88	36	22.40	46	10.03
7	4.83	17	7.63	27	7.62	37	8.62	47	1.75
8	5.90	18	4.48	28	9.88	38	5.57	48	5.38
9	4.93	19	7.10	29	8.60	39	4.00	49	8.70
10	6.73	20	8.43	30	14.88	40	18.08	50	16.57

sequence according to the method of place selection represents a fair sampling of the 60,000 places, not in some "local" region (say in the first part of the expansion only), but all the way across with each of the 50 subsets having their last digit terminate in the last 50 digits of the 60,000 place sample. Thus, the subsequences are fairly representative of the density of a given digit across the

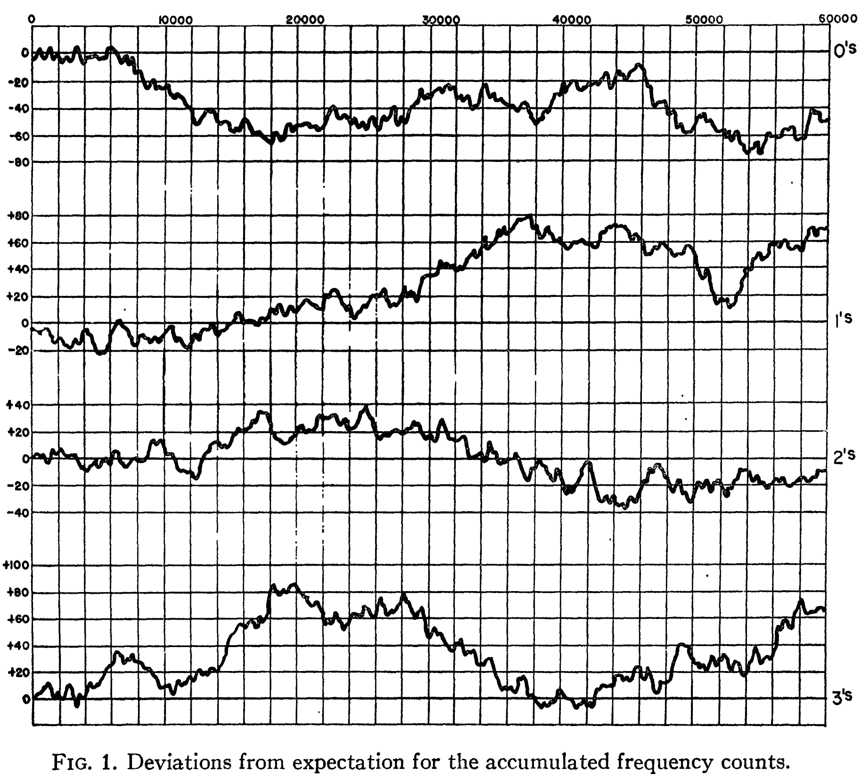


FIG. 1. Deviations from expectation for the accumulated frequency counts.

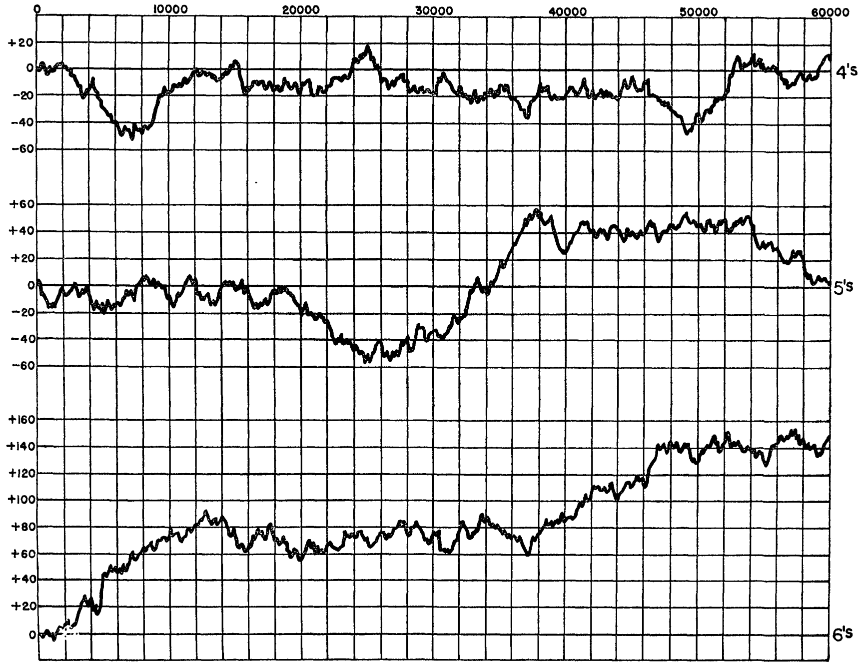

0
10000
20000
30000
40000
50000
60000
+20
0
-20
-40
-60
4's
+60
+40
+20
0
-20
-40
-60
5's
+160
+140
+120
+100
+80
+60
+40
+20
0
6's

FIG. 2

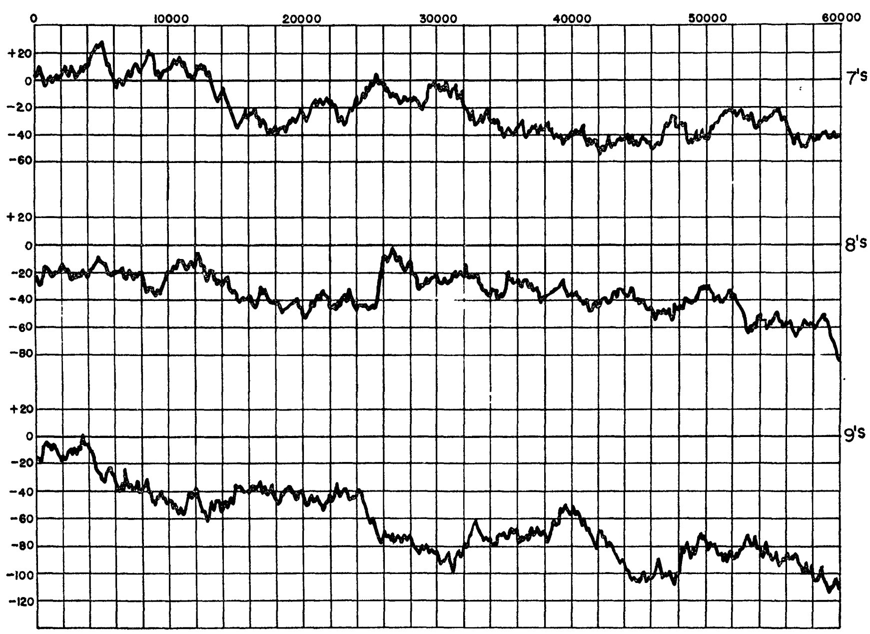

0
10000
20000
30000
40000
50000
60000
+20
0
-20
-40
-60
7's
+20
0
-20
-40
-60
-80
8's
+20
0
-20
-40
-60
-80
-100
-120
9's

FIG. 3

whole sample. It is easily seen that with an expectancy of 120 for each count that the data in Table 9 indicates remarkable agreement with the relative frequency of 1/10 in support of the von Mises concept of randomness for this sample of "e." The other counts not shown have approximately the same range. In Table 10, for 9 degrees of freedom, the acceptable range of χ^2 is such that $2.08 \leqq \chi^2 \leqq 21.67$, and we note that two subsequences, viz. $S=36$ and 47, have frequency counts whose χ^2 values are outside the acceptable range. The goodness-of-fit for the χ^2's in Table 10, was $\chi^2 = 6.2$ which is acceptable since $.2 < P(\chi^2 > 6.2) < .3$.

8. Empirical evidence on the "normality" of "e." In this section, we shall present data and some empirical observations on the behavior of the relative frequency counts for single and pairs of digits as their values approach 1/10 and $1/10^2$ for a large number of places. Further, we present graphical representation (Figs. 1, 2, 3) for each single digit which illustrates the qualitative and quantitative behavior of the distribution of digits with respect to the accumulated frequency counts as the number of digits in the expansion increases. (These interesting graphs were prepared with the kind assistance of Prof. Paul Scherer and students in the Industrial Arts department of the University of California at Goleta, California in 1957.)

Let the deviation from probability expectation be given by $D(n) = S_n^i - n/10$, where S_n^i is the count of the ith digit in n places for $i = 0, 1, \cdots, 9$. In Figures 1, 2, and 3, we present plots for each digit from 0 to 9 of $D(n)$ vs. n for $n = 100, 200, \cdots, 60{,}000$ with these data points joined by a "smooth" curve. On the vertical axis, the deviation from expectation is plotted in units which are actual counts, i.e., for example, on the zero graph, in the first 8000 places of "e," we find exactly 785 zeros and we expect 800, thus the data point is $(-15, 8000)$. The indicated horizontal units are each equal to 2000 places and the "zero" level ordinate represents the fact that the total accumulated counts are exactly equal to pure chance expectation. Since $D(n)/n = S_n^i/n - 1/10$, where S_n^i/n is the relative frequency for $i = 0, 1, \cdots, 9$, we note that $S_n^i/n > 1/10$ or $S_n^i/n < 1/10$ as $D(n) > 0$ or $D(n) < 0$, respectively. Thus, we may study from these graphs the oscillatory behavior of the relative frequency and mode of approach to 1/10, by noting whether $D(n) < 0$ or $D(n) > 0$ for each digit as n increases.

In addition, in Table 11, we have tabulated the numerical values of the relative frequencies of single digits for large values of n, i.e. for $n = 57{,}600$; 57,700; . . . ; 60,000 places. These may be studied in relation to Figures 1, 2, and 3 by noting $D(n) < 0$ or $D(n) > 0$ thus denoting a passage from below 1/10 to above and vice versa. We note various apparent particular types of approach to 1/10 for these relative frequencies. Let us stress the fact that the empirical hypotheses which we state based on this 60,000 place sample could be radically different if billions of places were studied. On the other hand, it is quite possible that a sample of this size is sufficiently large to indicate significant theoretical trends.

TABLE 11. The relative frequencies of single digits

$n/100$	0's	1's	2's	3's	4's	5's	6's	7's	8's	9's
576	.09891	.10095	.09967	.10035	.09991	.10045	.10262	.09932	.09901	.09830
577	.09889	.10095	.09965	.10099	.09998	.10043	.10253	.09931	.09903	.09823
578	.09889	.10098	.09969	.10106	.09991	.10047	.10247	.09927	.09905	.09820
579	.09895	.10100	.09969	.10111	.09988	.10043	.10249	.09921	.09903	.09822
580	.09890	.10102	.09969	.10117	.09984	.10036	.10255	.09926	.09905	.09819
581	.09888	.10095	.09971	.10120	.09991	.10033	.10248	.09926	.09904	.09824
582	.09895	.10089	.09972	.10127	.09986	.10022	.10246	.09923	.09900	.09838
583	.09903	.10091	.09972	.10118	.09988	.10021	.10244	.09928	.09900	.09834
584	.09909	.10103	.09966	.10111	.09995	.10012	.10241	.09928	.09907	.09827
585	.09911	.10104	.09967	.10108	.09991	.10015	.10239	.09933	.09908	.09822
586	.09913	.10109	.09964	.10106	.09995	.10009	.10242	.09933	.09909	.09819
587	.09910	.10114	.09964	.10109	.09988	.10014	.10237	.09935	.09915	.09814
588	.09908	.10119	.09968	.10107	.09991	.10005	.10243	.09934	.09915	.09810
589	.09907	.10117	.09971	.10110	.09990	.10008	.10243	.09932	.09910	.09815
590	.09914	.10120	.09973	.10108	.09995	.10008	.10239	.09925	.09910	.09812
591	.09909	.10120	.09971	.10110	.10005	.10012	.10228	.09931	.09902	.09812
592	.09905	.10111	.09973	.10110	.10005	.10014	.10230	.09929	.09900	.09823
593	.09904	.10108	.09983	.10108	.10012	.10012	.10224	.09929	.09902	.09825
594	.09896	.10118	.09980	.10109	.10008	.10008	.10227	.09936	.09892	.09822
595	.09897	.10116	.09983	.10113	.10010	.10008	.10232	.09928	.09891	.09824
596	.09898	.10119	.09980	.10112	.10017	.10012	.10237	.09930	.09879	.09824
597	.09899	.10116	.09978	.10116	.10020	.10008	.10240	.09930	.09878	.09819
598	.09898	.10115	.09978	.10115	.10018	.10008	.10241	.09932	.09868	.09826
599	.09896	.10114	.09978	.10115	.10018	.10008	.10244	.09932	.09861	.09833
600	.09903	.10115	.09980	.10112	.10017	.10005	.10248	.09928	.09860	.09832

The following possibilities are suggested by the numerical data in Table 11 and Figures 1, 2, and 3.

1. The digits 0, 2, 7, 8, and 9 after some initial variations have relative frequency ratios which approach 1/10 through an oscillating increasing sequence of values *less than* 1/10 for the remainder of the 60,000 places.

2. The digits 1, 3, 5, and 6 approach 1/10 from above by means of an oscillating decreasing sequence of values.

3. The digit 4 appears to approach 1/10 by means of a sort of "damped" oscillation above and below 1/10.

4. There appears to be a consistent "excess of sixes" as in the 1938 report of Fisher and Yates [7].

5. For a given value of n, it appears that the relative frequency of each digit differs from 1/10 by significantly variable amounts, i.e., they approach 1/10 at different "rates." For example, if $n=60{,}000$ and we choose as a unit the relative frequency for the 5's (the "closest" to 1/10 at $n=60{,}000$) less 1/10, then the other digits are such that we have for the zeros, -19.4; ones, $+23.0$; twos, -3.6; threes, $+22.4$; fours, $+3.4$; fives, 1.0; sixes, 49.6; sevens, -14.4; eights,

-28.4; nines, -33.6. These results clearly imply that

$$\left| S_n^i/n - 1/10 \right| < \epsilon(i, n),$$

where $\epsilon(i, n)$ can be made arbitrarily small *depending* on the choice of digit and n sufficiently large.

6. Each digit may have its own approximate "period" and for a sufficiently large sample (if not, the sixes show a "nonrandom" character!) perhaps they would exhibit an approach to 1/10 by means of a damped oscillation in *all* cases.

Certainly, the data, in any case, offer strong evidence for, at least, the "simple" normality of "e."

TABLE 12. Frequency table for pairs in 60,000 places of "e"

N	f	$N/60{,}000$
534–544	1	.00890–.00906
545–555	3	.00908–.00925
556–566	3	.00927–.00943
567–577	15	.00945–.00962
578–588	12	.00963–.00980
589–599	15	.00982–.00998
600–610	17	.01000–.01017
611–621	15	.01018–.01035
622–632	11	.01037–.01053
633–643	3	.01055–.01072
644–654	3	.01073–.01090
655–665	1	.01092–.01108
666–676	1	.01110–.01127

From the data for the serial test, we may derive two kinds of information on the frequency of pairs. First, in Table 12, we present a frequency table, in order to conserve space, for the total counts of the number of occurrences of all the 100 possible pairs from 00 to 99 in the 60,000 place sample. (Note: The actual total consecutive pairs across the whole 60,000 place sample may differ by a few counts out of the totals from those we tabulate since the serial test was based on counts in 12 separate consecutive blocks of 5000 digits in which pairs are counted in such a way as to neglect the possibility of a counted pair occurring with the last digit in one block and the first digit in the next block. These effects, however, will not alter the conclusions in any gross fashion.) Let N be the indicated ranges of the total counts of pairs in the 60,000 places and let f be the frequency of occurrences of those distinct pairs which fall in the particular class. We have also indicated under $N/60{,}000$, the values of the relative frequencies S_n^{ij}/n. Thus, noting the first entry, there was 1 pair out of the 100 possible pairs which had total counts in the range 534–544 for their occurrences in the sample.

We did not prepare any particular statistical analysis of this data but one can see by inspection that there is an approximately "normal" distribution around the most frequent (and expected!) range which is 600–610. This data suggests an interesting theoretical question about normal numbers. Can one prove that the total counts in some arbitrarily chosen frequency range of all possible blocks B_k consisting of k digits for some fixed k, are normally distributed (in a statistical sense) about the limit value $1/g^k$ where g is the base of representation in some block of digits X_n chosen somewhere in the infinite sequence of digits in a normal number?

TABLE 13. Normality data for pairs

$n/1000$	S_n^{00}	S_n^{00}/n	S_n^{55}	S_n^{55}/n	S_n^{99}	S_n^{99}/n
5	45	.0090	57	.0114	47	.0094
10	97	.0097	111	.0111	92	.0092
15	138	.0092	164	.0109	143	.0095
20	189	.0094	225	.0112	189	.0094
25	241	.0096	276	.0110	240	.0096
30	296	.0099	321	.0107	287	.0096
35	342	.0098	368	.0105	361	.0103
40	392	.0098	414	.0103	414	.0103
45	442	.0098	472	.0105	448	.0099
50	487	.0097	515	.0103	503	.0101
55	537	.0098	570	.0104	563	.0102
60	591	.0098	624	.0104	618	.0103

Since Table 12 does not show the variation of the relative frequency with the number of places, we have prepared Table 13 which shows the behavior of S_n^{ij}/n as n increases for the blocks 00, 55, and 99 as a sample where n increases by blocks of 5000 digits.

Again, we note the types of approaches to $1/10^2$ from above, below, and the "damped" oscillation for the pair 99. These data give clear empirical support for the normality of "e" for $1/10^2$.

Finally, Figures 1, 2, and 3 suggest a number of other approaches to the search for "prevailing regularities" in a transcendental such as "e," i.e. perform a "spectral analysis" in order to detect any dominant "periods" in the distribution, consider the digit sequence as a "random walk" with such ideas as the probability of long leads for a given digit, the arc sine law (use $p=1/10$ and $q=9/10$), the number of returns to the origin etc. [11, p. 84]. To date, little has been done with these techniques in either empirical or theoretical investigations of the digits in irrationals.

(1) The data produced by D. Wheeler in 1952 were loaned to the author by R. T. Gregory, now at the University of Texas. On a research grant from the University of California at Goleta, Calif. in 1956, the accumulated frequency counts were computed for every 100 places by the IBM Corp. of Los Angeles. The Kendall-Smith tests were performed at the Western Data Processing Center

(see WDPC Rep. no. 1, 1959, p. 44) at the University of California in Los Angeles in 1957 with assistance of Mr. S. Dunin.

(2) Daniel Shanks of the David Taylor Model Basin, Washington, D. C., kindly made a comparison check of our accumulated frequency counts with his 1961 calculation of "e" to 100,265 D. We found a difference of two digits in the 56,435 and 56,436th place of our hand copied sample. These corrections have been made throughout the various frequency studies contained in this paper.

References

1. I. Niven, Irrational numbers, Carus Monograph No. 11, Math. Assoc. of Amer., 1956.

2. R. von Mises, Probability, statistics and truth, Macmillan, New York, 1957.

3. N. Metropolis, G. Reitwiesner, and J. von Neumann, Statistical treatment of values of the first 2000 decimal places of 'e' and calculated on the Eniac, Math. Tables and Other Aids to Computation, 4 (1950) 11–15, 109–111.

4. R. K. Pathria, A statistical analysis of 2500 decimal places of 'e' and '$1/e$,' Proc. Nat. Inst. Sci., India, Part A (4) 27 (1961) 270–282.

5. ———, A statistical study of randomness among the first 10,000 digits of π, Math. Comp., 16 (1962) 188–197.

6. H. Geiringer, On the statistical investigation of transcendental numbers, Studies in mathematics and mechanics, Academic Press, New York (1954) 310–322.

7. R. Fisher and F. Yates, Statistical tables for biological, agricultural, and medical research, Oliver and Boyd, London, 1938.

8. H. Poincaré, La science et l'hypothèse, E. Flammarion, Paris, 1918.

9. J. Franel, À propos des Tables des Logarithmes, Vierteljschr. Naturforsch. Ges. Zürich, 1917, 286–295.

10. W. Schmidt, On normal numbers, Pacific J. Math., 10 (1960) 661–672.

11. W. Feller, An introduction to probability theory and its applications, 2nd ed., Wiley, New York, 1959.

12. R. Stoneham, The reciprocals of integral powers of primes and normal numbers, Proc. Amer. Math. Soc., 15 (1964) 200–208.

ON SECOND-ORDER RECURRENCES

OSWALD WYLER, University of New Mexico

1. Introduction. Various papers and problems related to Fibonacci numbers and to some more general second-order recurrences of integers have appeared in this MONTHLY in recent years. Thus a study of the general second-order recurrence of integers,

$$A_{n+2} = PA_{n+1} - QA_n \qquad (n = 0, 1, 2, \cdots), \tag{1}$$

with arbitrary "initial values" A_0 and A_1, should be of interest.

A particular case of (1) is the sequence of Lucas numbers, given by

$$U_{n+2} = PU_{n+1} - QU_n, \qquad U_0 = 0, \qquad U_1 = 1. \tag{2}$$

If we put $P=1$, $Q=-1$ in (2), the well-known Fibonacci numbers result.

In the present paper, we consider P and Q in (1) and (2) as fixed. We call a sequence (A_n) satisfying (1) a *recurrence*. In the first part of the paper, we define addition and multiplication of recurrences in such a way that recurrences form a commutative ring, with the Lucas recurrence (U_n) as unit element. This allows us to formulate some general identities, which may be new, involving two recurrences and their product. Many of the known identities for Lucas numbers or more general recurrences follow immediately from our identities. Others may be obtained easily by the same method.

If we reduce (1) and (2) modulo a positive integer M, we obtain periodic sequences. We denote the primitive period of (U_n) modulo M by $K(M)$ or just by K. If Q and M are relatively prime, then there is a positive integer $k=k(M)$ such that $U_n \equiv 0 \pmod{M}$ if and only if $n \equiv 0 \pmod{k}$, and $K(M)$ is a multiple of $k(M)$. We call $K(M)$ the *period* and $k(M)$ the *rank* of (U_n) modulo M. The second part of our paper is concerned with these numbers and their relations.

In the last two sections, we apply our results to Fibonacci numbers, and we add some remarks concerning topics omitted from this paper.

2. Preliminaries and notations. It will be convenient to consider (1) and (2) as relations in a commutative ring $\mathcal{R}$ with unit element. We denote the unit element of $\mathcal{R}$ by I, and we replace the last part of (2) by $U_1=I$. In this way, we can distinguish between an integer u and the element uI of $\mathcal{R}$. There is a nonnegative integer M, called the *characteristic* of $\mathcal{R}$, such that $uI=0$, for an integer u, if and only if $u \equiv 0 \pmod{M}$. Thus equations in $\mathcal{R}$ replace congruences of integers modulo M.

For convenience, we shall denote integers by lower case letters and elements of $\mathcal{R}$ by capital letters, except that H, K and M denote integers.

Lehmer [3] generalized (2) by assuming only that P^2 and Q (instead of P and Q) are integers. Since all results of this paper remain valid for Lehmer's generalization of Lucas numbers, we adopt the assumption that there are two relatively prime integers q and r such that $Q=qI$ and $P^2=rI$.

We form an extension $\mathcal{R}[\omega]$ of $\mathcal{R}$ by adjoining to $\mathcal{R}$ an element ω such that $\omega^2-P\omega+Q=0$, but $A+\omega B\neq 0$ if $B\neq 0$. This can be done e.g. by considering the quotient ring of the polynomial ring $\mathcal{R}[X]$ modulo the polynomial X^2-PX+Q, and letting ω be the equivalence class of the polynomial X in this quotient ring.

Every element of the ring $\mathcal{R}[\omega]$ has a unique representation $A+\omega B$, with A and B in $\mathcal{R}$. We obtain a one-to-one correspondence between recurrences and $\mathcal{R}[\omega]$ by associating with any recurrence (A_n) the binomial $A_1-\omega A_0$ in $\mathcal{R}[\omega]$. In this correspondence, the Lucas recurrence (U_n) is associated with the unit element I of $\mathcal{R}[\omega]$.

We add and multiply recurrences by adding and multiplying the corresponding binomials in $\mathcal{R}[\omega]$. With these operations, recurrences form a commutative ring (in fact, an algebra over $\mathcal{R}$), with the Lucas recurrence (U_n) as unit element.

In this ring, we have $(A_n)+(B_n)=(S_n)$ and $(A_n)(B_n)=(C_n)$, where $S_n=A_n+B_n$ for all n, and $C_1-\omega C_0=(A_1-\omega A_0)(B_1-\omega B_0)$.

It will be convenient to put $\bar{\omega}=P-\omega$, so that $\omega+\bar{\omega}=P$ and $\omega\bar{\omega}=Q$. We note that $(\omega-\bar{\omega})^2=dI=P^2-4Q$, where $d=r-4q$. We call the integer d the *discriminant* of the Lucas recurrence (U_n).

3. Identities. Let (A_n) be a recurrence. We call

$$N(A) = (A_1)^2 - A_0A_2 = (A_1 - \omega A_0)(A_1 - \bar{\omega}A_0) \tag{3}$$

the *norm* of (A_n). In particular, $N(U)=I$ for the unit recurrence (U_n). It is easily proved by induction that

$$(A_{n+1})^2 - A_nA_{n+2} = Q^nN(A) \tag{4}$$

for all n. We note also that $N(C)=N(A)N(B)$ for $(C_n)=(A_n)(B_n)$.

It is easily proved by induction that

$$A_{n+1} - \omega A_n = \bar{\omega}^n(A_1 - \omega A_0) \tag{5}$$

for all n. Similarly, $A_{n+1}-\bar{\omega}A_n=\omega^n(A_1-\bar{\omega}A_0)$.

Let (A_n) and (B_n) be recurrences, and let $(C_n)=(A_n)(B_n)$. Then

$$\begin{aligned} C_{m+n+1} - \omega C_{m+n} &= \bar{\omega}^{m+n}(A_1 - \omega A_0)(B_1 - \omega B_0) \\ &= (A_{m+1} - \omega A_m)(B_{n+1} - \omega B_n) \end{aligned}$$

by (5) and the definition of (C_n), and it follows that

$$C_{m+n} = A_mB_{n+1} + A_{m+1}B_n - PA_mB_n; \tag{6}$$

$$C_{m+n+1} = A_{m+1}B_{n+1} - QA_mB_n. \tag{7}$$

It is easily verified that $\omega^m = U_{m+1}-\bar{\omega}U_m$ for all m. Thus we have:

$$\begin{aligned} Q^m(A_{n+1} - \omega A_n) &= (\omega\bar{\omega})^m\bar{\omega}^n(A_1 - \omega A_0) \\ &= (U_{m+1} - \bar{\omega}U_m)(A_{m+n+1} - \omega A_{m+n}), \end{aligned}$$

and it follows that

$$Q^mA_n = U_{m+1}A_{m+n} - U_mA_{m+n+1}. \tag{8}$$

Among the many identities which can be deduced from (4), (6) and (8), we mention only a few, without proofs.

$$A_{m+n} = A_mU_{n+1} + A_{m+1}U_n - PA_mU_n, \tag{9}$$

$$A_{m+h}B_n - A_mB_{n+h} = U_h(A_{m+1}B_n - A_mB_{n+1}), \tag{10}$$

$$A_{m+h}A_{n+h} - A_hA_{m+n+h} = U_mU_nQ^hN(A), \tag{11}$$

$$A_{m+h}B_{n+h} - Q^hA_mB_n = U_hC_{m+n+h}, \tag{12}$$

where again $(C_n)=(A_n)(B_n)$.

Other identities which follow from the ones stated above can be found e.g.

in [2]. By specializing, identities for the recurrence (U_n) of Lucas numbers, and the recurrence (V_n) of Lucas numbers of the second kind, can be obtained. With regard to the recurrence (V_n), the following facts are needed. We have $V_n = 2U_{n+1} - PU_n$ for all n. The element of $\mathfrak{R}[\omega]$ corresponding to (V_n) is $\bar{\omega} - \omega$, and $(V_n)^2 = d(U_n) = (P^2 - 4Q)(U_n)$ for $d = r - 4q$.

4. Rank and period. Preliminaries. We prove two theorems, and we state without proof some related results which we shall not need and which can also be found, with proofs, elsewhere (see e.g. [**3**]).

THEOREM 1. *If Q is not a zero divisor in $\mathfrak{R}$, then there is an integer k such that $U_n = 0$ if and only if n is a multiple of k.*

REMARK. $k = k(M)$ if M is the characteristic of $\mathfrak{R}$.

Proof. As $N(U) = I$, it follows from (4) that U_{m+1} is not a zero divisor if $U_m = 0$. By (9), $U_{m+n} = U_{m+1}U_n$ if $U_n = 0$, and thus $U_n = 0$ if and only if $U_{m+n} = 0$, and the Theorem follows.

THEOREM 2. *If $U_m = 0$, $U_{m+1} = I$, then $A_{m+n} = A_n$ for all recurrences (A_n) and all n. Conversely, if Q and $N(A)$ are not zero divisors, and if $A_{m+n} = A_n$, $A_{m+n+1} = A_{n+1}$ for some n, then $U_m = 0$, $U_{m+1} = I$.*

Proof. The first part follows directly from (9). If $A_{m+n} = A_n$ and $A_{m+n+1} = A_{n+1}$, then $U = U_m$, $V = U_{m+1}$ is a solution of the system

$$(A_{n+1} - PA_n)U + A_nV = A_n,$$
$$(A_{n+2} - PA_{n+1})U + A_{n+1}V = A_{n+1},$$

also by (9). The determinant of this system is $Q^nN(A)$ by (4) and hence not a zero divisor, so that $U = 0$, $V = I$ is the only solution.

COROLLARY. *If Q is not a zero divisor and (U_n) is periodic, with primitive period K, then any recurrence (A_n) has period K, and K is the primitive period of (A_n) if $N(A)$ is not a zero divisor. Moreover, K is a multiple of k.*

This follows immediately from Theorems 1 and 2.

If $\mathfrak{R}$ has positive characteristic M, then the number of elements of $\mathfrak{R}$ of the form uI or uP, u an integer, is finite. As all U_n are of one of these forms, the sequence (U_n) must be periodic. $Q = qI$ is not a zero divisor in $\mathfrak{R}$ if and only if q and M are relatively prime. If this is the case, it follows from the Corollary that $k = k(M)$ is positive. If q and M are not relatively prime, it can be shown that $k(M) = 0$.

Reverting for the moment to the Lucas recurrence $U_{n+2} = \sqrt{r}\,U_{n+1} - qU_n$ in the ring of integers, with $\sqrt{r}$ adjoined, we make the following remarks. If M is the product of relatively prime factors M_i, then $U_{m+n} \equiv U_n \pmod{M}$ if and only if $U_{m+n} \equiv U_n \pmod{M_i}$ for all M_i. It follows that $k(M)$ and $K(M)$ are the least common multiples of the numbers $k(M_i)$ and $K(M_i)$ respectively. Thus it is sufficient to consider the case $M = p^\alpha$ where p is a prime number.

If q is odd, then $k(2)=3$ if r is odd, $k(2)=2$ or $k(2)=4$ if r is even. $k(4)=2K(2)$ or $k(4)=k(2)$. If μ is the highest exponent such that $k(2^\mu)=k(4)$, then $k(2^\alpha)=2^{\alpha-\mu}k(4)$ for $\alpha\geqq\mu$.

For an odd prime number p and q not divisible by p, we have $k(p)\mid 2p$ if $dr=r(r-4q)$ is a multiple of p, and $k(p)\mid p-(dr/p)$ otherwise. If μ is the highest exponent such that $k(p^\mu)=k(p)$, then $k(p^\alpha)=p^{\alpha-\mu}k(p)$ for $\alpha\geqq\mu$.

5. Rank and period. Theorems. We assume in this section that $\mathfrak{R}$ has positive characteristic M, and that $Q=qI$ is not a zero divisor in $\mathfrak{R}$, i.e., that q and M are relatively prime. In this case, $k=k(M)$ and $K=K(M)$ are defined and positive, and there is a positive integer h such that $Q^m=I$, i.e. $q^m\equiv 1 \pmod{M}$, if and only if m is a multiple of h. We denote by H the least common multiple of h and k.

LEMMA. *If $U_m=0$, then $U_{mi+n}=(U_{m+1})^iU_n$ and $(U_{m+1})^2=Q^m$.*

This follows from (4) and repeated application of (9). We shall use this lemma and the results of §4 without further reference.

THEOREM 3. *Either $K=H$ or $K=2H$.*

Proof. $(U_{K+1})^2=Q^K=I$, so that K is a multiple of h, hence of H. On the other hand, $U_{2H+1}=(U_{H+1})^2=Q^H=I$, so that $2H$ is a multiple of K. Thus $K=H$ or $K=2H$.

We assume now that M is a power p^α of an odd prime number p. In this case, any element of $\mathfrak{R}$ of the form uI or uP, u an integer, is either regular (i.e. has a reciprocal in $\mathfrak{R}$) or nilpotent, and $2I$ is regular. Any element $aU_{n+1}-bPU_n$ or aU_n is of the form uI or uP, hence regular or nilpotent. The nilpotent elements of $\mathfrak{R}$ form an ideal of $\mathfrak{R}$.

THEOREM 4. *Let $h=2^ah'$ and $k=2^bk'$, where h' and k' are odd integers. If $a\neq b$, then $K=2H$. If $a=b>0$, then $K=H$.*

Proof. If $a>b$, then $H=ki$ with i even, and $U_{H+1}=(U_{k+1})^i=(Q^k)^{i/2}=Q^{H/2}$. But $H/2$ is not a multiple of h in this case, so that $U_{H+1}\neq I$ and hence $K\neq H$, i.e. $K=2H$.

If $b>0$, let $U_{k/2}=U$, $U_{(k/2)+1}=V$. Then $U_k=U(2V-PU)=0$, $U_{k+1}=V^2-QU^2$, and $U\neq 0$. If also $2V-PU\neq 0$, then U and $2V-PU$ are nilpotent according to the remarks preceding the Theorem. Since $2I$ is regular, it follows that V is nilpotent. But then U_{k+1} is nilpotent, contradicting $(U_{k+1})^2=Q^k$. Thus $PU=2V$. Now $V^2-PUV+U^2=Q^{k/2}$ by (4), and thus $Q^{k/2}=QU^2-V^2=-U_{k+1}$.

Let now $H=ki$. If $b\geqq a$, then i is odd, and $U_{H+1}=(U_{k+1})^i=(-Q^{k/2})^i=-Q^{H/2}$. If $b>a$, then $H/2$ is a multiple of h, and hence $U_{H+1}=-I$. Thus $K=2H$ in this case. If $a=b>0$, then $H/2$ is an odd multiple of $h/2$. The multiplicative group of elements uI of $\mathfrak{R}$, u and M relatively prime, is cyclic for $M=p^\alpha$ (see e.g. [**4**],

Theorem 65) and thus has only one element of order 2, namely $-I$. It follows that $Q^{h/2}=-I$ and hence also $Q^{H/2}=-I$. But then $U_{H+1}=I$, and $K=H$.

6. Applications to Fibonacci numbers. For Fibonacci numbers, $q=-1$, and hence $h=2$ for $M>2$. We have $k(2)=K(2)=3$, $k(4)=K(4)=6$, and $k(2^\alpha)=6\cdot 2^{\alpha-3}$, $K(2^\alpha)=2k(2^\alpha)$, for $\alpha\geqq 3$.

For $M=p^\alpha$, where p is an odd prime number, we have:

THEOREM 5. *If k is odd, then $K=4k$. If $k\equiv 2$ (mod 4), then $K=k$. If $k\equiv 0$ (mod 4), then $K=2k$.*

This follows immediately from Theorem 4.

Conversely, it follows from Theorem 5 that $K=k$ if $K\equiv\pm 2$ (mod 8), $K=4k$ if $K\equiv 4$ (mod 8), and $K=2k$ if $K\equiv 0$ (mod 8). Thus k and K determine each other for $M=p^\alpha$. $K(p)$ has been tabulated for $p<2000$ in [5], and this table furnishes also $k(p)$ by the remark just made.

If $k(p^2)\neq k(p)$, then $k(p^\alpha)=p^{\alpha-1}k(p)$ and $K(p^\alpha)=p^{\alpha-1}K(p)$ for all exponents $\alpha>1$. It is not known at present whether there is an odd prime number p such that $k(p^2)=k(p)$.

We mention without proof the following results. If p is an odd prime number, then $k(p)\mid p-(5/p)$. If $p\equiv 3$ or $p\equiv 7$ (mod 20), then $k\equiv 0$ (mod 4) and $K=2k$. If $p\equiv 11$ or $p\equiv 19$ (mod 20), then $k\equiv 2$ (mod 4) and $K\equiv k$. If $p\equiv 3$ or $p\equiv 7$ (mod 20) then k is odd and $K=4k$. Finally, $k(5)=5$ and $K(5)=20$.

For $p\equiv 1$ or $p\equiv 9$ (mod 20), all three cases of Theorem 5 may occur, as is shown by $p=29$, 41, 61, with $k=14$, 20, 15, and $K=14$, 40, 60. Ward [7] has given criteria for determining whether $k(p)$ is even or odd in this case, but there are at present no criteria to determine whether $k\equiv 0$ or $k\equiv 2$ (mod 4) if k is even.

7. Concluding remarks. We have not taken up in this paper the question whether a given recurrence (A_n) has zeros modulo a given integer M or not. For the case $M=p$, a prime number, Hall [1] has given a necessary and sufficient criterion for the existence of such zeros, and Ward [6] has shown that (A_n) has zeros modulo p for an infinite number of primes p.

We have not discussed the primitive period of (A_n) if $N(A)$ or Q is a zero divisor. For the case $M=p$, we remark the following: Q is a zero divisor only if $Q=0$, and then (1) leads to $A_{n+1}=P^nA_1$. If $N(A)=0$ and A_0 is regular, then $A_1=UA_0$ where $U^2-PU+Q=0$, and it follows that $A_n=A_0U^n$ for all n. If $Q\neq 0$, then U is regular, and the period of (A_n) is the order of U in the group of regular elements of $\mathfrak{R}$.

Theorem 4, our main result, does not always settle the question whether $K=H$ or $K=2H$. If h and k are both odd, or if $M=2^\alpha$, both possibilities may occur. Consider e.g. the Lucas recurrence $U_{n+2}=3U_{n+1}+2U_n$. For $p=11$, we have $h=5$, $k=3$, and $K=30=2H$. For $p=139$, we have $h=69$, $k=5$, and $K=345=H$. For the Fibonacci series, we have $K(2^\alpha)=2H(2^\alpha)$ for $\alpha\geqq 3$, and for the recurrence $U_{n+2}=7U_{n+1}-5U_n$, we have $K(2^\alpha)=H(2^\alpha)$ for $\alpha\geqq 2$.

References

1. Marshall Hall, Divisors of second order sequences, Bull. Amer. Math. Soc., 43 (1937) 78–80.
2. A. F. Horadam, A generalized Fibonacci sequence, this MONTHLY, 68 (1961) 455–459.
3. D. H. Lehmer, An extended theory of Lucas functions, Ann. Math., (2) 31 (1930) 419–448.
4. Trygve Nagell, Introduction to Number Theory, Stockholm and New York, 1951.
5. D. D. Wall, Fibonacci series modulo m, this MONTHLY, 67 (1960) 525–532.
6. Morgan Ward, Prime divisors of second order recurrences, Duke Math. J., 21 (1954) 607–614.
7. ——, The prime divisors of Fibonacci numbers, Pacific J. Math., 11 (1961) 379–386.

O-SEQUENCES AND O-NETS

L. C. ROBERTSON, University of California, Los Angeles AND
S. P. FRANKLIN, University of Florida

Levine [4] introduces the notion of an O-sequence and proves several theorems about their convergence, his main results being that O-sequences always converge in T_1 spaces and are eventually constant in Hausdorff spaces. In Section 1 we generalize his results, in so far as possible, to nets. The fact that every net has a universal subnet [3] provides a characterization of compactness and a brief proof of the Tychonoff Theorem. Proofs based on the same basic idea, expressed in terms of ultrafilters or ultraphalanxes are known [1], [5]. The authors wish to thank Professor R. H. Sorgenfrey for pointing them in the direction of universal nets.

Section 2 is devoted to spaces in which all O-sequences converge and includes two characterizations of such spaces, together with a corollary which helps clarify the role of T_1 as a sufficient condition for convergence. Also, spaces in which O-sequences are eventually constant are characterized and the place of such spaces with respect to various separation axioms (T_0, T_1, T_2) is exposed.

1. A net $\phi: D \to X$ from a directed set D into a topological space X is called an *O-net* iff whenever ϕ is frequently in an open subset U of X, ϕ is eventually in U. The following properties follow immediately from the definition.

(a) *ϕ is an O-net iff whenever ϕ is frequently in some closed subset F of X, ϕ is eventually in F.*

(b) *An O-net converges to each of its cluster points.*

(c) *ϕ is an O-net iff every continuous image of ϕ is an O-net, i.e., $f \circ \phi$ is an O-net for each continuous f.*

(d) *Every subnet of an O-net is an O-net.*

Since the set of cluster points of any net is closed, from (a) and (b) we have

(e) *An O-net either converges to each point of a residual* (*set of elements beyond some element*) *subset of its range or else fails to converge to any point of some such residual subset.*

A net $\phi: D \to X$ is called *universal* iff for each subset A of X, ϕ is eventually in A or eventually in $X \sim A$, the complement of A.

Every universal net is an O-net. The identity map from the natural numbers onto the natural numbers provided with the cofinite topology is a counterexample to the converse. (For a more interesting example let X be the set of ordinals less than the first uncountable ordinal and take as a base for the topology of X the sets $B \sim F$ where F is finite and either $B = X$ or $B = \{\alpha \mid \alpha < \alpha_0\}$ for some $\alpha_0 \in X$. The identity map $i: X \to X$ is an O-net which is not a universal net. Note that i is an example of a nonconvergent O-net in a T_1 space. However, Theorem 1 will show that any noncompact T_1 space must provide such an example.)

In marked contrast to Levine's result on O-sequences (see Section 2), there exist, in every infinite topological space, O-nets which fail to be eventually constant. These are obtained by choosing a universal subnet of any sequence of distinct points.

THEOREM 1. *If X is a topological space, the following conditions are equivalent*:
(i) *X is compact* (*not necessarily Hausdorff*).
(ii) *Every O-net $\phi: D \to X$ converges.*
(iii) *Every universal net converges.*

Proof. (i)$\Rightarrow$(ii) Since X is compact an O-net has a cluster point and, by (b), converges to it. (ii)$\Rightarrow$(iii) Every universal net is an O-net. (iii)$\Rightarrow$(i) If ϕ is any net in X, every universal subnet of ϕ must converge. Each such limit point is a cluster point of ϕ.

THEOREM 2 (Tychonoff). *The topological product of any family of compact spaces is compact.*

Proof. An O-net in the product space must converge since each coordinate projection of it is an O-net by (c) and hence converges.

2. An *O-sequence* is an O-net which is a sequence. In this section we will consider spaces in which O-sequences must converge. It is clear that every closed subspace of such a space has this property. The proof of Theorem 2 shows such spaces to be productive. This property is not preserved under continuous images, intersections or closure, nor is a bijective map of such a space into a Hausdorff space necessarily a homeomorphism. Also nothing can be said of subspaces of such a space since an arbitrary space has a compactification.

A sequence $\{S_n\}$ of subsets of a topological space X will be called a *J-sequence* in X iff (i) each S_j is the closure of a point $x_j \in X$, $\bar{x}_j = S_j$ and (ii) $\{S_n\}$ is strictly decreasing.

THEOREM 3. *O-sequences must converge in a space X iff every J-sequence in X has a nonempty intersection.*

LEMMA 1. *Suppose that* $\{x_n\} \subseteq X$ *is such that*

(i) $\{x_n\}$ *has no cluster point, and*

(ii) *for every subsequence* $\{y_n\}$ *of* $\{x_n\}$, $\{\bar{y}_n\}$ *fails to be a J-sequence. Then there exists a subsequence* $\{z_n\}$ *of* $\{x_n\}$ *and a sequence* $\{U_n\}$ *of open sets with* z_j *the first element of* $\{z_n\}$ *contained in* U_j *for each* j.

Proof. Let $P = \{x_j \in \{x_n\} \mid \{x_n\}$ is eventually outside of $\bar{x}_j\}$. If P were finite, then a subsequence $\{y_n\}$ of $\{x_n\}$ could be chosen with each $y_{j+1} \in \bar{y}_j$. Then it follows from (ii) that $\{\bar{y}_n\}$ is eventually constant. But $\bar{y}_m = \bar{y}_{m+j}$ for all j implies that $\{y_n\}$ converges to y_m, contradicting (i). Hence with P infinite define $\{z_n\} \subseteq P$ and $\{U_n\}$ by choosing $U_1 = X$, z_1 an arbitrary element of P,

$$U_{j+1} = X - \bigcup_{i=i}^{j} \bar{z}_k \quad \text{and} \quad z_{j+1} \in U_{j+1} \cap P$$

so that $\{z_n\}$ is a subsequence of $\{x_n\}$.

LEMMA 2. *If a sequence* $\{x_n\}$ *has no cluster points, then there exists a subsequence* $\{y_n\}$ *of* $\{x_n\}$ *and a sequence* $\{U_n\}$ *of open sets such that for each* j, y_j *is the last element of* $\{y_n\}$ *belonging to* U_j.

Proof. This is a direct application of the definition of cluster point.

Proof of the theorem. If $\{\bar{x}_n\}$ is any J sequence, then $\bigcap_{n=1}^{\infty} \bar{x}_n$ is the set of cluster points of the sequence $\{x_n\}$ which is by (1a) an O-sequence. Hence if O-sequences converge, J-sequences have nonempty intersections.

For the converse it suffices to show that sequences without cluster points are not O-sequences. Since the hypotheses of the lemmas are satisfied, by successive applications we may extract a subsequence $\{y_n\}$ and open sets isolating each point of $\{y_n\}$ from the others. Then $\{y_n\}$ is frequently but not eventually in the union of the even numbered sets.

COROLLARY. *If all points of* X *have compact closure, then O-sequences must converge.*

The above condition is not necessary as can be seen by adding to the real line an ideal point whose closure is the whole space.

Davis has shown [2] that topological spaces can be characterized by means of filters in $X \times X$. The next theorem provides such a characterization for spaces in which O-sequences converge.

THEOREM 4. *Every O-sequence in a topological space* X *must converge iff there exists a filter* $\mathfrak{F}$ *in* $X \times X$ *satisfying*

(i) $\Delta \subseteq \bigcap \mathfrak{F}$ (Δ *is the diagonal of* $X \times X$),

(ii) *for every* $x \in X$, $\{U(x) \mid U \in \mathfrak{F}\}$ *is the filter of neighborhoods of* x,

(iii) *if* $f \subseteq D^{-1}$ *is a function with domain* X *then* $\bigcap_{k=1}^{\infty} D^{-1} \circ f^{(k)}$ *is a relation with domain* X, *where* $D = \bigcap \mathfrak{F}$ *and* $f^{(k)}$ *is* $f \circ f \circ \cdots \circ f$, k *times.*

If an f-orbit is defined to be a sequence $\{f^{(n)}(x_0)\}$ *generated by a function*

$f \subseteq D^{-1}$ with domain X and by $x_0 \in X$, then (iii) *is equivalent to*

(iii′) *every f-orbit must converge.*

Proof. There exists a filter $\mathfrak{F}$ in $X \times X$ satisfying (i) and (ii) [2]. Suppose O-sequences in X converge. D^{-1} is the relation which maps each point of X into its closure, $x \rightarrow \bar{x}$. Also

$$\left[\bigcap_{k=1}^{\infty} D^{-1} \circ f^{(k)}\right](x) = \bigcap_{k=1}^{\infty} [(D^{-1} \circ f^{(k)})(x)].$$

But the sequence $\{(D^{-1} \circ f^{(k)})(x)\}$, $k=1, 2, \cdots$, is either eventually constant or else contains a J-subsequence. In either case

$$\left[\bigcap_{k=1}^{\infty} D^{-1} \circ f^{(k)}\right](x) \neq \phi.$$

On the other hand, this condition implies that any J-sequence $\{\bar{x}_n\}$ has a nonempty intersection. Let $f(x_j) = x_{j+1}$ for all $x_j \in \{x_n\}$ and $f(x) = x$ otherwise. Then $f \subseteq D^{-1}$ and $\{\bar{x}_n\} = \{D^{-1} \circ f^{(n)}(x_1)\}$.

The equivalence of (iii) and (iii′) follows from the fact that each f-orbit is an O-sequence, and each J-sequence is generated by an f-orbit.

A topological space is called an R_0 *space* iff $y \in \bar{x}$ is equivalent to $x \in \bar{y}$ [2]. Regular spaces (and therefore pseudometric spaces) and T_1 spaces are examples of R_0 spaces. Also any topological group (not necessarily T_0) is an R_0 space. (If e is the identity, $\bar{e}$ is a subgroup, $\bar{x} = \bar{e}x$, and $x \in \bar{e}$ implies $e \in \bar{x}$.)

THEOREM 5. *An O-sequence in an R_0 space converges to all but a finite number of the points of its range.*

Proof. Since there can be no J-sequences in an R_0 space we know that O-sequences converge. Deleting the set of cluster points of the sequence leaves an R_0 space whose intersection with the sequence must be finite. Although R_0 is sufficient, it is not necessary in order that O-sequences converge to almost all of their range.

THEOREM 6. *All O-sequences in X have finite range iff*

(i) *for every countably infinite subset A of X, there is an open set U in X such that both $A \cap U$ and $A \sim U$ are infinite.*

Furthermore, condition (i) *is equivalent to the following two conditions*:

(ii) *O-sequences in X converge, and*

(iii) *condition* (i) *holds for all compact countably infinite subsets of X.*

Proof. The first equivalence is clear from the definition of O-sequence, taking A to be the range. If O-sequences have finite range, they must have cluster points and (ii) follows. Of course (i) implies (iii). Given (ii) and (iii) let $\{x_n\}$ be an O-sequence converging to x. Since $\{x_n\} \cup \{x\}$ is compact, the range must be finite in order that (iii) not contradict the definition of O-sequence.

THEOREM 7. *O-sequences in X are eventually constant iff X is a T_0 space and condition* (i) *is satisfied.*

Proof. If O-sequences are eventually constant, (i) follows from Theorem 6. If x and y are distinct points such that $\bar{x}=\bar{y}$, then $x, y, x, y, \cdots$ is a non-eventually constant O-sequence. Hence no pair of distinct points of X can have the same closure, i.e. X is T_0. Conversely, an O-sequence with finite range (by Theorem 6) in a T_0 space cannot hit each of two distinct points infinitely many times.

An equivalent (and perhaps more useful) formulation of Theorem 7 can of course be obtained by requiring (ii) and (iii) in place of (i).

COROLLARY 1. *Let X be a T_1 space. Then O-sequences are eventually constant if and only if condition* (iii) *is satisfied.*

Proof. If X is T_1, then it is also T_0 and (ii) is satisfied.

COROLLARY 2. *Let X be an R_0 space. If O-sequences in X are eventually constant, then X must be a T_1 space.*

Proof. Eventual constancy implies that X is T_0, and Davis [2] shows that T_0 is equivalent to T_1 for R_0 spaces.

COROLLARY 3. *Let X be a homogeneous space (i.e., there exists a transitive group of homeomorphisms from X onto X). If O-sequences are eventually constant, then X must be a T_1 space.*

Proof. There are no J-sequences, and X is a T_0 space, so that some point must be a closed set. Thus X is a T_1 space, by homogeneity.

Levine [4] has shown that O-sequences are eventually constant in Hausdorff spaces. (This result can be deduced immediately from Theorem 5 by using uniqueness of limit points.) The above results therefore show that eventual constancy of O-sequences is a condition intermediate between T_0 and T_2, and, for R_0 spaces or homogeneous spaces, a condition between T_1 and T_2.

References

1. N. Bourbaki, Éléments de Mathématiques, Première Partie, Livre III, Chapitre I, Hermann, Paris, 1960, p. 102.

2. A. S. Davis, Indexed systems of neighborhoods for general topological spaces, this MONTHLY, 68 (1961) 886–894.

3. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955, p. 81.

4. N. Levine, A note on convergence in topological spaces, this MONTHLY, 67 (1960) 667–8.

5. J. W. Tukey, Convergence in Topology, Princeton, 1940, p. 75.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146.

NORMABILITY OF WEAKNORMED LINEAR SPACES

MICHAEL MANOVE, Swarthmore College and Harvard University

1. Introduction. The concept of a weaknormed real linear space is the generalization of the concept of a normed linear space which results when the homogeneity property of the norm, $\|\alpha x\| = |\alpha| \cdot \|x\|$ for any scalar α, is weakened by restricting α to the nonnegative real numbers. Some of the general properties of such spaces have been discussed by Leichtweiss [2]. Quasimetric spaces, which are related to weaknormed linear spaces in the same manner as metric spaces are related to normed linear spaces, are discussed by Ribeiro [3] and by Balanzat [1].

This note contains several theorems about topologies on weaknormed real linear spaces. We show that a weaknorm, in much the same manner as a norm, induces a topology on a real linear space, though the space with this topology is not necessarily a topological linear space, nor even a topological group. A real linear space with a topology is said to be weaknormable if a weaknorm can be assigned to the space such that the topology induced by the weaknorm is identical with the given topology. We shall state necessary and sufficient conditions for weaknormability of a real linear space with a topology, and the conditions necessary and sufficient for normability of a space with a topology induced by a weaknorm. Two examples are given of a weaknormed linear space with a topology that is not normable.

Throughout this paper, the function s denotes vector addition; that is, $s(x, y) = x + y$, where x and y are members of a vector space. The function m denotes multiplication of a vector by a nonnegative scalar; i.e., $m(\alpha, x) = \alpha x$, where α is a nonnegative scalar and x is a vector. R' denotes the set of nonnegative reals.

2. Quasi-topological linear spaces and weaknormability. A *quasi-topological linear space* (X, τ) is a linear space X with a topology τ such that the function $s: X \times X \to X$ is continuous in the usual product topology, and the function $m: R' \times X \to X$ is also continuous. A set S is said to be *semi-balanced* if $E \cdot S = S$, where E is the real interval $[0, 1]$. The standard theorem characterizing all topological linear spaces in terms of local properties ([4], Theorem 3.3-F, for example) holds for quasi-topological linear spaces as well, with "balanced" replaced by "semi-balanced." If $x_0 \in X$, $\alpha > 0$, then the functions $x \to x_0 + x$, $x \to \alpha x$ are homeomorphisms.

A *weaknorm* is a real valued functional $\|\cdot\|$ defined on a linear space, where $\|\cdot\|$ has the following properties:

(1) $\|x\| \geqq 0$, (2) $\|x\|+\|y\| \geqq \|x+y\|$,
(3) $\alpha\|x\| = \|\alpha x\|$ for $\alpha \geqq 0$, (4) $\|x\| = 0$ if and only if $x=0$.

A linear space X with its weaknorm $\|\cdot\|$ is called a *weaknormed linear space* and denoted by $(X, \|\cdot\|)$. By the remarks in the preceding paragraph, a weaknormed linear space is a quasi-topological linear space with the sets of the form $\{x: \|x\| < r\}$, $r > 0$, as an open neighborhood base at 0. A linear space with a topology is said to be *weaknormable* if a weaknorm can be assigned to the space such that the topology induced by the weaknorm is identical with the given topology.

If X is a linear space and $0 \in K \subset X$ and K is absorbing, the well-known *Minkowski functional* of K is given by $p(x) = \inf \{\alpha: \alpha > 0, x \in \alpha K\}$. For properties of p, see e.g. [**4**], pp. 134–35. A set S in a quasi-topological linear space is *bounded* if for every neighborhood U of 0, there exists $\gamma > 0$ such that $S \subset \gamma U$.

THEOREM 1. *A quasi-topological linear space X is weaknormable if and only if X is a T_1-space and there exists in X a convex and bounded open neighborhood K of* 0.

Proof. We prove sufficiency; the proof of necessity is standard. Let p be the Minkowski functional of K. It is easy to see that, as in the standard situation, p is a weaknorm for X. We prove that p induces the given topology as follows. Suppose $x \in K$. Then the pair $(1, x) \in m^{-1}(K)$, which is open since K is. Choose neighborhoods A of 1 and U of x with $A \cdot U \subset K$. A contains some $\alpha > 1$ for which $\alpha x \in K$; thus $p(x) \leqq \alpha^{-1} < 1$. This shows that $K = \{x: p(x) < 1\}$, so that $\{\gamma K: \gamma > 0\}$ is a base for the topology induced by p, which, by the boundedness of K, is thus identical with the given topology.

3. Normability of weaknormed linear spaces. A weaknormed linear space is *normable* if there exists a norm for the space such that the topology induced by the norm is identical with the topology induced by the weaknorm. The major theorem of this section states necessary and sufficient conditions for normability of a weaknormed linear space.

LEMMA. *Let $(X, \|\cdot\|)$ be a weaknormed linear space with the following property: for any sequence $\{x_n\}$ such that $\|x_n\| \to 0$, we also have $\|-x_n\| \to 0$. Then there exists $M > 1$ such that for all $x \in X$, we have $\|x\| \leqq M\|-x\|$.*

Proof. Suppose that no such M exists. Then for each $n > 0$ there exists $x_n \in X$ with $\|x_n\| > n\|-x_n\|$. Let $y_n = (n\|-x_n\|)^{-1} \cdot x_n$. Then $\|-y_n\| = 1/n$ and $\|y_n\| > 1$, contradicting the hypothesis.

THEOREM 2. *A weaknormed linear space $(X, \|\cdot\|)$ is normable if and only if, for any sequence $\{x_n\}$ in X, $\|x_n\| \to 0$ implies $\|-x_n\| \to 0$.*

Proof. Since multiplication by -1 is a homeomorphism in a normed linear space, the condition is necessary. To prove sufficiency we first note that the

function $p(x) = \max\{\|x\|, \|-x\|\}$ is a norm on X. Next, given $r>0$, we see that $\|x\| < r/M$ implies $p(x) < r$, where M is given by the lemma; so that the topology induced by $\|\cdot\|$ is the same as the topology induced by p.

Note that for any weaknormed linear space $(X, \|\cdot\|)$, the functional p, as defined above, is a norm on X, and the p-topology is stronger than the $\|\cdot\|$-topology. Later, we shall give some examples in which it is properly stronger.

THEOREM 3. *Let $(X, \|\cdot\|)$ be a weaknormed linear space which is not normable. If S is any sphere in X, the set $-S = \{x: -x \in S\}$ is unbounded.*

Proof. It suffices to prove that the negative of any sphere $U = \{x: \|x\| < r\}$ about 0 is unbounded. Choose a sequence $\{x_n\}$ in U such that $\|x_n\| \to 0$ while $\|-x_n\| \not\to 0$. Let $y_n = r(2\|x_n\|)^{-1} \cdot x_n \in U$. The sequence $\{-y_n\}$ is unbounded, and the theorem follows.

THEOREM 4. *Every finite-dimensional weaknormed linear space is normable.*

Proof. Let $(X, \|\cdot\|)$ be a finite-dimensional weaknormed linear space, and let $\{v_i\}$ $(i = 1, \cdots, n)$ be a basis for X. Consider a function f on the real space $l^1(n)$ into the non-negative reals, given by $f(\alpha) = \|\sum \alpha_i v_i\|$, where α denotes the n-tuple $(\alpha_1, \cdots, \alpha_n)$. The function f is continuous, for

$$\begin{aligned} |f(\alpha) - f(\beta)| &= \left| \|\textstyle\sum \alpha_i v_i\| - \|\sum \beta_i v_i\| \right| \\ &\leqq \max\left\{ \|\textstyle\sum (\alpha_i - \beta_i) v_i\|, \|\sum (\beta_i - \alpha_i) v_i\| \right\} \\ &\leqq \textstyle\sum |\alpha_i - \beta_i| \cdot \max\{\|v_i\|, \|-v_i\|\} \leqq N \cdot (\sum |\alpha_i - \beta_i|), \end{aligned}$$

where

$$N = \max_{1 \leqq i \leqq n} \max\{\|v_i\|, \|-v_i\|\}.$$

Consider the set $S = \{\alpha: \sum |\alpha_i| = 1\} \subset l^1(n)$. S is compact, so that the non-negative set $f(S)$ has a least member, say r. Should $r = 0$, then for some $\alpha \in S$ we would have $f(\alpha) = \|\sum \alpha_i v_i\| = 0$, implying $\sum \alpha_i v_i = 0$, a contradiction since $\{v_i\}$ are linearly independent. Therefore, $r > 0$. But for any $\sum \beta_i v_i$ in X, we have

$$\|\textstyle\sum \beta_i v_i\| \geqq r \cdot (\sum |\beta_i|)$$

since, if some $\beta_i \neq 0$, $\|\sum \beta_i v_i\| = (\sum |\beta_i|) \|\sum (\sum |\beta_j|)^{-1} \beta_i v_i\|$, where

$$\|\textstyle\sum (\sum |\beta_j|)^{-1} \beta_i v_i\| \in f(S).$$

Furthermore, $N \cdot (\sum |\beta_i|) \geqq \|\sum -\beta_i v_i\|$. The conclusion follows by Theorem 2.

That any finite-dimensional weaknormed linear space is topologically isomorphic to the Euclidean space of the same dimension, is now an immediate consequence of the theorem that all finite-dimensional normed linear spaces of a given dimension are topologically isomorphic (see [4], p. 95).

4. Examples of nonnormable weaknormed linear spaces.

Example 1. Let X be the set of all functions in $L^1[1, \infty)$ (identifying, in the usual way, functions equal almost everywhere). For any function $f \in X$, we de-

fine $\|f\| = \int_1^\infty f^*(t)dt$, where $f^*(t) = f(t)$ when $f(t) \geqq 0$, and $f^*(t) = |f(t)|/t$ when $f(t) < 0$. The space $(X, \|\cdot\|)$ is a weaknormed linear space. To show that X is not normable, consider the sequence $\{f_n\}$, where $f_n(t) = 0$ for $t < n+1$ and

$$f_n(t) = -(t-n)^{-2} \quad \text{for } t \geqq n+1.$$

It is clear that $\|f_n\| \to 0$, while $\|-f_n\| = 1$ for all n, and the conclusion follows by Theorem 2.

Example 2. Let (Y, μ) be a nonatomic measure space with $\mu(Y) < \infty$. With each real-valued measurable function f on Y (identifying, in the usual way, functions equal almost everywhere) we associate two functions f_+ and f_- defined as follows: $f_+(x) = \max\{f(x), 0\}$, and $f_-(x) = \max\{-f(x), 0\}$. Let

$$p(f) = \int f_+^2 d\mu + \int f_- d\mu.$$

Then the set $K = \{f: p(f) < \infty, p(-f) < \infty\}$ is a vector space; indeed, K contains the same functions as $L^2(Y)$.

We will consider the Minkowski functional of the set $S = \{f \in K: p(f) \leqq 1\}$. S contains 0, for $p(0) = 0 < 1$. We prove that S is absorbing. Take any function $f \in K$ where $f \neq 0$. Pick α such that $|\alpha| < \min\{1/p(f), 1/p(-f), 1\}$. Then, if $\alpha \geqq 0$, we have

$$p(\alpha f) = \alpha^2 \int f_+^2 d\mu + \alpha \int f_- d\mu \leqq \alpha p(f). \tag{1}$$

But $\alpha < 1/p(f)$, and thus, by (1), $p(\alpha f) \leqq \alpha p(f) < 1$. If $\alpha < 0$, we have by (1), $p(\alpha f) \leqq p(|\alpha|(-f)) = |\alpha| p(-f) < 1$. Therefore, $\alpha f \in S$; S is absorbing.

We shall show that S is convex. Suppose $f, g \in S$ and $\alpha + \beta = 1$, where $\alpha, \beta \geqq 0$. Then,

$$\begin{aligned} p(\alpha f + \beta g) &= \int (\alpha f + \beta g)_+^2 d\mu + \int (\alpha f + \beta g)_- d\mu \\ &\leqq \int (\alpha f_+ + \beta g_+)^2 d\mu + \int (\alpha f_- + \beta g_-) d\mu. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} (\alpha f_+ + \beta g_+)^2 &= \alpha^2 f_+^2 + \beta^2 g_+^2 + 2\alpha\beta f_+ g_+ \\ &\leqq \alpha^2 f_+^2 + \beta^2 g_+^2 + \alpha\beta f_+^2 + \alpha\beta g_+^2 = \alpha f_+^2 + \beta g_+^2. \end{aligned}$$

Hence,

$$\begin{aligned} p(\alpha f + \beta g) &\leqq \alpha \int f_+^2 d\mu + \beta \int g_+^2 d\mu + \alpha \int f_- d\mu + \beta \int g_- d\mu \\ &= \alpha p(f) + \beta p(g) \leqq 1, \end{aligned}$$

since $p(f)$, $p(g) \leqq 1$. It follows that S is convex. By the properties of Minkowski functionals, we see that $(K, \|\cdot\|)$ is a weaknormed linear space, where $\|\cdot\|$ is the Minkowski functional of S.

To show that K is not normable, it is sufficient, by Theorem 2, to show that for any $n>0$, there exists an $f \in K$ such that $\|f\| \geqq n\|-f\|$. Given n, define f as follows: $f(x)=n$ on some set of measure $1/n^2$, and $f(x)=0$ everywhere else. We have $p(f)=1$ and thus $\|f\|=1$. However, $p(-nf)=1$, so that $\|-nf\|=n\|-f\|=1 =\|f\|$. Thus f is as required, and the conclusion follows.

Acknowledgement. This research was supported by a grant of the Dolfinger-McMahon Foundation to Swarthmore College and carried out during the summer of 1962, while I was an undergraduate there. I want to thank Dr. Stevens Heckscher of Swarthmore, who supervised my work, for his many valuable suggestions respecting this research, and for his kind assistance in preparing it for publication.

References

1. M. Balanzat, On the metrization of quasimetric spaces, Gaz. Mat. Lisboa 12, no. 50 (1951) 91–94.

2. K. Leichtweiss, Sur les espaces de Banach autoadjoints. Séminaire C. Ehresmann, 1957/58, exp. no. 4, Fac. des Sci. de Paris, 1958.

3. H. Ribeiro, Sur les espaces à métrique faible, Portugal. Math., 4 (1943) 21–40, 65–68.

4. A. E. Taylor, Introduction to Functional Analysis. Wiley, New York, 1959.

HADAMARD MATRICES AND SOME GENERALISATIONS

J. H. E. Cohn, Bedford College, London

For any given integer, $n \geqq 2$, the matrix H_n is called a *Hadamard Matrix* if H_n is an $n \times n$ matrix with elements ± 1 which satisfies $H_n H_n' = nI_n$. It is known that such a matrix can exist only if either $n=2$ or n is a multiple of four. It has been conjectured that these necessary conditions are also sufficient, but this has never been proved. A list of the numerous partial results available is given in [**1**] and [**4**], and in [**3**] and [**5**] it has also been proved true for the cases $n=92$ and 156.

Generalisations are possible in several directions. Following [**1**], we say that $H(p;\, n)$ is a generalised Hadamard Matrix of order n, if $H(p;\, n)$ is an $n \times n$ matrix all of whose elements are pth roots of unity and such that $H\overline{H}' = nI_n$. In this notation the ordinary Hadamard Matrix is an $H(2;\, n)$.

Another generalisation, without departing from the elements ± 1 arises as follows. Let $g(n)$ equal, for each n, the value of the greatest $n \times n$ determinant with elements ± 1. Then it follows by Hadamard's Theorem that $g(n) \leqq n^{n/2}$, where equality is possible if and only if the matrix G_n associated with $g(n)$ is an $H(2;\, n)$. However, we may ask for all values of n, and not only for multiples of four, what the value of $g(n)$ is, and also ask for the structure of G_n. It is possible to show, in addition to the fact that $g(n)=n^{n/2}$ whenever an $H(2;\, n)$ exists, that $g(3)=4$, $g(5)=48$ and $g(6)=160$, and also it has been proved in [2] that given any positive ϵ, $g(n)>n^{(1/2-\epsilon)n}$ for all sufficiently large n. If the conjecture

about the existence of an $H(2;\, n)$ for all $n \equiv 0 \pmod 4$ were proved, then this could be sharpened to

$$g(n) > n^{(n-1)/2}, \quad n \text{ odd}$$
$$g(n) > n^{(n/2)-1}, \qquad n \equiv 2 \pmod 4.$$

It is well known that if an $H(2;\, n)$ exists for a given n, then an $H(2;\, 2n)$ also exists. Our first result is to show that the same is true if we require merely the existence of an $H(4;\, n)$. Secondly, it has been conjectured that it is always possible to obtain G_{n+1} from G_n by a "bordering process"; that is, by adding a suitable row and column to G_n. It will be shown in Theorem 4 that this conjecture is false. Theorems 2 and 3, which are used to prove this, may be of some interest in themselves.

THEOREM 1. *If there exists an $H(4;\, n)$ then there exists an $H(2;\, 2n)$.*

Proof. Since $H(4;\, n)$ contains only elements ± 1 and $\pm i$ we may write

$$H(4; n) = X + iY,$$

where X and Y are real $n \times n$ matrices with the following properties:

(i) every element of each of X and Y is 1, 0 or -1,

(ii) for each $r, s \leqq n$, exactly one of x_{rs}, y_{rs} is zero, and so

(iii) both $X+Y$ and $X-Y$ have all their elements ± 1.

Also $H\bar{H}' = nI_n$, i.e. $(X+iY)(X'-iY') = nI_n$. Hence

(iv) $XX'+YY' = nI_n$

(v) $XY'-YX' = 0$.

We now assert that

$$K_{2n} = \begin{pmatrix} X+Y & X-Y \\ -X+Y & X+Y \end{pmatrix}$$

is an $H(2;\, 2n)$. Certainly, by (iii) all the elements of K_{2n} are ± 1. Also

$$\begin{aligned} KK' &= \begin{pmatrix} X+Y & X-Y \\ -X+Y & X+Y \end{pmatrix} \begin{pmatrix} X'+Y' & -X'+Y' \\ X'-Y' & X'+Y' \end{pmatrix} \\ &= \begin{pmatrix} 2(XX'+YY') & 2(XY'-YX') \\ -2(XY'-YX') & 2(XX'+YY') \end{pmatrix} \\ &= \begin{pmatrix} 2nI_n & 0 \\ 0 & 2nI_n \end{pmatrix} = 2nI_{2n}. \end{aligned}$$

This concludes the proof.

THEOREM 2. *If an $H(2;\, n+m)$ exists, and $n>m$, then no n-rowed minor is an $H(2;\, n)$.*

For, let A_{nn} be any n-rowed minor. Then by rearrangement, if necessary, we obtain an $H(2;\, n+m)$, viz.:

$$\begin{pmatrix} A_{nn} & B_{nm} \\ C_{mn} & D_{mm} \end{pmatrix}.$$

Hence,

$$(n+m)I_{n+m} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & C' \\ B' & D' \end{pmatrix},$$

and it follows that

$$(n+m)I_n = AA' + BB'.$$

Now if A were an $H(2;\, n)$, we would have $AA' = nI_n$, which would give $BB' = mI_n$. But since B has only $m<n$ columns, $r(BB') \leqq r(B) \leqq m < n$, and so $BB' \neq mI_n$. This concludes the proof of this theorem, but in fact the method can be used to prove slightly more, namely

THEOREM 3. *In any $H(2;\, n+m)$ with $n>m$, the determinant of any n-rowed minor does not exceed $m^{m/2}(n+m)^{(n-m)/2}$.*

As before, let A be the minor in question. Then

$$AA' + BB' = (n+m)I_n.$$

Let the eigenvalues of BB' be $\mu_1, \mu_2, \cdots, \mu_n$. Then since $r(BB') \leqq m$, at most m of these can be nonzero. Let $\mu_{m+1} + \mu_{m+2} = \cdots = \mu_n = 0$. Also the eigenvalues λ_r of AA' can be so arranged that for each $r \leqq n$,

$$\lambda_r = n + m - \mu_r.$$

Then $\lambda_r = n+m$, for $(m+1) \leqq r \leqq n$. Now $\sum_1^n \lambda_r = \operatorname{tr}(AA') = n^2$. Hence

$$\sum_1^m \lambda_r = n^2 - (n-m)(n+m) = m^2.$$

Hence

$$\prod_1^m \lambda_r \leqq \left\{\frac{1}{m}\sum_1^m \lambda_r\right\}^m = m^m,$$

since the eigenvalues of AA' are nonnegative. Hence

$$|A|^2 = |AA'| = \prod_1^n \lambda_r \leqq m^m(n+m)^{(n-m)}$$

which concludes the proof.

THEOREM 4. *It is not always possible to obtain G_{n+1} from G_n by bordering.*

For, if it were, then G_{12} would contain G_8 as a minor, and since $H(2;12)$ and $H(2;8)$ both exist, this cannot be so, by Theorem 2. In fact G_8 does not contain G_6, for by Theorem 3, the highest possible value for a six-rowed minor of $H(2;8)$ is 128, and $g(6)=160$.

References

1. A. T. Butson, Generalised Hadamard Matrices, Proc. Amer. Math. Soc., 13 (1962) 894–898.

2. J. H. E. Cohn, On the value of determinants, Proc. Amer. Math. Soc., 14 (1963) 581–588.

3. S. W. Golomb and L. D. Baumert, The search for Hadamard Matrices, this MONTHLY, 70 (1963) 12–17.

4. Karl Goldberg, Hadamard Matrices of order cube plus one, Amer. Math. Soc. Not., Abstract 567-90, 7 (1960) 348.

5. L. D. Baumert and Marshall Hall, Jr., A new construction for Hadamard matrices, Bull. Amer. Math. Soc., 71 (1965) 169–170.

ON THE GENERALISATION OF HILBERT'S INEQUALITY

Q. A. M. M. YAHYA, Imperial College, London

By the well-known extended inequality of Hilbert for finite sequences $\{a_n\}$, $\{b_n\}$, (see [1], [2]), we have

$$\sum_{u=0}^{m}\sum_{v=0}^{m}\frac{a_u b_v}{2u+2v+1} \leqq \frac{\pi}{2}\left(\sum_{u=1}^{m} a_u^2\right)^{1/2}\left(\sum_{v=1}^{m} b_v^2\right)^{1/2}.$$

We shall sharpen this result as follows.

THEOREM. *Let $\{a_n\}$ and $\{b_n\}$ be two finite sequences of real and nonnegative numbers. Then we must have*

$$\sum_{u=0}^{m}\sum_{v=0}^{m}\frac{a_u b_v}{2u+2v+1} \leqq (m+1)\sin\frac{\pi}{2(m+1)}\left(\sum_{u=0}^{m} a_u^2\right)^{1/2}\left(\sum_{u=0}^{m} b_u^2\right)^{1/2}.$$

To prove this we shall need the following result.

LEMMA. *Let k, u and v be integers. If $k|u-v|<n$, then*

$$\sum_{r=0}^{n-1}\exp\left(2ik(u-v)r\pi/n\right) = \begin{cases} 0, & \text{if } u \neq v \\ n, & \text{if } u = v. \end{cases}$$

The lemma follows immediately. We shall now prove the theorem.

Let P be a closed regular polygon with an even number of sides n, $(n>2m)$ inscribed in the unit circle $|z|=1$ and with the vertices $V_r=e^{2ir\pi/n}$, where $V_n=V_0$. Let C_r, $0\leqq r\leqq n-1$ be the chord joining the points V_r and V_{r+1}. On the chord C_r we have

$$z = e^{i\theta}\cos(\pi/n)/\cos\left[(2r+1)(\pi/n)-\theta\right],\ 2r(\pi/n) \leqq \theta \leqq (2r+2)(\pi/n),$$

so that

$$|dz| = \left[\cos(\pi/n)\right] d\theta/\cos^2\left[(2r+1)(\pi/n)-\theta\right],$$

if θ is measured in the positive sense of the angle.

Let P_1 and P_2 be the parts of P above and below the real axis respectively. Let

$$f(z) = \sum_{u=0}^{m} a_u z^{2u} \quad \text{and} \quad g(z) = \sum_{v=0}^{m} b_v z^{2v}.$$

Thus

$$\sum_{u=0}^{m}\sum_{v=0}^{m} \frac{a_u b_v}{2u+2v+1} = \frac{1}{2}\int_{-1}^{+1} f(x)g(x)dx = \frac{1}{2}\int_{P_1} f(z)g(z)dz$$

by Cauchy's theorem. Now

$$\int_{P_1} f(z)g(z)dz \leqq \frac{1}{2}\int_{P} |f(z)|\,|g(z)|\,|dz|$$

$$\leqq \frac{1}{2}\left(\int_{P} |f(z)|^2 |dz|\right)^{1/2}\left(\int_{P} |g(z)|^2 |dz|\right)^{1/2}.$$

Then we have

$$\int_{P} |f(z)|^2 |dz| = \int_{P} f(z)f(\bar{z})\,|dz| = \sum_{r=0}^{n-1}\int_{C_r} f(z)f(\bar{z})\,|dz|$$

$$= \sum_{r=0}^{n-1}\int_{C_r}\left(\sum_{u=0}^{m} a_u z^{2u}\right)\left(\sum_{v=0}^{m} a_v \bar{z}^{2v}\right)|dz|$$

$$= \sum_{r=0}^{n-1}\int_{2\pi r/n}^{(2r+2)\pi/n} \sum_{u=0}^{m} a_u \frac{[\cos^{2u}(\pi/n)]e^{2iu\theta}}{\cos^{2u}[(2r+1)(\pi/n)-\theta]} \sum_{v=0}^{m} \frac{[\cos^{2v}(\pi/n)]e^{-2iv\theta}}{\cos^{2v}[(2r+1)(\pi/n)-\theta]}$$

$$\cdot\frac{\cos(\pi/n)d\theta}{\cos^2[(2r+1)(\pi/n)-\theta]},$$

$$= \sum_{r=0}^{n-1}\int_{2\pi r/n}^{(2r+2)\pi/n} \sum_{u=0}^{m}\sum_{v=0}^{m} a_u a_v \frac{[\cos^{2(u+v)}(\pi/n)]e^{2i(u-v)\theta}}{\cos^{2(u+v)}[(2r+1)(\pi/n)-\theta]}\cdot\frac{\cos(\pi/n)d\theta}{\cos^2[(2r+1)(\pi/n)-\theta]}$$

$$= \sum_{u=0}^{m}\sum_{v=0}^{m} a_u a_v \int_{-\pi/n}^{\pi/n} \frac{[\cos^{2(u+v)+1}(\pi/n)]e^{2i(u-v)[(\pi/n)-\phi]}}{\cos^{2(u+v)+2}\phi} \sum_{r=0}^{n-1} e^{4i(u-v)\pi r/n} d\phi$$

$$= n\sum_{u=0}^{m} a_u^2 \int_{-\pi/n}^{\pi/n} \frac{\cos^{4u+1}(\pi/n)}{\cos^{4u+2}\phi}\,d\phi,$$

where we have used the lemma.

Observing that

$$\int_{-\pi/n}^{\pi/n} \frac{d\phi}{\cos^{4u+2}\phi} \leqq \frac{1}{\cos^{4u}(\pi/n)}\int_{-\pi/n}^{\pi/n}\frac{d\phi}{\cos^2\phi} = 2\,\frac{\sin(\pi/n)}{\cos^{4u+1}(\pi/n)}$$

for $0 \leqq u \leqq m$, we obtain

$$\int_P |f(z)|^2 |dz| \leqq 2n \sin(\pi/n) \cdot \sum_{u=0}^{m} a_u^2.$$

Similarly,

$$\int_P |g(z)|^2 |dz| \leqq 2n \sin(\pi/n) \cdot \sum_{u=0}^{m} b_u^2.$$

Hence we obtain

$$\sum_{u=0}^{m} \sum_{v=0}^{m} \frac{a_u b_v}{2u + 2v + 1} \leqq \frac{1}{2} n \sin(\pi/n) \left(\sum_{u=0}^{m} a_u^2 \right)^{1/2} \left(\sum_{u=0}^{m} b_u^2 \right)^{1/2}$$

for $n > 2m$, n being even. Setting $n = 2m+2$, we obtain our theorem at once.

References

1. G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge, 1952.
2. G. Pólya und G. Szegö, Aufgaben und Lehrsätze aus der Analysis I, Berlin, 1925.

REMARK ON LOCALLY ALGEBRAIC OPERATORS

S. CATER, University of Oregon

In this note A is a continuous (linear) operator on a complex normed linear space V. As in [2] we say that A is *algebraic* on V if $p(A) = 0$ for some nonzero polynomial p (all polynomials mentioned are understood to have complex coefficients and A^0 will always denote the identity operator on V). We say that A is *locally algebraic* on V if for each $z \in V$ there is a polynomial $p_z \neq 0$ such that $p_z(A)z = 0$. It follows from Lemma 14 of [2, p. 41] that Theorem 15 of [2, p. 40] can be rephrased as follows:

If V admits a continuous locally algebraic operator which is not algebraic, then V is not complete.

This note presents a much sharper result concerning polynomials in A which we hope is of some intrinsic interest. Observe that if A is locally algebraic but not algebraic we can find vectors $z_n \in V$ ($n = 1, 2, \cdots$) such that there is no nonzero polynomial p with $p(A)z_n = 0$ for all n (employ [2, Lemma 14, p. 41]).

THEOREM. *Let A be a continuous locally algebraic operator on a complex normed linear space V. If z_n ($n = 1, 2, \cdots$) are unit vectors for which there is no nonzero polynomial p satisfying $p(A)z_n = 0$ for all n, then there exist polynomials p_n satisfying $\|p_n(A)\| < 2^{-n}$ such that the series $\sum p_n(A)z_n$ does not converge to any vector in V. In particular, V is not complete if A is not algebraic.*

Proof. Let z_n ($n = 1, 2, \cdots$) be unit vectors for which there exists no nonzero polynomial p satisfying $p(A)z_n = 0$ for all n. For each scalar c let V_c be the linear

manifold composed of all vectors annihilated by powers of $A-c$. As is proved in [2], V is the direct sum of the V_c over all scalars c. Let Z_n be the linear manifold spanned by the vectors $A^j z_n$ $(j=0, 1, \cdots)$. The restriction of A to Z_n is algebraic on Z_n, and Z_n is the direct sum of the linear manifolds $V_c \cap Z_n$ over all scalars c. It follows that one or both of the following statements are true. (Note that if neither I nor II holds then there exist scalars $c_1, \cdots, c_N$ and an integer $m>0$ such that $(A-c_1)^m(A-c_2)^m \cdots (A-c_N)^m$ annihilates all the vectors z_n, contrary to assumption.)

I. There are infinitely many distinct scalars c for which

$$\bigcup_{n=1}^{\infty} V_c \cap Z_n \neq (0).$$

II. There is a scalar c for which there is no index m satisfying $(A-c)^m(V_c \cap Z_n) = (0)$ for all n.

First we will assume I and complete the proof. Since A is continuous the set of all eigenvalues of A is bounded in the complex plane. We can select an increasing sequence of indices $[i_n]$ $(n=1, 2, \cdots)$ and unit vectors $v_n \in V_{c_n} \cap Z_{i_n}$ such that $(A-c_n)v_n=0$, the c_n are pairwise distinct, $[c_n]$ converges, and for all n, $c_n \neq \lim_m c_m = c$. Let q_n be a polynomial for which $v_n = q_n(A)z_{i_n}$.

By the Hahn-Banach Theorem there are continuous linear functionals f_n $(n=1, 2, \cdots)$ on V for which $f_n(v_n)=1$, and f_n annihilates the null space of $(A-c)^n(A-c_1)(A-c_2) \cdots (A-c_{n-1})$. Define inductively a sequence $[s_n]$ of positive scalars satisfying

(a) $s_n < \min(2^{-n}, 2^{-i_n}/\|q_n(A)\|)$,

(b) $s_n |f_m(A^j v_n)| < 2^{-n}s_m$ for $m=1, \cdots, n-1$, $j=0, \cdots, n$.

We claim that $\sum s_n v_n$ does not converge to any vector in V. The proof is by contradiction; assume $\sum s_n v_n = v \in V$. There is then a nonzero polynomial q for which $q(A)v=0$. Say $q(x)=(x-c)^t p(x)$, where $p(c)\neq 0$. Then $p(A)v$ is in the null space of $(A-c)^t$. Set

$$p(x) - p(c_m) = r_k x^k + \cdots + r_1 x + r_0(m)$$

and

$$R(m) = |r_k| + \cdots + |r_1| + |r_0(m)|.$$

By (b) $|f_m[p(A)-p(c_m)](s_n v_n)| < 2^{-n}s_m R(m)$ for $m \geqq k$, $n>m$. Consequently

$$|f_m[p(A) - p(c_m)]v| < R(m)s_m \sum_{n=m+1}^{\infty} 2^{-n} = 2^{-m}s_m R(m)$$

for $m \geqq k$ because $x-c_m$ divides $p(x)-p(c_m)$ and f_m annihilates the null space of $(A-c_n)(A-c_{n-1}) \cdots (A-c_1)$ if $n<m$.

Setting $j=0$ in (b) we have

$$s_m(1+2^{-m}) > \sum_{n=m}^{\infty} |s_n f_m(v_n)| \geqq |f_m(v)| \geqq s_m - \sum_{n=m+1}^{\infty} s_n |f_m(v_n)|$$
$$> s_m - \sum_{n=m+1}^{\infty} 2^{-n} s_m = s_m(1-2^{-m}).$$

We also have

$$p(c)f_m(v) = f_m(p(A)v) - f_m([p(A) - p(c_m)]v) - [p(c_m) - p(c)]f_m(v),$$
$$s_m(1-2^{-m})|p(c)| < |p(c)|\,|f_m(v)|$$
$$\leqq |f_m(p(A)v)| + |f_m[p(A) - p(c_m)]v| + |p(c_m) - p(c)|\,|f_m(v)|$$

and, for $m \geqq k$,

$$(1)\quad (1-2^{-m})|p(c)| < s_m^{-1}|f_m(p(A)v)| + 2^{-m}R(m) + |p(c_m) - p(c)|(1+2^{-m}).$$

(The author is indebted to Mr. Bernard Kripke for the inequality (1).)

For $m \geqq t$, $f_m(p(A)v) = 0$. As $m \to \infty$, $R(m)$ converges and $p(c_m) - p(c)$ converges to 0. From (1) it follows that $p(c) = 0$, contrary to the choice of p. This completes the proof that $\sum s_n v_n$ does not converge to any vector in V. Now set $p_{i_n} = s_n q_n$ and set $p_m = 0$ for all indices m not in the sequence $[i_n]$. We have $\|p_n(A)\| < 2^{-n}$, and $\sum p_n(A)z_n$ does not converge to a vector in V. This concludes the proof if I holds.

Now assume II holds and let c be the scalar required in II. Then we can select an increasing sequence of indices $[i_n]$ $(n = 1, 2, \cdots)$ and unit vectors $v_n \in V_c \cap Z_{i_n}$ such that $(A-C)^{n-1}v_n \neq 0 = (A-c)^n v_n$. Let q_n be a polynomial for which $v_n = q_n(A)z_{i_n}$.

By the Hahn-Banach Theorem there exist continuous linear functionals f_n on V for which $f_n(v_n) = 1$, and f_n annihilates the null space of $(A-c)^{n-1}$. Choose inductively a sequence $[s_n]$ of positive scalars satisfying (a) and (b) above.

We claim that $\sum s_n v_n$ does not converge to any vector in V. The proof is by contradiction; assume $\sum s_n v_n = v \in V$. There is a nonzero polynomial q such that $q(A)v = 0$. Set $q(x) = (x-c)^t p(x)$ so that $p(c) \neq 0$. Then $p(A)v$ is in the null space of $(A-c)^t$. Set

$$p(x) - p(c) = r_k x^k + \cdots + r_1 x + r_0$$

and

$$R = |r_k| + \cdots + |r_1| + |r_0|.$$

By essentially the same argument as in the preceding proof with c replacing c_m, R replacing $R(m)$, we have for $m \geqq k$,

$$(2)\qquad (1-2^{-m})|p(c)| \leqq s_m^{-1}|f_m(p(A)v)| + 2^{-m}R.$$

For $m > t$, $f_m(p(A)v) = 0$. Letting $m \to \infty$ we see that $p(c) = 0$, contrary to the

choice of p. This proves that $\sum s_n v_n$ does not converge to any vector in V. Set $p_{i_n} = s_n q_n$ and set $p_m = 0$ for all indices m not in the sequence $[i_n]$. Then $\|p_n(A)\| < 2^{-n}$ and $\sum p_n(A) z_n$ does not converge to any vector in V. This concludes the proof.

Finally, observe that our Theorem remains valid if K_n replaces 2^{-n}, where $[K_n]$ is any preassigned sequence of positive numbers. To see this make (a) read

$$s_n < \min (2^{-n}, K_{i_n}/\|q_n(A)\|).$$

We now present a generalization of the Uniform Boundedness Principle which is suggestive of our Theorem.

REMARK. *Let X and Y be complex normed linear spaces, let $\{T_\alpha\}$ be a family of continuous linear operators mapping X into Y such that*

$$\sup_\alpha \|T_\alpha x\| = B_x < \infty$$

for each $x \in X$, and let $[K_n]$ be a sequence of positive numbers. If x_n $(n = 1, 2, \cdots)$ are unit vectors in X for which $\sup_n B_{x_n} = \infty$, there exist nonnegative scalars $c_n < K_n$ such that the series $\sum c_n x_n$ does not converge to any vector in X.

The reader can easily prove this Remark by modifying slightly the argument in [1, Section 4].

References

1. I. S. Gál, On sequences of operations in complete vector spaces, this MONTHLY, 60 (1953) 527–538.

2. I. Kaplansky, Infinite Abelian Groups, University of Michigan Press, 1954.

REMARK ON A NONLINEAR CONVERGENCE-PRODUCING SERIES TRANSFORMATION

D. L. GOLDSMITH, Fordham University

1. Introduction. Let $\sum_{n=0}^{\infty} a_n$ be an arbitrary series of real numbers, with partial sums $\{A_n\}$. Following Lubkin ([2], p. 228) and Shanks ([3], p. 9) we let

$$B_n = A_n + \frac{a_{n+1}}{1 - (a_{n+1}/a_n)}, \qquad n \geqq 0. \tag{1.1}$$

If the $\{B_n\}$ are regarded as partial sums of a new series $\sum_{n=0}^{\infty} b_n$, then (1.1) represents a nonlinear transformation of series which we shall denote, following Shanks, by e_1. While (1.1) defines a relationship between partial sums,

$$B_n = e_1(A_n),$$

the relationship (which we shall denote by λ) between the individual terms of the corresponding series is easily calculated. Writing r_n instead of a_{n+1}/a_n, we see that (1.1) holds if and only if

(1.2) $$b_{n+1} = \lambda(a_{n+1}) = a_{n+1}\frac{r_{n+1} - r_n}{(1 - r_{n+1})(1 - r_n)}, \qquad n \geqq 0,$$

and

(1.2′) $$b_0 = \lambda(a_0) = \frac{a_0}{1 - a_1/a_0}.$$

In order that (1.1) make sense we shall consider, from now on, only series $\sum a_n$ such that $a_n \neq 0$ and $a_{n+1} \neq a_n$ for every $n \geqq 0$. A series with zero terms or consecutively equal terms may be treated by using higher order transforms (see [**3**], pp. 17–21).

Our aim in this note is to apply the λ transformation to certain classes of divergent series. To that end we formulate the following definition:

DEFINITION. *Let $\sum_{n=0}^{\infty} a_n$ be a divergent series, and suppose T is a transformation such that $b_n = T(a_n)$, $(n = 0, 1, 2, \cdots)$. T is said to be effective with respect to $\sum_{n=0}^{\infty} a_n$ provided $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} T(a_n)$ converges.*

2. **Main result.** The λ transformation is a smoothing operation in the sense that it will decrease the oscillation of certain types of series and give as limit the point about which the series oscillates. More precisely, we have the following

THEOREM. *Let $\sum_{n=0}^{\infty} a_n$ be a divergent real series with partial sums $\{A_n\}$, and let a_{n+1}/a_n be denoted by r_n. Suppose that*

(1) $$\lim_{n \to \infty} r_n = -1,$$

(2) $$\{r_n\} \text{ is eventually monotone,}$$

(3) $$\{A_n\} \text{ is bounded.}$$

Then $\sum_{n=0}^{\infty} \lambda(a_n)$ converges, i.e. λ is effective with respect to $\sum_{n=0}^{\infty} a_n$.

For example, the divergent geometric series $\sum_{n=0}^{\infty} (-1)^n$ is summed to $\frac{1}{2}$.

Our proof will lean heavily on the following lemma.

LEMMA. *If $\{r_n\}$ is a sequence such that* (i) $\lim_{n \to \infty} r_n = R \neq 1$, (ii) *$\{r_n\}$ is eventually monotone,* (iii) $r_n \neq 1 (n = 0, 1, 2, \cdots)$, *then the series*

$$\sum_{n=0}^{\infty} \frac{r_{n+1} - r_n}{(1 - r_{n+1})(1 - r_n)}$$

converges absolutely.

Proof. For convenience, set $x_n = r_{n+1} - r_n$ and $y_n = 1/(1 - r_{n+1})(1 - r_n)$. Then

$$\sum_{n=0}^{N} x_n = r_{N+1} - r_0, \quad \text{and} \quad \lim_{N \to \infty} \sum_{n=0}^{N} x_n = \sum_{n=0}^{\infty} x_n = R - r_0.$$

By conditions (i) and (ii) it is easily verified that $\{y_n\}$ is eventually monotone. Finally, since $\{y_n\}$ is bounded, we may apply Abel's theorem (see [1], page 315) to $\sum_{n=0}^{\infty} x_n y_n$ and, noting that $x_n y_n$ does not change sign for large n, we see that the lemma follows.

Proof of the Theorem. Let us set $c_{n+1}=(r_{n+1}-r_n)/(1-r_{n+1})(1-r_n)$. We know by the Lemma that $\sum_{n=0}^{\infty} c_n$ converges absolutely. Since $\sum_{n=0}^{\infty} a_n$ has bounded partial sums, the convergence of

$$\sum_{n=1}^{\infty} \lambda(a_n) = \sum_{n=1}^{\infty} a_n c_n$$

is again a consequence of Abel's theorem. This completes the proof.

References

1. K. Knopp, Theory and application of infinite series, Blackie, London, 1947.
2. S. Lubkin, A method of summing infinite series, J. Res. Nat. Bur. Stand., 48 (1952) 228–254.
3. D. Shanks, Nonlinear transformations of divergent and slowly convergent sequences, J. Math and Phys., 34 (1955) 1–42.

ON GROUP-COMMUTATORS OF 2×2 MATRICES

I. SINHA, Michigan State University

1. A pair of matrices A, B are defined to be 2-commutative in the additive sense if, denoting $AB-BA$ by $[A, B]$, we have

$$[[A, B], B] = 0 = [A, [A, B]].$$

In other words, the additive commutator $AB-BA$ commutes with both A and B. It is well known that if A and B are 2×2 matrices, then 2-commutativity in the additive sense merely degenerates into commutativity.

The purpose of this note is to prove a near analogue of this result for the multiplicative case.

Throughout this note we shall limit ourselves to algebraically closed fields of characteristic zero.

2. For a pair A, B of nonsingular matrices, we define the group-commutator as $ABA^{-1}B^{-1}$. Then A, B, will be called 2-commutative in the multiplicative sense if $ABA^{-1}B^{-1}$ commutes with both A and B.

The following result has been proved by the author in [2].

THEOREM 1. *Let A and B be nonsingular matrices such that* (i) $ABA^{-1}B^{-1}$ *commutes with A and B, and* (ii) *at least one of A and B does not have a complete set of the m-th roots of any scalar amongst its characteristic roots for m greater than* 1. *Then $C=ABA^{-1}B^{-1}-I$ is nilpotent, where I denotes the identity matrix.*

From Theorem 1 we obtain the following result.

THEOREM 2. *If A and B are 2×2 matrices which are 2-commutative in the multiplicative sense, then either A and B are commutative or they are anti-commutative; i.e. either $AB=BA$ or $AB=-BA$.*

Proof. By virtue of Theorem 1, either $C=ABA^{-1}B^{-1}-I$ is nilpotent or B has a complete set of roots of some scalar amongst its characteristic roots, which can then be only the two square roots of a number. Accordingly, we consider two distinct cases.

Case (i). Let C be nilpotent. Since C and B commute by hypothesis, we may assume them to be simultaneously reduced to triangular form (see [1]). Thus, let

$$C+I=\begin{bmatrix}1 & c\\ 0 & 1\end{bmatrix},\qquad B=\begin{bmatrix}b_{11} & b_{12}\\ 0 & b_{22}\end{bmatrix},\qquad A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}.$$

Then from $BC=CB$ it follows that

$$\begin{bmatrix}b_{11} & c\cdot b_{11}+b_{12}\\ 0 & b_{22}\end{bmatrix}=\begin{bmatrix}b_{11} & b_{12}+c\cdot b_{22}\\ 0 & b_{22}\end{bmatrix},$$

so that

$$c\cdot b_{11}=c\cdot b_{22}. \tag{1}$$

It also follows from $AC=CA$ that

$$\begin{bmatrix}a_{11} & c\cdot a_{11}+a_{12}\\ a_{21} & c\cdot a_{21}+a_{22}\end{bmatrix}=\begin{bmatrix}a_{11}+c\cdot a_{21} & a_{12}+c\cdot a_{22}\\ a_{21} & a_{22}\end{bmatrix},$$

so that $c\cdot a_{21}=0$, and

$$c\cdot a_{11}=c\cdot a_{22}. \tag{2}$$

Now if $c=0$, then $C+I=I=ABA^{-1}B^{-1}$ and hence A, B are already commutative.

On the other hand, if we assume that $c\neq 0$, then from (1) and (2) we conclude that $b_{11}=b_{22}$, $a_{21}=0$, and $a_{11}=a_{22}$. Thus we may assume that,

$$A=\begin{bmatrix}\lambda & a\\ 0 & \lambda\end{bmatrix}\quad\text{and}\quad B=\begin{bmatrix}\mu & b\\ 0 & \mu\end{bmatrix}.$$

Hence

$$A^{-1}=\begin{bmatrix}\dfrac{1}{\lambda} & \dfrac{-a}{\lambda^2}\\ 0 & \dfrac{1}{\lambda}\end{bmatrix}\quad\text{and}\quad B^{-1}=\begin{bmatrix}\dfrac{1}{\mu} & \dfrac{-b}{\mu^2}\\ 0 & \dfrac{1}{\mu}\end{bmatrix},$$

so that $ABA^{-1}B^{-1}=I$ contrary to our assumption that $c\neq 0$.

Thus, in either case, A and B commute when C is nilpotent.

Case (ii). Next suppose C is not nilpotent. Then B must have the two square-roots of a scalar λ as its characteristic roots by virtue of Theorem 1. We can therefore assume B to be in its canonical form,

$$B = \begin{bmatrix} +\lambda & 0 \\ 0 & -\lambda \end{bmatrix}.$$

Considering A, B to be a pair of linear-transformations of a 2-dimensional vector-space V, if it were a decomposable pair, then each indecomposable component of V must be 1-dimensional and hence A, B will be diagonable, and therefore a commutative pair. In that case $C = ABA^{-1}B^{-1} - I$ will be nilpotent contrary to our assumption.

We may therefore assume A, B to be an indecomposable pair. Then from the commutativity of B and C we can assume that C is diagonable (see [1]) and has the form

$$C + I = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix},$$

where $\det(C+I) = 1$ implies that $\mu_1 \neq 0$, $\mu_2 \neq 0$.

If $\mu_1 \neq \mu_2$ then V is a C-decomposable space. As C and A commute, every C-invariant component of V is also A-invariant (see [1]). Thus again A will be automatically in the diagonal form and hence will commute with B. This will again imply that C is nilpotent contrary to our assumption. Hence $\mu_1 = \mu_2$.

Since $\det(C+I) = 1$, either $\mu_1 = \mu_2 = 1$, or $\mu_1 = \mu_2 = -1$.

In the first case C is nilpotent contrary to our assumption and in the latter case $ABA^{-1}B^{-1} = -I$, so that $AB = -BA$.

References

1. N. Jacobson, Lectures on Abstract Algebras, vol. II, Van Nostrand, Princeton, N. J., 1951.

2. I. Sinha, Dissertation, 1962, entitled "Reduction of Sets of Matrices to Triangular Forms." University of Wisconsin, Madison, Wisconsin.

"Integrals," as shown by Cauchy,
"Of regular functions of z
Are really quite dull
For their value is null
On simple closed curves in B.V."

"In ten pigeon holes," said Dirichlet,
"Eleven fat pigeons waited."
The Master then went on to say,
"At least two of them are mated."

JAMES P. BURLING

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

A SPECTRAL MAPPING THEOREM FOR POLYNOMIALS

H. A. GINDLER, San Diego State College and University of Pittsburgh

Consider a linear transformation T whose domain $D(T)$ and range $R(T)$ lie in a complex Banach space X. We denote by $[X]$ the class of all linear transformations T for which $D(T)=X$ and T is continuous (bounded). Also, we denote the resolvent set of T by $\rho(T)$ and the spectrum of T by $\sigma(T)$. If λ is a complex number, we write $\lambda-T$ instead of $\lambda I-T$ and if $\lambda\in\rho(T)$ we write $R(\lambda; T)$ for $(\lambda-T)^{-1}$. If p is a polynomial of degree n, the operator $p(T)$ is defined on D_n, the domain of T^n, in the usual way.

In [1] (p. 204) one finds the fine-point spectral mapping theorem for a certain class of analytic functions. It is interesting to see how this theorem applies to a polynomial p. If $T\in[X]$, then p is analytic on $\sigma(T)$; hence p satisfies the hypotheses of the fine-point spectral mapping theorem. If $T\notin[X]$, then, since p is not analytic at ∞, the theorem does not apply to p. Nevertheless, a more general theorem is true, as we propose to show in this paper. An interesting feature of the proof is the fact that it does not rely at all on the operational calculus. The author has succeeded in generalizing the fine-point spectral mapping theorem to a wider class of functions. The proof in the more general case, however, relies very heavily on the operational calculus.

DEFINITION. *The properties P_i $(i=1, 2, 3)$ for the operator T are defined as follows, on the assumption that T is linear, with domain and range in X:*

P_1: T is not one-to-one.

P_2: The range of T is not dense in X.

P_3: There exists a sequence of unit vectors x_n for which $\|Tx_n\|\to 0$ as $n\to\infty$.

The following lemma will be needed. We employ the notation introduced earlier.

LEMMA. *Let X be a Banach space and let T be a closed linear operator from $S\subset X$ to X. If $\{y_k\}$ is a sequence of unit vectors in the domain of T for which $(\alpha-T)y_k\to 0$ as $k\to\infty$, then, for each positive integer n and each β in $\rho(T)$, $\|[R(\beta; T)]^{n-1}y_k\|$ is bounded away from zero.*

Proof. The proof is by induction. The case $n=1$ is trivial, so we may assume $n>1$ and that $\|[R(\beta; T)]^{n-2}y_k\|$ is bounded away from zero. Then, writing $x_k=[R(\beta; T)]^{n-1}y_k$, we have

$$\begin{aligned}\|(\beta-\alpha)x_k\| &\geqq \|(\beta-T)x_k\| - \|(\alpha-T)x_k\| \\ &= \|[R(\beta;T)]^{n-2}y_k\| - \|[R(\beta;T)]^{n-1}(\alpha-T)y_k\|.\end{aligned}$$

The second term on the right hand side of this equation goes to zero, while the first term on the right is bounded away from zero. It follows that $\|x_k\|$ is bounded away from zero, as was to be proved.

THEOREM. *Let X be a Banach space, let T be a closed linear operator with nonempty resolvent set and let p be a polynomial of degree $n>0$. If $\alpha\in\sigma(T)$ and if $\alpha-T$ has the property P_i ($i=1$, 2 or 3), then $p(\alpha)-p(T)$ has the property P_i also. Moreover, if $\mu\in\sigma(p(T))$ and if $\mu-p(T)$ has the property P_i ($i=1$, 2 or 3) then there is an $\alpha\in\sigma(T)$ such that $p(\alpha)=\mu$ and $\alpha-T$ has the property P_i.*

Proof. If $\alpha\in\sigma(T)$, write $p(\alpha)-p(\lambda)=(\alpha-\lambda)q(\lambda)$, where q is a polynomial of degree $n-1$ in the complex number λ. Then the fact that $[p(\alpha)-p(T)]x=(\alpha-T)q(T)x$ for all x in D_n shows that the range of $p(\alpha)-p(T)$ is contained in the range of $\alpha-T$. Thus, if $\alpha-T$ has the property P_2, then $p(\alpha)-p(T)$ also has the property P_2. Similarly, $[p(\alpha)-p(T)]x=q(T)(\alpha-T)x$ for all x in D_n implies that $[p(\alpha)-p(T)]x$ will be zero whenever $(\alpha-T)x=0$. This yields the corresponding assertion concerning the property P_1.

To see that the property P_3 is also inherited, suppose that y_k, $k=1, 2, \cdots$, are unit vectors in the domain of T for which $(\alpha-T)y_k\to 0$ as $k\to\infty$ and suppose $\beta\in\rho(T)$. Letting $x_k=[R(\beta;T)]^{n-1}y_k$, we see that $x_k\in D_n$ for each k and that

$$[p(\alpha)-p(T)]x_k = q(T)(\alpha-T)x_k = q(T)[R(\beta;T)]^{n-1}(\alpha-T)y_k.$$

By rewriting $q(T)$ in powers of $\beta-T$ and multiplying through by $[R(\beta;T)]^{n-1}$ we obtain $q(T)[R(\beta;T)]^{n-1}$ as a polynomial in $R(\beta;T)$. Hence, $q(T)[R(\beta;T)]^{n-1}\in[X]$. Since $(\alpha-T)y_k\to 0$ as $k\to\infty$, by continuity so does the expression on the right side of the preceding equation. It follows that $[p(\alpha)-p(T)]x_k\to 0$ as $k\to\infty$. Moreover, the lemma shows that the $\|x_k\|$ are bounded away from zero. Hence, $p(\alpha)-p(T)$ has the property P_3 if $\alpha-T$ has the property P_3. This completes the proof of the first assertion of the theorem.

To prove the converse, suppose that $\mu\in\sigma(p(T))$ and write

$$\mu - p(\lambda) = \alpha_0(\alpha_1-\lambda)\cdots(\alpha_n-\lambda),$$

where $p(\alpha_i)=\mu$ for all $i=1, 2, \cdots, n$ and α_0 is a suitable complex number. Without loss of generality assume $\alpha_0=1$ so that

$$[\mu-p(T)]x = (\alpha_1-T)\cdots(\alpha_n-T)x = \prod_{i=1}^{n}(\alpha_i-T)x \tag{1}$$

for all $x\in D_n$. From equation (1) we see that $\mu-p(T)$ must be one-to-one if each α_i-T is one-to-one. Hence if $\mu-p(T)$ has the property P_1, so has α_i-T, for some i. An operator A fails to have property P_3 if and only if there exists a positive constant m such that $\|Ax\|\geqq m\|x\|$ for each x in the domain of A. From this

and equation (1) we can see that if each of $\alpha_1 - T, \cdots, \alpha_n - T$ fails to have property P_3, say with $\|(\alpha_i - T)x\| \geqq m_i\|x\|$ for each x in the domain of T, then $\|[\mu - p(T)]x\| \geqq m_1 m_2 \cdots m_n\|x\|$ for each x in D_n, so that $\mu - p(T)$ fails to have property P_3.

It remains to show that property P_2 of $\mu - p(T)$ is inherited from some pre-image point of μ. The proof is similar to that in [1] (p. 205). For x in D_n, let $y = (\beta - T)^n x$, where $\beta \in \rho(T)$. Using $x = [R(\beta; T)]^n y$, equation (1) now takes the form

$$(2) \qquad [\mu - p(T)]x = \prod_{i=1}^{n} (\alpha_i - T)[R(\beta; T)]^n y = \left(\prod_{i=1}^{n} q_i\right) y,$$

where $q_i = (\alpha_i - T)R(\beta; T)$ for $i = 1, \cdots, n$. Each q_i is continuous, for the formula

$$q_i = (\alpha_i - \beta + \beta - T)R(\beta; T) = (\alpha_i - \beta)R(\beta; T) + I$$

shows that each q_i is a sum of continuous operators.

Now, if f is any continuous function and if S is a subset of X, then $f(\overline{S}) \subset \overline{f(S)}$, where the bar denotes set closure. If, in addition, $\overline{S} = X$ and if the range of f is dense in X, then $X = \overline{f(X)} = f(\overline{S}) \subset \overline{f(S)}$, so that f maps dense subsets of X into dense subsets of X. Hence, if each q_i in equation (2) has a dense range, it follows by mathematical induction, from what has just been shown, and equation (2), that $\mu - p(T)$ also has this property. Hence, if $\mu - p(T)$ has the property P_2, the above argument shows that q_i has the property P_2 for some i. From this one readily obtains that $\alpha_i - T$ also has this property.

Reference

1. E. Hille and R. S. Phillips, Functional analysis and semi-groups, Amer. Math. Soc. Coll. Publ., vol. XXXI, Revised ed. 1957.

ON THE REMAINDERS OF CERTAIN QUADRATURE FORMULAS

WALTER JENNINGS, U. S. Naval Postgraduate School

The remainder terms of many formulas for numerical integration, such as those of Newton-Cotes and Legendre-Gauss, have as a factor a derivative of even order of the integrand function. Thus a polynomial of highest degree for which one of these formulas is exact is necessarily of odd degree. In particular, Simpson's rule, while based on parabolas, is exact for all cubics. The usual derivations [1] of remainder formulas do not offer a simple explanation for this oddness. It seems not to have been remarked that it is an immediate consequence of the symmetrical nature of these formulas.

Many procedures are available for constructing the unique *interpolating polynomial* $I_n(x)$ of degree n or less corresponding to an arbitrary function $F(x)$ and $n+1$ distinct numbers $x_0 < x_1 < x_2 < \cdots < x_n$ of its domain of definition.

In particular, Lagrange remarked that

$$I_n(x) = \sum_{i=0}^{n} L_i(x)F(x_i), \quad \text{where} \quad L_i(x) = \frac{\Pi_{(i)}(x)}{\Pi_{(i)}(x_i)}$$

with

$$\Pi_{(i)}(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n).$$

Thus

$$F(x) = I_n(x) + R_{n+1}(x), \tag{1}$$

where $R_{n+1}(x_i) = 0$, $i = 0, 1, 2, \cdots, n$. If $F(x)$ is integrable on the interval $[a, b]$ then

$$\int_a^b F(x)dx = \sum_{i=0}^{n} A_iF(x_i) + \int_a^b R_{n+1}(x)dx, \tag{2}$$

where $A_i = \int_a^b L_i(x)dx$. Thus the A_i depend only on the numbers $a, b; x_0, x_1, \cdots, x_n$. The summation on the right of equation (2) is called an *interpolatory quadrature formula* and the integral on the right the *remainder*. If $a = x_0 - \alpha$, $b = x_n + \alpha$, where α is arbitrary, and if all the x_i are such that $x_i - x_0 = x_n - x_{n-i}$, the interpolatory quadrature formula is said to be *symmetrical*. A simple calculation shows that $A_i = A_{n-i}$, $i = 0, 1, 2, \cdots, n$, for any symmetrical formula. In particular, all the Newton-Cotes and Legendre-Gauss formulas are symmetrical. An interpolatory quadrature formula is said to be *exact* if the remainder is zero. If such a formula is exact for functions $f(x)$ and $g(x)$ it is necessarily exact for $af(x) + bg(x)$, where a and b are arbitrary constants.

We now suppose a particular symmetrical interpolatory quadrature formula $\sum_{i=0}^{n} A_iF_i$ exact for all polynomials of degree $\leqq 2k$. An arbitrary polynomial of degree $2k+1$ can be written in the form $\sum_{i=0}^{2k+1} a_i(x-\mu)^i$, where $\mu = (x_0 + x_n)/2$. But

$$\int_{x_0-\alpha}^{x_n+\alpha} (x - \mu)^{2k+1}dx = 0 = \sum_{i=0}^{2k+1} A_i(x_i - \mu)^{2k+1}.$$

Thus an arbitrary polynomial of degree $2k+1$ can be written as a linear combination of functions for which the given formula is exact and so it is exact for all polynomials of degree $2k+1$. We shall say that a quadrature formula is of *order N* if it is exact for all polynomials of degree $\leqq N$ but is not exact for some polynomial of degree $N+1$. We have thus established the following

THEOREM. *A symmetrical interpolatory quadrature formula is necessarily of odd order.*

Reference

1. V. I. Krylov, Approximate Calculation of Integrals, translated from the Russian by A. H. Stroud, Macmillan, New York, 1962.

ATTEMPT AT A STRONGEST VECTOR TOPOLOGY

E. O. THORP, New Mexico State University

A set A in E is circled if $aA \subset A$ for all scalars a such that $|a| \leqq 1$. A set is radial at 0 if it contains a line segment through 0 in each direction. A vector topology for E is a topology such that addition and scalar multiplication are each jointly continuous. A local base is a fundamental system of neighborhoods of 0.

If E is an infinite dimensional real or complex linear space, the family of all circled sets radial at 0 is not a local base for a vector topology for E. This is observed in [1, I, Section 1, no. 9, ex. 6] and the example is repeated in [2, 5D], where the infinite dimensionality of E is essential to the argument.

We offer here the sharper observation that if E is a real or complex linear space, the family of all circled sets radial at 0 is not a local base for a vector topology for E iff dim $E \geqq 2$.

First consider the case $E = E^2$, where E^2 is either real or complex two dimensional Euclidean space and $\{u_1, u_2\}$ is the usual orthonormal basis. For each x in E such that $|x| = 1$, let $f(x)$ be the magnitude $|(x, u_1)|$ of the inner product (x, u_1). Let A consist of a union of ray segments, one for each x such that $|x| = 1$. If $f(x)$ is 0 or irrational, let $[0, x] \subset A$. If $f(x) = p/q$ is a nonzero rational in lowest terms, let $[0, x/q] \subset A$. The fact that A is circled follows from $|(x, u_1)| = |(e^{i\theta}x, u_1)|$. Clearly A is radial at 0.

There is no set C radial at 0 with $C + C \subset A$. Suppose that there is such a C. Then there is an $\epsilon > 0$ such that $[0, \epsilon u_1]$ and $[0, \epsilon u_2]$ are in C. Therefore $C + C$ contains all ray segments $[0, x]$ where

$$x = t_1 u_1 + t_2 u_2, \qquad 0 \leqq t_1, t_2 \leqq \epsilon.$$

Letting

$$y = \frac{t_1 u_1 + t_2 u_2}{\sqrt{(t_1^2 + t_2^2)}}, \quad \text{we have} \quad f(y) = \frac{t_1}{\sqrt{(t_1^2 + t_2^2)}}.$$

Let q be a prime such that $1/q < \epsilon$. If $t_1 = \epsilon/q$ and $t_2 = \epsilon\sqrt{(1 - 1/q^2)}$, then $f(y) = 1/q$. Hence the ray in A in the y direction is $[0, y/q]$. Now $|y| = \epsilon$ so y is not in A. But y is in $C + C$. Thus $C + C \not\subset A$.

In the general case dim $E \geqq 2$, let e_1 and e_2 be linearly independent vectors in E. Then the subspace $F = \operatorname{sp}(e_1, e_2)$ spanned by e_1 and e_2 is isomorphic to E^2 under the map I such that $Iu_1 = e_1$, $Iu_2 = e_2$. If $B = I(A)$, B as a subspace of F has all the properties proved above for A. Let G be a subspace complementary to F. Then $G + A$ is radial at 0 in E and is circled.

There is no set C in E which is radial at 0 and such that $C + C \subset A + G$ (and thus the circled sets radial at 0 do not yield a vector topology). Suppose that there is such a C. Then $D = C \cap F$, considered as a subset of F, is radial at 0 and $D + D \subset A$. But this contradicts our result for E^2.

If dim $E \leqq 1$, the family of circled sets radial at 0 is a base for the Euclidean topology.

REMARK. In the real case for E^2 the proof is simpler and more intuitive if we let $f(x)$ be the angle between x and e_1.

References

1. N. Bourbaki, Espaces vectoriels topologiques, Actualités Sci. Ind., 1189 and 1229, Hermann, Paris, 1953 and 1955.

2. J. L. Kelley, Linear topological spaces, Van Nostrand, Princeton, N. J., 1963

ON SEQUENCES OF POWER SERIES WITH RESTRICTED COEFFICIENTS

O. SHISHA, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio

1. We begin with the following result, for which an elementary proof is provided.

THEOREM 1. *Let S be a subset of the disk $|z|<1$. For $k=1, 2, \cdots$, let*

$$P_k(z) \equiv \sum_{j=0}^{\infty} a_j^{(k)} z^j$$

converge in $|z|<1$, and for each $z \in S$ let the sequence $P_k(z)$ converge to some $f(z)$. Suppose also that all $|a_j^{(k)}|$ are smaller than a constant $M(<\infty)$ and that all $a_j^{(k)}$ belong to some closed set C. Then there exists a power series $\sum_{j=0}^{\infty} a_j z^j$, convergent in $|z|<1$, with all $a_j \in C$, such that $f(z)=\sum_{j=0}^{\infty} a_j z^j$ throughout S.

2. Examples. Suppose that for $k=1, 2, \cdots$, $\sum_{j=0}^{\infty} a_j^{(k)}$ converges, and that $P_k(z) \equiv \sum_{j=0}^{\infty} a_j^{(k)} z^j$ converges to a function $f(z)$ in $S \cup \{1\}$, where S is a subset of $|z|<1$.

A. Assume that all $a_j^{(k)}$ are $\geqq 0$. Then since the sequence $(\sum_{j=0}^{\infty} a_j^{(k)})_{k=1}^{\infty}$ is bounded, all hypotheses of Theorem 1 hold, with C being the ray $\text{Re}(z) \geqq 0$.

B. More generally, assume that all $a_j^{(k)}$ belong to a closed sector C with vertex at the origin, whose angular measure ϕ satisfies $0 \leqq \phi < \pi$. Let $M_1(<\infty)$ be an upper bound for all $|\sum_{j=0}^{\infty} a_j^{(k)}|$. Let $j(\geqq 0)$ and $k(\geqq 1)$ be given integers. Then $a_j^{(k)} \leqq M_1$ if $\phi \leqq \pi/2$, and $|a_j^{(k)}| \leqq M_1/\sin\phi$ if $\phi > \pi/2$, which shows that the hypotheses of Theorem 1 hold. In proving these inequalities we may assume

$$a_j^{(k)} \neq 0, \qquad b = \sum_{\substack{n=0 \\ n \neq j}}^{\infty} a_n^{(k)} \neq 0.$$

Let $\alpha (0 \leqq \alpha < \pi)$ be the angle between the vectors $a_j^{(k)}$ and b. If $\phi \leqq \pi/2$, then $\alpha \leqq \pi/2$, and therefore $|a_j^{(k)}| < |\sum_{n=0}^{\infty} a_n^{(k)}| \leqq M_1$. If $\phi > \pi/2$ and $\alpha > \pi/2$, then by the law of sines

$$\left|\sum_{n=0}^{\infty} a_n^{(k)}\right| \Big/ \left| a_j^{(k)} \right| \geqq \sin\alpha \geqq \sin\phi,$$

and so $|a_j^{(k)}| \leqq M_1/\sin\phi$. If $\phi > \pi/2$ but $\alpha \leqq \pi/2$, then again

$$|a_j^{(k)}| < \left|\sum_{n=0}^{\infty} a_n^{(k)}\right| \leqq M_1 < M_1/\sin\phi.$$

3. Proofs of Theorem 1. Before we give an elementary proof of Theorem 1 (using only basic properties of sequences and series) we point out that one can prove the theorem readily using a basic property of normal families. In fact, consider the set Φ of all functions which throughout $|z|<1$ are of the form $\sum_{n=0}^{\infty}\alpha_n z^n$, with all $|\alpha_n|<M$. Then Φ is clearly locally uniformly bounded in $|z|<1$, and consequently, Φ is a normal family in that disk. Hence there exists a subsequence $P_{n_k}(z)$ of $P_n(z)$ converging throughout $|z|<1$ to some $F(z)\equiv\sum_{j=0}^{\infty}a_j z^j$, uniformly in every compact subset of $|z|<1$. For $j=0, 1, 2, \cdots$,

$$a_j = F^{(j)}(0)/j! = \lim_{k\to\infty}\left[P_{n_k}^{(j)}(0)/j!\right] = \lim_{k\to\infty} a_j^{(n_k)} \in C.$$

Obviously, throughout S,

$$f(z) = \lim_{k\to\infty} P_{n_k}(z) = \sum_{j=0}^{\infty} a_j z^j.$$

Here is a second proof of Theorem 1. We choose, by the diagonal process, a subsequence $(h_k)_{k=0}^{\infty}$ of $(1, 2, \cdots)$ such that

$$(a_j^{(h_k)})_{k=0}^{\infty}$$

converges (say, to a_j) for $j=0, 1, 2, \cdots$. Then $|a_j| \leqq M$, $a_j \in C$ $(j=0, 1, 2, \cdots)$. Clearly $\sum_{j=0}^{\infty} a_j z^j$ converges in $|z|<1$; we shall prove that it equals $f(z)$ throughout S. Let $|z|<1$, and let ϵ be a positive number. Choose a $J_1(\geqq 1)$ such that

$$\sum_{j=J_1}^{\infty} |z|^j < \frac{\epsilon}{4M}\cdot$$

Choose a $k_1 \geqq 0$ such that if $k \geqq k_1$ and $0 \leqq j \leqq J_1 - 1$ then

$$|a_j^{(h_k)} - a_j| < \frac{\epsilon}{2J_1}\cdot$$

If $J \geqq J_1$ and $k \geqq k_1$, then

$$\sum_{j=0}^{J} |(a_j^{(h_k)} - a_j)z^j| = \sum_{j=0}^{J_1-1} |a_j^{(h_k)} - a_j|\,|z|^j + \sum_{j=J_1}^{J} |a_j^{(h_k)} - a_j|\,|z|^j$$
$$< \sum_{j=0}^{J_1-1} \frac{\epsilon}{2J_1} + \sum_{j=J_1}^{J} 2M|z|^j < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, if $k \geqq k_1$ then

$$\left|\sum_{j=0}^{\infty} a_j^{(h_k)} z^j - \sum_{j=0}^{\infty} a_j z^j\right| \leqq \sum_{j=0}^{\infty} \left| (a_j^{(h_k)} - a_j) z^j \right| \leqq \epsilon.$$

We have thus shown that

$$\lim_{k\to\infty} P_{h_k}(z) = \sum_{j=0}^{\infty} a_j z^j$$

throughout $|z| < 1$. Since in S, $\lim_{k\to\infty} P_{h_k}(z) = f(z)$, therefore $f(z) = \sum_{j=0}^{\infty} a_j z^j$ throughout S.

4. For subsequent use we prove here the following simple

LEMMA. *Let C be a closed sector as in example* B. *Then there exists a constant $\mu(0<\mu<\infty)$ such that if $\alpha_0, \alpha_1, \cdots, \alpha_p$ are arbitrary points of C, then*

$$\sum_{j=0}^{p} |\alpha_j| \leqq \mu \left| \sum_{j=0}^{p} \alpha_j \right|.$$

In fact, μ can be taken as 2 if $\phi \leqq \pi/2$, and as $2/\sin\phi$ if $\phi > \pi/2$.

Proof. Let the rays U, V form the boundary of C, and let u, v be vectors starting at the origin and of unit length, along U and V respectively. Let $\alpha_0, \alpha_1, \cdots, \alpha_p$ be points of C, and let $\alpha_j = \xi_j u + \eta_j v (\xi_j > 0, \eta_j > 0, j = 0, 1, \cdots, p)$. Then

$$\sum_{j=0}^{p} \alpha_j = \left(\sum_{j=0}^{p} \xi_j\right) u + \left(\sum_{j=0}^{p} \eta_j\right) v.$$

If $\phi \leqq \pi/2$, then clearly $\left|\sum_{j=0}^{p} \alpha_j\right| \geqq \sum_{j=0}^{p} \xi_j$, $\left|\sum_{j=0}^{p} \alpha_j\right| \geqq \sum_{j=0}^{p} \eta_j$, and therefore

$$2\left|\sum_{=0}^{p} \alpha_j\right| \geqq \sum_{=}^{p} (\xi_j + \eta_j) \geqq \sum_{j=0}^{p} |\alpha_j|.$$

If $\phi > \pi/2$, then by the law of sines,

$$\left|\sum_{j=0}^{p} \alpha_j\right| \geqq \left(\sum_{j=0}^{p} \xi_j\right) \sin\phi, \qquad \left|\sum_{j=0}^{p} \alpha_j\right| \geqq \left(\sum_{j=0}^{p} \eta_j\right) \sin\phi,$$

and therefore $(2/\sin\phi)\left|\sum_{j=0}^{p} \alpha_j\right| \geqq \sum_{j=0}^{p} (\xi_j + \eta_j) \geqq \sum_{j=0}^{p} |\alpha_j|$.

5. We complement Theorem 1 by the following

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold, where C is the ray* $\mathrm{Re}(z) \geqq 0$ *or a closed sector as in example* B. *Suppose that S contains a sequence (x_n) of nonnegative reals such that $x_n \to 1$ and such that $f(x_n)$ converges (say, to $f(1)$). Then in the conclusion of Theorem* 1, *S can be replaced by $S \cup \{1\}$.*

Proof. Let the power series $\sum_{j=0}^{\infty} a_j z^j$ satisfy the conclusions of Theorem 1. We shall prove that $f(1)=\sum_{j=0}^{\infty} a_j$. Suppose first that C is the ray $\operatorname{Re}(z)\geqq 0$. If $\sum_{j=0}^{\infty} a_j$ diverged, then for some J,

$$\sum_{j=0}^{J} a_j > f(1)+1,$$

and therefore, for all $n \geqq$ some n_0,

$$\sum_{j=0}^{J} a_j x_n^j > f(1)+1$$

and therefore

$$f(x_n) = \sum_{j=0}^{\infty} a_j x_n^j > f(1)+1,$$

contradicting the relation $f(x_n)\to f(1)$. Thus, $\sum_{j=0}^{\infty} a_j$ converges, and by Abel's continuity theorem,

$$f(1) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sum_{j=0}^{\infty} a_j x_n^j = \sum_{j=0}^{\infty} a_j. \tag{1}$$

Let us now assume the more general hypothesis on C. Let $\mu(0<\mu<\infty)$ be such that if $\alpha_0, \alpha_1, \cdots$ are points of C for which $\sum_{j=0}^{\infty} \alpha_j$ converges, then

$$\sum_{j=0}^{\infty} |\alpha_j| \leqq \mu \left| \sum_{j=0}^{\infty} \alpha_j \right|$$

(see the Lemma). Again, if $\sum_{j=0}^{\infty} |a_j|$ diverged, then for some K, $\sum_{j=0}^{K} |a_j| > \mu(|f(1)|+1)$, and therefore for all $n \geqq$ some n_1,

$$\sum_{j=0}^{K} |a_j| x_n^j > \mu(|f(1)|+1) \text{ and consequently } \sum_{j=0}^{\infty} |a_j| x_n^j > \mu(|f(1)|+1).$$

Hence $|f(x_n)| = \left|\sum_{j=0}^{\infty} a_j x_n^j\right| > |f(1)|+1 (n=n_1, n_1+1, \cdots)$, contradicting the relation $f(x_n)\to f(1)$. Again (1) gives the desired conclusion.

6. Remarks. I. Suppose that for $k=1, 2, \cdots$, $\sum_{j=0}^{\infty} a_j^{(k)}$ converges, and that

$$P_k(z) \equiv \sum_{j=0}^{\infty} a_j^{(k)} z^j$$

converges uniformly to some $f(z)$ on $S\cup\{1\}$, where S is a subset of $|z|<1$. Let C (containing all $a_j^{(k)}$) be the ray $\operatorname{Re}(z)\geqq 0$ or a closed sector as in example B. Assume also, that 1 is a limit point of the intersection of S with the real axis. As shown in Section 2, all hypotheses of Theorem 1 hold. Let $(x_n)_{n=1}^{\infty}$ be a sequence of nonnegative real points of S converging to 1. Consider the functions $P_k(z)$ restricted to the set $T=\{1, x_1, x_2, \cdots\}$. They are continuous on T, and

the uniform convergence implies that $f(1)=\lim_{n\to\infty} f(x_n)$. By Theorem 2, the conclusion of Theorem 1 holds, with S replaced by $S\cup\{1\}$.

II. Let $(P_k(x))_{k=1}^{\infty}$ be a sequence of polynomials whose coefficients are $\geqq 0$, converging uniformly to some $f(x)$ on $[0, 1]$. By remark I, there exist $a_0, a_1, \cdots$, all $\geqq 0$ such that $f(x)=\sum_{j=0}^{\infty} a_j x^j$ throughout $[0, 1]$. Thus we can conclude (the known fact) that f is absolutely monotonic in $[0, 1]$, i.e. continuous in $[0, 1]$ and satisfying $f(x)\geqq 0, f'(x)\geqq 0, f''(x)\geqq 0, \cdots$, throughout $(0, 1)$.

III. Let f be a function with domain $[0, 1]$, absolutely monotonic in $[0, 1]$. Then for $k=1, 2, \cdots$, the Bernstein polynomial

$$B_k(x) \equiv \sum_{j=0}^{k} f\left(\frac{j}{k}\right)\binom{k}{j} x^j(1-x)^{k-j}$$

is known to have all its coefficients $\geqq 0$. Since $B_k(x)$ converges uniformly to $f(x)$ on $[0, 1]$ it follows, by Remark II, that there exist $a_0, a_1, \cdots$, all $\geqq 0$, such that $f(x)=\sum_{j=0}^{\infty} a_j x^j$ throughout $[0, 1]$. Thus we have the (well-known) result that f can be defined at all points of the disk $|z|<1$ outside $[0, 1)$ so as to become holomorphic in that disk.

COMPACT SPACES AND LOCALLY COMPACT SUBSPACES

RICHARD LAATSCH, Miami University, Oxford, Ohio

In a recently published text the reader is asked to prove that an open subset of a compact space is locally compact (in the relative topology) [1]. The proposition is not true, but the error may well be typographical since the proposition is made true by making the space Hausdorff. The following space provides a counterexample with several additional properties.

Using conventional interval notation, consider the set of real numbers $K=[-1, 1]$. Let the open subsets of K be the empty set, K, and all intervals of the forms $[-1, b)$, (a, b), and $(a, 1]$ with $a<0<b$. Thus every nonempty open set contains 0, and the closure of every nonempty set contains -1 or 1. The space K is compact since, given any open covering of K, any two sets which cover -1 and 1, respectively, will cover K; but no open set (a, b) is locally compact since the relative closure of any open subset of (a, b) is (a, b).

As to other properties of the space K, K is trivially separable since $\{0\}$ is dense in K. Thus the only continuous functions $f: K\to R$ are constant because $f(0)$ and continuity on K fix the values of f everywhere. Also K satisfies the second axiom of countability (using the same base as in the usual topology) but is not metrizable since it satisfies separation axiom T_0 but not T_1.

Reference

1. H. L. Royden, Real Analysis, Macmillan, New York, 1963, Exercise 9.16a, p. 147.

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N. W., Washington, D. C. 20005

COORDINATION OF MATHEMATICS WITH ENGINEERING CURRICULA

R. D. LARSSON, Dean of Instruction, Mohawk Valley Community College

Wherever curricula in the sciences or engineering are offered, the problem of the mathematics department playing the role of a service department is a major concern. Viewpoints vary from institution to institution as to the responsibilities and prerogatives of the mathematicians in curriculum development.

One point of view is that mathematics is mathematics and it makes little difference whether the students taking a calculus course, for example, are studying to be civil engineers or physicists or even mathematicians. It is held that a rigorous course in calculus will prepare the student to use it in whatever type of application may be forthcoming in his later work in other courses. Advocates of this approach would ask the science and engineering departments to indicate the broad areas of mathematics desired and then leave it to the mathematics department to design the courses to be offered, without regard to applications.

I would like to make one observation here. There is a tendency in some quarters to feel that a rigorous course means to eliminate all manipulative work and concentrate only on theory. For example, some would eliminate techniques of integration in the calculus. To do this renders a disservice to the student and his engineering instructors, as such a facility cannot be acquired overnight. To include it will not lower the level of any such course.

A second point of view is that the mathematics courses must truly serve the other departments by being directly oriented towards the curriculum involved, with a maximum of applications to better motivate the work and prepare the student for its continuing use in other courses. It is from this philosophy that departments of applied mathematics, or separate service departments, have evolved in some cases.

I believe that there should be extensive consultations between the mathematics department and the departments which it attempts to serve, but the final responsibility for the nature of the courses to be offered and the staffing of those courses should rest solely in the hands of the mathematics department.

Some might argue that the professors of science and engineering know exactly which topics in mathematics they want their students to learn and need only to submit such a list to the mathematics department. Such an argument is not sound for several reasons.

First, no real mathematics course is just a collection of topics. It is an orderly development of basic concepts with some unifying thread that binds them to-

gether just like a well tailored garment. Not just any collection of topics can be woven together in this fashion.

Second, if one tries this approach in practice, he will find that each engineering department will have its own idea as to the makeup of the course it wants. At the level following the calculus, it is quite likely that a mathematics department would have to offer a separate course for the students in each engineering and science curriculum in order to satisfy the demand. In some cases this would be highly impractical for economic reasons. In addition, there is a real question as to the possibility of inherent dangers in excessive specialization at a low level. Can we pinpoint, at the undergraduate level, the precise mathematical topics needed in the future by the students in each separate department of engineering? I think not.

As a matter of fact, I have observed the following through actual experience. Suppose a mathematics department asks an engineering department to make suggestions as to the makeup of a mathematics course at a certain level. You will find that the engineering faculty will make almost as many different suggestions of topics as if they were members of separate departments. I recall one man who insisted that a course in statistics was the only one of any importance for civil engineers. It turned out that his doctoral thesis was based primarily on the use of statistics. Another man in an electrical department was sold on a course in theory of functions of a real variable. He had taken one as a graduate student. In each case the man mentioned was positive and sincere in his belief, but his ideas were not shared by others in the same department. The final decision on which course or courses should be offered may not be easy. One would hope that the decision would not go simply to the most vocal proponent of certain topics.

But let us face two very important facts. The professor in charge of a given engineering course wants to have the mathematics, which the course demands, available when he needs it and not later than that. The student wants to be able to apply his mathematical knowledge when called upon to do so. We are only too familiar with complaints of other departments that certain mathematical topics are not taught early enough or that the students cannot apply their mathematics in engineering problems.

The real difficulty is that too many faculty members have little more than a superficial knowledge, at best, of the makeup of the courses in other departments. There is too much of a tendency for departments to exist in relative isolation from one another in this respect.

One of the most helpful committee experiences which I have had in education was to serve on a committee which was charged with studying the correlation between mechanics, mathematics and physics, as well as whether or not the mechanics courses satisfied the needs of the engineering departments of that college under a proposed new curriculum. I remember that it soon became evident that we could not arrive at any sound judgments until we had available weighted outlines for each course, showing not only the point at which a topic

was introduced, but the time allotted to it. We all had general opinions, based on hearsay evidence in most cases. We soon found out whether or not they were valid.

I will not take time to detail the complete study, which ran over a period of several months, but indicate some of the items considered.

First, two departments found evidence that there should be some additional work included, such as statistical mechanics. This question was referred to the departments concerned for specific recommendations as to the possibilities of including such material, how much should be included, the position in the program and whether it should be given in mathematics, physics or mechanics.

Second, we found, counter to earlier general impressions, that the correlation between physics, mathematics and mechanics would be good. The criteria were that the prerequisite courses for each individual course in mechanics contained the desired subject matter with the proper time allotted to it and in the proper sequence so that, in the case of concurrent courses, the topics would be covered early enough in the prerequisite course to be of use in the mechanics course.

Third, it became clear that the correlation of the mechanics courses with some of the engineering courses would not be good. In the revised curriculum dynamics would have followed fluid dynamics, which would have made it difficult to teach fluid dynamics and would have prevented the use of dynamics in some electrical courses offered earlier. But dynamics had been shifted in order to make greater use of differential equations. It appeared that the best solution would be to shift the dynamics back, but this probably would make necessary a shift in the materials testing and fluid dynamics laboratories, which might require changes in other departments. Problems like this are striking illustrations of why each department should be fully cognizant of the needs and programs of the others.

Fortunately, a few years later, a shift in the mathematics courses which brought calculus into the first semester of the first year, followed later by a consolidation of the mechanics courses, alleviated many of these difficulties. But again this was not so much a unilateral decision on the part of the departments concerned, as a realization of the favorable impact which it would have on the courses in engineering and science.

But when you come right down to it, the determining factor in the coordination of mathematics and engineering is the professor in the classroom. Here coordination will rise or fall despite all the other efforts to which I have referred.

Just a few years ago I was invited to teach a section in dynamics for the civil engineering department, followed by a section of mechanics of materials the next semester and then dynamics again the semester after that. The purpose was to determine whether or not these courses were making maximum and most effective use of the mathematics which the students had had up to that point.

This was quite a challenge, as I had never taken a course in dynamics. In the case of the materials course, I had audited such a course some twelve years

before. I was to cover the same material as the other sections, teach it my own way and then have my students take a common final examination with the students from the other sections.

I had the opportunity of frequent consultations with some members of the civil engineering department. In the case of the materials course, I used six different texts to give me a broad picture of the way various topics might be presented. It took a great deal of time to make an adequate preparation.

There were some obvious deficiencies in the use of mathematics and the texts did not help matters. I hold no brief either for or against the use of vectors, but I do say that one should exploit them fully or not at all. To use the vector notation, without the tools, is misleading to the student. One might just as well be working with components only. Unit vectors, especially in polar coordinates where they are variables, are very helpful in kinematics. To struggle through the derivation of the Coriolis acceleration using rectangular coordinates, as the author did, seemed a waste of space, time and knowledge.

It also seemed awkward, for example, to solve a homogeneous, constant coefficient differential equation by the use of the independent variable missing method, especially when the students had just studied the easier approach which made the problem trivial.

I found too much evidence that problems were designed to eliminate the need for integration wherever possible. It certainly is true that graphical methods are helpful in some problems; that formulas are useful in many cases and that approximations are often required. But the fundamental principles are the most important elements of any topic and too often they are forgotten or minimized in the haste to simplify the examples to be used. Why do we require calculus if we do not intend to use it freely?

This teaching experience was very helpful to me, but I do not recommend it as a general procedure. My inadequacies, especially in the engineering uses of some of the material, must have been a handicap to the students. In the work on welds and rivets, I felt that I was giving lectures that lacked depth, although technically correct. Any successes which I had in final grades were probably balanced out by some of these other factors.

On the basis of the sum total of my experiences in the coordination of mathematics and engineering, I make the following recommendation, and I feel that it can be generalized to cover all departments.

Wherever one department offers a service course to the students of another department, I recommend that the service department provide a faculty member as a part-time consultant to that other department as needed.

He could assist the instructors who are teaching courses for which the service course is a prerequisite, by acquainting them with the material which the students should know; suggesting the best ways in which prerequisite material could be used for a particular topic, perhaps including a guest lecturer as needed; to advise in the development of the course outline as to the timing of prerequisite

material when offered concurrently, (calculus and physics are a good example of this); to aid in the selection of new textbooks, from the point of view of the best use of prerequisite material by the author.

I further believe that the use of consultants can work the other way as well. How often we find the same tried and true applications of engineering in our mathematics texts, some of them rather artificial; and they are frequently taught with little real understanding by the instructor. A consultant from the engineering department concerned could suggest fresh material, better motivation and perhaps lecture as needed.

On the undergraduate level we expect faculty members to be fairly familiar with the courses offered in their own department. It is not reasonable to expect them to be as familiar with courses in other departments. It is no reflection on any engineering faculty member to say that a mathematician might assist him in making the best use of the mathematics needed in his work. The converse is also true. Wherever curricula stray across departmental boundaries, there must be close and continuing consultations in both directions.

Presented at a meeting of the Mathematics Division at the Annual Meeting of the American Society for Engineering Education held at the University of Maine in June, 1964.

CAREER CHOICES OF SCIENCE TALENT SEARCH HONOR STUDENTS

The Honors Group of the nation's top science competition, the Science Talent Search, has been announced by Dr. Watson Davis, director of Science Service. The Search is conducted by Science Clubs of America, a Science Service activity.

The 300 most promising science students from the 1965 high school graduating classes come from communities scattered from the Atlantic Seaboard to Alaska. The 300 students receiving honors are 15 to 18 years old and go to school in 174 communities in 38 states and the District of Columbia. The outstanding student-scientists include 77 girls and 223 boys.

Career preferences of the group cover the entire range of science from aeronautical engineering to zoology. First choice of the group is medical science, chosen by 41 of the 300 Honors Group members. One of these plans to be a veterinarian and another aspires to a dental career. A close second is physics, with 39. Education ranks next, with 38 planning to teach at high school or college level. Biochemistry is next with 28, and chemistry follows closely with 27 selecting that field. Mathematics claims 26, with engineering and biology following with 20 and 19 respectively. The remainder are scattered in almost every scientific field.

Only four are "undecided," while 11 plan non-science careers. These include the ministry, politics and one writing novels.

PROCEEDINGS OF THE INTERNATIONAL CONFERENCE ON MODERN SCHOOL MATHEMATICS

Teacher training programs in mathematics, most of which are outmoded and inadequate in the U.S., must be radically reformed if this country is to make progress in developing good teachers of modern mathematics. This was one of the conclusions of an International Conference on Modern School Mathematics held in November, 1963, in Athens, Greece. The proceedings of the meeting, sponsored by the Organization for Economic Co-Operation and Development (O.E.C.D.), have been published in book form under the title, *Mathematics To-Day: A Guide for Teachers*. It is available in the U.S. through the McGraw-Hill Book Company, O.E.C.D. Unit, TMIS Annex, 351 W. 41st Street, New York, New York. Price: $3.00.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

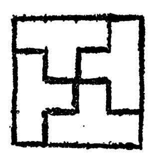
All solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before September 30, 1965.

E 1785. *Proposed by Robert Bowen, University of California, Berkeley*

Let $T_m = \{(a_1, \cdots, a_m): a_i \text{ is a positive integer, } a_1 = 1,\ a_i \leqq 1 + \max_{k<i} a_k\}$, and $t(m)$ be the number of elements in T_m. Set $g_1(x) = 1$ and $g_{k+1}(x) = (x-1)g_k(x) + g_k(x+1)$. Show that $t(m) = g_m(1)$.

E 1786. *Proposed by Robert Spira, Duke University*

A rectangle can be made up of T-shaped tetrominoes (made of unit squares) if and only if each side of the rectangle is divisible by 4. (The simplest case is the 4×4.)



E 1787. *Proposed by Kenneth Kloss, National Bureau of Standards*

Prove the identity

$$\sum_{1 \leq i \leq j \leq n} x_i x_j = \sum_{k=1}^{n} \frac{k+1}{2k}\left(x_k + \frac{x_{k+1} + \cdots + x_n}{k+1}\right)^2.$$

E 1788. *Proposed by Joel Pitcairn, Bryn Athyn, Pa.*

Let J be the open interval $[t: 0<t<1]$. We call a subset of J a C-set in case it contains $1-t$ and ts whenever it contains t and s.

(1) Let E be a C-set. Prove (a) if E is nonvoid, then E is dense in J; and (b) if $\operatorname{int}(E)$ is nonvoid, then $E=J$.

(2) Write t' for $1-t$. Suppose X is a real vector space, $A \subset X$, and f a real-valued function on A. (a) Let E be the set of all those $t \in J$ such that $x, y \in A$ implies $tx+t'y \in A$. (Thus A is midpoint convex iff $\frac{1}{2} \in E$.) Prove that E is a C-set. (b) Suppose A is convex, and let E be the set of those $t \in J$ such that $x, y \in A$ implies $f(tx+t'y) \leq tf(x)+t'f(y)$. Prove that E is a C-set.

E 1789. *Proposed by Richard A. Bell, Kansas City, Mo.*

Suppose that $g(x)$ has its first $n+1$ derivatives defined and continuous in $[-1, 1]$. Define $y(x)=g(x)/x$ for $x \neq 0$ and $y(0)=g'(0)$. If $g(0)=0$, prove that $y^{(n)}(0)=d^n y/dx^n\big|_{x=0}$ exists and equals $g^{(n+1)}(0)/(n+1)$.

E 1790. *Proposed by Alvin Hausner, City College of New York*

For what real numbers m, $n \neq 0$ do there exists functions $y=f(x)$, positive and having continuous derivatives for all real x, such that the mth power of the length of arc of the curve $y=f(x)$ between any two points is proportional to the nth power of the area under the curve between these points? Determine the functions in those cases where they exist.

E 1791. *Proposed by R. G. Buschman, State University of New York at Buffalo*

(a) Consider the set S of circles such that (i) the center of every circle of S is interior to the unit circle and (ii) the point $(1, 0)$ is not interior to any circle of S. Show that the union of the interiors of the circles of the set S is the interior of a cardioid.

(b) Consider the set S' such that (i) holds and (ii) is replaced by (ii') none of the points representing the nth roots of unity are interior to any circle of S'. What figure replaces the cardioid of (a)?

E 1792. *Proposed by A. M. Gleason, Harvard University*

Let f be a function from the reals into the reals, differentiable at every point. Suppose that, for every r, the set of points x where $f'(x)=r$ is closed. Prove that f' is continuous.

E 1793. *Proposed by Walter Bluger, Dominion Bureau of Statistics, Ottawa, Canada*

The projection of a solid body onto three mutually perpendicular planes

yields a unit disk and two areas with square boundaries. Describe the possible shapes of the body.

E 1794. *Proposed by Alan Schwartz, University of Wisconsin*

Let G be a finite abelian group, multiplicatively written; and let n be the order of G. Then n is odd if and only if each element of G is a square.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Skeleton Long Division

E 1691 [1964, 553]. *Proposed by the late Harry Langman, Clarkson College of Technology*

In the following skeleton long division each dash represents a digit and the eight-digit block under the brace represents a repetend. Given that the dividend is a multiple of three, reconstruct the division and show that it is unique.

```
              - - . - - - - - - - - -
- - - ) - - - - - - .
          - - -
          - - - -
          - - - -
                -   - - -
                    - - -
                      - - -
                      - - -
                      - - - -
                      - - - -
                        - - - -
                        - - - -
                            - - -
                            - - -
                              - - -
                              - - -
                                  -
```

Solution by Burnett R. Toskey, Seattle University, Seattle, Washington. The first division shows that the first digit in the dividend is 1. The third division shows that the second remainder, and hence also the last remainder, is 1, so that the reciprocal of the divisor must have an eight digit repetend, i.e., must equal

a fraction with denominator $99,999,999=(9)(11)(73)(101)(137)$. Thus, the divisor must be 101, 137, $(9)(73)=657$, $(9)(101)=909$, or $(11)(73)=803$. But the third division also shows that some "digit" multiple of the divisor is a 3-digit number $9\_\_$, and this rules out 657 and 803. Of the remaining candidates, 101 and 909 give 1-digit remainders at the fourth division (line 10 of the skeleton long-division above), so the divisor must be 137 and the repetend 00729927. Next, the first digit of the quotient must be 7, and the second digit 8 or 9, yielding the two possible dividends 10687 and 10824. The first of these is not a multiple of 3, so the unique division is $137\overline{)10824}$.

Also solved by A. N. Aheart, R. R. Andrews and H. R. Haman (jointly), Merrill Barneby, L. W. Beineke, D. N. Bloom, Walter Bluger, R. A. Carlson, D. I. A. Cohen, C. L. Dotton, D. L. Dye, E. T. Frankel, Murray Geller, Anton Glaser, Gregory Gruska, D. M. Hancasky, B. A. Hausmann, S.J., Stephen Hoffman, J. A. H. Hunter, Edgar Karst, T. M. Little, D. C. B. Marsh, Franklin Mohr, J. B. Muskat, Walter Penney, Donald Quiring, Azriel Rosenfeld, Robin Sibson, Jr., K. Soni, Ira Sterbakov, Joseph Sullivan, C. W. Trigg, Simon Vatriquant, Gary Venter, D. R. Wilder, R. L. Wilson, and the proposer.

Wilder found a similar problem in W. W. Rouse Ball, *Mathematical Recreations and Essays*, Macmillan, 1942, p. 22, attributed to Professor Schuh of Delft.

Number of Relations from One Set onto Another

E 1692 [1964, 554]. *Proposed by Roy Feinman, Rutgers University*

Let A and B be sets containing m and n elements respectively, with $1\leqq m\leqq n$. How many relations are there from A to B whose domain is A and whose range is B?

I. *Solution by Stephen Hoffman, Trinity College, Hartford, Connecticut.* If S is a k-membered set, the number of relations from A to S whose domains include a specific member of A is 2^k-1, (the empty relation is excluded!), so that the number of relations on A to S is $(2^k-1)^m$. If $B=\{b_1, b_2, \cdots, b_n\}$ let A_i be the set of all relations on A to $B-\{b_i\}$ $(i=1, 2, \cdots, n)$, so that $\bigcup_{i=1}^n A_i$ is the set of all relations on A to B, each of whose ranges misses some member of B. It follows that the number of relations on A onto B is $(2^n-1)^m-\nu(\bigcup_{i=1}^n A_i)$, where $\nu(T)$ is the number of members in the finite set T. If $1\leqq i_1<i_2<\cdots<i_k\leqq n$, then $A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}$ is the set of all relations on A to an $(n-k)$-membered set, and the number of such relations is accordingly $(2^{n-k}-1)^m$. Then

$$\begin{aligned}\nu\left(\bigcup_{i=1}^n A_i\right) &= \sum_{k=1}^n (-1)^{k-1} \sum_{1\leq i_1<i_2<\cdots<i_k\leq n} \nu(A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k})\\ &= \sum_{k=1}^n (-1)^{k-1}\binom{n}{k}(2^{n-k}-1)^m = \sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n}{j}(2^j-1)^m,\end{aligned}$$

and the required number of relations is then $\sum_{j=0}^n (-1)^{n-j}\binom{n}{j}(2^j-1)^m$.

II. *Solution by N. J. Fine, Pennsylvania State University.* Let $f(m, n)$ be the required number of relations, with $m\geqq 0$, $n\geqq 0$. Given a set C with M ele-

ments and a set D with N elements, we count all the relations between C and D, that is, all the subsets of $C\times D$. The number of these is 2^{MN}. But they can also be classified by the number m of elements in the domain and the number n of elements in the range. To each m and n there are $\binom{M}{m}\binom{N}{n}$ pairs (A, B) with $A\subset C$, $B\subset D$, $\#A=m$, $\#B=n$, that can serve as domain and range. For each pair there are, by definition, $f(m, n)$ relations having that pair as domain and range. Hence

$$\begin{aligned} 2^{MN} &= \sum_{m,n\geq 0}\binom{M}{m}\binom{N}{n}f(m, n) = \sum_{m,n\geq 0}\binom{M}{m}\binom{N}{n}E_1^m E_2^n f(0, 0) \\ &= (E_1+1)^M(E_2+1)^N f(0, 0), \end{aligned}$$

where E_1 and E_2 are the shift operators: $E_1f(x, y)=f(x+1, y)$, $E_2f(x, y)=f(x, y+1)$, and $E_i^{m+1}f(x, y)=E_i\{E_i^m f(x, y)\}$. Therefore,

$$\begin{aligned} f(m, n) &= E_1^m E_2^n f(0, 0) = \{(E_1+1)-1\}^m\{(E_2+1)-1\}^n f(0, 0) \\ &= \sum_{M,N\geq 0}(-1)^{m-M}(-1)^{n-N}\binom{m}{M}\binom{n}{N}(E_1+1)^M(E_2+1)^N f(0, 0) \\ &= \sum_{M,N\geq 0}(-1)^{m-M}(-1)^{n-N}\binom{m}{M}\binom{n}{N}2^{MN}. \end{aligned}$$

Also solved by Shair Ahmad, D. M. Hancasky, R. A. Jacobson, A. M. Kriegsman, E. S. Langford, D. C. B. Marsh, B. R. Toskey, and the proposer.

Most solvers observed that $(2^n-1)^m=\sum_{k=1}^n\binom{n}{k}f(m, k)$ with $f(m, n)$ as in Solution II above, and then verified the final result by mathematical induction.

Simson Line of a Triangle

E 1693 [1964, 554]. *Proposed by F. J. Servedio, Iona College, New Rochelle, N. Y.*

Given three concurrent circles such that their other three points of intersection are collinear, prove that their centers are concyclic with the point of concurrency.

I. *Solution by C. W. Trigg, San Diego, California.* Let the circles with centers at O_1, O_2, O_3 be concurrent at A, and the pair (O_1), (O_2) also at B, (O_1), (O_3) at C, and (O_2), (O_3) at D. O_3O_1 and O_3O_2 are perpendicular to CA and AD, respectively, so $\angle O_1O_3O_2+\angle CAD=180°$. In circle (O_1), $\angle ACB=\frac{1}{2}\widehat{AB}=\angle AO_1O_2$ (since O_1O_2 is the perpendicular bisector of AB). In O_2, $\angle ADB=\frac{1}{2}\widehat{AB}=\angle AO_2O_1$. Now C, B, D are collinear, so triangles ACD and AO_1O_2 are similar, and $\angle CAD=\angle O_1AO_2$. It follows that $\angle O_1O_3O_2+\angle O_1AO_2=180°$, and $O_1O_3O_2A$ is inscriptible.

II. *Solution by D. P. Ambrose, University of Colorado.* Denote the circles by WBC, WCA, WAB, where A, B, C, are collinear, and denote the centers of the circles by O_A, O_B, O_C respectively. The line of centers O_BO_C perpendicularly bisects WA at A' say. Define points B' and C' similarly. Then since A, B, C are

collinear, A', B', C' are collinear. Thus the feet of the perpendiculars from W to the sides of the triangle $O_AO_BO_C$ are collinear, and hence by the converse of the Simson (pedal) line theorem, W, O_A, O_B, O_C are concyclic.

Also solved by D. P. Ambrose (another solution), Leon Bankoff, J. Basile, D. I. A. Cohen, D. L. Dye, Geoffrey Elsten and Roberta Reiff (jointly), S. H. Greene, H. Guggenheimer, Hajna Janos, W. R. McEwen, D. C. B. Marsh, Walter Penney, Stanton Philipp, J. M. Quoniam, P. T. A. Riddy, H. D. Ruderman, Horvath Sandor, W. M. Self (two solutions), Robin Sibson, Jr., R. P. Soni, Helen Strassburg, Van de Vyle, Clery Vanhelleputte (three solutions), Simon Vatriquant, Hazel S. Wilson, and the proposer.

McEwen and Vanhelleputte point out that the result is a theorem in section 287 of N. A. Court, *College Geometry*, 2nd ed., and Ambrose cites E. A. Maxwell, *Geometry for Advanced Pupils*, pp. 52–53.

A Known Ratio

E 1694 [1964, 554]. *Proposed by Omar Khayyam, Jr., University of California at Berkeley*

Let R, r, P, p respectively represent the circumradius, inradius, perimeter, and perimeter of the orthic triangle of a given acute triangle. Prove that $P/p = R/r$.

Solution by Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California. The result is not new. According to N. A. Court, *College Geometry*, Johnson Publishing Co. (1925), page 89, the perimeter of the orthic triangle is equal to the doubled area of the given triangle divided by the circumradius of the given triangle. Thus, $P=2\Delta/R=rp/R$ or $P/p=R/r$.

Other references: N. A. Court, *College Geometry*, Barnes and Noble, 1952, page 100, Theorem 199; R. A. Johnson, *Modern Geometry*, (Dover reprint), p. 191.

Also solved by A. N. Aheart, D. P. Ambrose, W. J. Blundon, D. I. A. Cohen, Michael Goldberg, K. K. Gorowara, D. M. Hancasky, Ned Harrell, Andrew Jarosak, Geoffrey Kandall, N. F. Lindquist, W. R. McEwen, D. C. B. Marsh, Stanton Philipp, J. M. Quoniam, Simeon Reich, V. K. Rohatgi, Horvath Sandor, M. S. R. K. Sastry, C. W. Trigg, Simon Vatriquant, Hazel S. Wilson, and the proposer.

Morley Triangles

E 1695 [1964, 554]. *Proposed by J. B. Reynolds, Sugar Run, Pa.*

If the trisectors of the exterior angles of a triangle are drawn so that those adjacent to each side intersect, prove that the intersections are the vertices of an equilateral triangle.

Solution by D. C. B. Marsh, Colorado School of Mines. We need this simple identity:

$$\frac{\sin\phi}{\sin\phi/3} \equiv 4\sin\frac{\pi-\phi}{3}\sin\frac{2\pi-\phi}{3}\cdot$$

Denote both the vertices and interior angles of the given triangle by A, B, C; denote the length of the side opposite A, B, C by a, b, c; let the trisectors adjacent to sides AB, BC, CA meet at C', A', B'. Elementary geometry gives $\angle ACB' = (\pi - C)/3$, $\angle C'BA = (\pi - B)/3$, $\angle CB'A = (2\pi - B)/3$, $\angle AC'B = (2\pi - C)/3$, and $\angle B'AC' = (2\pi + A)/3$. Applying the Sine Law to triangles ACB', ABC', and ABC, we find

$$AB' = \frac{b \sin \{(\pi - C)/3\}}{\sin \{(2\pi - B)/3\}}, \qquad AC' = \frac{c \sin \{(\pi - B)/3\}}{\sin \{(2\pi - C)/3\}}, \qquad \frac{b}{c} = \frac{\sin B}{\sin C}.$$

These ratios, in conjunction with the identity of line 1, yield

$$\frac{AB'}{AC'} = \frac{\sin B/3}{\sin C/3}.$$

With $\angle B'AC' = (2\pi + A)/3$ in triangle $B'AC'$, it follows that $\angle AC'B' = B/3$ and $\angle C'B'A = C/3$. Thus, $\angle B'C'A' = \angle AC'B - \angle AC'B' - \angle A'C'B = \pi/3$. Appealing to symmetry, we have $\angle C'A'B' = \angle A'B'C' = \pi/3$ as well, showing that the intersection of the trisectors as described are vertices of an equilateral triangle.

Comment by Leon Bankoff, 6360 Wilshire Boulevard, Los Angeles, California. To the best of my knowledge, the first published solution to this problem appeared in the *Educational Times*, July 1, 1908, p. 307, problem 16381. It was proposed by E. J. Ebden and was solved by M. Satyanarayana. The proposal included three other related problems, one of which concerns the configuration associated with what is known today as the Morley Theorem. The solution published in 1908 is a direct trigonometrical approach and is based on an idea quite a bit different from the solution I submitted.

The reference to the *Educational Times* problem was sent to me some time ago by J. A. H. Hunter and H. S. M. Coxeter, along with an additional solution (indirect geometrical) by M. T. Naraniengar, published in the *Educational Times*, New Series 15, 1909, page 47.

Comment by Robin Sibson, Sr., Sutton, Surrey, England. As long ago as February, 1938, W. J. Dobbs gave an exhaustive description of the complete Morley figure in the *Mathematical Gazette*, indicating which of the triangles obtainable from the trisectors of the angles of a triangle are equilateral and which, in general, are not. An article of mine in the same journal for May 1960, gives an analytical method for dealing with the same question, which is easily applicable to the present problem.

Also solved by D. P. Ambrose, Leon Bankoff, Leonard Carlitz, Michael Goldberg, D. M. Hancasky, Stephen Hoffman, Geoffrey Kandall, Simon Vatriquant, and the proposer.

Carlitz found the problem in H. S. M. Coxeter, *Introduction to Geometry*, p. 24.

An Odd Polynomial

E 1696 [1964, 555]. *Proposed by D. I. A. Cohen, Princeton University, and Ralph Greenberg, University of Pennsylvania*

Prove that a polynomial $P(x)$ which assumes real values for real x and imaginary values for imaginary x must be linear.

I. *Solution by Charles Chouteau, West Virginia State College, Institute, West Virginia.* The example $P(x)=x^3$ shows that the conclusion of the problem is not obtained if "imaginary" means "pure imaginary." We therefore interpret "imaginary" as "nonreal." Writing $P(x)=a(x-x_1)(x-x_2)\cdots(x-x_n)$, it is then clear that the zeros of $P(x)$ are real. Substitution of any other real x shows that a is real, so $P(x)$ is a real polynomial. It is possible (translate vertically!) to choose a real number b so that $P(x)-b=0$ has no more than one real root. If $n>1$, then, $P(x)=b$ for some nonreal value of x.

II. *Solution by M. J. Pascual, Maggs Research Center, Watervliet Arsenal, N. Y.* The conclusion of the problem should read "must be nonzero, real, and odd." If $P(x)=\sum_{k=0}^{n}(a_k+ib_k)x^k$ with the a_k and b_k real, then $\sum_{k=0}^{n}b_kx^k=0$ for all real x, so that $b_0=b_1=\cdots=b_n=0$. For $x=iy$ with y real and nonzero,

$$P(iy)=\sum_{2k\leq n}(-1)^k a_{2k}y^{2k}+i\sum_{2k+1\leq n}(-1)^k a_{2k+1}y^{2k+1}.$$

Since $P(iy)$ is imaginary for y real, it follows that $a_0=a_2=\cdots=a_{[n/2]}=0$ and $a_{2k+1}\neq 0$ for some k (and $P(x)$ has no imaginary zeros!).

The condition that $P(x)$ be nonzero, real, and odd is not sufficient for $P(x)$ to assume real values for real x and imaginary values for imaginary x. Consider, for example, $P(x)=x+x^3$; here, $P(\pm i)=0$.

Also solved by Shair Ahmad, E. R. Barnes, William Becker, W. Bluger, J. A. Burslem, Leonard Carlitz, Roy Feinman, N. J. Fine, D. M. Mead, W. D. Fryer, Anton Glazer, Michael Goldberg, A. J. Goldman, Victor Goodman, S. H. Greene, Fred Gross, D. M. Hancasky, Stephen Hoffman, Yasuhiko Ikebe, Erwin Just and Norman Schaumberger (jointly), A. J. Karson, P. L. Kingston, Murray Lieb, D. C. B. Marsh, Norman Miller, J. M. O'Neil, M. J. Pascual, H. D. Ruderman, Horvath Sandor, Donna Jean Seaman, Robin Sibson, Jr., Edward Thiel, B. R. Toskey, Andy Vince, Edward Walter, R. E. Whitney, and J. E. Wilkins, Jr.

In the spirit of the first solution above, Gross shows that any entire function $f(z)$ which is real if and only if z is real, must be linear. His argument is based on the Great Picard Theorem. Seaman, on the other hand, uses the Symmetry Principle to prove that an entire function $f(z)$ which is real for real z and imaginary for imaginary z, has a Taylor Series $\sum_{n=0}^{\infty}a_nz^{2n+1}$ with the a_n real.

Common Divisors of Certain Binomial Coefficients

E 1697 [1964, 555]. *Proposed by Harley Flanders, Purdue University*

Let $2^r\leq n<2^{r+1}$ and $0\leq k\leq n-2^r$. Show that the binomial coefficients

$$\binom{n}{k}\quad\text{and}\quad\binom{n-2r}{k}$$

have the same parity.

Solution by E. S. Langford, Autonetics Division, North American Aviation, Anaheim, California. There is a misprint:

$$\binom{n-2r}{k} \text{ should be } \binom{n-2^r}{k}.$$

We make the following generalization: *Let p be a prime and suppose that $p^r \leqq n < p^{r+1}$ and $0 \leqq k \leqq n - p^r$. Then p^s, $s \geqq 0$, divides $\binom{n}{k}$ if and only if it divides*

$$\binom{n-p^r}{k}.$$

(Note that this is stronger than the stated result in case $p=2$.)

Let $t(n-j)$ $(j=0, 1, \cdots, k-1)$ be the highest power of p dividing $n-j$. Since $n-j<p^{r+1}$, it is clear that $t(n-j) \leqq r$, and it follows immediately that $t(n-p^r-j) \geqq t(n-j)$. Conversely, since $n \geqq p^r$ and, hence, $n-p^r-j<p^{r+1}$, we have that $t(n-p^r-j) \leqq t(n-j)$, and hence equality. But clearly $t(\mu\nu)=t(\mu)+t(\nu)$, so that $t((n)_k)=t((n-p^r)_k)$ and, then,

$$t\left(\binom{n}{k}\right) = t\left(\binom{n-p^r}{k}\right).$$

Also solved by D. I. A. Cohen, N. J. Fine, S. H. Greene, D. M. Hancasky, D. C. B. Marsh, Walter Penney, Stanton Philipp, and the proposer.

Minimum of $\prod_1^{10} p_k - \sum_1^{10} p_k^2$

E 1698 [1964, 555]. *Proposed by D. L. Silverman, National Security Agency, Fort Meade, Md.*

Solve the equation $984+\sum_{k=1}^{10} p_k^2 = \prod_{k=1}^{10} p_k$, where the p_k are primes.

I. *Solution by J. E. Yeager, Temple University, Philadelphia, Pennsylvania.* Let L and R denote the left and right sides, respectively, of the given equation. Furthermore, in all congruences considered, the modulus will be 8. Now, any prime p is either 2 or congruent to 1, 3, 5, or 7, giving $p^2 \equiv 4$ or $p^2 \equiv 1$, respectively. If n is the number of $p_k=2$, then n is even since, otherwise, L is odd while R is even. If $n=0$, $L \equiv 2$ while $R \equiv 1$, 3, 5, or 7. If $n=2$, $L \equiv 0$ while $R \equiv 4$. Since $8 \mid 4n$,

$$L = 984 + 4n + \sum_{p_k \text{ odd}} p_k^2 \equiv 4n + (10-n) \equiv 2 - 2n.$$

If $n>3$, then $n=10$ since $R \equiv 0$. That $n=10$ is a solution can be verified by inspection.

II. *Solution by Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania.* The equation is satisfied for $p_1=p_2= \cdots =p_{10}=2$. For $p_k=2+a_k>2$,

we have

$$984 + \sum_{k=1}^{10} (2 + a_k)^2 = 2^{10} + 2^2 \sum_{k=1}^{10} a_k + \sum_{k=1}^{10} a_k^2$$

$$< 2^{10} + 2^9 \sum a_i + 2^8 \sum_{i<j} a_i a_j + \cdots + a_1 a_2 \cdots a_{10}$$

$$= \prod_{k=1}^{10} (2 + a_k).$$

Also solved by A. N. Aheart, Joseph Arkin, F. W. Arcuri and P. L. Kingston (jointly), Merrill Barneby, Joseph Bechely, Ralph Bennett, W. J. Blundon, J. A. Burslem, W. Bluger, Allan Chuck and Peter Goldstein (jointly), D. I. A. Cohen, J. R. Curry, M. J. De Leon, N. J. Fine, W. D. Fryer, Michael Goldberg, A. J. Goldman, S. H. Greene, D. M. Hancasky, Stephen Hoffman, Hajna Janos and Horvath Sandor (jointly), J. E. Jean, Jr., Kenneth Kramer, J. L. Lebbert, D. C. B. Marsh, J. B. Muskat, W. I. Nissen, Jr., Stanton Philipp, S. W. Reyner, J. A. Schneer, J. J. Schneider, Robin Sibson, Jr., Joseph Sullivan, A. M. Vaidya, T. Vanden Eynden, Van de Vyle, Simon Vatriquant, R. L. Wilson, and the proposer.

Complete Residue Systems Modulo n

E 1699 [1964, 555]. *Proposed by Azriel Rosenfeld, Yeshiva University*

Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be permutations of $1, \cdots, n$. Prove that:
(1) $a_1+b_1, \cdots, a_n+b_n$ can be incongruent modulo n if and only if n is odd,
(2) $a_1b_1, \cdots, a_nb_n$ can never be incongruent modulo n if $n>2$.

Solution by N. J. Fine, Pennsylvania State University. (1) For n odd, choose $a_j=b_j=j$, $(j=1, \cdots, n)$. Then $\{a_j+b_j\}=\{2j\}$ is a complete residue system mod n.

If n is even, then for any complete residue system $\{a_j\}$ mod n,

$$\sum_{j=1}^{n} a_j \equiv \sum_{j=1}^{n} j \equiv \frac{n(n+1)}{2} \equiv \frac{n}{2} \pmod{n}.$$

Thus, if $\{b_j\}$ is another complete residue system mod n,

$$\sum_{j=1}^{n} (a_j + b_j) \equiv 0 \pmod{n}.$$

Since $n/2 \not\equiv 0 \pmod{n}$, $\{a_j+b_j\}$ is not a complete residue system mod n.

(2) Suppose that $\{a_j\}$, $\{b_j\}$, and $\{a_jb_j\}$ are complete residue systems mod n, $n>2$. If $4|n$, the $n/2$ odd products a_jb_j exhaust all the odd a_j and b_j, so the remaining products are divisible by 4, and the residue 2 (mod n) is never achieved. This contradiction shows that $n=qm$, where $q=p$ or $q=2p$, p is an odd prime, and m is odd.

Now let K be the product of all those elements in a complete residue system mod n which are prime to q. Modulo q, K is independent of the choice of such a

system, and indeed

$$K \equiv \prod_{\substack{j=1\\(j,q)=1}}^{n} j \equiv \left(\prod_{\substack{j=1\\(j,q)=1}}^{q} j \right)^m \pmod{q}.$$

If $q=p$, Wilson's theorem shows that $K \equiv (-1)^m \equiv -1 \pmod{q}$. If $q=2p$, then

$$\begin{aligned}\prod_{\substack{j=1\\(j,2p)=1}}^{2p} j &= 1\cdot 3\cdot 5 \cdots (p-2)(p+2)(p+4)\cdots(2p-1)\\ &\equiv 1\cdot 3\cdot 5\cdots(p-2)\cdot 2\cdot 4\cdots(p-1) \pmod{p}\\ &\equiv (p-1)! \equiv -1 \pmod{p}.\end{aligned}$$

Since the product is $\equiv -1 \pmod 2$, we again have $K \equiv (-1)^m \equiv -1 \pmod{q}$. But

$$K \equiv \prod_{(a_jb_j,q)=1} a_jb_j \equiv \prod_{(a_j,q)=1} a_j \prod_{(b_j,q)=1} b_j \equiv K^2 \pmod{q}.$$

This contradiction proves the theorem.

Also solved by W. J. Blundon, Leonard Carlitz, D. I. A. Cohen, E. S. Langford, D. C. B. Marsh, and the proposer.

Carlitz points out that conclusion (2) was proved by Chowla and Vijayaraghavan, Quarterly J. Math., Oxford, 19 (1948) 193.

An Almost-Defined Sequence

E 1700 [1964, 555]. *Proposed by Jon Petersen, Saskatchewan Power Corporation, Regina, Saskatchewan*

Given that $T_0=1$, $T_{n+1}=\sum_{k=1}^{n} T_kT_{n-k}$. Express T_n as a function of n.

I. *Solution by William D. Fryer*, 5120 *Ledge Lane, Williamsville, N. Y.* The problem is ill-defined; probably intended was

$$T_{n+1} = \sum_{k=0}^{n} T_kT_{n-k}.$$

Assuming this and letting $F(s) = \sum_{n=0}^{\infty} T_ns^n$ be the generating function, one gets $(1/s)[F(s)-1]=F^2(s)$, a quadratic for $F(s)$ with one acceptable solution (bounded at $s=0$): $F(s)=[1-(1-4s)^{\frac{1}{2}}]/2s$. Expansion by the binomial form gives for coefficient of s^n the expression

$$T_n = -\frac{1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

as desired answer. The problem has probability implications, and is similar to problems in Feller's book.

II. *Solution by Sylvan H. Greene, The Franklin Institute of Pennsylvania.* Since T_1 is not defined by the given recurrence formula, either that formula should be $T_n=\sum_{k=0}^{n} T_kT_{n-k}$ (Interpreting $\sum_{k=1}^{0}$ as zero would give $0=T_1=T_2=\cdots$!) or T_1 should be considered arbitrary. We take the latter point of view and assume that $T_0=\lambda$ and $T_1=\mu$ are both arbitrary, and that $T_{n+1}=\sum_{k=1}^{n} T_kT_{n-k}$ for $n=1, 2, 3, \cdots$.

If $f(x)=\sum_{n=0}^{\infty} T_nx^n$, the usual power-series manipulations lead to $xf^2(x)-(1+\lambda x)f(x)+(\lambda+\mu x)=0$, whence

$$f(x)=\frac{1+\lambda x-[(1-\lambda x)^2-4\mu x^2]^{1/2}}{2x}.$$

(The positive square root leads to an f with a pole at 0). Now,

$$[(1-\lambda x)^2-4\mu x^2]^{1/2}=\{1-x[2\lambda+(4\mu-\lambda^2)x]\}^{1/2}$$
$$=1-\lambda x-\sum_{p=1}^{\infty} x^{p+1}\sum_{j=[p/2]}^{p}\binom{j+1}{p-j}\binom{2j}{j}\frac{(4\mu-\lambda^2)^{p-j}\lambda^{2j+1-p}}{2^p(j+1)},$$

where we have used the formulas

$$(1+u)^{1/2}=\sum_{r=0}^{\infty}\binom{\frac{1}{2}}{r}u^r \text{ and } \binom{\frac{1}{2}}{r}=(-1)^{r-1}\binom{2r-2}{r-1}\Big/(2^{r-1}r),$$

applied the Binomial Theorem, inverted the order of summation, and made several changes of summing index. Substituting into our expression for $f(x)$, we find that

$$T_n=\frac{1}{2^{n+1}}\sum_{j=[n/2]}^{n}\binom{j+1}{n-j}\binom{2j}{j}\frac{(4\mu-\lambda^2)^{n-j}\lambda^{2j+1-n}}{j+1}$$

for $n=2, 3, 4, \cdots$ (and $T_0=\lambda$, $T_1=\mu$).

Also solved by Randy Barron, W. J. Blundon, A. J. Goldman, R. P. Kelisky, E. S. Langford W. S. Lawton, Viktors Linis, D. C. B. Marsh, Stanton Philipp, H. J. Ricardo (two solutions), O. G. Ruehr, M. G. Murdeshwar and V. K. Rohatgi (jointly), Robin Sibson, Jr., J. E. Wilkins, Jr., J. Williams, David Zeitlin, and the proposer.

Kelisky points out the similarity of this problem with Problem 5178 [1964, 217].

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before November 30, 1965.

5225. *Correction* (vol. 71, 1964, p. 802): Replace "continuous derivatives" by "continuous derivative."

5244. *Correction* (vol. 71, 1964, p. 1047): Replace "Marvin Marcus" by "Solomon Marcus."

5290. *Proposed by Daniel I. A. Cohen, Princeton University*

Let $\tau(a)$ be the number of divisors of a, and let $\lambda(a)$ be the number of distinct prime divisors of a. Prove that

$$\tau(n) = \sum_{k^2|n} 2^{\lambda(n/k^2)}, \tag{1}$$

$$\sum_{d|n} (d, n/d) = \sum_{k^2|n} k2^{\lambda(n/k^2)}. \tag{2}$$

5291. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Establish the formula

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} = \int_0^1 x^{-x} dx,$$

and generalize to $\sum z^n/(n+1)^{n+1}$.

5292. *Proposed by Eldon Hansen, Lockheed Missiles and Space Co., Palo Alto, California*

Obtain the Taylor series expansion for arcsec x about a point $x=a$.

5293. *Proposed by Martin J. Cohen, Beverly Hills, California*

Find a function f such that $\sum f(d)f(n/d)=1$ for every positive integer n, where the sum is taken over all d which divide n (including 1 and n).

5294. *Proposed by Frank Dean, Long Beach, California*

Let f and g be functions analytic at the point p. Show that the conditions:

$$|f'(p)|^2 > |f(p)f''(p)|, \qquad |g'(p)|^2 > |g(p)g''(p)|$$

guarantee the existence of a point q such that:

$$|f(q)| > |f(p)| \quad \text{and} \quad |g(q)| > |g(p)|.$$

5295. *Proposed by M. V. Subbarao, University of Alberta, Canada*

Let a and k be arbitrary positive integers, and b, c positive integers such that $3b \geqq 5c$. Let $N=(kn^b(n^c+1))!$. If $t(N)$ denotes the number of divisors of N, show that for all sufficiently large positive integers n, $t(N)\equiv 0 \pmod{(1+k)^a}$.

5296. *Proposed by J. E. Wetzel, University of Illinois*

The theorem of Weierstrass asserts that any continuous function f on an interval $[a, b]$ can be uniformly approximated with arbitrary accuracy by a polynomial.

(a) Let $x_1, \cdots, x_n$ be distinct points of the interval $[a, b]$. Show that the

polynomial p can be chosen to pass through all the points $(x_1, f(x_1)), \cdots, (x_n, f(x_n))$ in addition to approximating f uniformly on $[a, b]$.

(b) Let $x_1, \cdots, x_n$ be distinct points not in the interval $[a, b]$ and $y_1, \cdots, y_n$ arbitrary numbers. Show that the polynomial p can be chosen to pass through all the points $(x_1, y_1), \cdots, (x_n, y_n)$ in addition to approximating f uniformly on $[a, b]$.

5297. *Proposed by S. Lajos, K. Marx University of Economics, Budapest, Hungary*

Prove that any two-sided ideal of a two-sided ideal of a (von Neumann) regular ring A is again a two-sided ideal of A.

5298. *Proposed by S. Lajos, K. Marx University of Economics, Budapest, Hungary*

A subsemigroup A is said to be an (m, n)-ideal of the semigroup S if $A^mSA^n \subseteq A$, where m, n are nonnegative integers and A^m is suppressed if $m=0$. Prove that a semigroup S is a group if and only if it has no proper (m, n)-ideal, where m, n are fixed positive integers.

5299. *Proposed by S. Baron, McGill University, Montreal*

Find a set of necessary and sufficient conditions on a topological space (X, τ) such that for any topological space (Y, τ') and function $f: X \to Y$, f is continuous if and only if: $x_n \to x$ in X implies $f(x_n) \to f(x)$ in Y.

SOLUTIONS OF ADVANCED PROBLEMS

On Polygons Similar to a Given Polygon

5185 [1964, 326]. *Proposed by Carl Evans, Cornell Aeronautical Laboratory, Buffalo, N. Y.*

Prove: Through any three noncollinear points of the plane there are uncountably many polygons similar to a given simple closed polygon, with each of the three points on a different side.

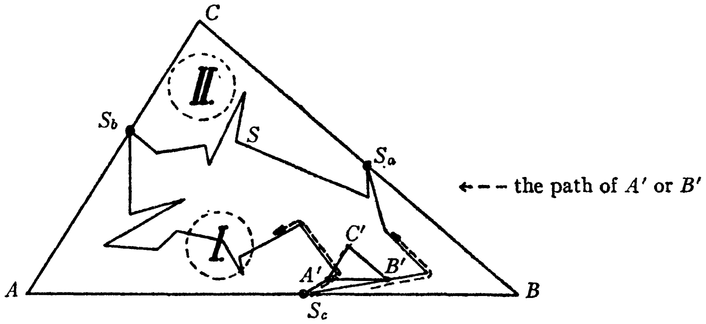
Solution by Horváth Sándor, University of Technology, Budapest, Hungary. We prove the following equivalent proposition: On the perimeter of a fixed polygon P, there are a continuum of triples of points, A, B, C, each on a different side of P, such that each triangle ABC is similar to a fixed given triangle $A_0B_0C_0$.

We observe first that there is a continuum set T of triangles ABC similar to $\triangle A_0B_0C_0$, no one of the triangles having sides parallel to another member of T, containing and inscribing P in such a way that no vertex of P coincides with a vertex of $\triangle ABC$ and such that each side of $\triangle ABC$ contains at most one vertex of P.

To construct a member of T, start with a line L_1 containing a vertex of P and having all of P on one side of L_1. Then find two lines L_2, L_3 such that the triangle formed by the lines L_1, L_2, L_3 is similar to $\triangle A_0B_0C_0$ and minimal with

respect to the containment of P. If $\triangle L_1L_2L_3$ does not satisfy the conditions of T, then a slight rotation of L_1 followed by slight motions of L_2, L_3 will produce a continuum of triangles in set T. We refer to the members of T as supporting triangles.

Let $\triangle ABC$ be a supporting triangle. Let S_a be the vertex of P on BC; S_b, S_c similarly defined. Let p be the part of the perimeter of P between S_a and S_b which does not contain S_c. p divides ABC into two connected pieces I and II (See figure). Form a triangle $A'B'C'$ similar to $\triangle ABC$ with A', B' on each of the



sides of P which meet at S_c, and with C' in I. Now move triangle $A'B'C'$ parallel to itself by having A', B' move on the perimeter of P, separating with respect to S_c. It is possible that one of the vertices A', B' will be retracing some of its path if the other vertex moves back toward AB in its path. But ultimately B', say, will arrive at S_a and then C' will be in domain II.

Thus the path of C' is in $\triangle ABC$ and is a continuous broken line with points in domains I and II. Therefore, for some position A', B' we will have C' on p, and C' is separated from A' and B' by vertices S_b and S_a, respectively. If one of the vertices of $\triangle A'B'C'$ is at a vertex of P, then a small rotation of the supporting triangle ABC will change $\triangle A'B'C'$ into, say, $A''B''C''$ similar to the new supporting triangle, and A'', B'', C'' will be properly on different sides of P. This completes the construction and, clearly, we have a continuum set of possibilities.

Also solved by H. S. M. Coxeter, Sylvan H. Greene, and the proposer.

Decomposition Characteristic of Abelian Groups

5200 [1964, 561]. *Proposed by T. J. Head, Iowa State University*

If a group G is the set theoretic union of a family of proper normal subgroups each two of which have only the identity in common, then G is abelian. (Abelian groups which are such unions were described by J. W. Young in *On the partitions of a group and the resulting classification*, Bull. Amer. Math. Soc., 33 (1927), 453–461.)

Solution by David Appleyard, University of Wisconsin. Let $G=\bigcup N_i$, $i\in I$. First suppose $a\in N_i$, $b\in N_j$, $i\neq j$. Let $c=aba^{-1}b^{-1}=a(ba^{-1}b^{-1})=(aba^{-1})b^{-1}$. By

the first representation $c\in N_i$, and by the second, $c\in N_j$. Hence $c=1$, and $ab=ba$.

Now suppose $a, b\in N_i$. Since N_i is proper, let $g\in G$, $g\notin N_i$. Then $ga\notin N_i$ and, by the above, g and ga commute with every element of N_i. In particular,

$$1 = (ga)(ba)(ga)^{-1}(ba^{-1}) = aba^{-1}b^{-1},$$

and $ab=ba$.

Also solved by T. H. Brooks, P. R. Chernoff, Charles Chouteau, R. A. Cunninghame-Green (England), M. J. DeLeon, Charles Fefferman, N. J. Fine, G. J. Ford, G. S. Glazer, W. J. Hartman, Stephen Hoffman, J. E. Humphreys, A. J. Kaison, John B. Kelly, Suzanne E. Kidd, Hayon Kim, W. M. Lambert, Jr., Robert Levy, M. D. Mavinkurve (India), M. W. McWaters, José Morgado (Brazil), Azriel Rosenfeld, Camilo Schmidt, R. Sibson, Jr. (England), J. D. Tarwater, B. R. Toskey, Fred Wabnik, S. L. Warner, W. C. Waterhouse, J. Williams, R. L. Wilson, Jr., Robert Lee Wilson, and the proposer.

No Matrix Map from Convergent to Null Sequences

5201 [1964, 561]. *Proposed by A. Wilansky, Lehigh University.*

On p. 181 of Banach's book, an isomorphism between c and c_0 is given: these are the spaces of convergent and null sequences. Show that no such isomorphism (continuous or not) can be given by a matrix map $\{x_n\}\in c\to\{\sum_k a_{nk}x_k\}\in c_0$.

Note. The proof of this theorem appears as Corollary 1 of the proposer's article *Topological divisors of zero and Tauberian theorems*, Transactions of the Amer. Math. Soc., 113 (1964) 244. The proposer and editors had hoped a simpler solution would be contributed.

A Multilinear System of Equations

5202 [1964, 561]. *Proposed by David R. Hayes, Duke University*

Let K be a field, let n be a positive integer and let $\{h_{ij,k}\colon 1\leqq i, j, k\leqq n\}$ be a set of n^3 elements of K. Show that the system of n^2 equations

$$X_iX_j = \sum_{k=1}^{n} h_{ij,k}X_k \qquad 1 \leqq i, j \leqq n,$$

has at most $n+1$ solutions in K.

I. *Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands.* The theorem is obvious if $n=1$. We proceed by induction. Let $n>1$, and assume there is a solution $(X_1, \cdots, X_n)=(a_1, \cdots, a_n)\neq(0, \cdots, 0)$. Without loss of generality $a_1\neq 0$. Put $X_1/a_1=Z_1$, $X_2-a_2X_1/a_1=Z_2$, $\cdots$, $X_n-a_nX_1/a_1=Z_n$. We easily derive a set of equations

$$Z_iZ_j = \sum_{k=1}^{n} h'_{ij,k}Z_k \qquad 1 \leqq i, j \leqq n.$$

A solution is $(Z_1, \cdots, Z_n)=(1, 0, \cdots, 0)$. Therefore, the equations with both $i>1, j>1$ do not contain Z_1. By the induction hypothesis, this set has at most n solutions in $Z_2, \cdots, Z_n$. Unless $(Z_2, \cdots, Z_n)=(h'_{12,1}, \cdots, h'_{1n,1})$ there is at

most one value of Z_1 in agreement with this solution $(Z_2, \cdots, Z_n)$; if, for example, $Z_2 \neq h'_{12,1}$ then the equation for Z_1Z_2 determines Z_1 uniquely. If $(Z_2, \cdots, Z_n) = (h'_{12,1}, \cdots, h'_{1n,1})$ then there are at most two values of Z_1, according to the quadratic equation for Z_1^2. Thus the number of solutions can be at most $n+1$.

An example of a set $\{h_{ij,k}\}$ producing exactly $n+1$ solutions is provided by the system $X_iX_j = \delta_{ij}X_i$ $(1 \leqq i, j \leqq n)$. The solutions are $(0, \cdots, 0), (1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$.

II. *Solution by N. J. Fine, Pennsylvania State University.* Define the matrices $H_i = ((h_{ij,k}))_{j,k=1}^n$. If $X = (X_1, \cdots, X_n)'$ is a nonzero solution of the system, then $H_iX = X_iX$ $(i = 1, \cdots, n)$. Thus X is a common eigenvector of all the H_i, and its ith component is the associated eigenvalue. Let $X^{(\alpha)}$ $(\alpha = 1, \cdots, r)$ be a set of distinct nonzero solutions. We shall show that this set is linearly independent. Suppose that $\sum_{\alpha=1}^r c_\alpha X^{(\alpha)} = 0$, with no $c_\alpha = 0$, and that r is minimal, $r \geqq 2$. Then $X^{(r)} \neq X^{(1)}$, so there is some i such that $X_i^{(r)} \neq X_i^{(1)}$. Then

$$
\begin{aligned}
0 = (H_i - X_i^{(r)}) \sum_{\alpha=1}^r c_\alpha X^{(\alpha)} &= \sum_{\alpha=1}^r c_\alpha (X_i^{(\alpha)} - X_i^{(r)}) X^{(\alpha)} \\
&= \sum_{\alpha=1}^{r-1} c_\alpha (X_i^{(\alpha)} - X_i^{(r)}) X^{(\alpha)}.
\end{aligned}
$$

This contradicts the minimality of r. Hence $r = 1$, and $c_1X^{(1)} = 0$, a contradiction. This proves our assertion that the $X^{(\alpha)}$ are linearly independent. Hence there are at most n of them.

Also solved by Harley Flanders, and by the proposer.

Unboundedness of $\sum 1/n \sin 1/n$ on $(-\infty, \infty)$

5203 [1964, 561]. *Proposed by D. J. Newman, Yeshiva University and L. A. Shepp, Bell Telephone Laboratories*

Is $\sum_{n=1}^\infty (1/n) \sin (x/n)$ a bounded function of x on the whole line?

Solution by M. D. Mavinkurve, Siddharth College, Bombay, India. The function is not bounded on the whole line. Let $f(x) = \sum (1/n) \sin (x/n)$. The convergence is uniform on every finite interval but not on the whole line. The series of derivatives $\sum (1/n^2) \cos (x/n)$ converges uniformly on the whole line. Its sum function $\phi(x)$, as the uniform limit of a sequence of periodic functions, is almost periodic. Therefore if $f(x)$ were bounded on the whole line, $f(x) = \int_0^x \phi(x)dx$ would be almost periodic; and it would have an average value

$$
\lim_{t \to \infty} \frac{1}{2t} \int_{-t}^{t} f(x)dx = 0. \tag{1}
$$

But the average value may also be evaluated as

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f(x)dx = \lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{\infty}\left(1-\cos\frac{t}{n}\right) = \lim_{t\to\infty}\frac{2}{t}\sum_{n=1}^{\infty}\sin^2\frac{t}{2n}\cdot$$

By letting t vary through a sequence of values $k\pi$ $(k=1, 2, \cdots)$, this limit is

$$\lim_{k\to\infty}\frac{2}{k\pi}\sum_{n=1}^{\infty}\sin^2\frac{k\pi}{2n}\cdot$$

For a fixed integer k, $\sin(k\pi/2n)>k/n$ if $n>k$. Therefore

$$\sum_{n=1}^{\infty}\sin^2\frac{k\pi}{2n}>\sum_{n=k+1}^{2k}\frac{k^2}{n^2}>\frac{k}{4};$$

or

$$\lim_{k\to\infty}\frac{2}{k\pi}\sum_{n=1}^{\infty}\sin^2\frac{k\pi}{2n}\geqq\frac{1}{2\pi},$$

which contradicts (1).

Also solved by Paul Erdös, Sanford L. Segal, and the proposers.

Erdös and Segal refer to Hardy and Littlewood, *On Lambert Series*, Proc. London Math. Soc., 41 (1936) 257, ff., where it is proved that the sum equals $\Omega((\log\log x)^{\epsilon})$, where $\epsilon\geqq\frac{1}{2}$. The conjecture that $\epsilon\geqq 1$ is still unresolved.

Representation of an Irrational Number

5204 [1964, 561]. *Proposed by Ralph Greenberg, University of Pennsylvania*

Let a be an integer >1, and let $q_1, q_2, \cdots$ be a sequence of positive integers satisfying $q_{n+1}>2q_n$ for $n=1, 2, \cdots$. Prove that $\prod_{n+1}^{\infty}(1+1/a^{q_n})$ is irrational.

I. *Solution by John B. Kelly, Arizona State University.* From $q_{n+1}-q_n-q_n\geqq 1$ it follows immediately that $q_{n+1}-(q_1+q_2+\cdots+q_n)\geqq n+1$.
Letting $s=\prod_{n=1}^{\infty}(1+1/a^{q_n})$, we have

$$\begin{aligned}s &= 1+\frac{1}{a^{q_1}}+\left(\frac{1}{a^{q_2}}+\frac{1}{a^{q_1+q_2}}\right)+\cdots+\left(\frac{1}{a^{q_n}}+\cdots+\frac{1}{a^{q_1+\cdots+q_n}}\right)\\ &\quad+\left(\frac{1}{a^{q_{n+1}}}+\cdots+\frac{1}{a^{q_1+\cdots+q_{n+1}}}\right)+\cdots.\end{aligned}\tag{1}$$

Each term of the series (1) yields a digit 1 at a certain place in the "decimal" expansion of s to the base a. All other digits are zero. It follows that there is a block of zeros between the $(q_1+\cdots+q_n)$-th place and the q_{n+1}-th place, and this block is of length $\geqq n+1$. Thus the expansion of s to the base a is nonterminating and contains arbitrarily long blocks of zeros. Thus, this expansion cannot be periodic and must be irrational.

II. *Solution by Stanton Philipp, Long Beach State College, California.* This problem is an instance of a theorem generally attributed to Cantor. Let $a_1, a_2, \cdots$ be a sequence of integers such that $a_1 > 1$, $a_{n+1} \geqq a_n^2$ for $n = 1, 2, \cdots$. Then $\prod_{n=1}^{\infty} (1+1/a_n)$ is rational if and only if there exists an integer N such that $a_{n+1} = a_n^2$ for all $n \geqq N$. See Schlömilch's Zeitschrift, XIV (1869) 152.

III. *Solution by K. A. Post, Technological University, Eindhoven, Netherlands.* The partial products of $\prod_{n=1}^{\infty} (1+a^{-q_n})$ can be considered partial sums of an infinite series $\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{-1}$ satisfying the conditions of a problem of Erdös, (cf. Wiskundige Opgaven met de Oplossingen, 20 (1955–1959), Groningen, problem 147).

Also solved by Robert Breusch, L. Carlitz, N. J. Fine, Hajna János (Hungary), and the proposer.

Topology on a Vector Space

5205 [1964, 561]. *Proposed by James Duemmel, Malmstrom Air Force Base, Montana*

For a nontrivial linear space X (over the real or the complex field) with algebraic dual X', let $T(X, M')$ denote the weak topology generated on X by the subspace M' of X'. Is there a pair (X, M') such that $T(X, M')$ is the discrete topology?

Solution by Robert Bowen, University of California, Berkeley. Let $y(\neq 0) \in X$, and let X be linear over the complex field C. Now $T(X, M')$ is the topology generated by inverse images of open sets of C under elements of M'. If $\{y\}$ were open in $T(X, M')$, we would have for some integer n, $\{y\} = \bigcap_{k=1}^{n} A_k$, where $A_k = f_k^{-1}\{z: |z - f_k(y)| < r_k\}$, $f_k \in M'$, and $r_k > 0$. Since $|f_k(zy) - f_k(y)| = |z-1|$ $|f_k(y)|$, it follows that for some $s_k > 0$, $B_k = \{zy: |z-1| < s_k\} \subseteq A_k$. Now $y \in B_k$, so $\bigcap_{k=1}^{n} B_k = \{y\}$, and $\{1\} = \bigcap_{k=1}^{n} \{z: |z-1| < s_k\}$, clearly an impossibility. Thus $T(X, M')$ cannot be discrete.

This argument, in substance, applies to the real field also.

Also solved by W. C. Waterhouse, and by the proposer.

A Strictly Increasing Sum

5206 [1964, 562]. *Proposed by Robert Spira, Duke University*

Show that $\sum_{m=2}^{n} (-1)^m \binom{n}{m} \log m$ is an increasing function of n.

I. *Solution by Irving Gerst, State University of New York, Long Island Center.* Letting $s(n)$ represent the given sum and using the representation

$$\log m = \int_0^{\infty} \frac{(e^{-t} - e^{-mt})}{t}\, dt, \qquad m > 0,$$

we have

$$s(n) = \int_0^\infty \sum_{m=1}^{n} (-1)^m \binom{n}{m} \frac{(e^{-t} - e^{-mt})}{t} dt = \int_0^\infty \frac{1 - e^{-t} - (1 - e^{-t})^n}{t} dt,$$

whence it follows that $s(n)$ is increasing.

II. *Solution by W. C. Waterhouse, Harvard University.* Set $f_n(x) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \log(m+x)$. Then $s(n) - s(n+1) = f_n(1)$. Now

$$f_n'(x) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{1}{m+x} = \frac{n!}{x(x+1)\cdots(x+n)} > 0, \qquad x > 0,$$

and

$$f_n(x) = f_n(x) - \sum_{m=0}^{n} (-1)^m \binom{n}{m} \log x = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \log\left(1 + \frac{m}{x}\right) \to 0$$

as $x \to \infty$. Thus $f_n(1) = s(n) - s(n+1) < 0$.

III. *Solution by Robert Breusch, Amherst College.* We have

$$s(n+1) - s(n) = (-1)^{n+1} \Delta^n \log 1,$$

where $\Delta^n f(x) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} f(x+m)$. If $f(x)$ is n-times differentiable it is known that $\Delta^n f(x) = f^{(n)}(\xi)$, $x < \xi < x+n$. Thus, for some $\xi > 1$, $s(n+1) - s(n) = (n-1)!/\xi^n > 0$.

Also solved by W. A. Al-Salam and L. Carlitz, A. J. Bosch (Netherlands), N. G. de Bruijn (Netherlands), N. J. Fine, Eldon Hansen, John B. Kelly, A. E. Livingston, A. E. Livingston and E. L. Whitney, Patricia A. Ralicki, and Mrs. K. Soni.

Maximum of Certain Rational Functions in Unit Circle

5207 [1964, 562, 1046]. *Proposed by H. S. Shapiro, New York University and D. J. Newman, Yeshiva University*

Let $P(z)$, $Q(z)$ be polynomials with complex coefficients and degree not exceeding $n-1$. Show that

$$\operatorname*{Max}_{|z| \leq 1} \left| z^n - \frac{P(z)}{Q(z)} \right| \geqq 1.$$

Solution by Peter M. Gibson, North Carolina State College, Raleigh. We discard the trivial case in which $Q(z)$ has zeros in $|z| \leq 1$. It follows from Rouché's theorem that if $f(z)$, $g(z) (\neq 0)$ are analytic for $|z| \leq 1$ and $f(z)$ has fewer than n zeros in $|z| < 1$, then

$$\max_{|z|\leqq 1}\left|z^n - \frac{f(z)}{g(z)}\right| = \max_{|z|=1}\left|\frac{z^n g(z) - f(z)}{z^n g(z)}\right| \geqq 1.$$

Also solved by Robert Breusch, M. D. Mavinkurve (India), K. A. Post (Netherlands), and by Q. I. Rahman and Q. G. Mohammad (India).

Infinite Product Representation

5208 [1964, 562]. *Proposed by L. Carlitz, Duke University*

Let $\tau(n)$ denote the number of divisors of n and let $k\geqq 1$. Show that

$$\sum_{n=1}^{\infty}\frac{(\tau(n))^k}{n^s} = \zeta(s)\prod_p A_k(p^{-s}),$$

where the product is taken over all primes p, and $A_k(x)$ is defined by means of $A_0(x)=1$, $(A(x)-1)^k = xA^k(x)$, where, after expansion, $A^k(x)$ is replaced by $A_k(x)$.

Solution by D. Suryanarayana, Andhra University, Waltair, India. Consider

$$\frac{1}{\zeta(s)}\sum_{n=1}^{\infty}\frac{(\tau(n))^k}{n^s} = \prod_p (1-p^{-s})\prod_p\left\{\sum_{\alpha=0}^{\infty}(\alpha+1)^k p^{-\alpha s}\right\} = \prod_p A_k(p^{-s}),$$

where $A_k(x) = \sum_{\alpha=0}^{\infty}\{(\alpha+1)^k - \alpha^k\}x^\alpha$. Now

$$\begin{aligned}(A(x)-1)^k &= \sum_{\gamma=0}^{k}(-1)^\gamma\binom{k}{\gamma}A_{k-\gamma}(x)\\ &= \sum_{\gamma=0}^{k}(-1)^\gamma\binom{k}{\gamma}\left[\sum_{\alpha=0}^{\infty}\{(\alpha+1)^{k-\gamma}-\alpha^{k-\gamma}\}x^\alpha\right]\\ &= \sum_{\alpha=0}^{\infty}\left[\sum_{\gamma=0}^{k}(-1)^\gamma\binom{k}{\gamma}\{(\alpha+1)^{k-\gamma}-\alpha^{k-\gamma}\}\right]x^\alpha\\ &= \sum_{\alpha-1}^{\infty}[\{(\alpha+1)-1\}^k - \{\alpha-1\}^k]x^\alpha = xA_k(x),\end{aligned}$$

as was to be proved.

Also solved by N. J. Fine, Irving Gerst, M. G. Greening, S. E. Payne, Stanton Philipp, M. V. Subbarao, and the proposer.

The original version of the problem misstated the recursion formula for $A_k(x)$. Gerst and Subbarao corrected the formula to read: $1+x(A(x)+1)^k = x+A_k(x)$, which is equivalent to the form above, used by all other solvers.

material, but the format makes it difficult for them to know where to resume reading.

There are nice sequences of problems on maxima and minima, related rates, "Δx, Δy" notation, and the use of set language (though after introducing this language the author rarely uses it). The most outstanding positive aspect is the series of problems involving the differential equations of motion (all with variables separable). These make clear the relation between the constants of integration and the initial conditions.

W. B. HOUSTON, JR., Antioch College

BRIEF MENTION

Linear Programming, 2nd ed. By Saul I. Gass. McGraw-Hill, New York, 1964. xii+280 pp. $8.95.

First published in 1958. This revised and updated edition includes new sections on sensitivity analysis, integer programming and the decomposition algorithm.

Recent Advances in Mathematical Programming. Edited by Robert L. Graves and Philip Wolfe. McGraw-Hill, New York, 1963. viii+350 pp. $11.95.

Report of a symposium held at Chicago in June 1962, at which 43 papers were presented. 23 of these are given in full, the rest as abstracts.

General Stochastic Processes in the Theory of Queues. By Vaclav E. Beneš. Addison-Wesley, Reading, Mass., 1963. viii+188 pp. $5.75.

Les principes de l'analyse dimensionelle. By René Saint-Guilhem. Gauthier-Villars, Paris, 1962. 79 pp.

An Introduction to Phase-integral Methods. By J. Heading. Wiley, New York, 1963. 160 pp. $4.50.

Renewal Theory. By D. R. Cox. Wiley, New York, 1962. ix+142 pp. $4.50.

A Select Bibliography on Digital Computer Education. By S. P. Page. Hertfordshire County Council Technical Library and Information Service, 1963. 28 pp.

Lectures on Modern Mathematics, vol. 1. Edited by T. L. Saaty. Wiley, New York, 1963. ix+174 pp. $5.75.

Expository lectures by eminent mathematicians, each delineating a substantial research area for an audience of mathematicians not specialists in that area: A Glimpse into Hilbert Space, by P. R. Halmos; Some Applications of the Theory of Distributions, by Laurent Schwartz; Numerical Analysis, by A. S. Householder; Algebraic Topology, by Samuel Eilenberg; Lie Algebras, by Irving Kaplansky; Representations of Finite Groups, by Richard Brauer.

Introduction to Mathematical Sociology. By James S. Coleman. Free Press of Glencoe, Macmillan, New York, 1964. xv+553 pp. $9.95.

Tables of the Negative Binomial Probability Distribution. By Williamson and Bretherton. Wiley, New York, 1963. 275 pp. $13.50.

Der Mathematikunterricht. Edited by Eugen Loffler. Ernst Klett Verlag, Stuttgart. Jahrgang 9, Heft 2, June 1963. "Philosophie im Mathematikunterricht II." 106 pp. 6.80 DM. Jahrgang 9, Heft 4, November 1963. "Axiomatik und Geometrieunterricht." 100 pp. 6.60 DM.

Smoothing, Forecasting, and Prediction of Discrete Time Series. By Robert Goodell Brown. Prentice-Hall, Englewood Cliffs, N. J., 1963. 468 pp. $13.25.

For readers directly concerned with practical applications.

Matrix Methods of Structural Analysis. Edited by F. de Veubeke. Macmillan, New York, 1964. 342 pp. $15.00.

A collection of papers presented to the Structures and Materials Panel of AGARO.

Ergodic Theory. Edited by Fred B. Wright. Academic Press, New York, 1963. xi+315 pp. $8.00.

Eighteen research papers presented at a symposium held at Tulane University, October 1961.

Multistage Inventory Models and Techniques. Edited by Herbert E. Scarf, Dorothy M. Gilford, and Maynard W. Shelly. Stanford University Press, Stanford, California, 1963. vi+225 pp. $7.50.

This is the first volume in the Office of Naval Research Monograph Series on Mathematical Methods in Logistics. It includes seven papers.

Programming and Coding the IBM 709, 7090, and 7094 Computers. By Philip M. Sherman. Wiley, New York, 1963. xiv+136 pp. $1.95.

A booklet to accompany the author's book, "Programming and Coding Digital Computers."

Some Aspects of the Relative Efficiencies of Indian Languages. By B. S. Ramakrishna, K. K. Nair, V. N. Chiplunkar, B. S. Atal, V. Ramachandran, and R. Subraminian. Indian Institute of Science, Bangalor 12, 1963. ix+90 pp.

A study from the point of view of information theory.

The Mathematical Theory of Communication. By Claude E. Shannon and Warren Weaver. University of Illinois Press, Urbana, Illinois, 1963. 118 pp. $.95. Republication. First publication in 1949.

Random Essays on Mathematics, Education, and Computers. By John Kemeny. Prentice-Hall, Englewood Cliffs, N. J., 1964. x+163 pp. $4.95.

Elements of Differential Equations. By Wilfred Kaplan. Addison-Wesley, Reading, Mass., 1964. 270 pp. $7.50.

A short version of the author's "Ordinary Differential Equations."

International Journal of Computer Mathematics, vol. I, no. 1, May 1964. Peter H. Friedlander, Editor. Gordon and Breach Science Publishers, New York, 1964, 89 pp.

Räumliche Probleme der Elastizitätstheorie. By A. I. Lurje. Akademie-Verlag, Berlin, 1964. xi+504 pp. DM 65.00.

Tables for Normal Sampling with Unknown Variance. By Jerome Bracken and Arthur Schleifer, Jr. Div. of Res., Harvard Business School, Boston, 1964. xiii+193 pp. $8.00.

Introduction to Electronic Computers: Problem Solving with the IBM 1620. By Fred J. Gruenberger and Daniel D. McCracken. Wiley, New York, 1963. vii+170 pp. $2.95.

New Directions in Mathematics. By John G. Kemeny, Robin Robinson, and Robert W. Ritchie. Prentice-Hall, Englewood Cliffs, N. J., 1963. 124 pp. $4.95.

Einführung in die Mathematik für Studenten der Wirtschaftswissenschaften. By Friedrich Sommer. Springer-Verlag, Berlin, 1963. viii+232 pp. DM 29.80.

Second Hungarian Mathematical Congress, vols. I and II, Budapest, August 24–31, 1960. Akadémiai Kiadó, Budapest, 1961. 330 pp. and 261 pp.

Equations Différentielles. Formulaire de Mathématiques à l'usage des Physiciens et des Ingénieurs. Fascicule VI. Centre National de la Recherche Scientifique, 15, Quai Anatole-France, Paris 7ème. 78 pp. 10 F.

Differential Equations, 4th ed. By Max Morris and Orley E. Brown. Prentice-Hall, Englewood Cliffs, N. J., 1964. viii+366 pp. $8.50.

The Critical Approach to Science and Philosophy. Mario Bunge, Editor. Free Press of Glencoe (Macmillan), New York, 1964. xv+480 pp. $9.95.

The Universal Encyclopedia of Mathematics. By James R. Newman. Simon and Schuster, New York, 1964. 715 pp. $8.95.

A translation of a German compendium, for the use of high school and college students, which encompasses many branches of mathematics from arithmetic through the calculus.

Calculus with Analytic Geometry, 3rd ed. By Johnson and Kiokemeister. Allyn and Bacon, Boston, Mass., 1964. 798 pp. $14.60.

Cambridge Conference on School Mathematics, Goals for School Mathematics. Houghton Mifflin, Boston, Mass., 1963. x+102 pp. $1.00.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Associate Secretary, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Victor Klee, University of Washington, has been awarded an honorary Doctor of Science degree by Pomona College.

Professor Rufus Oldenburger, Director of the Automatic Control Center at Purdue University, received the American Society of Mechanical Engineers 1964 Machine Design Award.

Professor Anice Seybold, North Central College, represented the Association at the inauguration of V. Raymond Edman as First Chancellor of Wheaton College on January 8, 1965.

Humboldt State College: Associate Professor O. M. Klose has been promoted to Professor; Assistant Professor J. E. Householder has been promoted to Associate Professor.

Oregon State University: Dr. R. B. Crittenden, University of Oregon, has been appointed Assistant Professor; Associate Professor P. M. Anselone has been promoted to Professor.

Professor H. S. M. Coxeter is on leave from the University of Toronto to serve as Distinguished Visiting Professor of Mathematics at Florida Atlantic University.

Associate Professor V. E. Hoggatt, Jr., San Jose State College, has been promoted to Professor.

Dr. Jane A. Slezak, Rensselaer Polytechnic Institute, has been appointed Assistant Professor of Chemistry at the College of St. Rose.

Professor Emeritus Edward B. Beaty, Oregon State University, died on September 3, 1964. He was a member of the Association for 44 years.

Professor Emeritus Philip Franklin, Massachusetts Institute of Technology, died on January 27, 1965. He was a member of the Association for 46 years.

Dr. Douglas L. Guy, Silver Spring, Maryland, died on August 18, 1964. He was a member of the Association for 16 years.

Associate Professor Earl C. Rex, George Pepperdine College, died on August 24, 1964. He was a member of the Association for 23 years.

Professor Emeritus D. H. Richert, Bethel College, died on November 28, 1964. He was a charter member of the Association.

Professor Evelyn C. Rusk, Wells College, died on December 5, 1964. She was a member of the Association for 37 years.

Professor Emeritus Charles H. Sisam, Colorado College, died on December 4, 1964. He was a charter member of the Association.

Professor Emeritus W. T. Stratton, Kansas State University, died on December 2, 1964. He was a charter member of the Association.

Mr. Richard R. Swift, Racine, Wisconsin, died on October 10, 1964. He was a member of the Association for 7 years.

Professor Emeritus Edwin B. Wilson, Harvard University, died on December 28, 1964. He was a charter member of the Association.

ARLINGTON STATE COLLEGE

A relativity conference will be given at Arlington State College from June 14 to July 2 (3 weeks). It is sponsored by the National Science Foundation and some stipends will be available. Most lectures will be restricted both in material and in presentation to that which could reasonably be given to undergraduates, although some lectures will go beyond this to give additional depth. The geometry of spacetime will be emphasized in establishing the fundamentals of relativity; in addition to special relativity there will be some cosmology and nonmathematical general relativity.

Letters of application and inquiry should be submitted as soon as possible to the conference director, Dr. Jason Ellis, Department of Physics, Arlington State College, Arlington, Texas.

ASSOCIATION OF MATHEMATICS TEACHERS OF NEW YORK STATE

The Association of Mathematics Teachers of New York State is conducting a Summer Workshop at LeMoyne College in Syracuse, New York, August 22–26. Three sections (Senior High, Junior High and Elementary) will each have two professors for six lectures. For information write to: Mrs. Barbara S. Mohan, Box 381, West Sand Lake, N. Y. 12196.

THE TWELFTH-GRADE PRE-COLLEGE MATHEMATICS PROGRAM: AN NCTM PUBLICATION

This booklet is a report of the joint meeting of the Association and the NCTM which was held at Denver on January 30, 1965. It will be available for 50 cents from the National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington, D. C. 20036.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 6

CONTENTS

JUNE–JULY 1965

THE AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to F. R. OLSON, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00. For nonmembers the subscription price is \$8.00 (\$10.00 effective August–September 1965).

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

PERMANENTS

MARVIN MARCUS AND HENRYK MINC, University of California, Santa Barbara

1. Introduction. If A is an n-square matrix then the *permanent* of A is defined by

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{1.1}$$

where the summation extends over S_n, the symmetric group of degree n. This matrix function appears naturally in any combinatorial setting where a count of the number of systems of distinct representatives of some configuration is required [A **14**]. In general the permanent is an appropriate invariant for matrices that arise in combinatorial investigations where the problem is essentially unaltered by a relabeling of the items under consideration. For example, the total number of derangements ("le problème des rencontres") of n distinct items is given by per $(J-I_n)$, where J is the n-square matrix with every entry equal to 1, and I_n is the n-square identity matrix. The initial ordering of the items is clearly immaterial. This is just a reflection of the more general fact that permutations of the rows and columns leave the permanent unaltered. In fact if T is a linear operator on the space of n-square matrices, $n>2$, then per $(T(A))=$ per A for all A if and only if either $T(A)=DPAQL$ or $T(A)=DPA^TQL$, where P and Q are n-square permutation matrices, and D and L are n-square diagonal matrices satisfying per $(DL)=1$. This theorem is a relatively recent discovery [B **18**].

As we shall subsequently see, the permanent is related to other more familiar matrix invariants, usually via inequalities. For example, Schur [B **51**] proved that per $A \geqq \det A$ for any positive semi-definite hermitian matrix. However, as Pólya observed [B **46**], no uniform affixing of $\pm$ signs to the elements of a matrix can convert the permanent into the determinant. In fact, quite a general result along these lines is known [B **19**]. Namely, there is no linear operation on matrices T: $A \rightarrow T(A)$ such that per $T(A)=\det A$ for all A (again except for $n=2$). This result, of course, terminates further efforts at investigating the permanent by relating it in some simple way to the more tractable determinant function.

The name "permanent" seems to have originated in Cauchy's memoir of 1812 [B **3**]. Cauchy's "fonctions symétriques permanentes" designate any symmetric function. Some of these, however, were permanents in the sense of the definition (1.1). Joachimstal [B **9**] points out that the sum $\sum \alpha_1\beta_2 \ldots \mu_m$ "tantum a determinante differt, quod omnes ejus termini sunt positivi." Hammond [B **7**] refers to the function (1.1) as an "alternate determinant." As far as we are aware the name "permanent" as defined in (1.1) was introduced by Muir [B **38**].

In the period 1855–1918 various results involving permanents were announced (see references). Many of these are very special identities involving permanents and determinants. Of these the one which apparently created the

most interest is due to Borchardt [B **1**]: if $A=[a_{ij}]$, $a_{ij}=1/(t_i-\alpha_j)$, and $B=[b_{ij}]$, $b_{ij}=a_{ij}^2$, then det $B=$ det A per A.

Another interesting and more general identity is due to Muir [B **40**]. Let M be the totality of $(mn)!/(m!)^n n!$ permutations $\sigma\in S_{mn}$ for which

$$\sigma(m(t-1)+1)<\sigma(mt+1), \qquad t=1,\cdots,n-1,$$

and

$$\sigma(km+i)<\sigma(km+j), \quad k=0,\cdots,n-1, \quad 1\leqq i<j\leqq m.$$

Let P_σ denote the mn-square permutation matrix corresponding to σ. If X is an mn-square matrix partitioned into n^2 m-square nonoverlapping submatrices X_{ij} then define $B(X)$ to be the n-square matrix whose (r, s) entry is det X_{rs}. Then for any mn-square matrix A

$$\det A=\sum_{\sigma\in M}\epsilon(\sigma)\det B(AP_\sigma) \qquad \text{if } m \text{ is odd,}$$

$$\det A=\sum_{\sigma\in M}\epsilon(\sigma)\operatorname{per} B(AP_\sigma) \qquad \text{if } m \text{ is even.}$$

Much of the modern interest in the permanent stems from three results obtained in this century: Pólya's problem [B **46**], Schur's result [B **51**], both previously mentioned, and the unresolved conjecture due to van der Waerden [B **55**]. The current status of van der Waerden's conjecture is discussed in the next section.

2. Properties of the permanent. In order to minimize the number of indices we introduce some notation for sequence sets. Let r and n be positive integers. The set $\Gamma_{r,n}$ is the totality of n^r sequences $\omega=(\omega_1,\cdots,\omega_r)$ for which $1\leqq\omega_i\leqq n$, $i=1,\cdots,r$. If $r\leqq n$ then $Q_{r,n}$ will denote the subset of $\Gamma_{r,n}$ consisting of those $\binom{n}{r}$ sequences ω for which $\omega_1<\omega_2<\cdots<\omega_r$. The set $G_{r,n}$ is the totality of $\binom{n+r-1}{r}$ sequences in $\Gamma_{r,n}$ for which $\omega_1\leqq\omega_2\leqq\cdots\leqq\omega_r$. If $A=[a_{ij}]$ is an n-square matrix and $\omega,\tau\in\Gamma_{r,n}$ then $A[\omega|\tau]$ is the r-square matrix whose (s, t) entry is $a_{\omega_s\tau_t}$. If $\omega,\tau\in Q_{r,n}$ then $A[\omega|\tau)$ is the $r\times(n-r)$ submatrix lying in rows ω and outside columns τ of A. Similarly for $A(\omega|\tau]$ and $A(\omega|\tau)$. If $\omega\in G_{r,n}$ then $\mu(\omega)$ will denote the product of the factorials of the multiplicities of the distinct integers in ω, e.g., $\mu(2, 4, 4, 4, 5, 5, 7)=3!2!$.

Permanents do possess some properties analogous to those of determinants:

(a) the permanent is a multilinear function of the rows and columns;
(b) per $A^*=\overline{\text{per } A}$;
(c) if P and Q are permutation matrices then per $PAQ=$ per A;
(d) if D and G are diagonal matrices then per $DAG=$ per D per A per G;
(e) (Laplace expansion theorem)

$$\operatorname{per} A=\sum_{\omega\in Q_{r,n}}\operatorname{per} A[1,\cdots,r\,|\,\omega]\operatorname{per} A(r+1,\cdots,n\,|\,\omega);$$

(f) (Cauchy-Binet theorem) if A is $m\times n$ and B is $n\times m$, $m\leqq n$,

$$\operatorname{per} AB = \sum_{\gamma\in G_{m,n}} \frac{1}{\mu(\gamma)} \operatorname{per} A[1, \cdots, m \mid \gamma] \operatorname{per} B[\gamma \mid 1, \cdots, m].$$

Properties (a)–(e) are immediate consequences of the definition (1.1). The proof of property (f) is somewhat more involved. The most useful property that determinants have is invariance under addition of a multiple of a row to another row. This property is conspicuous by its absence from the above list. Its failure makes the computation of any particular permanent difficult. The following formula due to Ryser [A **14**] makes the evaluation of certain permanents feasible by high speed computing devices. Let A be an n-square matrix and let A_r denote a matrix obtained from A by replacing some r columns of A by zeros. Let $S(X)$ be the product of the row sums of the matrix X. Then

$$(2.1)\quad \operatorname{per} A = S(A) - \sum S(A_1) + \sum S(A_2) - \cdots + (-1)^{n-1} \sum S(A_{n-1}),$$

where $\sum S(A_r)$ denotes the sum over all $\binom{n}{r}$ replacements of r of the columns of A by zeros. Formula (2.1) was used by Nikolai [B **45**] for computing permanents of incidence matrices for certain (v, k, λ)-configurations.

Let $\mathfrak{A}$ be a set of v distinct items $x_1, \cdots, x_v$ and suppose $\mathfrak{A}_1, \cdots, \mathfrak{A}_v$ are subsets of $\mathfrak{A}$. We define the v-square *incidence matrix* $A=[a_{ij}]$ for this configuration by $a_{ij}=1$ if $x_i\in\mathfrak{A}_j$ and $a_{ij}=0$ if $x_i\notin\mathfrak{A}_j$. A *system of distinct representatives* is an ordered v-tuple $(x_{\sigma(1)}, \cdots, x_{\sigma(v)})$, $x_{\sigma(t)}\in\mathfrak{A}_t$, $\sigma\in S_v$. Hence per A is just the number of such systems. In case each $\mathfrak{A}_i$ contains exactly k items and $\mathfrak{A}_i\cap\mathfrak{A}_j$, $i\neq j$, contains λ items, $0<\lambda<k<v$, then we have what is called a (v, k, λ)-configuration. The matrix A then satisfies $AA^T=A^TA=(k-\lambda)I_n+\lambda J$ and it can be shown that

$$(2.2)\qquad |\det A| = k(k-\lambda)^{(v-1)/2}.$$

From (2.2) it is seen that $|\det A|$ depends only on the parameters v, k and λ and not on the configuration. The question posed and answered in the negative by Nikolai [B **45**] is whether per A similarly depends only on the parameters v, k, λ.

In answer to Montmort's "problème des rencontres" [A **14**] mentioned in section 1 we can compute

$$\operatorname{per}(J - I_v) = v!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^v \frac{1}{v!}\right).$$

In fact,

$$\operatorname{per}(zI_v + J) = v! \sum_{r=0}^{v} \frac{z^r}{r!}\cdot$$

The matrix $J-I_v$ is one of the few (0, 1)-circulants [B **35**] for which an explicit formula for the permanent is available. We describe a few other examples of

permanents that can be directly computed. Let P_v be the v-square permutation matrix with ones in the first superdiagonal. Then of course per $P_v=1$. Also, per $(P_v+P_v^2)=2$ and [B **35**]

$$\begin{aligned}
\operatorname{per}(P_v + P_v^2 + P_v^3) &= \operatorname{per}(P_{v-1} + P_{v-1}^2 + P_{v-1}^3) + \operatorname{per}(P_{v-2} + P_{v-2}^2 + P_{v-2}^3) - 2 \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^v + \left(\frac{1-\sqrt{5}}{2}\right)^v + 2 \\
&= \operatorname{tr}(C_2^v) + 2,
\end{aligned}$$

where

$$C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Moreover,

$$\begin{aligned}
\operatorname{per}(P_v + P_v^2 + P_v^3 + P_v^4) &= \operatorname{per}(P_{v-1} + P_{v-1}^2 + P_{v-1}^3 + P_{v-1}^4) \\
&\quad + \operatorname{per}(P_{v-2} + P_{v-2}^2 + P_{v-2}^3 + P_{v-2}^4) + \operatorname{per}(P_{v-3} + P_{v-3}^2 + P_{v-3}^3 + P_{v-3}^4) - 4 \\
&\quad = 2(\operatorname{tr}(C_3^v) + 1),
\end{aligned}$$

where

$$C_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is known that unfortunately no analogous recurrence formula for per $(\sum_{v=1}^{k} P_v^i)$, $k>4$, exists [B **35**]. It is possible to show that if such a recurrence were to hold for $k>5$ then the van der Waerden conjecture would fail.

A remarkable result for (0, 1)-circulants was recently obtained by Tinsley [B **54**]. Let A be a (0, 1)-circulant with k ones, $k \geqq 3$, in each row and column. Then

$$\operatorname{per} A \geqq |\det A|,$$

with equality if and only if after suitable permutations of the rows and columns A can be reduced to a direct sum of 7-square (0, 1)-circulants of the form $I_7+P_7+P_7^3$. Then

$$\operatorname{per} A = |\det A| = 24^{v/7}.$$

A *doubly stochastic* n-square matrix $A=[a_{ij}]$ is one with nonnegative entries each of whose row sums and column sums is 1. Thus $J_n=(1/n)J$ is doubly stochastic. These matrices appear in a wide variety of contexts ranging from probability theory to the theory of eigenvalue inequalities. The results on doubly

stochastic matrices are discussed extensively in [A **7**; B **11**; B **20**; B **37**]. Notice that the above (0, 1)-circulants are all multiples of doubly stochastic matrices as are the incidence matrices for the (v, k, λ)-configurations. There is not much known about upper and lower bounds for the permanent of a general doubly stochastic matrix. The following bounds were recently obtained and constitute examples of results relating the permanent with the more familiar matrix invariants.

If A is an n-square doubly stochastic matrix then

$$\text{per } A \leqq \left(\frac{\rho(A)}{n}\right)^{1/2}, \tag{2.3}$$

where $\rho(A)$ is the rank of A. Equality holds in (2.3) if and only if A is a permutation matrix [B **22**]. If A happens to be normal as well then (2.3) can be improved to

$$\text{per } A \leqq \frac{\rho(A)}{n}, \tag{2.4}$$

where the inequality is strict unless A is a permutation matrix or $n=2$ and $A=J_2$ [B **22**].

Once again if A is doubly stochastic with h eigenvalues of modulus 1 then [B **23**]

$$\text{per } A \geqq \frac{1}{(n-h+1)^{n-h+1}}. \tag{2.5}$$

If A is also indecomposable then

$$\text{per } A \geqq \left(\frac{h}{n}\right)^n. \tag{2.6}$$

Equality holds in (2.5) or (2.6) if and only if A is a permutation matrix.

The best known conjecture on the lower bound for the permanent of an n-square doubly stochastic matrix is the van der Waerden conjecture:

$$\text{per } A \geqq \frac{n!}{n^n} \tag{2.7}$$

with equality if and only if $A=J_n$. We list six results that represent all that is currently known about (2.7). Let Ω_n denote the set of doubly stochastic n-square matrices and let Z_n be the subset of Ω_n consisting of those matrices with strictly positive entries.

I. *If* per $A=\min_{S\in\Omega_n}$ per S *then* $A^k\in Z_n$ *for some* k [B **26**].

II. *If* per $A=\min_{S\in\Omega_n}$ per A *then* per $A(i|j)=$ per A *if* $a_{ij}>0$ *and* per $A(i|j)=$ per $A+\beta$ *if* $a_{ij}=0$, *where* $\beta\geqq 0$ *is independent of* i *and* j [B **26**].

III. *If* per $A=\min_{S\in\Omega_n}$ per S *and* $A\in Z_n$ *then* $A=J_n$ [B **26**].

IV. *If $A \neq J_n$ and A is in a sufficiently small neighborhood of J_n then* [B **26**]

$$\text{per } A > \text{per } J_n.$$

V. *For $n \leqq 3$,* $\min_{S \in \Omega_n}$ per $S = n!/n^n$ *and the minimum is assumed uniquely for $S = J_n$* [B 26].

VI. *If $A \in \Omega_n$ is positive semi-definite symmetric then* [B **28**; B **29**; B **33**; B 22]

$$\text{per } A \geqq \frac{n!}{n^n}$$

with equality if and only if $A = J_n$.

To conclude we quote a recently obtained upper bound for the permanent of a general (0, 1)-matrix. If A is a (0, 1)-matrix with row sums $r_1, \cdots, r_n$ then [B **34**]

$$\text{per } A \leqq \prod_{i=1}^{n} \left(\frac{r_i + 1}{2} \right),$$

with equality if and only if A is a permutation matrix.

3. Recent developments. In this section we describe a relatively new approach to the permanent function. The central idea is to represent the permanent as an inner product on the symmetry class of completely symmetric tensors. At the outset this may seem like an unnecessarily sophisticated approach to a matrix function which does not appear much more complicated than the determinant. In order that the reader may appreciate the difficulties involved we list a number of results recently obtained by these techniques. It seems unlikely that a frontal attack on any of these inequalities would succeed.

THEOREM 1. *If A is an n-square positive semi-definite hermitian matrix with row sums $r_1, \cdots, r_n$ and if $r = \sum_{i=1}^n r_i \neq 0$ then*

$$\text{per } A \geqq n! \prod_{i=1}^{n} |r_i|^2/r^n. \tag{3.1}$$

Equality holds in (3.1) *if and only if* (i) *A has a zero row or* (ii) $\rho(A) = 1$, [B **22**].

Notice that as a direct corollary to Theorem 1 we obtain the result VI at the end of section 2.

THEOREM 2 (Permanent analogue of the Hadamard determinant theorem [B **21**; B **14**; B **15**]). *If A is an n-square positive semi-definite hermitian matrix then*

$$\text{per } A \geqq \prod_{i=1}^{n} a_{ii} \tag{3.2}$$

with equality if and only if A is a diagonal matrix or A has a zero row.

THEOREM 3 (Schur [B **51**]). *If A is an n-square positive semi-definite hermitian matrix then*

$$\text{per } A \geqq \det A \tag{3.3}$$

with equality if and only if A is diagonal or A has a zero row.

THEOREM 4. *If A is an n-square normal matrix with eigenvalues $\alpha_1, \cdots, \alpha_n$ then* [B **22**]

$$|\text{per } A| \leqq \frac{1}{n} \sum_{i=1}^{n} |\alpha_i|^n. \tag{3.4}$$

THEOREM 5. *If A is an $m \times n$ matrix and B is an $n \times m$ matrix then*

$$|\text{per } AB|^2 \leqq \text{per } AA^* \text{ per } B^*B. \tag{3.5}$$

Equality holds in (3.5) *if and only if either* (i) *A has a zero row or* (ii) *B has a zero column or* (iii) *$A = DPB^*$, where D is a diagonal matrix and P is a permutation matrix* [B **29**].

We introduce the preliminary material on tensor spaces that will suffice to exhibit the technique of proof used in establishing these results.

Let V be an n-dimensional unitary space with inner product (x, y). We let $V^{(m)}$ denote the space of *m-contravariant tensors* [A **8**, Chap. 16]. This is most easily defined as follows. Let $M_m(V)$ be the space of m-multilinear functionals on V, i.e., the space of complex valued functions $\phi(x_1, \cdots, x_m)$, $x_i \in V$, $i = 1, \cdots, m$, linear in each x_i separately. For example, if V is the space of complex n-tuples and $m = n$, $x_i = (x_{i1}, \cdots, x_{in})$, then the multilinear functional ϕ defined by $\phi(x_1, \cdots, x_n) = \det(x_{ij})$ is in $M_n(V)$. The space $V^{(m)}$ is now defined as the dual space of $M_m(V)$, i.e., $V^{(m)}$ is the space of all complex-valued linear functionals on $M_m(V)$. There are certain distinguished m-contravariant tensors, the *decomposable* tensors, that arise from elements of V: if $x_i \in V$, then $f = x_1 \otimes \cdots \otimes x_m$ is the element of $V^{(m)}$ whose value on any $\phi \in M_m(V)$ is given by

$$f(\phi) = \phi(x_1, \cdots, x_m).$$

The decomposable tensor f is called the *tensor product* of the x_i, $i = 1, \cdots, m$.

It is easy to show that if $e_1, \cdots, e_n$ is a basis of V then the n^m tensor products $e_{\omega_1} \otimes \cdots \otimes e_{\omega_m}$, $\omega \in \Gamma_{m,n}$, constitute a basis of $V^{(m)}$. An inner product can be introduced in $V^{(m)}$ in terms of the inner product in V. The defining relation is given in terms of its values on pairs of tensor products,

$$(x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) = \prod_{i=1}^{m} (x_i, y_i). \tag{3.6}$$

Certain special operators $P(\sigma)$ on $V^{(m)}$, called *permutation operators*, are defined in terms of elements $\sigma \in S_m$:

$$(3.7) \qquad P(\sigma)x_1 \otimes \cdots \otimes x_m = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}.$$

The *completely symmetric operator* on $V^{(m)}$ is now defined by

$$(3.8) \qquad T_m = \frac{1}{m!} \sum_{\sigma \in S_m} P(\sigma).$$

Thus T_m maps $V^{(m)}$ into a subspace of itself. The range of T_m is denoted by $V_{(m)}$ and is called the *symmetry class* of completely symmetric tensors [A **4**, p. 217]. If $x_1, \cdots, x_m$ are in V we set

$$(3.9) \qquad x_1 * \cdots * x_m = T_m(x_1 \otimes \cdots \otimes x_m);$$

$x_1 * \cdots * x_m$ is called the *symmetric* product of the x_i, $i = 1, \cdots, m$. It is clear from the definition of the symmetric product that if $\sigma \in S_m$ then

$$P(\sigma)x_1 * \cdots * x_m = x_1 * \cdots * x_m.$$

The operator T_m is hermitian with respect to the inner product (3.6) and is also idempotent:

$$(3.10) \qquad T_m = T_m^2 = T_m^*.$$

We are now ready to bring the permanent into this picture. Let $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$ be arbitrary vectors in V and set $a_{ij} = (x_i, y_j)$. Then by (3.9), (3.10), (3.6), and (1.1) in succession we compute

$$\begin{aligned}
(x_1 * \cdots * x_m, y_1 * \cdots * y_m) &= (T_m x_1 \otimes \cdots \otimes x_m, T_m y_1 \otimes \cdots \otimes y_m) \\
&= (T_m^* T_m x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) \\
&= (T_m x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} (P(\sigma)x_1 \otimes \cdots \otimes x_m, y_1 \otimes \cdots \otimes y_m) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} (x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}, y_1 \otimes \cdots \otimes y_m) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^m (x_{\sigma^{-1}(i)}, y_i) \\
&= \frac{1}{m!} \sum_{\sigma \in S_m} \prod_{i=1}^m (x_i, y_{\sigma(i)}) \\
&= \frac{1}{m!} \operatorname{per} A.
\end{aligned}$$

We have then

$$(3.11) \qquad (x_1 * \cdots * x_m, y_1 * \cdots * y_m) = \frac{1}{m!} \operatorname{per} A,$$

where $A = (a_{ij}) = ((x_i, y_j))$.

To illustrate the use of the identity (3.11) we prove Theorem 1. Set $m=n$. Since A is positive semi-definite hermitian it is a Gram matrix based on some set of vectors $x_1, \cdots, x_n$: $a_{ij}=(x_i, x_j)$. From (3.11) and the Cauchy-Schwarz inequality applied to the inner product in $V^{(m)}$ we have

$$(3.12)\qquad \frac{1}{n!}\operatorname{per} A = (x_1 * \cdots * x_n, x_1 * \cdots * x_n) \geqq \left|\left(x_1 * \cdots * x_n, \frac{u}{\|u\|}\right)\right|^2,$$

where u is any nonzero vector in $V^{(m)}$. Let $v=\sum_{i=1}^n x_i$ so that

$$(v, v) = \sum_{i,j=1}^n (x_i, x_j) = \sum_{i,j=1}^n a_{ij} = r.$$

Since $r\neq 0$, $v\neq 0$ and in fact $\|v\|^2=r>0$. We set $u=v * \cdots * v$ and compute, by (3.11), that

$$\|u\|^2 = (v * \cdots * v, v * \cdots * v) = \frac{1}{n!}\operatorname{per} B,$$

where $b_{ij}=r$, $i,j=1, \cdots, n$. Hence $\|u\|^2=r^n$. Returning to (3.12) we have

$$\frac{1}{n!}\operatorname{per} A \geqq \left| (x_1 * \cdots * x_n, v * \cdots * v)\right|^2/r^n = \left|\frac{1}{n!}\operatorname{per}((x_i, v))\right|^2/r^n.$$

Now $(x_i, v)=(x_i, \sum_{j=1}^n x_j)=\sum_{j=1}^n (x_i, x_j)=\sum_{j=1}^n a_{ij}=r_i$. Hence $\operatorname{per}((x_i, v)) = n!\prod_{i=1}^n r_i$ and therefore

$$\frac{1}{n!}\operatorname{per} A \geqq \prod_{i=1}^n |r_i|^2/r^n,$$

the inequality (3.1). The discussion of the cases of equality is a bit involved and we omit it. Notice that if A is doubly stochastic then $r_1= \cdots =r_n=1$, $r=n$, and (3.1) implies VI in section 2.

We refer the reader to [B **29**] in which a very similar argument is used to prove Theorem 5. We illustrate the use of (3.5) in proving (3.3). Since A is positive semi-definite hermitian it can be factored, $A=TT^*$, in which T is upper triangular. Notice that, for such a T, per $T=\det T$. Thus

$$\begin{aligned}\det A &= \det TT^* = \det T \det T^*\\ &= \operatorname{per} T \operatorname{per} T^*\\ &\leqq |\operatorname{per} TI_n|\,|\operatorname{per} I_nT^*|\\ &\leqq \sqrt{\operatorname{per} TT^*}\sqrt{\operatorname{per} TT^*} = \operatorname{per} A.\end{aligned}$$

The inequality (3.3) could have been obtained from (3.2) and the Hadamard determinant theorem. However the proof of (3.2) is quite involved [B **14**; B **15**] and will be omitted.

4. Conjectures and problems. In this section we list several conjectures and problems that involve the permanent function. The conjectures in this field, that have been announced at various times, seem to separate into two quite unsatisfactory classes. In the first class are those statements that sound plausible but appear at present beyond reach. In the second class are those conjectures that seemed for a time undeniably true and for which counterexamples were eventually discovered. We list some conjectures that are as yet unclassified.

CONJECTURE 1 (van der Waerden). *If A is an n-square doubly stochastic matrix then*

$$\operatorname{per} A \geqq \frac{n!}{n^n}$$

with equality if and only if $A = J_n$ [B **55**; B 26; B **28**; B **33**].

CONJECTURE 2 (H. J. Ryser). *If A is an n-square doubly stochastic matrix then*

$$\operatorname{per} AA^T \leqq \operatorname{per} A. \tag{4.1}$$

Until just recently it was conjectured [A **14**; p. 59] that

$$\operatorname{per} AB \leqq \min \{\operatorname{per} A, \operatorname{per} B\} \tag{4.2}$$

when both A and B are n-square doubly stochastic matrices. We are indebted to the referee for making available to us a surprising counter-example to (4.2) discovered by W. B. Jurkat of Syracuse University:

$$A = \frac{1}{24}\begin{bmatrix} 11 & 5 & 8 \\ 13 & 11 & 0 \\ 0 & 8 & 16 \end{bmatrix}, \qquad B = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$24^3 \operatorname{per} A = 3808, \qquad 24^3 \operatorname{per} AB = 3840.$$

Of course, if (4.1) were known then Conjecture 1 would follow immediately from VI in section 2. (Morris Newman recently communicated to us a counter-example to Conjector 2.)

CONJECTURE 3 (H. J. RYSER). *If A is an mk-square* (0, 1)*-matrix with k ones in each row and column then*

$$\operatorname{per} A \leqq \sum_1^m {}^{\cdot} J, \tag{4.3}$$

where $\sum_1^m \cdot J$ denotes the direct sum of J taken m times and J is k-square [A **14**].

CONJECTURE 4 (H. Minc). *If A is an n-square* (0, 1)*-matrix with row sums r_i, $i = 1, \cdots, n$, then*

$$\operatorname{per} A \leqq \prod_{i=1}^n (r_i!)^{1/r_i}. \tag{4.4}$$

Clearly (4.4) implies (4.3) [B **34**].

CONJECTURE 5 (H. J. Ryser). *If the totality of v-square* (0,1)*-matrices with k ones in each row and column contains incidence matrices of* (v, k, λ)*-configurations then the permanent is minimal in this totality for one of these incidence matrices.*

CONJECTURE 6 (H. Minc). *Let R_k denote the class of all v-square* (0, 1)*-matrices with k ones in each row and column. Then for a fixed v*

$$\min_{A \in R_k} \operatorname{per}\left(\frac{1}{k} A\right)$$

is monotone decreasing in k.

CONJECTURE 7 (M. Marcus and M. Newman). *If A is n-square doubly stochastic then*

$$\operatorname{per}(I_n - A) \geqq 0.$$

This is known to be true in case A is symmetric as well [B **29**]. (R. A. Brualdi and Morris Newman announced the affirmative resolution of Conjecture 7.)

CONJECTURE 8 (M. Marcus and M. Newman). *If A is positive semi-definite n-square hermitian and* $1 \leqq k \leqq n$ *then*

$$\operatorname{per} A \geqq \operatorname{per} A[1, \cdots, k \mid 1, \cdots, k] \operatorname{per} A[k+1, \cdots, n \mid k+1, \cdots, n].$$

This is known for $k=1$ [B **14**, B **15**].

CONJECTURE 9 (M. Marcus). *Let A be an mk-square positive semi-definite hermitian matrix partitioned as follows:*

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}$$

in which each A_{ij} is k-square. Let B be the m-square matrix whose (i, j) *entry is* per A_{ij}. *Then*

$$\operatorname{per} A \geqq \operatorname{per} B. \tag{4.5}$$

If A is positive definite then equality holds in (4.5) *if and only if* $A = \sum_{i=1}^{m} A_{ii}$.

Note that Conjecture 9 implies Conjecture 8. For suppose (4.5) holds. Assume without loss of generality in Conjecture 8 that $k \leqq n-k$. Let C be the $2(n-k)$-square matrix: $C = I_{n-2k} \dotplus A$. Partition C as follows:

$$C = \begin{bmatrix} I_{n-2k} \dotplus A_k & C_{12} \\ C_{21} & A_{n-k} \end{bmatrix},$$

where $A_k = A[1, \cdots, k|1, \cdots, k]$ and $A_{n-k} = A[k+1, \cdots, n|k+1, \cdots, n]$. Clearly C is poritive semi-definite hermitian and (4.5) implies

$$\begin{aligned} \text{per } A &= \text{per } C \geqq \text{per} \begin{bmatrix} \text{per }(I_{n-2k} + A_k) & \text{per } C_{12} \\ \text{per } C_{21} & \text{per } A_{n-k} \end{bmatrix} \\ &= \text{per } A_k \text{ per } A_{n-k} + |\text{per } C_{12}|^2 \\ &\geqq \text{per } A_k \text{ per } A_{n-k}. \end{aligned}$$

CONJECTURE 10 (M. Marcus). *Let $A = [a_{ij}]$ and $a_{ij} > 0$, $i, j = 1, \cdots, n$. If the $n!$ terms $\prod_{i=1}^n a_{i\sigma(i)}$ in the expansion of* per A *take on at most r different values then $\rho(A) \leqq r$.*

This conjecture is known to be true for $n < 5$ [unpublished results of H. Minc and R. Westwick].

CONJECTURE 11 (M. Marcus and M. Newman). *Let $A \in \Omega_n$. There exists no positive number β, independent of i, j, such that*

$$\text{per } A(i|j) = \text{per } A, \qquad a_{ij} \neq 0$$

and

$$\text{per } A(i|j) = \text{per } A + \beta, \qquad a_{ij} = 0.$$

(See section 2, II.)

CONJECTURE 12 (M. Marcus). *Let Δ_n be the group of n-square nonsingular matrices of the form PD, where P is a permutation matrix and D is a diagonal matrix. Show that Δ_n is a maximal group on which the permanent is multiplicative. In other words, Δ_n cannot be a proper subgroup of a group G in which* per AB $=$ per A per B *for all $A, B \in G$.*

CONJECTURE 13 (M. Marcus). *If A is doubly stochastic and $f(z) =$* per $(zI_n - A)$ *then the zeros of $f(z)$ lie in or on the boundary of the unit disc $|z| \leqq 1$.*

PROBLEM 1. *Find the maximum value of $f(U) =$* per (U^*AU) *as U runs over all n-square unitary matrices. Here A is a fixed n-square positive semi-definite hermitian matrix* [B **29**].

It is known [B **22**] that $f(U) \leqq \text{tr}(A^n)/n$, but in general this bound is not achievable.

PROBLEM 2. *Let H be a subgroup of S_n and let χ be a character of degree 1 of H. Under what conditions on χ and H will the inequality*

$$\sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \leqq \text{per } A$$

hold for all positive semi-definite hermitian A? [B **51**].

The work of the first author was supported in part by the National Science Foundation under Grant G. P. 1085 and the Air Force Office of Scientific Research under Grant AFOSR 698-65.

The work of the second author was supported in part by the Air Force Office of Scientific Research under Grant AFOSR 432-63.

References

A. Books

1. A. C. Aitken, Determinants and matrices, 4th ed., Edinburgh, 1946.

2. F. Faà Di Bruno, Théorie des formes binaires, Turin, 1876, 38–50.

3. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, 2nd ed. Cambridge, 1952, 44–51.

4. Nathan Jacobson, Lectures in abstract algebra, vol. 2, Van Nostrand, Princeton, N. J., 1953.

5. Dudley E. Littlewood, The theory of group characters and matrix representations of groups, 2nd ed., Oxford, 1950.

6. J. S. Lomont, Applications of finite groups, New York, 1959.

7. Marvin Marcus and Henryk Minc, A survey of matrix theory and matrix inequalities, Boston, 1964.

8. George D. Mostow, Joseph H. Sampson, and Jean-Pierre Meyer, Fundamental structures of algebra, McGraw-Hill, New York, 1963, chap. 16.

9. Thomas Muir, The theory of determinants, vol. I–IV, Dover, New York, 1960.

10. ———, Contributions to the history of determinants, 1900–1920, Glasgow, 1930.

11. Thomas Muir, and William H. Metzler, A treatise on the theory of determinants, Dover, 1960.

12. Oystein Ore, Theory of graphs, Amer. Math. Soc., Coll. Publ. vol. 38, Providence, 1962, p. 128.

13. John Riordan, An introduction to combinatorial analysis, Wiley, New York, 1958, p. 184.

14. Herbert John Ryser, Combinatorial mathematics, Carus Math. Monograph no. 14, 1963.

15. H. W. Turnbull, The theory of determinants, matrices and invariants, 3rd ed., Dover, New York, 1960, p. 14, 251.

16. J. H. M. Wedderburn, Lectures on matrices, Amer. Math. Soc., Coll. Publ. vol 17, Providence, 1934, 75–76.

B. Papers

1. C. W. Borchardt, Bestimmung der symmetrischen Verbindungen vermittelst ihrer erzeugenden Funktion, Monatsb. Akad. Wiss., Berlin, 1855, 165–171; Crelle's J., 53, 193–198; Ges. Werke, 97–105.

2. E. R. Caianiello, Regularization and renormalization, I, Nuovo Cimento, 13 (1959) 637–661.

3. A. L. Cauchy, Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des transpositions opérées entre les variables qu'elles renferment, J. de l'Éc. Polyt., 10 (1812) 29–112.

4. A. Cayley, Note sur les normales d'une conique, Crelle's J., 54 (1857) 182–185; Collected Math. Papers, vol. 4, 74–77.

5. F. Ferber, Sur un symbole analogue aux déterminants, Bull. Soc. Math. de France, 27 (1899) 285–288.

6. ———, Application du symbole des déterminants positifs, Bull. Soc. Math. de France, 28 (1900) 128–130.

7. J. Hammond, Question 6001, Educ. Times, 32 (1879) 179.

8. J. Horner, Notes on determinants, Quart. J. Math, 8 (1865), 157–162.

9. F. Joachimsthal, De aequationibus quarti et sexti gradus quae in theoria linearum et superficierum secundi gradus occurunt, Crelle's J., 53 (1856) 149–172.

10. M. Lerch, Krátký důkaz Borchardtovy vety determinantni, Časopis Pěst. Mat. 23 (1893) 76–78.

11. Marvin Marcus, Some properties and applications of doubly stochastic matrices, this Monthly, 67 (1960) 215–221.

12. ———, Linear operations on matrices, this Monthly, 69 (1962) 837–847.

13. ———, An inequality connecting the P-condition number and the determinant, Numer. Math., 4 (1962) 350–353.

14. ———, The permanent analogue of the Hadamard determinant theorem, Bull. Amer. Math. Soc., 69 (1963) 494–496.

15. ———, The Hadamard theorem for permanents, Proc. Amer. Math. Soc., 15 (1964) 967–973.

16. ———, The use of multilinear algebra for proving matrix inequalities, Proc. of Conference on Matrix Theory, U. of Wisc., Madison, Oct. 14–16th, 1963.

17. Marvin Marcus, and William R. Gordon, Inequalities for subpermanents, Illinois J. Math., 8 (1964) 607–614.

18. Marvin Marcus, and F. C. May, The permanent function, Canad. J. Math., 14 (1962) 177–189.

19. Marvin Marcus, and Henryk Minc, On the relation between the determinant and the permanent, Illinois J. Math., 5 (1961) 376–381.

20. ———, Some results on doubly stochastic matrices, Proc. Amer. Math. Soc., 76 (1962) 571–579.

21. ——— ,The pythagorean theorem in certain symmetry classes of tensors, Trans. Amer. Math. Soc., 104 (1962) 510–515.

22. ———, Inequalities for general matrix functions, Bull. Amer. Math. Soc., 70 (1964) 308–313.

23. ———, Diagonal products in doubly stochastic matrices, Quart. J. Math. Oxford, (in press).

24. Marvin Marcus, Henryk Minc, and Benjamin Moyls, Some results on non-negative matrices, J. Res. Nat. Bur. Standards, 65 B (1961) 205–209.

25. Marvin Marcus, Henryk Minc, and B. N. Moyls, Permanent preservers on the space of doubly stochastic matrices, Canad. J. Math., 14 (1962) 190–194.

26. Marvin Marcus, and Morris Newman, On the minimum of the permanent of a doubly stochastic matrix, Duke Math. J., 26 (1959) 61–72.

27. ——, Permanents of doubly stochastic matrices, Amer. Math. Soc., Proc. Symp. Appl. Math., vol. 10 (1960) 169–174.

28. ——, The permanent function as an inner product, Bull. Amer. Math. Soc., 67 (1961) 223–224.

29. ——, Inequalities for the permanent function, Ann. of Math., 75 (1962) 47–62.

30. Marvin Marcus, and R. Ree, Diagonals of doubly stochastic matrices, Quart. J. Math. Oxford Ser. (2), 10 (1959) 296–302.

31. Marvin Marcus, and Robert C. Thompson, The field of values of the Hadamard product, Arch. Math., 14 (1963) 283–288.

32. N. S. Mendelsohn, Permutations with confined displacements, Canad. Math. Bull., 4 (1961) 29–38.

33. Henryk Minc, A note on an inequality of M. Marcus and M. Newman, Proc. Amer. Math. Soc., 14 (1963) 890–892.

34. ———, Upper bounds for permanents of (0,1)-matrices, Bull. Amer. Math. Soc., 69 (1963) 789–791.

35. ———, Permanents of (0, 1)-circulants, Canad. Math. Bull., 7 (1964) 253–263.

36. Henryk Minc, and Leroy Sathre, Some inequalities involving $(r!)^{1/r}$, Proc. Edinburgh Math. Soc., 14 (1964) 41–46.

37. L. Mirsky, Results and problems in the theory of doubly stochastic matrices, Z. Wahrschein., 1 (1963) 319–334.

38. T. Muir, On a class of permanent symmetric functions, Proc. Roy. Soc. Edinburgh, 11 (1882) 409–418.

39. ———, A relation between permanents and determinants, Proc. Roy. Soc. Edinburgh, 22 (1897) 134–136.
40. ———, On a development of a determinant of the *mn*-th order, Trans. Roy. Soc. Edinburgh, 3 (1899) 623–628.
41. ———, Question 6001, Educ. Times, 65 (1912) 139.
42. ———, Note on double alternant, Trans. Roy. Soc. S. Africa, 3 (1912) 177–185.
43. ———, Question 18015, Educ. Times, 68 (1915) 238.
44. ———, Determinants whose elements are alternating numbers, Messenger of Math., 45 (1915) 21–27.
45. Paul J Nikolai, Permanents of incidence matrices, Math. Comput., 14 (1960) 262–266.
46. G. Polya, Aufgabe 424, Arch. Math. Phys. (3), 20 (1913) 271.
47. V. Řehořovsky, O vytvornjici funkci Borchardt-ove, Časopis Pěst. Mat. 11 (1882) 111–120.
48. H. J. Ryser, Compound and induced matrices in combinatorial analysis, Amer. Math. Soc., Proc. Symp. Appl. Math., 10 (1960) 149–167.
49. ———, Matrices of zeros and ones, Bull. Amer. Math. Soc., 66 (1960) 442–464.
50. ———, Matrices of zeros and ones in combinatorial mathematics, Proc. of Conference on Matrix Theory, U. of Wisc., Madison, Oct. 14–16, 1963.
51. I. Schur, Über endliche Gruppen und Hermitesche Formen, Math. Z., 1 (1918) 184–207.
52. R. F. Scott, Notes on determinants, Messenger of Math., 8 (1879) 182–187.
53. Morton L. Slater, and Robert J. Thompson, A permanent inequality for positive functions on the unit square, Pacific J. Math., 14 (1964) 1069–1078.
54. M. F. Tinsley, Permanents of cyclic matrices, Pacific J. Math., 10 (1960) 1067–1082.
55. B. L. van der Waerden, Aufgabe 45, Jber. Deutsch. Math. Verein., 35 (1926) 117.
56. A. Young, The expansion of the *n*-th power of a determinant, Messenger of Math., 33 (1903) 113–116.

THE CONSTRUCTION OF ORTHOGONAL AND QUASI-ORTHOGONAL NUMBER SETS

H. W. GOULD, West Virginia University

1. Introduction. There has always been great interest in the discovery of inversion relations for series and associated orthogonality relations. The references at the end of this paper will give some idea of current activity in this field and each bears an intimate relationship to our work here. The reader will find it of interest to compare some of our results with the relations in Riordan's recent paper [6]. Several results of Riordan were influenced by the orthogonality relations found by the writer [2]. Theorem 3 below is in turn based on an extension of ideas in [2] which are not unlike some new results found by Carlitz [1].

We shall distinguish between *orthogonality* and *quasi-orthogonality*. Such a distinction has been made by S. Tauber [9] and we shall follow his nomenclature. Associated with two number sets A_n^m and B_n^m are the polynomials

$$(1.1) \qquad A_n(x) = \sum_{k=0}^{n} A_n^k x^k, \qquad B_n(x) = \sum_{k=0}^{n} B_n^k x^k,$$

and the number sets are called *quasi-orthogonal* in case

$$(1.2)\qquad \sum_{s=m}^{n} A_n^s B_s^m = \delta_n^m = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

Tauber gives an equivalent formulation in terms of triangular matrices of the numbers A_n^m and B_n^m. Also we take $A_n^m = 0 = B_n^m$ for $n < 0$, $m < 0$, or $n < m$.

By contrast, we should call the pair A and B *orthogonal* in case

$$(1.3)\qquad \sum_{s=m}^{n} A_s^n B_s^m = \delta_n^m.$$

In the present paper we develop three theorems which allow one to construct any number of *quasi-orthogonal* relations involving binomial coefficients. Some may wish to call these *orthogonal* relations instead.

The distinction between A_s^n and A_n^s needs elaboration. It is familiar that

$$(1.4)\qquad \sum_{s=m}^{n} (-1)^{s+m} \binom{n}{s}\binom{s}{m} = \binom{n}{m} \sum_{s=0}^{n-m} (-1)^s \binom{n-m}{s} = \delta_n^m,$$

but no correspondingly simple result of the form

$$(1.5)\qquad \sum_{s=m}^{n} \binom{n}{s}\binom{m}{s} w(s)$$

for a weight function w is evident.

Similarly, in the case of the Stirling numbers $s(m, n)$, $S(m, n)$, of first and second kinds respectively, we have [6; (6)]

$$(1.6)\qquad \sum_k s(n, k)S(k, m) = \sum_k S(n, k)s(k, m) = \delta_n^m,$$

but no correspondingly simple result for a summation with k and m interchanged.

It seems appropriate to go further and call attention to the following unsolved problem. For various classical polynomials $P_n(x)$ (e.g., Legendre) we have

$$(1.7)\qquad \int_a^b P_n(x)P_m(x)w(x)dx = \delta_n^m,$$

and although one may define $P_x(m)$ for real x, little seems known about the possibility of having a quasi-orthogonal relation of the form

$$(1.8)\qquad \int_a^b P_n(x)P_x(m)w(x)dx = \delta_n^m.$$

2. S-polynomials; Laguerre and Hermite numbers. We consider first the S-polynomials of Tauber [11], who has considered the situation when one assumes

the expansion

$$S(x, n) = \sum_{k=0}^{n} A_n^k x^k \tag{2.1}$$

together with the inverse expansion

$$x^n = \sum_{k=0}^{n} B_n^k S(x, k), \tag{2.2}$$

and with the condition that $B_0^0 S(x, 0) = 1$.

Since $\{1, x, x^2, x^3, \cdots\}$ forms a linearly independent set of functions, it is necessary to suppose that $B_n^n \neq 0$ for all integers $n \geqq 0$ in order to guarantee the expansion (2.2). Then, substitution of (2.1) into (2.2) leads Tauber to the quasi-orthogonality

$$\sum_{s=m}^{n} B_n^s A_s^m = \delta_n^m. \tag{2.3}$$

Similarly, it is of interest to note that substitution of (2.2) into (2.1) leads to the symmetrical counterpart of (2.3), with A and B interchanged,

$$\sum_{s=m}^{n} A_n^s B_s^m = \delta_n^m. \tag{2.4}$$

If we were dealing with ordinary orthogonality of the sort supposed in (1.5) then (2.3) would be identical with (2.4), whereby we have another general distinction which arises in the nature of quasi-orthogonality. Examination of the proofs of some of the inversion relations of the author [2] will reveal that a similar situation holds there. This is generally reflected by variations in the method of proof for the relation (2.4), say, as opposed to proof of (2.3). One may consider, for example the proofs of Carlitz [1] for the general pair of inverse relations which he has recently found. Also, cf. Riordan [6; (16), (17), (18)].

Tauber [11] shows that certain recurrence relations must hold for the A and B numbers in the S-polynomial expansions (2.1) and (2.2). From this he is able to determine sets of quasi-orthogonal numbers in the case of the usual Laguerre and Hermite polynomials. We shall present a different approach, avoiding the use of recurrence relations entirely. The situation is that we may always arrive at new quasi-orthogonal number sets on the basis of such sets as we already know. Indeed, a fundamental result in this direction is found at once from relation (1.4). We have

THEOREM 1. *Let Q_s $(s=0, 1, 2, \cdots)$ be an arbitrary sequence of real or complex numbers such that neither Q_s nor Q_s^{-1} is zero for any relevant value of the index s. Then the numbers*

$$A_n^s = (-1)^s \binom{n}{s} Q_s, \qquad B_s^m = (-1)^m \binom{s}{m} Q_s^{-1} \tag{2.5}$$

form a set of quasi-orthogonal numbers in the sense that they satisfy both relations (2.3) *and* (2.4).

Proof. The proof is essentially trivial for we have

$$A_n^s B_s^m = (-1)^{s+m} \binom{n}{s} \binom{s}{m}$$

and

$$B_n^s A_s^m = Q_m Q_n^{-1} (-1)^{s+m} \binom{n}{s} \binom{s}{m},$$

so that by summing and using (1.4) we have the desired relations.

The theorem may seem innocuous enough, or somewhat transparent, but has wide application. Thus, to apply it to the study of the Laguerre polynomials, we first recall [8; 99–100] that an explicit formula for Laguerre polynomials is

$$(2.6) \qquad L_n^{(a)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+a}{n-k} \frac{x^k}{k!}, \qquad L_n^{(0)}(x) = L_n(x).$$

When $a=0$ we may write (following Tauber's notation)

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} = \sum_{k=0}^{n} L_n^k x^k$$

so that

$$(2.7) \qquad A_n^s = (-1)^s \binom{n}{s} \frac{1}{s!} = L_n^s \text{ (of Tauber)}$$

and

$$(2.8) \qquad B_s^m = (-1)^m \binom{s}{m} s! = J_s^m \text{ (of Tauber)}.$$

Here the choice is easy since Q_s involves only the parameter s. Thus $Q_s = 1/s!$.

When $a \neq 0$ we may use a slight modification of Theorem 1 and one solution is afforded by the following choice:

$$(2.9) \qquad A_n^s = (-1)^s \binom{n+a}{n-s} \frac{1}{s!} = (-1)^s \binom{n}{s} \frac{\Gamma(n+a+1)}{n!\Gamma(a+s+1)}$$

and

$$(2.10) \qquad B_s^m = (-1)^m \binom{s}{m} \frac{m!\Gamma(a+s+1)}{\Gamma(m+a+1)} = (-1)^m \binom{a+s}{s-m} s!.$$

Thus, for the general Laguerre polynomial $L_n^{(a)}(x) = A_n(x)$, the associated $B_n(x)$

polynomial in (1.1) may be given by

$$B_n(x) = n! \sum_{k=0}^{n} (-1)^k \binom{a+n}{n-k} x^k. \tag{2.11}$$

In case $a=0$ this has the simple form $n!(1-x)^n$. The series in (2.11) is not summable in closed form in general. When a is a nonpositive integer such that $0 \leq a+n \leq n$, then the binomial theorem may be applied.

It is evident how Theorem 1 or a slight modification of it may be applied then to deduce classes of quasi-orthogonal numbers. It is necessary only to invoke the basic relation (1.4). The numbers Q_s involved in (2.9) depend upon n as well as s. We may still choose Q_s^{-1} for use in (2.10) but then B_s^m would depend upon the parameter n. It is convenient to go from $Q_s(n)$ to $Q_s^{-1}(m)$ in such a case, as we have done above.

As we have remarked earlier, we are not concerned with recurrence relations; it may be of interest, however, to indicate that the L and J numbers of (2.7) and (2.8) satisfy simple recurrence relations. In fact

$$kL_n^k = kL_{n-1}^k - L_{n-1}^{k-1} \tag{2.12}$$

and

$$J_n^k = nJ_{n-1}^k - nJ_{n-1}^{k-1} \tag{2.13}$$

which may be contrasted with the relations

$$nL_n^k = (2n-1)L_{n-1}^k - L_{n-1}^{k-1} - (n-1)L_{n-2}^k \tag{2.14}$$

and

$$J_n^k = -kJ_{n-1}^{k-1} + (2k+1)J_{n-1}^k - (k+1)J_{n-1}^{k+1} \tag{2.15}$$

as found by Tauber [**11**]. Relations (2.12) and (2.13) follow from nothing more sophisticated than the basic recurrence relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

and formulas (2.7) and (2.8).

Now, if one examines the set of quasi-orthogonal numbers which arise in the case of Hermite polynomials, and in particular the final quasi-orthogonality found by Tauber [**11**] for his H and K Hermite numbers, it appears that a variation of (1.4) and our Theorem 1 are involved. Tauber's discussion involves the parity of the various indices occurring in the summations. This naturally arises in the fact that the Hermite polynomial $H_n(x)$ is an even function when n is even, and an odd function when n is odd. But this is a special case which falls out from a generalization of Hermite polynomials considered in [**3**; 58].

We need a simple number-theoretic function which takes account of a natural generalization of parity. We make the

DEFINITION. *Let $r \geqq 1$ be an integer. Then for any integer k*

$$(2.16)\qquad g_r(k) = g(k) = \begin{cases} 1 & \text{if } r \mid k, \\ 0 & \text{if } r \nmid k. \end{cases}$$

When $r=2$ one often makes use of the formula $g(k)=\frac{1}{2}(1+(-1)^k)$ in operations with series where certain terms are to be deleted.

We need to observe that $g(k+rs)=g(k)$ for any integer s (negative, zero, or positive). Note that g could be expressed in terms of the roots of unity.

We may now state our next result:

THEOREM 2. *Let Q_s $(s=0, 1, 2, \cdots)$ be an arbitrary sequence of real or complex numbers such that neither Q_s nor Q_s^{-1} is zero for any relevant value of the index s. Then the numbers*

$$(2.17)\qquad A_n^s = (-1)^{(n-s)/r} \frac{n!}{\left(\dfrac{n-s}{r}\right)!} Q_s g(n-s), \qquad r \geqq 1,$$

and

$$(2.18)\qquad B_s^m = \frac{1}{m!\left(\dfrac{s-m}{r}\right)!} Q_s^{-1} g(s-m)$$

form a set of quasi-orthogonal numbers in the sense that they satisfy both relations (2.3) *and* (2.4).

Proof. To verify (2.4) we have

$$\begin{aligned}
\sum_{s=m}^{n} A_n^s B_s^m &= \sum_{s=m}^{n} (-1)^{(n-s)/r} \frac{n!}{m!\left(\dfrac{s-m}{r}\right)!\left(\dfrac{n-s}{r}\right)!} g(n-s)g(s-m) \\
&= \sum_{s=0}^{n-m} (-1)^{(n-m-s)/r} \frac{n!}{m!\left(\dfrac{s}{r}\right)!\left(\dfrac{n-m-s}{r}\right)!} g(n-m-s)g(s) \\
&= \sum_{s=0}^{[(n-m)/r]} (-1)^{(n-m)/r}(-1)^s \frac{n!}{m!s!\left(\dfrac{n-m}{r}-s\right)!} g(n-m-rs) \\
&= (-1)^{(n-m)/r} g(n-m) \frac{n!}{m!} \sum_{s=0}^{[(n-m)/r]} (-1)^s \frac{1}{s!\left(\dfrac{n-m}{r}-s\right)!} \\
&= 0 \quad \text{if } r \nmid (n-m).
\end{aligned}$$

Let then $n-m=pr$ so that $n=m+pr$. The summation becomes then

$$(-1)^p \frac{(m+pr)!}{m!} \sum_{s=0}^{p} (-1)^s \frac{1}{s!(p-s)!}$$

$$= (-1)^p \frac{(m+pr)!}{m!p!} \sum_{s=0}^{p} (-1)^s \binom{p}{s} = (-1)^p \frac{(m+pr)!}{m!p!} \delta_0^p$$

which equals 0 when $p \neq 0$, i.e. when $n > m$; and equals 1 when $p=0$, i.e. when $n=m$. Thus our assertion is correct that (2.4) follows.

We shall omit the corresponding steps to establish relation (2.3) because the general pattern is the same. Theorem 1 is implied by Theorem 2 when $r=1$.

Theorem 2 is far less trivial than Theorem 1; yet is based again on the simplest kind of quasi-orthogonality flowing from the binomial theorem.

To apply our theorem to the Hermite polynomials, we recall [8; 105] the explicit formula

$$H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k}. \tag{2.19}$$

Then we have

$$H_n(x) = \sum_{k=0}^{n} (-1)^{k/2} \frac{n!}{\left(\frac{k}{2}\right)!\,(n-k)!} (2x)^{n-k} \cdot \frac{1+(-1)^k}{2}$$

$$= \sum_{k=0}^{n} (-1)^{(n-k)/2} \frac{n!}{\left(\frac{n-k}{2}\right)!\,k!} (2x)^k \cdot \frac{1+(-1)^{n-k}}{2}$$

$$= \sum_{k=0}^{n} H_n^k x^k,$$

so that we may take the coefficient of x^k to be

$$H_n^k = (-1)^{(n-k)/2} \frac{n!}{\left(\frac{n-k}{2}\right)!\,k!} 2^k \cdot \frac{1+(-1)^{n-k}}{2} \tag{2.20}$$

which is in agreement with Tauber's notation. Applying Theorem 2, with $r=2$, we see that we may take $Q_k = 2^k/k!$ and then a set of numbers quasi-orthogonal to the H_n^k will be given by

$$K_k^m = \frac{k!}{m!\left(\frac{k-m}{2}\right)!\,2^k} \cdot \frac{1+(-1)^{k-m}}{2}. \tag{2.21}$$

Except that Tauber [11] states the parity in words, relation (2.21) is precisely the same as the K_k^m which he found. Indeed, since the (finite) Taylor series expansion of $H_n(x)$ in a linear series or powers of x and the inverse expansion of x^n in a linear series of Hermite polynomials have each a unique set of coefficients, it is evident that our results must agree with those of Tauber.

Now it is of interest to apply Theorem 2 in its full generality to the generalization of Hermite polynomials studied in [3; 58]. There the exponential generating function

$$e^{tx+ht^r} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \overset{r}{g}_n(x, h) \tag{2.22}$$

led to a set of polynomials with the explicit formula

$$\overset{r}{g}_n(x, h) = \sum_{k=0}^{[n/r]} (-1)^k \frac{n!}{k!(n-rk)!} h^k x^{n-rk}. \tag{2.23}$$

In Theorem 2, choose $Q_s = 1/s!$. Then we have the set of quasi-orthogonal numbers

$$A_n^s = (-1)^{(n-s)/r} \frac{n!}{\left(\frac{n-s}{r}\right)!\, s!} g(n-s) \tag{2.24}$$

and

$$B_s^m = \frac{s!}{m!\left(\frac{s-m}{r}\right)!} g(s-m). \tag{2.25}$$

Corresponding to (2.24) we have the polynomial

$$\begin{aligned} A_n(x, r) &= \sum_{s=0}^{n} A_n^s x^s = \sum_{s=0}^{n} (-1)^{(n-s)/r} \frac{n!}{\left(\frac{n-s}{r}\right)!\, s!} x^s g(n-s) \\ &= \sum_{s=0}^{n} (-1)^{s/r} \frac{n!}{\left(\frac{s}{r}\right)!(n-s)!} x^{n-s} g(s) = \sum_{s=0}^{[n/r]} (-1)^s \frac{n!}{s!(n-rs)!} x^{n-rs}, \end{aligned}$$

whence we have by comparison with (2.23)

$$A_n(x, r) = \overset{r}{g}_n(x, -1). \tag{2.26}$$

In the same way, the polynomial $B_n(x, r)$ corresponding to (2.25) is seen to be

$$B_n(x, r) = \overset{r}{g}_n(x, 1). \tag{2.27}$$

We wish to observe that the generating function (2.22) then yields

$$e^{2tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} A_k(x, r) B_{n-k}(x, r) \tag{2.28}$$

so that the value of r has no effect on the value of the inner convolution on the right. Since $A_k(x, 1) = (x-1)^k$ and $B_{n-k}(x, 1) = (x+1)^{n-k}$, then we have incidentally found the convolution summation

$$\sum_{k=0}^{n} \binom{n}{k} A_k(x, r) B_{n-k}(x, r) = (2x)^n \tag{2.29}$$

valid for all real x and arbitrary r. This brings out then still another remarkable connection between the polynomials from which the quasi-orthogonal numbers A_n^s and B_s^m flow in our general example. Also, relation (2.29) may be viewed as a way of expanding x^n in terms of the A and B polynomials.

It is expected that other remarkable consequences flow from Theorem 2. Moreover, just as Theorems 1 and 2 are based upon the basic orthogonality formula (1.4) so we may expect to find similar generalizations which come from other types of orthogonal formulas, for example the orthogonal relation for Stirling numbers of first and second kinds.

3. The Humbert polynomials. We wish to turn next to a consideration of a generalization of Legendre and Gegenbauer polynomials which traces back to a suggestion of Pierre Humbert in 1921. An account of this and a study of new inverse series relations implied by generalizations of the Humbert polynomials is given by the author in [4]. We summarize some findings from this paper and show how they lead to a general set of quasi-orthogonal numbers analogous for Legendre polynomials to the set in our Theorem 2 for Hermite polynomials.

For our purposes we shall define the Humbert polynomial $P_n(r, x, p)$ by the expansion

$$(1 - rxt + t^r)^p = \sum_{n=0}^{\infty} t^n P_n(r, x, p), \qquad (r \geqq 1, \text{ integral}) \tag{3.1}$$

for an arbitrary real number p. Then $P_n(2, x, -\frac{1}{2}) = P_n(x)$ is the usual Legendre polynomial and $P_n(2, x, -\nu) = C_n^\nu(x)$ is the Gegenbauer polynomial. Also we may observe that $P_n(2, x, -1) = U_n(x)$ is called a Tchebycheff polynomial, so that expansion (3.1) is general enough to include many interesting examples.

The Humbert polynomials $P_n(r, x, p)$ with $r \geqq 3$ do not satisfy a three-term recurrence relation of the kind which is necessary [8; 42] for the polynomials to be orthogonal over any interval $a \leqq x \leqq b$ in the way that the Legendre and Gegenbauer polynomials are. However, it is still possible to obtain interesting expansions of functions in terms of the Humbert polynomials. In fact, it has been found in [4] that

$$P_n(r, x, p) = \sum_{k=0}^{[n/r]} \binom{p - n + rk}{k} \binom{p}{n - rk} (-rx)^{n-rk} \tag{3.2}$$

with the corresponding inverse expansion

$$(-rx)^n \binom{p}{n} = \sum_{k=0}^{[n/r]} (-1)^k \binom{p - n + k}{k} \frac{p + rk - n}{p + k - n} P_{n-rk}(r, x, p). \tag{3.3}$$

The derivation of (3.3) depends on a general inversion theorem which extends one of the results in [2].

Now it is clear that (3.2) and (3.3) correspond to the general class of expansions (2.1) and (2.2) considered by Tauber. Thus it is of interest to see what the corresponding quasi-orthogonal sets may be. Again we make use of the number-theoretic function defined by (2.16). There is no difficulty to see that relations (3.2) and (3.3) may be cast in the alternative forms

$$P_n(r, x, p) = \sum_{k=0}^{n} \left\{ \begin{matrix} p - k \\ \dfrac{n - k}{r} \end{matrix} \right\} \left\{ \begin{matrix} p \\ k \end{matrix} \right\} (-rx)^k g_r(n - k) \tag{3.4}$$

and

$$x^n = \sum_{k=0}^{n} (-1)^{(n-k)/r} \left\{ \begin{matrix} p - n + \dfrac{n - k}{r} \\ \dfrac{n - k}{r} \end{matrix} \right\} \frac{p - k}{p - n + \dfrac{n - k}{r}} \cdot \frac{P_k(r, x, p) g_r(n - k)}{(-r)^n \binom{p}{n}}. \tag{3.5}$$

From these two relations the exact form of the quasi-orthogonal numbers A_n^k and B_n^k as in (2.1) and (2.2) is immediately evident. However, just as was the case for Theorems 1 and 2 it is evident that a generalization may be stated. We have

THEOREM 3. *Let Q_s $(s = 0, 1, 2, \cdots)$ be an arbitrary sequence of real or complex numbers such that neither Q_s nor Q_s^{-1} is zero for any relevant value of the index s. Then the numbers*

$$A_n^s = \left\{ \begin{matrix} p - s \\ \dfrac{n - s}{r} \end{matrix} \right\} Q_s g_r(n - s) \tag{3.6}$$

and

$$B_s^m = (-1)^{(s-m)/r} \left\{ \begin{matrix} p - s + \dfrac{s - m}{r} \\ \dfrac{s - m}{r} \end{matrix} \right\} \frac{p - m}{p - s + \dfrac{s - m}{r}} Q_s^{-1} g_r(s - m) \tag{3.7}$$

with $r \geqq 1$ an integer, and p an arbitrary real parameter, form a set of quasi-orthogonal numbers in the sense that they satisfy both relations (2.3) *and* (2.4).

The proof of this result follows from the validity of the expansions (3.2) and (3.3). It will be seen on a critical examination that Theorem 3 is essentially equivalent to the theorem embodied in the pair of inverse relations (7.5) and (7.6) of the author's paper [4]. For the sake of completeness we quote them here:

$$F(n) = \sum_{k=0}^{[n/r]} \binom{p - n + rk}{k} f(n - rk) \tag{3.8}$$

if and only if

$$f(n) = \sum_{k=0}^{[n/r]} (-1)^k \binom{p - n + k}{k} \frac{p + rk - n}{p + k - n} F(n - rk). \tag{3.9}$$

We remark that Theorem 3 implies (3.4) and (3.5) when

$$Q_s = (-r)^s \binom{p}{s}.$$

We have thus found general theorems which include special cases dealing with the classical Laguerre, Hermite, Legendre, Gegenbauer, and Humbert polynomials in terms of the notion of quasi-orthogonality.

We wish to end our present discussion by showing that Theorem 3 also includes some inversion theorems found by Stanton and Sprott [7]. Indeed, choose $r=1$ and replace Q_s by $(-1)^s Q_s$. Then the numbers

$$A_n^s = (-1)^s \binom{p - s}{n - s} Q_s \quad \text{and} \quad B_s^m = (-1)^m \binom{p - m}{s - m} Q_s^{-1} \tag{3.10}$$

form a quasi-orthogonal number set provided that neither Q_s nor $1/Q_s$ is zero. In terms of (2.1) and (2.2) this means that

$$S(x, n) = \sum_{k=0}^{n} (-1)^k \binom{p - k}{n - k} Q_k x^k \tag{3.11}$$

if and only if

$$x^n = \sum_{k=0}^{n} (-1)^k \binom{p - k}{n - k} Q_n^{-1} S(x, k). \tag{3.12}$$

Write $Q_k x^k = f(k)$ and $F(k) = S(x, k)$. Then the last two relations may be written in quite general form as

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{p - k}{n - k} f(k) \tag{3.13}$$

if and only if

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{p-k}{n-k} F(k), \tag{3.14}$$

where F and f are any indeterminate number sequences in the sense that, given f, F is determined by (3.13) or conversely, given F, f is determined by (3.14). But this pair of inverse relations, when p is a nonnegative integer, is precisely the third theorem of Stanton and Sprott [7] which is equivalent to their first theorem. Relations (3.13) and (3.14) are also immediately evident from (3.8) and (3.9).

Research supported by National Science Foundation Grant GP-482.

References

1. L. Carlitz, Some inversion formulas, Amer. Math. Soc. Notices, 10 (1963) 575, Abstract 63T-329. Rend. Circ. Mat. Palermo, (2) 12 (1963) 183–199.

2. H. W. Gould, A new convolution formula and some new orthogonal relations for inversion of series, Duke Math. J., 29 (1962) 393–404.

3. H. W. Gould and A. T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *ibid.*, 51–63.

4. H. W. Gould, Inverse series relations and other expansions involving Humbert polynomials. To appear in Duke Math. J.

5. C. Jordan, Calculus of Finite Differences, Budapest, 1939. Reprinted by Chelsea, New York, 1950.

6. John Riordan, Inverse relations and combinatorial identities, this MONTHLY, 71 (1964) 485–498.

7. R. G. Stanton and D. A. Sprott, Some finite inversion formulae, Math. Gazette, 46 (1962) 197–202.

8. G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Coll. Publ., Rev. ed., 23 (1959).

9. S. Tauber, On quasi-orthogonal numbers, this MONTHLY, 69 (1962) 365–372.

10. ———, On multinomial coefficients, this MONTHLY, 70 (1963) 1058–1063.

11. ———, On two classes of quasi-orthogonal numbers, this MONTHLY, next article.

ON TWO CLASSES OF QUASI-ORTHOGONAL NUMBERS

SELMO TAUBER, Portland State College

1. Introduction. The definition of quasi-orthogonal numbers can be found in [1], p. 32. Classes of such numbers were studied in [2]. We recall that two sets of numbers A_n^s and B_s^m are said to be quasi-orthogonal if

$$\sum_{s=m}^{n} A_n^s B_s^m = \delta_n^m, \tag{1}$$

where δ_n^m is the Kronecker-δ, provided that

$$A_n^s = 0,\ B_s^m = 0, \quad \text{for,} \quad n < 0, m < 0, s < 0, s > n, m > s. \tag{2}$$

In this paper we shall study two more classes of such numbers which lead to some classical examples.

2. The S-polynomials. Let M_k and N, $k=0, 1, 2, \cdots, n$ be functions of the variable n which is either a positive integer or zero, such that $N(n)\neq 0$ for all $n\geqq 0$. For given p and q we define

$$S(x, n) = \sum_{k=1}^{q} M_k(n)S(x, n-k) + N(n)xS(x, n-p), \tag{3}$$

$$S(x, 0) = M_0(0) = \text{const.} \tag{4}$$

$S(x, n)$ is a polynomial of degree n and can be written

$$S(x, n) = \sum_{m=0}^{n} A_n^m x^m; \tag{5}$$

conversely let

$$x^n = \sum_{m=0}^{n} B_n^m S(x, m), \tag{6}$$

with

$$1 = B_0^0 S(x, 0) = B_0^0 M_0(0). \tag{7}$$

Since (2) is to be satisfied we can write

$$x^n = \sum_{s=0}^{n} B_n^s S(x, s) = \sum_{s=0}^{n} B_n^s \sum_{m=0}^{s} A_s^m x^m = \sum_{s=0}^{n} B_n^s \sum_{m=0}^{n} A_s^m x^m$$

$$x^n = \sum_{m=0}^{n} x^m \sum_{s=m}^{n} B_n^s A_s^m,$$

so that

$$\sum_{s=m}^{n} B_n^s A_s^m = \delta_n^m, \tag{8}$$

i.e. the numbers A and B are quasi-orthogonal, since we may assume that condition (2) is satisfied.

According to (3) and (5) we have

$$S(x, n) = \sum_{m=0}^{n} A_n^m x^m = \sum_{k=1}^{q} M_k(n)S(x, n-k) + N(n)xS(x, n-p)$$

$$= \sum_{k=1}^{q} M_k(n) \sum_{m=0}^{n-k} A_{n-k}^{m} x^{m} + N(n) \sum_{m=0}^{n-p} A_{n-p}^{m} x^{m+1}$$

$$= \sum_{k=1}^{q} M_k(n) \sum_{m=0}^{n} A_{n-k}^{m} x^{m} + N(n) \sum_{m=0}^{n} A_{n-p}^{m} x^{m+1}$$

$$= \sum_{m=0}^{n} x^{m} \sum_{k=1}^{q} M_k(n) A_{n-k}^{m} + N(n) \sum_{m=0}^{n} A_{n-p}^{m} x^{m+1}.$$

Equating the coefficients of x^m, we obtain

$$A_n^{m} = \sum_{k=1}^{q} M_k(n) A_{n-k}^{m} + N(n) A_{n-p}^{m-1}, \tag{9}$$

which gives a recurrence relation for the A-numbers.

On the other hand, according to (6)

$$x^{n+1} = \sum_{m=0}^{n+1} B_{n+1}^{m} S(x, m) = x \cdot x^{n} = \sum_{m=0}^{n} B_n^{m} x S(x, m). \tag{10}$$

But

$$S(x, m + p) = \sum_{k=1}^{q} M_k(m + p) S(x, m + p - k) + N(m + p) x S(x, m);$$

hence

$$xS(x, m) = \left[S(x, m + p) - \sum_{k=1}^{q} M_k(m + p) S(x, m + p - k) \right] \Big/ N(m + p). \tag{11}$$

Substituting (11) into (10) we obtain

$$\begin{aligned} x^{n+1} &= \sum_{m=0}^{n+1} B_{n+1}^{m} S(x,m) \\ &= \sum_{m=0}^{n} B_n^{m} \left[S(x, m + p) - \sum_{k=1}^{q} M_k(m + p) S(x, m + p - k) \right] \Big/ N(m + p). \end{aligned}$$

By equating the coefficients of $S(x, m)$ we have

$$B_{n+1}^{m} = B_n^{m-p} / N(m) - \sum_{k=1}^{q} B_n^{m-p+k} M_k(m + k) / N(m + k),$$

or, changing n into $n-1$

$$B_n^{m} = [1/N(m)] B_{n-1}^{m-p} - \sum_{k=0}^{q} [M_k(m + k)/N(m + k)] B_{n-1}^{m+k-p}, \tag{12}$$

which is the recurrence relation for the B-numbers. We can state these results as follows:

THEOREM I. *If A_n^m and B_n^m are quasi-orthogonal numbers, i.e. numbers satisfying* (1) *and* (2), *and if the numbers A_n^m satisfy the relation* (9)

$$A_n^m = \sum_{k=1}^{q} M_k(n) A_{n-k}^m + N(n) A_{n-p}^{m-1},$$

where $N(n) \neq 0$, for all positive integral values of n and $n=0$, then the numbers B_n^m satisfy the relation (12)

$$B_n^m = [1/N(m)] B_n^{m-p} - \sum_{k=1}^{q} [M_k(m+k)/N(m+k)] B_{n-1}^{m+k-p}.$$

Conversely, if the numbers B_n^m satisfy (12), *the numbers A_n^m satisfy* (9).

3. Example I: *Laguerre Numbers.* We call Laguerre numbers the coefficients of the simple Laguerre polynomials. For the properties of the Laguerre polynomials we refer to [**3**], pp. 200–215, and [**4**], Chapter V. In [**3**] p. 214, (18) we find the following recurrence relation for Laguerre polynomials:

$$nL_n(x) = (2n - 1 - x) L_{n-1}(x) - (n-1) L_{n-2}(x), \tag{13}$$

with $L_0(x) = 1$. By writing

$$L_n(x) = \sum_{m=0}^{n} L_n^m x^m,$$

and equating in (13) the coefficients of equal powers of x we obtain

$$nL_n^m = (2n-1) L_{n-1}^m - L_{n-1}^{m-1} - (n-1) L_{n-2}^m. \tag{14}$$

We see immediately that (13) and (14) are of the same form as (3) and (9), provided that we assume that $n > 0$, which we can do since for $n=0$ both (13) and (14) are identically verified. We thus have $p=1$, $q=2$, $M_1(n) = (2n-1)/n$, $M_2(n) = -(n-1)/n$, $N(n) = -1/n$. Let J_n^m be the numbers that are quasi-orthogonal to the numbers L_n^m. According to (12) the numbers J_n^m satisfy the relation

$$J_n^m = -mJ_{n-1}^{m-1} + (2m+1) J_{n-1}^m - (m+1) J_{n-1}^{m+1}, \tag{15}$$

with $L_0^0 = J_0^0 = 1$. It is easily verified that

$$L_n^m = (-1)^m (n)_{n-m} \binom{n}{m} \Big/ n!, \qquad J_n^m = (-1)^m n! \binom{n}{m}; \tag{16}$$

thus

$$\sum_{s=m}^{n} L_n^s J_s^m = \sum_{s=m}^{n} \left[(-1)^s (n)_{n-s} \binom{n}{s} \Big/ n! \right] \left[(-1)^m s! \binom{s}{m} \right] = \delta_n^m,$$

or

$$\sum_{s=m}^{n} (-1)^{s+m}(n)_{n-s}\binom{n}{s}\binom{s}{m} s!/n! = \delta_n^m. \tag{17}$$

4. Example II: *Hermite Numbers.* According to [3] pp. 187–196, [4] Chapter V, and [5] pp. 117–122, Hermite polynomials can be defined by the recurrence relation

$$H_n(x) = -2(n-1)H_{n-2}(x) + 2xH_{n-1}(x), \tag{18}$$

with $H_0(x) = 1$. We set

$$H_n(x) = \sum_{m=0}^{n} H_n^m x^m, \tag{19}$$

where $H_n^m = 0$, for $n - m = 1 \pmod 2$, i.e. for n and m not of same parity. By identifying equal powers of x in (18) we obtain

$$H_n^m = -2(n-1)H_{n-2}^m + 2H_{n-1}^{m-1}. \tag{20}$$

Comparing (18) and (20) to (3) and (9), we see that $p=1$, $q=2$, $M_1(n)=0$, $M_2(n) = -2(n-1)$, $N(n)=2$. Let thus K_n^m be the numbers that are quasi-orthogonal to the numbers H_n^m, which we call the *Hermite Numbers.* According to the general theory established earlier we have, for the K-numbers,

$$K_n^m = (1/2)K_{n-1}^{m-1} + (m+1)K_{n-1}^{m+1}. \tag{21}$$

It will be easily verified that

$$\begin{aligned} H_n^m &= (-1)^{(n-m)/2} 2^m (n)_{n-m} \Big/ \left(\frac{n-m}{2}\right)!, \\ K_n^m &= \binom{n}{m}(n-m)!/2^n\left(\frac{n-m}{2}\right)!. \end{aligned} \tag{22}$$

Since both n and m have same parity we assume that s is even if both n and m are even, and odd if both n and m are odd. Thus n, m, and s have always the same parity in the following expressions:

$$\sum_{s=m}^{n} H_n^s K_s^m = \delta_n^m,$$

hence we obtain after simplifications

$$\sum_{s=m}^{n} (-1)^{(n-s)/2}(n)_{n-s}\binom{s}{m}(s-m)! \Big/ \left(\frac{n-s}{2}\right)!\left(\frac{s-m}{2}\right)! = \delta_n^m. \tag{23}$$

References

1. K. S. Miller, An Introduction to the Calculus of Finite Differences and Difference Equations, Holt, Rinehart and Winston, New York, 1960.
2. S. Tauber, On quasi-orthogonal numbers, this MONTHLY, 69 (1962) 365–372.
3. E. D. Rainville, Special Functions, Macmillan, New York, 1960.
4. G. Szegö, Orthogonal polynomials, Amer. Math. Soc. Coll. Publ., 23(1959).
5. H. Margenau and G. M. Murphy, The Mathematics of Physics and Chemistry, Stechert-Hafner, New York, 1956.

SET EQUATIONS

A. W. GOODMAN, University of Kentucky

1. Introduction. The problem of solving a system of simultaneous equations in several real or complex variables, suggests the analogous problem of solving such a system when the unknowns are regarded as sets. Such a system will not always be solvable. In this paper, we consider two special systems and then proceed to slightly more general systems. In each case we obtain necessary and sufficient conditions for the system to have a solution.

2. Intersections in pairs. Let us consider first the simple system

$$X_i \cap X_j = G_{ij} \qquad 1 \leqq i < j \leqq n, \tag{1}$$

where the G_{ij} are given sets, and we are to determine the sets X_i, $i=1, 2, \cdots, n$ such that each of the $n(n-1)/2$ equations in the system (1) is satisfied. Observe that in (1) we require that $i \neq j$, because otherwise the equation $X_i \cap X_i = G_{ii}$ would force the solution $X_i = G_{ii}$ and the system (1) would be trivial. Further, since $X_i \cap X_j = X_j \cap X_i$, there is no loss of generality in requiring that $i<j$. For convenience we set $G_{ji} = G_{ij}$ for all pairs $i<j$.

Assume first that the system (1) has a solution. Then for each fixed index i

$$X_i \supset G_{i\alpha} \qquad \alpha = 1, 2, \cdots n, \qquad \alpha \neq i, \tag{2}$$

and hence

$$X_i \supset \bigcup_{\substack{\alpha=1\\ \alpha\neq i}}^{n} G_{i\alpha}. \tag{3}$$

Using (3) in (1) we see that

$$G_{ij} = X_i \cap X_j \supset \left(\bigcup_{\substack{\alpha=1\\ \alpha\neq i}}^{n} G_{i\alpha} \right) \cap \left(\bigcup_{\substack{\beta=1\\ \beta\neq j}}^{n} G_{\beta j} \right). \tag{4}$$

Now each of the two unions that appear in (4) contains the set G_{ij} and hence the containing sign may be replaced by the equality sign. Finally, using a distributive law, we can write (4) in the form

$$G_{ij} = \bigcup (G_{i\alpha} \cap G_{\beta j}), \tag{5}$$

where the union is over all pairs (α, β) such that $\alpha \neq i$ and $\beta \neq j$. Hence (5) is a necessary condition for the system (1) to have a solution. But it is also a sufficient condition, because if (5) is satisfied we can obtain a solution by setting

$$X_i = \bigcup_{\substack{\alpha=1 \\ \alpha \neq i}}^{n} G_{i\alpha} \qquad i = 1, 2, \cdots, n. \tag{6}$$

The solution is certainly not unique, for we can always select a new element that lies outside the set

$$G \equiv \bigcup G_{ij} \qquad 1 \leqq i < j \leqq n, \tag{7}$$

and adjoin it to one of the sets X_i defined in (6), to obtain a new system of solutions to (1). The containing sign in (3) shows, however, that the system of solutions (6) is a minimal system, i.e., each X_i is as small as possible. We have proved the following result.

THEOREM 1. *A necessary and sufficient condition that the system of set equations* (1) *be solvable for X_i, $i=1, 2, \cdots, n$, is that equation* (5) *holds for all pairs i, j such that $i \neq j$. When this condition is satisfied there is a unique minimal solution given by* (6), *and these solutions are subsets of the set G determined by the given sets G_{ij}* (*equation* (7)).

COROLLARY. *A necessary and sufficient condition that the system* (1) *be solvable is that for each pair i, j, with $i \neq j$,*

$$G_{ij} \supset G_{i\alpha} \cap G_{\beta j}, \qquad \begin{matrix} \alpha = 1, 2, \cdots, n, \alpha \neq i, \\ \beta = 1, 2, \cdots, n, \beta \neq j. \end{matrix} \tag{8}$$

The corollary allows us to test a given system by the following mechanical device. We arrange the given sets G_{ij} in a matrix, thus:

$$\mathcal{G} \equiv \begin{bmatrix} \phi & G_{12} & G_{13} & \cdots & G_{1n} \\ G_{21} & \phi & G_{23} & \cdots & G_{2n} \\ G_{31} & G_{32} & \phi & \cdots & G_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ G_{n1} & G_{n2} & G_{n3} & \cdots & \phi \end{bmatrix}$$

where for convenience we put ϕ, the empty set, in the main diagonal. The condition (8) then states that the intersection of any two sets selected from $\mathcal{G}$, one from a row and one from a column, must lie in the set determined by the inter-

section of that row and column (provided that they do not intersect on the main diagonal). In any practical problem this mechanical test is rather easy to apply.

THEOREM 2. *If the system* (1) *has a solution, it is unique in the set* G.

Proof. Let $\{X_1, X_2, \cdots, X_n\}$ be the system of solutions defined by (6). Suppose that for a fixed i we could obtain a new solution by replacing X_i with $Y_i = X_i \cup g$ where $g \in G$, and $g \notin X_i$. It follows from (6) that $g \notin G_{i\alpha}$, $\alpha = 1, 2, \cdots, n$. But, since $g \in G$, $g \in G_{jk}$ for some suitable pair j, $k \neq i$. This implies that $g \in X_j$, and hence $Y_i \cap X_j = G_{ij} \cup g$. Since G_{ij} does not contain g the new system is not a solution.

3. Unions in pairs. It is a simple matter to transform a theorem on intersections into a theorem on unions. Let H_{ij} be given sets and consider the system of equations

$$Y_i \cup Y_j = H_{ij} \qquad 1 \leqq i < j \leqq n. \tag{9}$$

As before we assume $H_{ij} = H_{ji}$, and we set

$$H = \bigcup H_{ij} \qquad 1 \leqq i < j \leqq n. \tag{10}$$

We take complements with respect to H and set $G_{ij} = \mathcal{C} H_{ij}$, $X_i = \mathcal{C} Y_i$. Then the system (9) reduces to the system (1). The set G defined by equation (7) is not necessarily identical with H, however, for if h is an element that occurs in each H_{ij} it will be absent in G_{ij}, and hence in G. Thus we have $G \subseteq H$, and $G = H$ if and only if

$$\bigcap H_{ij} = \phi \qquad 1 \leqq i < j \leqq n. \tag{11}$$

These considerations prove the following theorem.

THEOREM 3. *A necessary and sufficient condition that the system* (9) *be solvable for* Y_i, $i = 1, 2, \cdots, n$, *is that for all pairs* i, j, *with* $i \neq j$,

$$H_{ij} = \bigcap (H_{i\alpha} \cup H_{\beta j}), \tag{12}$$

where the intersection is taken over all pairs (α, β) *such that* $\alpha \neq i$ *and* $\beta \neq j$. *When this condition is satisfied a set of solutions is given by*

$$Y_i = \bigcap_{\substack{\alpha=1 \\ \alpha \neq i}}^{n} H_{i\alpha}. \tag{13}$$

Further, if the intersection in (11) *is empty the solution is unique, but if the intersection is not empty then there is more than one system of solutions. In any case the system generated by* (13) *is a maximal system.*

COROLLARY. *A necessary and sufficient condition that the system* (9) *be solvable is that for each pair* i, j, *with* $i \neq j$,

$$H_{ij} \subset H_{i\alpha} \cup H_{\beta j}, \qquad \begin{array}{l} \alpha = 1, 2, \cdots, n, \alpha \neq i, \\ \beta = 1, 2, \cdots, n, \beta \neq j. \end{array} \tag{14}$$

These conditions can be tested mechanically by using a matrix as in Section 2, but with the obvious modifications dictated by (14).

4. Generalizations. The generalizations seem to impose a complicated system of subscripts, but we can gain some simplification if we denote a set by $X(\alpha)$ in place of X_α, the standard subscript notation.

Our first objective is to generalize Theorems 1 and 2 to the case where each intersection may involve more than two sets. Let $I(1), I(2), \cdots, I(n)$, be n index sets

$$I(k) \subset \{1, 2, \cdots, m\}, \tag{15}$$

where the members of the set may be written

$$I(k) = \{\alpha\} = \{\alpha_1^{(k)}, \alpha_2^{(k)}, \cdots, \alpha_{t_k}^{(k)}\} \tag{16}$$

for $k=1, 2, \cdots, n$. Let $G(k)$, $k=1, 2, \cdots, n$ be n given sets. We wish to solve the system of n equations

$$\bigcap_{\alpha \in I(k)} X(\alpha) = G(k), \qquad k = 1, 2, \cdots, n, \tag{17}$$

for the unknown sets $X(\alpha)$, $\alpha=1, 2, \cdots, m$, in the set

$$G = \bigcup_{k=1}^{n} G(k). \tag{18}$$

To this end we must introduce the new index sets $J(\alpha)$,

$$J(\alpha) = \{k \mid \alpha \in I_k\} \qquad \alpha = 1, 2, \cdots, m, \tag{19}$$

and this amounts to examining the set of equations (17) vertically rather than horizontally.

Assume now that the system (16) has a solution. If $\alpha \in I(k)$, then we see from (17) that $X(\alpha) \supset G(k)$. Since this is true for each such k, $X(\alpha)$ contains the union and hence by (19)

$$X(\alpha) \supset \bigcup_{k \in J(\alpha)} G(k), \qquad \alpha = 1, 2, \cdots, m. \tag{20}$$

Using (20) in (17), we find that for $k=1, 2, \cdots, n$,

$$G(k) = \bigcap_{\alpha \in I(k)} X(\alpha) \supset \bigcap_{\alpha \in I(k)} \left(\bigcup_{\beta \in J(\alpha)} G(\beta) \right). \tag{21}$$

Here the containing sign can be replaced by the equal sign, because each term of the intersection contains $G(k)$. Indeed for each $\alpha \in I(k)$, equation (19) states that $k \in J(\alpha)$ and hence $G(k)$ is in the union. If we interchange union and intersection signs in (21) we can write

$$G(k) = \bigcup_k{}^* \left(\bigcap{}^* G(\beta)\right), \qquad k = 1, 2, \cdots, n, \tag{22}$$

where, by definition,

$$\bigcap{}^* G(\beta) = G(\beta_1) \cap G(\beta_2) \cap \cdots \cap G(\beta_{t_k}), \tag{23}$$

where

$$\beta_1 \in J(\alpha_1^{(k)}), \qquad \beta_2 \in J(\alpha_2^{(k)}), \qquad \cdots, \qquad \beta_{t_k} \in J(\alpha_{t_k}^{(k)}) \tag{24}$$

and $\bigcup_k^*$ denotes the union of all intersections in (23) in which the β's run through all combinations satisfying (24). Hence equation (22) together with (23) and (24) give a necessary condition for (17) to be solvable. The condition is easily seen to be sufficient, and we can take as minimal solutions

$$X(\alpha) = \bigcup_{k \in J(\alpha)} G(k) \qquad \alpha = 1, 2, \cdots, m. \tag{25}$$

THEOREM 4. *A necessary and sufficient condition that the system* (17), *of n equations in m unknown sets* $X(\alpha)$, *be solvable is that equation* (22) *be satisfied for* $k = 1, 2, \cdots, n$.

When (22) is satisfied, the minimal system of solutions is given by (25).

COROLLARY. *A necessary and sufficient condition that the system* (17) *be solvable is that, for* $k = 1, 2, \cdots, n$,

$$G(k) \supset G(\beta_1) \cap G(\beta_2) \cap \cdots \cap G(\beta_{t_k})$$

for every combination of $\beta_1, \beta_2, \cdots, \beta_{t_k}$ *satisfying* (24).

Just as in the simpler system (1) the solution is not unique for we can always add an element not in G to any one of the sets $X(\alpha)$. But even if we restrict ourselves to the set G, the solution need not be unique, as the following example shows.

For the system

$$\begin{aligned} X(1) \cap X(2) &= \{a, b\} \\ X(2) \cap X(3) &= \{b, c, d\} \\ X(3) \cap X(4) &= \{b, d, e\} \end{aligned} \tag{26}$$

the J sets given by (19) are $J(1) = \{1\}$, $J(2) = \{1, 2\}$, $J(3) = \{2, 3\}$, $J(4) = \{3\}$. By Theorem 4, equation (25), the minimal solutions are:

$$X(1) = \{a, b\}, X(2) = \{a, b, c, d\}, X(3) = \{b, c, d, e\} \quad \text{and} \quad X(4) = \{b, d, e\}.$$

But the system $Z(1) = X(1) \cup \{e\}$, $Z(2) = X(2)$, $Z(3) = X(3)$, $Z(4) = X(4) \cup \{a\}$ is also a solution of (26). It follows that the system of equations (26) has at least 4 different systems of solutions in the set $G = \{a, b, c, d, e\}$.

We can formulate a condition for the uniqueness of a solution but it is rather awkward. Let $\{X(\alpha), \alpha = 1, 2, \cdots, m\}$ be a minimal system of solutions and let $e \in G$ be such that $e \notin X(\beta)$ for some fixed β. We examine each equation in (17) that contains the set $X(\beta)$. If in each such equation e is not an element of

some $X(\gamma)$, $\gamma \neq \beta$, that occurs in the equation, then the solution is not unique, for we can replace $X(\beta)$ by $X(\beta) \cup \{e\}$ in the minimal solution, and obtain a new solution. On the other hand, suppose that for each e in G, and each $X(\beta)$ with $e \in X(\beta)$, there is some equation involving $X(\beta)$ with the property that every other $X(\alpha)$ in that equation already contains e. It is easy to see that if this condition is satisfied, then the solution is unique.

By considering complements it is easy to see that Theorem 4 and its corollary imply the following result.

THEOREM 5. *A necessary and sufficient condition that the system of n equations*

$$\bigcup_{\alpha \in I(k)} Y(\alpha) = H(k) \qquad k = 1, 2, \cdots, n, \tag{27}$$

in the m unknown sets $Y(1), Y(2), \cdots, Y(m)$ *be solvable is that for each index* $k = 1, 2, \cdots, n$,

$$H(k) \subset H(\beta_1) \cup H(\beta_2) \cup \cdots \cup H(\beta_{t_k}) \tag{28}$$

for every combination of the β'*s satisfying the condition* (24). *When this condition is satisfied a maximal system of solutions is given by*

$$Y(\alpha) = \bigcap_{k \in J(\alpha)} H(k) \qquad \alpha = 1, 2, \cdots, m. \tag{29}$$

It is clear that in general the solution need not be unique.

5. Translation by a set. Let R be a fixed set. If $\{X^*(\alpha), \alpha = 1, 2, \cdots, m\}$ is a solution of (17), then the system $Z(\alpha) = X^*(\alpha) \cup R$, $\alpha = 1, 2, \cdots, m$, is a solution of

$$\bigcap_{\alpha \in I(k)} X(\alpha) = G(k) \cup R \qquad k = 1, 2, \cdots, n. \tag{30}$$

We can also subtract a common set. Thus if $R \subset G(k)$, for each $k = 1, 2, \cdots, n$, then by (25) R is contained in each set of the solution, and it is clear that the system $Z(\alpha) = X^*(\alpha) - R$, $\alpha = 1, 2, \cdots, m$, is a solution of

$$\bigcap_{\alpha \in I(k)} X(\alpha) = G(k) - R \qquad k = 1, 2, \cdots, n. \tag{31}$$

Hence in solving a system such as (17) we can always simplify matters by subtracting $R = \bigcap G(k)$, if it is not empty. This occurs in the example.

The same type of translation, addition or subtraction of a set R, can be applied to the system (27) and its solution.

6. Polynomial equations. Let $p_k(x)$ be a polynomial in the variables $x_1, x_2, \cdots, x_m$, in which each coefficient is one. Let $p_k(X)$ denote the combination of sets that is obtained by replacing: x_α by $X(\alpha)$, addition by union, and multiplication by intersection, in $p_k(x)$.

It is not easy to state a general theorem about the solution for the sets $X(\alpha)$ of a system of equations

$$p_k(X) = H(k) \qquad k = 1, 2, \cdots, n. \tag{32}$$

Our previous work gives a method, however, for solving such a system. Indeed, each collection of intersections

$$\bigcap_{\beta \in I(\alpha)} X(\beta) \tag{33}$$

that occurs in (32) can be replaced by a new unknown set $Y(\alpha)$, and this substitution reduces (32) to the form (27). Hence, if (32) is solvable, the system obtained by the substitution must also be solvable, and this can be tested by the criterion of Theorem 5. Assuming that we have solutions $Y(\alpha)$ we proceed to examine the solvability of the system

$$\bigcap_{\beta \in I(\alpha)} X(\beta) = Y(\alpha) \qquad \alpha = 1, 2, \cdots, q, \tag{34}$$

where q is the number of different collections of intersections that occur in (32). This system is of the form covered by Theorem 4.

7. Concluding remarks. It is natural to try to introduce coefficients in the polynomials treated in Section 6. Here the coefficients would naturally take the form of known sets. It seems to be difficult to give an adequate discussion of the set analog of the system of linear equations

$$\sum_{\alpha=1}^{m} a_{k\alpha} x_\alpha = b_k \qquad k = 1, 2, \cdots, n.$$

In the simple case $m = 1$, however, we have the following result.

THEOREM 6. *Let A_k and G_k, $k = 1, 2, \cdots, n$, be given sets. A necessary and sufficient condition that the system*

$$A_k \cap X = G_k \qquad k = 1, 2, \cdots, n \tag{35}$$

be solvable is that $A_k \supset G_k$ for $k = 1, 2, \cdots, n$, and that

$$(A_k - G_k) \cap G_l = \phi \tag{36}$$

for all pairs k, l, with $k \neq l$. If these two conditions are satisfied then the solution is given by $X = \bigcup_{k=1}^{n} G_k$, and this solution is unique in the set $A = \bigcup_{k=1}^{n} A_k$.

The proof is left to the reader.

A SIMPLE DERIVATION OF JORDAN CANONICAL MATRICES OVER ARBITRARY FIELDS

SYLVAN WALLACH, C. W. Post College of Long Island University

1. In [1] Cater gave a rather simple derivation of the Jordan canonical form of a matrix representing a linear operator on a finite-dimensional vector space over an algebraically closed field. Cater's derivation just missed being elementary in that he made use of factor spaces, a tool not always available in first courses in linear algebra. There is presented below an extension of Cater's method to finite-dimensional vector spaces over *arbitrary* fields. The procedure is completely elementary. Because the usual derivations of the Jordan canonical form are beyond the scope of many undergraduate courses, the method used here is of interest. Standard results on minimal polynomials, elementary divisors, and invariant factors can be derived easily as corollaries.

2. Let T be a linear operator on an n-dimensional vector space V over an arbitrary field $\mathfrak{F}$, let $\{\alpha\}$ be an index set and let $P=\{f(\alpha, k, \lambda)\}$ be the set of irreducible monic polynomials of degree k in an indeterminate λ, with coefficients in $\mathfrak{F}$. Subspaces of V are defined as follows: $V(\alpha, k)$ is the set of vectors x such that $xf(\alpha, k, T)^j=0$ for some positive integer j; for each k, $V(k)$ is the smallest space containing all the spaces $V(\alpha, k)$, $\alpha\in\{\alpha\}$; and V' is the smallest space containing all the spaces $V(k)$.

Henceforth it is assumed that all spaces mentioned are not trivial. It will be shown that $V=V'$; V' is the direct sum of the subspaces $V(k)$; each $V(k)$ is the direct sum of the subspaces $V(\alpha, k)$, and each $V(\alpha, k)$ is the direct sum of cyclic subspaces generated by the operator $f(\alpha, k, T)$. The set of dimensions of any cyclic decomposition of $V(\alpha, k)$ by $f(\alpha, k, T)$ is unique and suitable choices of bases in $V(\alpha, k)$ yield generalizations of the Jordan matrix representation of a linear operator on a vector space over an algebraically closed field.

3. First, three lemmas are proved.

LEMMA 1. *Let W be a subspace of V. If $z\in V$, W is invariant under T and there exist relatively prime polynomials $f(\lambda)$, $g(\lambda)$, such that $zf(T)\in W$ and $zg(T)\in W$, then $z\in W$.*

Proof. There exists a polynomial $s(\lambda)$ of least degree such that $zs(T)\in W$. It is evident that $s(\lambda)$ divides both $f(\lambda)$ and $g(\lambda)$; hence $s(\lambda)$ is a scalar and $z\in W$.

LEMMA 2. *The set of spaces $\{V(k), k=1, 2, \cdots\}$ is independent.*

Proof. The proof proceeds by induction. Assume that every collection of subspaces of $\{V(k), k=1, 2, \cdots\}$ containing $m-1$ distinct subspaces is independent and suppose there exists a nontrivial linear relation

$$(1)\quad x_1+\cdots+x_{m-1}+x_m=0;\quad x_j\in V(k_j),\qquad j=1, \cdots, m; k_i\neq k_j \text{ if } i\neq j.$$

None of the vectors x_j is the zero vector because of the inductive hypothesis.

There exists a polynomial $g(\lambda)$ of least degree such that $x_m g(T)=0$. Let $g(\lambda)=f(\lambda)s(\lambda)$ such that $f(\lambda)$ is irreducible. Then $y_m=x_m s(T)\in V(k_m)$, $y_m\neq 0$, and $y_m f(T)=0$. Operate on (1) with $s(T)$ to give $y_1+\cdots+y_{m-1}+y_m=0$, $y_j\in V(k_j)$, $y_j\neq 0$. Because $y_m f(T)=0$, it follows from the inductive hypothesis that $y_j f(T)=0$, $j=1,\cdots,m-1$. Thus $y_j\in V(h)$, $j=1,\cdots,m$, where h is the degree of $f(\lambda)$. In a moment it will be shown that $y_j\in V(h)\cap V(k_j)$, $y_j\neq 0$, implies $h=k_j$. But $h\neq k_j$ for some $j=1,\cdots,m$ since the k_j's are distinct. This contradiction will prove the lemma.

Let $x\in V(k)\cap V(l)$ with $k\neq l$. There exist relatively prime polynomials $f(\lambda)$, $g(\lambda)$, such that $xf(T)=xg(T)=0$, where $f(\lambda)$ is a product of irreducible polynomials of degree k, $g(\lambda)$ is a product of irreducible polynomials of degree l. Lemma 1 with $W=0$ implies that $x=0$. Thus if $k\neq l$, $V(k)\cap V(l)=0$.

LEMMA 3. *The set of spaces* $\{V(\alpha, k)\}$, *k fixed, α arbitrary, is independent.*

Proof. Suppose that there exists a linear relation

$$x_0 = x_1+\cdots+x_r\neq 0,\qquad x_j\in V(\alpha_j,k),\qquad j=0,\cdots,r. \tag{2}$$

It may be assumed that $x_0 f_0=0$, where $f_0=f(\alpha_0,k,T)$ and $f(\alpha_0,k,\lambda)$ is irreducible of degree k. There exist monic irreducible polynomials $f_i=f(\alpha_i,k,\lambda)$ such that $x_i f_i^{m_i}=0$, $i=1,\cdots,r$. From (2) follows

$$x_0 f_0 = x_0 f_1^{m_1} f_2^{m_2}\cdots f_r^{m_r}=0. \tag{3}$$

Observe that $x_0 f_i = x_0(f_i-f_0)=x_0 g_i$ where $g_i=f(\alpha_i,k,\lambda)-f(\alpha_0,k,\lambda)$ is a polynomial in λ of degree $l_i<k$. Thus equations (3) imply that

$$x_0 g_1^{m_1}\cdots g_r^{m_r}=0. \tag{4}$$

The polynomial $g_1^{m_1}\cdots g_r^{m_r}$ contains no irreducible factor of degree greater than $k-1$. This fact, (4), and $x_0\neq 0$ together imply the existence of a polynomial $s(\lambda)$ such that $x_0 s(T)\neq 0$ and $x_0 s(T)\in V(l)$ for some $l<k$. But $x_0 s(T)\in V(k)$ also. According to Lemma 2, $V(l)\cap V(k)=0$. This contradiction proves Lemma 3. We will now prove

THEOREM 1. *The n-dimensional vector space V is the direct sum of the set of subspaces* $\{V(\alpha,k)\}$.

Proof. All that remains to be proved is that $V\subset V'$. Let $z\in V$. Let $n(z)$ be the degree of a polynomial $f(\lambda)$ of least degree such that $zf(T)=0$. If $n(z)=1$, then $z(T+a)=0$ for some $a\in\mathfrak{F}$, and $z\in V'$. Suppose that $z\in V'$ for all z such that $n(z)\leq m-1$. Let $n(z)=m$, and let $z\,f(T)=0$. If $f(\lambda)$ is a power of an irreducible polynomial, then $z\in V'$. Otherwise $f(\lambda)$ contains two relatively prime factors $s(\lambda)$ and $t(\lambda)$, each of degree greater than zero. By the inductive hypothesis $zs(T)$ and $zt(T)\in V'$. By Lemma 1, $z\in V'$.

4. Consider now a particular subspace $V(\alpha, k)$. The operator $f(\alpha, k, T)$ is nilpotent on $V(\alpha, k)$. The following theorem is needed.

THEOREM 2. *Let N be a nilpotent operator of order p on an n-dimensional vector space W over a field F. Then W is the direct sum of cyclic subspaces generated by N. The set of dimensions of these cyclic subspaces is unique.*

Proof. The theorem is trivially true for $p=1$, in which case $N=0$. Assume the theorem true for nilpotent operators of order $p-1$. N induces a nilpotent transformation, denoted also by N, of order $p-1$ on R, the range space of N. According to the inductive hypothesis $R = R_1 \oplus \cdots \oplus R_j$, where $R_i = L(x_i, x_iN, \cdots, x_iN^{r_i-1})$, $i=1, \cdots, j$, $x_iN^{r_i}=0$, $L(y_1, \cdots, y_n)$ is the subspace spanned by the vectors $y_1, \cdots, y_n$, and r_i is the dimension of R_i. Now $x_i \in R$ implies the existence of a vector $y_i \in W$ such that $y_iN = x_i$. The set of vectors

$$S_1 = \{y_1, y_1, N \cdots, y_1N^{r_1}, y_2, y_2N, \cdots, y_j, \cdots y_jN^{r_j}\}$$

is linearly independent, as is evident from the following chain of implications, in which degree $p_i(\lambda)$ may be assumed equal to or less than r_i.

$$\begin{aligned}\sum y_ip_i(N) = 0 &\Rightarrow \sum x_ip_i(N) = 0 \Rightarrow x_ip_i(N) = 0\\ &\Rightarrow p_i(\lambda) = c_i\lambda^{r_i} \Rightarrow \sum c_iy_iN^{r_i} = 0\\ &\Rightarrow \sum c_ix_iN^{r_i-1} = 0 \Rightarrow c_i = 0 \Rightarrow p_i(\lambda) = 0.\end{aligned}$$

The subset $\{y_iN^{r_i}\}$ of S_1 can be extended to a basis

$$S_2 = \{y_1N^{r_1}, \cdots, y_jN^{r_j}, z_1, \cdots, z_k\}$$

of the null space Z of N. The set of vectors $S_1 \cup S_2$ is a basis for W. The proof of linear independence follows the pattern above. It will now be shown that $W \subset L(S_1 \cup S_2) = W'$, the subspace spanned by $S_1 \cup S_2$. Let $y \in W$. If $yN=0$, then $y \in Z \subset W'$. If $yN \in R$, then $yN = \sum x_ip_i(N) = (\sum y_ip_i(N))N = y'N$ and $y' \in W'$. Thus $(y-y')N=0$, $y-y'=z \in Z \subset W'$, and $y=y'+z \in W'$. It follows that $W = Z_1 \oplus \cdots \oplus Z_k \oplus W_1 \oplus \cdots \oplus W_j$, where

$$Z_i = L(z_i) \quad \text{and} \quad W_i = L(y_i, y_iN, \cdots, y_iN^{r_i}), \qquad i = 1, \cdots, j.$$

The decomposition of W into a direct sum of cyclic subspaces is not unique. However, a simple inductive proof on the dimension of W shows that the set of dimensions is unique. If the dimension of W is 1, the assertion is trivially true. Assume it true for all spaces of dimension less than n. Let dimension of W be n and let $W = W_1 \oplus \cdots \oplus W_j$ be a direct sum of cyclic subspaces having dimensions $n_1, n_2, \cdots, n_j$. Then $WN = W_1N \oplus \cdots \oplus W_jN$ has dimension $n-j$, and according to the inductive hypothesis, every decomposition of WN into a direct sum of cyclic subspaces consists of subspaces of dimensions $n_1-1, \cdots, n_j-1$. Thus every decomposition of W consists of subspaces of dimensions $n_1, \cdots, n_j$.

5. Consider a subspace $V = V(\alpha, k)$ of V and the corresponding irreducible polynomial

$$f(\lambda) = f(\alpha, k, \lambda) = \lambda^k - a_{k-1}\lambda^{k-1} \cdots - a_0.$$

Let V_{i1} be a cyclic subspace of V of dimension r, with basis

$$\{x, xf(T), \cdots, xf(T)^{r-1}\}.$$

Writing f instead of $f(T)$, we see that the set of vectors

$$\{x, xf, \cdots, xf^{r-1}, xT, xTf, \cdots, xTf^{r-1}, \cdots, xT^{k-1}, \cdots, xT^{k-1}f^{r-1}\}$$

is a linearly independent set in V which spans the direct sum $V_i = V_{i1} \oplus \cdots \oplus V_{ik}$ of k of the cyclic subspaces of V. The set is independent because $g(\lambda) = f(\lambda)^r$ is the polynomial of least degree such that $g(T)$ annihilates x. Evidently the cyclic subspaces into which V is decomposed can be associated in disjoint sets, each containing k cyclic subspaces, in the manner shown above. Thus $V = V_1 \oplus \cdots \oplus V_m$ where V_i is the direct sum of k subspaces cyclic with respect to a nilpotent operator which is a polynomial in T. The spaces V_{ij} are not invariant under T, hence are not useful in selecting bases for matrix representations of T. Rather, the invariant subspaces V_i must be used. Various generalizations of the Jordan form of a matrix over an algebraically closed field are obtained by selecting bases in the subspaces V_i. Each rule for choosing a basis gives rise to a canonical matrix representation.

6. An obvious and simple way to choose a basis $\{e_1, \cdots, e_{kr}\}$ for a subspace V_i is the following

$$\left\{\begin{array}{l} e_1 = x, \quad e_2 = xT, \cdots, e_k = xT^{k-1} \\ e_{k+1} = xf, \quad e_{k+2} = xfT, \cdots, e_{2k} = xfT^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ e_{(r-1)k+1} = xf^{r-1}, \cdots, e_{rk} = xf^{r-1}T^{k-1}. \end{array}\right. \tag{5}$$

The matrix of T, restricted to the subspace V_i of V, is shown in Figure 1.

$$B_i = \begin{bmatrix} PN & & & & & \\ & PN & & & & \\ & & P & & & \\ & & & \ddots & & \\ & & & & N & \\ & & & & & P \end{bmatrix}$$

FIG. 1

$$P = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & \cdots & \cdots & \cdots & a_{k-1} \end{bmatrix}$$

FIG. 2

There are r blocks P on the diagonal, where P is the $k \times k$ companion matrix of f

shown in Figure 2. The $k \times k$ matrix N has 1 in the lower left hand corner and zeroes elsewhere. The basis used by Jacobson [2, p. 71] is

$$
\begin{cases} e_1 = xf^{r-1}, \cdots, e_k = xf^{r-1}T^{k-1} \\ \cdot \\ \cdot \\ \cdot \\ e_{(r-1)k+1} = x, \cdots, e_{rk} = xT. \end{cases} \tag{6}
$$

The matrix of T on V_i with respect to the basis (6) differs from Figure 1 only in having the blocks N below the blocks P instead of above them.

7. A less refined matrix representation of T is obtained by choosing the basis $\{x, xT, \cdots, xT^{rk-1}\}$ for V_i. The matrix representing T, restricted to V_i, is the companion matrix of the elementary divisor $f(\lambda)^r$. The matrix representing T on V has the companion matrices of the elementary divisors on the diagonal and zeroes elsewhere.

8. Now let $f(\alpha, k_\alpha, \lambda)$, $\alpha = 1, \cdots, r$, be the distinct irreducible polynomials to which correspond subspaces $V(\alpha, k_\alpha)$ such that if $x \in V(\alpha, k_\alpha)$, then $xf(\alpha, k_\alpha T)^j = 0$ for some positive integer j. The space V is decomposed into a direct sum as follows:

$$V = V(1, k_1) \oplus \cdots \oplus V(r, k_r); \; V(\alpha, k_\alpha) = V_1(\alpha, k_\alpha) \oplus \cdots \oplus V_{m_\alpha}(\alpha, k_\alpha);$$

and

$$V_i(\alpha, k_\alpha) = {}_{i1}(\alpha, k_\alpha) + \cdots \oplus V_{ik_\alpha}(\alpha, k_\alpha).$$

The matrix representing T has r blocks $A_1, \cdots, A_r$ on the diagonal, each block A_α has blocks $B_{\alpha j}$, $j = 1, \cdots, m_\alpha$ on its diagonal, and each block $B_{\alpha j}$, has the form shown in Figure 1, with P the companion matrix of $f(\alpha, k_\alpha, \lambda)$. The blocks N are unique only if the field $\mathfrak{F}$ is algebraically closed in which case N is the scalar 1.

9. The essential uniqueness of the matrix representation of T described in the preceding paragraph is by now almost obvious. Only those blocks P can occur which are companion matrices of polynomials $f(\alpha, k, \lambda)$ because the blocks determine the polynomials and conversely. The sizes of the blocks $B_{\alpha j}$ occurring in any block A_α are unique according to Theorem 2. All that remains subject to choice are the order in which the blocks occur and the structure of the blocks N.

References

1. S. Cater, An elementary development of the Jordan canonical form, this MONTHLY, 69 (1962) 391–393.

2. N. Jacobson, Lectures in Abstract Algebra, II, Van Nostrand, Princeton, N.J., 1953.

GENERALIZED POWERS

GLORIA OLIVE, Anderson College

1. Introduction. The function which we shall write as

$$\text{(I)} \qquad N_h^k(b) \equiv \lim_{z\to b} \prod_{i=0}^{h-1} \frac{z^{k-i}-1}{z^{i+1}-1}$$

can be traced back to Gauss [4]. Various names have been given to it (e.g., "Gaussian expression" and "q-number" [when q replaces b]) and its properties have motivated several papers (e.g., Jackson [6], Szegö [9], Carlitz [3], Al-Salam [1], and Gould [5]).

The identity $N_h^k(1)=\binom{k}{h}$ motivates our development; for it implies that $N_h^k(b)$ is a generalized binomial coefficient and therefore it is reasonable to suspect that $N_h^k(b)$ could be used to generate "generalized powers" [i.e., $\sum_{h=0}^k N_h^k(b)c_{(b)}^h = (c+1)_{(b)}^k$] just as $\binom{k}{h}$ can be used to generate ordinary powers [i.e., $\sum_{h=0}^k \binom{k}{h}c^h=(c+1)^k$].

The purpose of this paper is to create and investigate the generalized power $M_{(b)}^E$ (read "M to the E to the base b") with *medial* M, *exponent* E, and *base* b representing complex numbers. First we shall need some preliminaries.

1.1. *Notations.* Unless otherwise indicated we will let b, t, x, y, z represent complex numbers; g, h represent integers; c, d, i, j, k, q, r represent non-negative integers; and n represent a positive integer. Using these symbols we now let $0^0=1$ and $\prod_{i=0}^{-n} f(i)=0^{n-1}$ when $f(i)$ is a function of i.

1.2. *Some properties of* $N_h^k(b)$. The following list of properties will prove useful in establishing some of the theorems for generalized powers:

$$(1.2.1) \qquad N_0^k(b) = 1$$

$$(1.2.2) \qquad N_k^k(b) = 1$$

$$(1.2.3) \qquad N_h^k(b) = 0 \text{ if } h<0 \text{ or } h>k$$

$$(1.2.4) \qquad N_{ph'+h''}^{pk'+k''}(b) = \binom{k'}{h'} N_{h''}^{k''}(b) \text{ if } b \text{ has period } p \text{ and } 0 \leqq h'', k'' < p$$

$$(1.2.5) \qquad N_h^k(1) = \binom{k}{h}.$$

2. Generalized powers. Now that the preliminaries are over we will proceed to develop generalized powers in three stages. At the first stage both the medial and exponent will be restricted to non-negative integers, at the second stage the medial will be permitted to be complex but the exponent will remain a non-negative integer, and at the third stage both the medial and exponent will be complex.

2.1. *Stage* 1: $c^k_{(b)}$. These generalized powers will be defined recursively by means of

$$0^k_{(b)} \equiv 0^k \tag{II}$$

and

$$(c+1)^k_{(b)} \equiv \sum_{h=0}^{k} N^k_h(b) c^h_{(b)}. \tag{III}$$

We now list some direct consequences of these definitions:

$$1^k_{(b)} = 1 \tag{2.1.1}$$

$$c^0_{(b)} = 1 \tag{2.1.2}$$

$$c^1_{(b)} = c \tag{2.1.3}$$

$$c^k_{(1)} = c^k. \tag{2.1.4}$$

The following property is somewhat less immediate.

$$(c+d)^k_{(b)} = \sum_{h=0}^{k} N^k_h(b) c^{k-h}_{(b)} d^h_{(b)}. \tag{2.1.5}$$

Proof. In order to establish (2.1.5) by finite induction we note that it holds for $d=0$ [by (II) and (1.2.1)], and proceed to show that it holds for $d+1$ whenever it holds for d.

Since $c^k_{(b)}$ is well-defined, $[c+(d+1)]^k_{(b)} = [(c+d)+1]^k_{(b)}$. Therefore (III) yields

$$[c+(d+1)]^k_{(b)} = \sum_{j=0}^{k} N^k_j(b)(c+d)^j_{(b)}.$$

After applying the inductive assumption on d and rearranging terms, we obtain

$$[c+(d+1)]^k_{(b)} = \sum_{i=0}^{k} \sum_{j=i}^{k} N^k_j(b) N^j_i(b) c^{j-i}_{(b)} d^i_{(b)}.$$

If we now let $j=k-h+i$, we obtain

$$[c+(d+1)]^k_{(b)} = \sum_{h=0}^{k} N^k_h(b) c^{k-h}_{(b)} \cdot \sum_{i=0}^{h} N^h_i(b) d^i_{(b)}$$

after rearranging terms again and factoring. Another application of (III) now yields the desired result and the proof is complete.

Since (2.1.5) reduces to the binomial formula when $b=1$, it is a *generalized binomial formula.*

In addition to the ordinary-looking properties above, generalized powers

also have laws of exponents—when the base is a root of unity. For example when b has period p we have:

$$c_{(b)}^{qp+r} = c^{q}\cdot c_{(b)}^{r} \tag{2.1.6}$$

$$c_{(b)}^{qp} = c^{q} \tag{2.1.7}$$

$$c_{(b)}^{qp}\cdot c_{(b)}^{r} = c_{(b)}^{qp+r} \tag{2.1.8}$$

$$(c^{q})_{(b)}^{rp} = (c^{qr})_{(b)}^{p} = c_{(b)}^{qrp} \tag{2.1.9}$$

$$(cd)_{(b)}^{qp} = c_{(b)}^{qp}\cdot d_{(b)}^{qp}. \tag{2.1.10}$$

In order to prove (2.1.6) we will need

LEMMA 2.1. *Formula* (2.1.6) *is valid when* $r<p$.

Proof. In order to establish this lemma by finite induction we note that it holds for $c=0$ [by (II)], and proceed to show that it holds for $c+1$ whenever it holds for c.

By (III), (1.2.4), and (1.2.3) we have

$$(c+1)_{(b)}^{qp+r} = \sum_{j=0}^{qp+r} N_j^{qp+r}(b)c_{(b)}^{j} = \sum_{h=0}^{q}\sum_{i=0}^{r}\binom{q}{h}N_i^{r}(b)c_{(b)}^{hp+i}$$

when $j=hp+i$ and i, $r<p$. If we use the inductive assumption on c and factor, the above equation becomes

$$(c+1)_{(b)}^{qp+r} = \sum_{h=0}^{q}\binom{q}{h}c^{h}\cdot\sum_{i=0}^{r}N_i^{r}(b)c_{(b)}^{i}.$$

After applying the binomial formula and (III) the desired result is obtained, and the lemma follows.

We now use the lemma by letting $r=jp+i$ with $i<p$ to obtain

$$c_{(b)}^{qp+r} = c_{(b)}^{(q+j)p+i} = c^{q+j}\cdot c_{(b)}^{i} = c^{q}(c^{j}\cdot c_{(b)}^{i}) = c^{q}\cdot c_{(b)}^{r}.$$

Thus, Formula (2.1.6) is established.

The remaining four laws of exponents (2.1.7)—(2.1.10) are corollaries of (2.1.6).

2.2. *Stage* 2: $x_{(b)}^{k}$. In order to establish generalized powers with complex medials we need the following lemma which follows from the identity $c_{(b)}^{k}=\sum_{j=0}^{c-1}[(j+1)_{(b)}^{k}-j_{(b)}^{k}]$ and (III).

LEMMA 2.2. *If c and k are positive integers, then*

$$c_{(b)}^{k} = \sum_{h=0}^{k-1} N_h^{k}(b)\sum_{j=0}^{c-1} j_{(b)}^{h}. \tag{2.2.1}$$

If we now let $k=1, 2, 3, \cdots$ in (2.2.1) we find a pattern developing. This leads to

$$\overset{k}{c}_{(b)} = \sum_{j=0}^{k} R_j^k(b)\binom{c}{j}, \tag{2.2.2}$$

where $R_0^k(b)=0^k$, and $R_n^k(b)$ is the sum of all products of the form

$$\prod_{i=0}^{n-1} N_{[i]}^{[i+1]}(b)$$

with $[i]$ representing a nonnegative integer, $[0]=0$, $[n]=k$, and $[i]<[i+1]$ (which is proved in [8] pp. 30, 31).

One feature of the right hand member of (2.2.2) is that it has meaning when c is complex. It is therefore reasonable to let

$$\overset{k}{x}_{(b)} \equiv \sum_{j=0}^{k} R_j^k(b)\binom{x}{j}. \tag{IV}$$

A first question is "Do the formulas of Section 2.1 remain valid when all variable medials are converted to complex variables?" The answer is "yes". For if each generalized power with a variable medial is expressed in polynomial form by use of (2.2.2), an identity is obtained by [**10**, p. 66] which holds when each variable medial component is converted to a complex variable; and the affirmative answer results after an application of (IV). Thus we have

THEOREM 2.2. *The formulas of Section* 2.1 *are valid when c and d are complex variables.*

Another question is "Can (IV) be used to obtain new results?" The answer to this is also "yes." For example $(-1)^{k}_{(b)}$ now has meaning and in [**8**, p. 16] it is proved that

$$(-1)^{k}_{(b)} = (-1)^{k} b^{\binom{k}{2}}. \tag{2.2.3}$$

Also, by using (2.2.3) and (2.1.5) we can obtain

$$(-x)^{k}_{(b)} = (-1)^{k}_{(b)}\overset{k}{x}_{(1/b)} \quad \text{if } b \neq 0. \tag{2.2.4}$$

In order to discover additional properties it would seem natural to investigate $x^2_{(b)}$. This is easy to do since

$$\overset{2}{x}_{(b)} = \frac{1+b}{2}x^2 + \frac{1-b}{2}x \qquad \text{[by (IV)]}.$$

Using direct substitution we obtain

$$\overset{2}{x}_{(b)} = \overset{1}{x}_{(b)}\cdot\overset{1}{x}_{(b)} \quad \text{iff } x(x-1)(b-1)=0, \tag{2.2.5}$$

$$(xy)^{2}_{(b)} = \overset{2}{x}_{(b)}\cdot\overset{2}{y}_{(b)} \quad \text{iff } xy(x-1)(y-1)(b^2-1)=0, \tag{2.2.6}$$

$$(2.2.7) \qquad x^{2}_{(b)} = 0 \qquad \text{iff } x = 0 \text{ or } x = \frac{b-1}{b+1}\cdot$$

(Thus in every base other than ± 1, there exists a nonzero number whose "square" is zero. Or, given any $x \neq 1$, there exists a b such that $x^{2}_{(b)} = 0$.)

$$(2.2.8) \qquad x^{2}_{(b)} = 1 \quad \text{iff } x = 1 \text{ or } x = \frac{-2}{1+b}\cdot$$

(Thus given any $x \neq 0$, there exists a b such that $x^{2}_{(b)} = 1$. That is, any number different from zero can be a "square root" of 1, if b is chosen properly.) Also

$$(2.2.9) \qquad x^{2}_{(b)} \geqq 0 \quad \text{for all real } x \text{ iff } b = 1.$$

On the basis of (2.2.5)–(2.2.9) it does not seem likely that $x^{k}_{(b)}$ will have any more ordinary-looking properties.

2.3. *Stage* 3: $x^{t}_{(b)}$. Just as Stage 1 was based on an extension of a binomial formula, this stage will be based on an extension of the binomial series.

We first consider

$$(\text{V}) \qquad S(b, t, x) \equiv \sum_{j=0}^{\infty} N^{t}_{j}(b) x^{j}_{(b)}.$$

One may object to this definition on the grounds that $N^{t}_{j}(b)$ has not been defined. To remedy this situation we replace k by t in (I) and define z^{y} to be its principal value [7, p. 423]. With this extended definition (1.2.1) and (1.2.5) are valid when t replaces k, and (1.2.4) is valid when g replaces k.

On the basis of (1.2.3) and the generalized binomial formula we have

$$(2.3.1) \qquad S(b, k, x) = (1 + x)^{k}_{(b)}.$$

Therefore it is reasonable to let

$$(\text{VI}) \qquad (1 + x)^{t}_{(b)} \equiv S(b, t, x) \quad \text{when} \quad S(b, t, x) \text{ converges.}$$

At this point we would like to state the circumstances under which $S(b, t, x)$ converges, but we shall not do this—because we do not know. Instead, our discussion will contain one expected result, two unexpected results, and a conjecture.

Since $S(1, t, x)$ is the binomial series expansion of $(1+x)^{t}$, and a convergent binomial series converges to its principal value, we have

$$(2.3.2) \qquad (1 + x)^{t}_{(1)} = (1 + x)^{t} \text{ for all } t \text{ iff } |x| < 1.$$

In [8, pp. 25, 26] it is proved that formulas (2.1.6)–(2.1.10) are valid when q is a negative integer, provided that $|c-1| < 1$ and $|d-1| < 1$. A consequence of this extension of (2.1.6) (letting $q = -1$, $r = 1$, and $c = 1+x$) is

$$(1+x)_{(b)}^{1-p} = 1 \quad \text{if } b \text{ has period } p \text{ and } |x| < 1. \tag{2.3.3}$$

Thus the exponent $1-p$ can operate like the exponent zero.

By observing

$$0_{(b)}^{-n} = \prod_{j=1}^{n} (1 - b^{-j})^{-1} \quad \text{if } |b| > 1 \tag{2.3.4}$$

(which is proved in [8, pp. 27, 28]) we note that negative powers of zero can be meaningful.

We conclude this section with

CONJECTURE 2.3. *$(1+x)_{(b)}^{t}$ exists for all t only if $|x| < |b|$.*

3. Some side results. Although our development of generalized powers has not yet been responsible for any practical applications, it has yielded functions with curious-looking properties as well as some by-products which we will now consider.

3.1. *An application.* Since the number of subgroups of order b^h in an elementary Abelian group of order b^k is $N_h^k(b)$ [2, p. 111], it follows immediately that $\sum_{h=0}^{k} N_h^k(b) = (1+1)_{(b)}^{k} = 2_{(b)}^{k}$ is the total number of subgroups in an elementary Abelian group of order b^k [by (2.1.1) and (III)].

3.2. *A nilpotent-type property.* Since a generalized power with base zero can exist only if the exponent is a nonnegative integer [by (VI) and (I)], a complete characterization for base zero is given by

$$x_{(0)}^{k} = (-1)^{k} \binom{-x}{k}. \tag{3.2.1}$$

This formula is a direct consequence of (IV), Vandermonde's convolution, and elementary algebra; for they jointly yield

$$x_{(0)}^{k} = \sum_{j=0}^{k} \binom{k-1}{j-1} \binom{x}{j} = \binom{x+k-1}{k} = (-1)^{k} \binom{-x}{k}.$$

Using Formula (3.2.1) we can obtain

$$(q+1)_{(0)}^{r} = (r+1)_{(0)}^{q} = \binom{q+r}{r}, \tag{3.2.2}$$

$$(-r)_{(0)}^{r} = (-1)^{r}, \tag{3.2.3}$$

and

$$(-r)_{(0)}^{k} = 0 \quad \text{when} \quad k > r. \tag{3.2.4}$$

Thus generalized powers with base zero have a property which looks like nilpotence.

3.3. *Generalized Stirling numbers.* If $T_i^k(b)$ is a function such that

$$\sum_{i=0}^{k} T_i^k(b) x_{(b)}^i = \binom{x}{k}. \tag{3.3.1}$$

Formula (2.1.4) can be used to establish that $k!T_i^k(1) = S_i^k$ is a Stirling number of the first kind. Thus $k!T_i^k(b) = S_i^k(b)$ is a *generalized Stirling number of the first kind* and $S_i^k(1) = S_i^k$. By (IV) and (2.1.4), $(j!)^{-1}R_j^k(1) = s_j^k$ is a Stirling number of the second kind. Thus $(j!)^{-1}R_j^k(b) = s_j^k(b)$ is a *generalized Stirling number of the second kind* and $s_j^k(1) = s_j^k$.

Since a generalized Stirling number of the second kind is a polynomial, one does not question its existence. However, in the case of generalized Stirling numbers of the first kind, we have [using (3.4.6), Cramer's rule, and (3.4.3)]:

THEOREM 3.3. *If* $0 < i \leqq k$, *then* $S_i^k(b)$ *fails to exist iff* b *has period* p *such that* $2 \leqq p \leqq k$.

Three identities for generalized Stirling numbers (assuming $S_i^k(b)$ exists when $i > 0$) are:

$$\sum_{i=0}^{k} S_i^k(b) = 0 \quad \text{if } k > 1, \tag{3.3.2}$$

$$\sum_{j=i}^{k} S_i^j(b) s_j^k(b) = \sum_{j=i}^{k} s_i^j(b) S_j^k(b) = 0^{|k-i|}, \tag{3.3.3}$$

and

$$j!s_j^k(b) = \sum_{i=0}^{j} (-1)^{i+j} \binom{j}{i} i_{(b)}^k. \tag{3.3.4}$$

We note that (3.3.2) follows from (3.3.1); the orthogonality relation (3.3.3) from both (3.3.1) and (IV); and the inversion relation (3.3.4) from both (IV) and Newton's formula.

3.4. *P-polynomials.* If the terms of (IV) are juggled, we find that

$$x_{(b)}^k = \sum_{i=0}^{k} R_i^k(b) \binom{x}{i} = \sum_{j=0}^{k} \left[\sum_{i=j}^{k} S_j^i s_i^k(b) \right] x^j.$$

This equation yields

$$x_{(b)}^k = \sum_{j=0}^{k} P_j^k(b) x^j, \tag{3.4.1}$$

where the *P-polynomials* are given by

$$P_j^k(b) = \sum_{i=j}^{k} S_j^i s_i^k(b) \quad \text{if } 0 \leqq j \leqq k. \tag{3.4.2}$$

Thus a P-polynomial is a linear combination of generalized Stirling numbers of the second kind with Stirling numbers of the first kind as coefficients. A direct consequence is

$$P_k^k(b) = (k!)^{-1} \prod_{i=1}^{k} N_1^i(b). \tag{3.4.3}$$

Thus all the zeros of $P_k^k(b)$ are roots of unity. We will next consider the question "Do other P-polynomials have all their zeros on the unit circle?"

Since the coefficients of $P_j^k(b)$ are real (in fact, they are rational), the following theorem provides a necessary condition for all the zeros of $P_j^k(b)$ to lie on the unit circle.

THEOREM 3.4. *The coefficients of the polynomial $P_j^k(b)$ are symmetric or antisymmetric according as $j+k$ is even or odd.*

Proof. We will consider two cases: (1) $j=0$ and (2) $j=n$. Except for the statement "Case (1) is trivial, for it follows directly from the formula $P_0^k(b)=0^k$," the proof is devoted to Case (2).

Since (3.4.1), (2.2.4), and (2.2.3) yield

$$\sum_{j=0}^{k} P_j^k(b)(-x)^j = (-x)^k_{(b)} = (-1)^k b^{\binom{k}{2}} \sum_{j=0}^{k} P_j^k\left(\frac{1}{b}\right)x^j,$$

we have

$$P_j^k(b) = (-1)^{j+k} b^{\binom{k}{2}} P_j^k\left(\frac{1}{b}\right) \tag{3.4.4}$$

after equating the coefficients of x^j. By joint use of (3.4.1) and (3.2.1) we obtain

$$P_n^k(0) > 0. \tag{3.4.5}$$

Now (3.4.4) and (3.4.5) together establish that the degree of $P_n^k(b)$ is $\binom{k}{2}$. Thus, another application of (3.4.4) completes the proof.

As noted above, Theorem 3.4 provides a necessary condition for all the zeros of $P_n^k(b)$ to lie on the unit circle. However, it is not sufficient (e.g., consider b^2+4b+1).

A guess (which has been verified for $n=k$, $n=k-1$, and $k<7$) at an answer to our question concerning the zeros of P-polynomials is given in

CONJECTURE 3.4. *All the zeros of $P_n^k(b)$ lie on the unit circle.*

Since $P_0^k(b)\geqq 0$, $P_n^k(0)>0$, and $\sum_{j=0}^k P_j^k(b)=1$ [by (3.4.1) and (2.1.1)], Conjecture 3.4 (if true) would establish that $P_j^k(b)$ is a probability function when $-1\leqq b\leqq 1$.

Three other properties of P-polynomials are

$$\sum_{i=j}^{k} P_j^i(b) S_i^k(b) = S_j^k \tag{3.4.6}$$

which follows from (3.4.2) and (3.3.3),

$$\sum_{h=j}^{k} s_j^h P_h^k(b) = s_j^k(b) \tag{3.4.7}$$

which follows from (IV) and (3.4.1), and

$$\sum_{i=j}^{k} P_j^i(b) N_i^k(b) = \sum_{i=j}^{k} \binom{i}{j} P_i^k(b) \tag{3.4.8}$$

which is proved in [8, p. 19].

4. Conclusion. It seems likely that the development of generalized powers could be extended by including topics such as (1) the set of polynomials in x which are generated when $x_{(b)}^k$ is expressed as a polynomial in b; (2) the use of generalized powers in generalizing the various functions of powers (e.g., polynomials, series, and integrals); (3) a generalized calculus founded on a generalized Taylor series; (4) the use of generalized powers in the construction of probability functions (e.g., (3.4.1) with $x_{(b)}^k = 1$); (5) the solution of various problems involving generalized powers (e.g., the following generalization of Fermat's last theorem "If $n > 2$, find integers x, y, z and base b such that $x_{(b)}^n + y_{(b)}^n = z_{(b)}^n$"); and (6) an extension in which the medial, exponent, and base are not all restricted to complex numbers.

This paper is essentially the author's Ph.D. thesis at Oregon State University (1963) and was written while an NSF Science Faculty Fellow.

The kindness of the late Professor C. C. MacDuffee is gratefully acknowledged and this paper is dedicated to his memory.

References

1. W. A. Al-Salam, q-Bernoulli numbers and polynomials, Math. Nach., 17 (1958) 239–260.
2. W. Burnside, Theory of Groups of Finite Order, Cambridge, London, 1911.
3. L. Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc., 76 (1954) 332–350.
4. K. F. Gauss, Summatio quarundam serierum singularium, Werke, Vol. 2, 16.
5. H. W. Gould, The q-Stirling numbers of first and second kinds, Duke Math. J., 28 (1961) 281–290.
6. F. H. Jackson, q-difference equations, Amer. J. Math., 32 (1910) 305–314.
7. K. Knopp, Theory and Application of Infinite Series, Hafner, New York, 1951.
8. G. Olive, Generalized Powers, Oregon State University Ph.D. thesis, 1963.
9. G. Szegö, Ein Beitrag zur Theorie der Thetafunktionen, S-B. Preuss. Akad. Wiss., Phys.-Math. Klasse, (1926) 242–252.
10. B. L. van der Waerden, Modern Algebra, Vol. I, Ungar, New York, 1949.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146.

PARTIALLY ORDERED SPACES AND METRIC SPACES

RALPH DEMARR, University of Washington

In a partially ordered space it is possible to introduce a notion of convergence which is similar to that defined for real numbers. This convergence is called order convergence. In this note we will show that every metric space can be embedded in a partially ordered space so that convergence in the metric space can be obtained from order convergence. We will also show that a standard fixed point theorem in complete metric spaces can be obtained as a special case of a fixed point theorem in partially ordered spaces.

A partially ordered space is an abstract set X with a binary relation $\leqq$, called a partial order, which satisfies three conditions:

1) $x \leqq x$ for all $x \in X$;
2) $x \leqq y$ and $y \leqq z$ imply $x \leqq z$ for all $x, y, z \in X$;
3) $x \leqq y$ and $y \leqq x$ imply $x = y$ for all $x, y \in X$.

If M is a nonempty subset of X, then inf M denotes the infimum or greatest lower bound of M and sup M denotes the supremum or least upper bound of M. Thus, inf M (if it exists) is defined to be that element $v \in X$ such that $v \leqq x$ for all $x \in M$ and if $w \leqq x$ for all $x \in M$, then $w \leqq v$. Similarly, sup M (if it exists) is defined to be that element $u \in X$ such that $x \leqq u$ for all $x \in M$ and if $x \leqq w$ for all $x \in M$, then $u \leqq w$. It is easy to show that sup M and inf M are unique if they exist.

A sequence $\{x_n\}$ of elements from X is said to converge to $x \in X$ if there exist two sequences $\{y_n\}$ and $\{z_n\}$ such that $y_1 \leqq y_2 \leqq \cdots \leqq x \leqq \cdots \leqq z_2 \leqq z_1$, $y_n \leqq x_n \leqq z_n$, and $\sup \{y_n\} = x = \inf \{z_n\}$. Convergence defined in this way is called order convergence (o-convergence). If the sequence $\{x_n\}$ o-converges to x, we will write $o\text{-}\lim x_n = x$. The reader should note that if X is taken to be the set of real numbers with the order $\leqq$ defined as it usually is, then o-convergence is the same as the usual convergence for real numbers.

Now let S be a metric space, where $d(\cdot, \cdot)$ will denote the metric. We then define X to be the collection of ordered pairs (p, λ), where $p \in S$ and λ is a real number. The set X will be made into a partially ordered space as follows: if $x = (p, \lambda)$ and $y = (q, \mu)$, then $x \leqq y$ iff $d(p, q) \leqq \mu - \lambda$. We now verify that $\leqq$ has the desired properties of a partial order. Since $d(p, p) = 0 \leqq \lambda - \lambda$, we see that $x \leqq x$. Setting $z = (r, \eta)$ and assuming $x \leqq y$ and $y \leqq z$, we have

$$d(p, r) \leqq d(p, q) + d(q, r) \leqq (\mu - \lambda) + (\eta - \mu) = \eta - \lambda;$$

hence, $x \leqq z$. If $x \leqq y$ and $y \leqq x$, then $d(p, q) \leqq \mu - \lambda$ and $d(p, q) \leqq \lambda - \mu$; hence, $\lambda = \mu$ and $d(p, q) = 0$ (i.e., $p = q$) and, therefore, $x = y$.

If we define X_0 as the subset of X consisting of all elements of the form $(p, 0)$, then it is clear that there is a one-to-one correspondence between S and X_0. The correspondence $p \rightarrow (p, 0)$ embeds S in X.

THEOREM 1. *Let $\{p_n\}$ be a sequence in S and let $\{x_n\}$, where $x_n = (p_n, 0)$, be the corresponding sequence in X. Then* $\lim p_n = p \in S$ *(i.e.,* $\lim d(p_n, p) = 0$*) iff* o-$\lim x_n = x = (p, 0)$.

Proof. Suppose $\lim p_n = p$. Define $\lambda_n = \sup \{d(p_k, p) : k \geqq n\}$ and then set $y_n = (p, -\lambda_n)$, $z_n = (p, \lambda_n)$. It is clear that $y_1 \leqq y_2 \leqq \cdots \leqq x \leqq \cdots \leqq z_2 \leqq z_1$ and that $y_n \leqq x_n \leqq z_n$. Now if $w = (q, \mu) \leqq z_n$ for every n, then $d(p, q) \leqq \lambda_n - \mu$ for every n, but since $\lim \lambda_n = 0$, we have $d(p, q) \leqq 0 - \mu$, which means $w \leqq x$. Hence, $x = \inf \{z_n\}$. In the same way we may show that $x = \sup \{y_n\}$. Thus, we have shown that o-$\lim x_n = x$.

Now suppose o-$\lim x_n = x$. Thus, there exist two sequences $\{y_n\}$ and $\{z_n\}$ such that $y_1 \leqq y_2 \leqq \cdots \leqq x \leqq \cdots \leqq z_2 \leqq z_1$, $y_n \leqq x_n \leqq z_n$, and $\sup \{y_n\} = x = \inf \{z_n\}$. Setting $y_n = (q_n, \lambda_n)$, we see that $d(p, q_n) \leqq -\lambda_n$ and $d(p_n, q_n) \leqq -\lambda_n$; hence, $d(p, p_n) \leqq -2\lambda_n$. Therefore, to show that $\lim p_n = p$, we need only show that $\lim \lambda_n = 0$. Let us assume the contrary, i.e., that $\lim \lambda_n = 2\alpha < 0$. Since $\lambda_1 \leqq \lambda_2 \leqq \cdots$, there must exist an integer k such that $3\alpha \leqq \lambda_n \leqq 2\alpha$ for all $n \geqq k$. If we set $w = (q_k, \alpha)$, then for all $n \geqq k$, we have

$$d(q_k, q_n) \leqq \lambda_n - \lambda_k \leqq -\alpha \leqq \alpha - \lambda_n,$$

which means that $y_n \leqq w$. Since $y_1 \leqq y_2 \leqq \cdots$, we see that $y_n \leqq w$ for all n. But since $\alpha < 0$, we see that the relation $x \leqq w$ is false, which contradicts the fact that $\sup \{y_n\} = x$. Thus, we must have $\lim \lambda_n = 0$; hence, $\lim p_n = p$.

In a partially ordered space it is possible to introduce various notions of completeness. We will be interested only in the following

DEFINITION. *A partially ordered space X is said to be Dedekind σ-complete if for every sequence $\{y_n\}$, where $y_1 \leqq y_2 \leqq \cdots \leqq u$,* $\sup \{y_n\}$ *exists and if for every sequence $\{z_n\}$, where $v \leqq \cdots \leqq z_2 \leqq z_1$,* $\inf \{z_n\}$ *exists.*

The terminology here is due to McShane [2, pp. 9–11].

THEOREM 2. *Let S be a metric space and let X be the partially ordered space constructed from S as above. Then X is Dedekind σ-complete iff S is a complete metric space.*

Proof. Assume first that S is a complete metric space. Now let $\{y_n\}$ be a sequence from X such that $y_1 \leqq y_2 \leqq \cdots \leqq u \in X$. If we set $y_n = (p_n, \lambda_n)$ and $u = (q, \mu)$, then $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \mu$. Thus, $\lim \lambda_n = \lambda$ exists. Since $d(p_k, p_n) \leqq |\lambda_k - \lambda_n|$, $\{p_n\}$ is a Cauchy sequence; hence, $\lim p_n = p$. It is easily seen that $d(p, p_n) \leqq \lambda - \lambda_n$, which means that $y_n \leqq y = (p, \lambda)$ for all n. Now if $y_n \leqq w = (r, \eta)$ for all n, then $d(p_n, r) \leqq \eta - \lambda_n$ for all n; hence, $d(p, r) \leqq \eta - \lambda$, which means $y \leqq w$.

Thus, $\sup \{y_n\} = y$. The proof that every bounded monotonically decreasing sequence from X has an infimum is similar and will therefore be omitted. Hence, X is Dedekind σ-complete.

We will now show that if X is Dedekind σ-complete, then S is a complete metric space. Let $\{p_n\}$ be a Cauchy sequence from S. By a well-known process we may select a sequence $\{p_n'\}$ such that $d(p_{n+1}', p_n') \leqq 1/2^n$ and $\lim d(p_n', p_n) = 0$ (this is usually done by taking $\{p_n'\}$ as a subsequence of $\{p_n\}$). Therefore, if we define $\lambda_n = 2/2^n$ and $y_n = (p_n', -\lambda_n)$ and $z_n = (p_n', \lambda_n)$, then $y_1 \leqq y_2 \leqq \cdots \leqq \cdots \leqq z_2 \leqq z_1$. Hence, $\inf \{z_n\}$ exists; we will set $v = (q, \mu) = \inf \{z_n\}$. Since $y_n \leqq v \leqq z_n$ for every n, we have $d(p_n', q) \leqq \mu + \lambda_n$ and $d(p_n', q) \leqq \lambda_n - \mu$; hence, $d(p_n', q) \leqq \lambda_n$. Therefore, $\lim p_n' = \lim p_n = q$, which means that S is a complete metric space.

We will now show that a fixed point theorem in complete metric spaces can be obtained as a special case of a fixed point theorem in partially ordered spaces, but first we need some preliminaries. If X is a partially ordered space, then a chain L is a nonempty subset of X such that if $x, y \in L$, then $x \leqq y$ or $y \leqq x$.

LEMMA 3. *Let S be a complete metric space and let X be the partially ordered space constructed from S as above. If L is a chain in X and is bounded above, then* $\sup L$ *exists.*

Proof. Let $u = (q, \lambda)$ be an upper bound for L. Then, for each $(p, \mu) \in L$, we have $\mu \leqq \lambda$ and thus by the familiar properties of the real numbers there exists an increasing sequence $\{y_n\}$ in L, with $y_n = (p_n, \mu_n)$, such that $\lim \mu_n = \sup \{\mu: (p, \mu) \in L\}$. It is clear that for each $y \in L$ there exists k such that $y \leqq y_k$. By Theorem 2, $\sup \{y_n\}$ exists. By the properties of the sequence $\{y_n\}$ we see that $\sup L = \sup \{y_n\}$.

If X is a partially ordered space, then a mapping $F: X \to X$ is said to be isotone if $F(x) \leqq F(y)$ whenever $x \leqq y$. Now let S be any metric space and let X be the partially ordered space constructed from S as above. If $f: S \to S$ and is such that $d(f(p), f(q)) \leqq \alpha d(p, q)$ for all $p, q \in S$, then f determines an isotone mapping $F: X \to X$ as follows: if $x = (p, \lambda) \in X$, then $F(x) = (f(p), \alpha\lambda)$. It is easy to verify that F is isotone. We note that if the mapping F has a fixed point, then so does f. In case S is a complete metric space and $\alpha < 1$, then it is well known that f has a fixed point. To obtain this result in a different way we proceed as follows: take any $p_0 \in S$ and then define λ_0 so that $d(p_0, f(p_0)) = (1 - \alpha)\lambda_0$. Define $x_0 = (p_0, -\lambda_0)$ and $x_1 = (p_0, \lambda_0)$. Using the definition of λ_0, we see that $x_0 \leqq F(x_0) \leqq F(x_1) \leqq x_1$. We will now show that the above conditions guarantee that F has a fixed point in X; hence, f has a fixed point in S. The generalization comes from the fact that in the proof one only uses the fact that F is isotone. The fact that F and f are related as above is irrelevant. Also, the fact that F is continuous (in any sense) is not used.

THEOREM 4. *Let S be a complete metric space and let X be the partially ordered space constructed from S as above. Let $F: X \to X$ be isotone. If there exist $x_0, x_1 \in X$ such that $x_0 \leqq F(x_0) \leqq F(x_1) \leqq x_1$, then F has a fixed point.*

Proof. Define $M=\{x: x\leqq F(x) \text{ and } x_0\leqq x\leqq x_1\}$. Since $x_0\in M$, M is nonempty. By Zorn's lemma there exists a maximal chain $L\subset M$. It is clear that $x_0\in L$. By Lemma 3 sup L exists; thus, define $u=\sup L$. Since $x\leqq u$ for all $x\in L$, we have $x\leqq F(x)\leqq F(u)$ for all $x\in L$. Therefore, $u\leqq F(u)$. Since $u\leqq x_1$ and L is a maximal chain in M, we must have $u\in L$. Since F is isotone and $u\leqq x_1$, we have $F(u)\leqq F(x_1)\leqq x_1$, and since $F(u)\leqq F(F(u))$, we have $F(u)\in M$. Hence, by the maximality of L and the fact that $u\leqq F(u)$, we must have $F(u)=u$. Q.E.D.

We should note that fixed point theorems for isotone mappings have been obtained by Tarski; see [1, p. 54] or [3].

References

1. G. Birkhoff, Lattice Theory, rev. ed., Amer. Math. Soc. Coll. Pub., 25 (1948).
2. E. J. McShane, Order-preserving maps and integration processes, Annals of Math. Studies, No. 31, 1953.
3. A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pac. Jour. of Math., 5 (1955) 285–309.

ON THE ZEROS OF POLYNOMIALS

Q. G. Mohammad, Jammu and Kashmir University (India)

The main purpose of this paper is to apply Schwarz's lemma to the study of the location of the zeros of a class of polynomials.

Throughout this paper $P(z)=a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$. Let

$$M=\max_{|z|=1}\left|a_1z^{n-1}+\cdots+a_n\right|=\max_{|z|=1}\left|a_nz^{n-1}+\cdots+a_1\right|.$$

We prove

Theorem 1. *All the zeros of* $P(z)$ *lie in* $|z|\leqq M/|a_0|$ *if* $|a_0|\leqq M$.

Proof. Consider $p(z)=a_0+a_1z+\cdots+a_nz^n$, then

$$|p(z)|\geqq|a_0|-|a_1z+a_2z^2+\cdots+a_nz^n|. \tag{1}$$

The definition of M and Schwarz's lemma imply that

$$|a_1z+a_2z^2+\cdots+a_nz^n|\leqq M|z|\quad\text{for } |z|\leqq 1.$$

Hence, (1) implies that if $|z|\leqq 1$ then

$$|p(z)|\geqq|a_0|-M|z|. \tag{2}$$

Thus in $|z|\leqq 1$, $|p(z)|>0$ if $|z|<|a_0|/M$ ($\leqq 1$ since $|a_0|\leqq M$ by hyp.); hence $p(z)$ does not vanish for $|z|<|a_0|/M$. Consequently all the zeros of $p(z)$ lie in $|z|\geqq|a_0|/M$. As $P(z)=z^np(1/z)$, we conclude that all the zeros of $P(z)$ lie in $|z|\leqq M/|a_0|$. This proves the theorem.

Remark 1. If $|a_0|\geqq M$, it follows easily from (2) that all the zeros of $P(z)$ lie in $|z|\leqq 1$.

Remark 2. The limit in Theorem 1 is attained by $P(z)=-nz^n+z^{n-1}+\cdots+z+1$.

REMARK 3. The bound obtained in Theorem 1 is in general better than the traditional $(|a_1|+|a_2|+\cdots+|a_n|)/|a_0|$.

COROLLARY 1. *If* $a_k \geqq 0$, $k=1, 2, \cdots, n$ *and* $|a_0| \leqq a_1+a_2+\cdots+a_n$, *then all the zeros of* $P(Z)$ *lie in* $|z| \leqq (a_1+a_2+\cdots+a_n)/|a_0|$.

In particular all the zeros of $R(z)=\pm S_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ lie in $|z| \leqq 1$, where $S_0=a_1+a_2+\cdots+a_n$.

A well-known theorem of Enestrom and Kakeya ([1], p. 106) states that if $a_0 \geqq a_1 \geqq a_2 \geqq \cdots \geqq a_{n-1} \geqq a_n > 0$, then all the zeros of $P(z)$ lie in $|z| \leqq 1$.

We show that it can be deduced easily from Corollary 1. Let

$$
\begin{aligned}
F(z) &= (1-z)P(z) = (1-z)(a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n) \\
&= -a_0z^{n+1} + (a_0-a_1)z^n + (a_1-a_2)z^{n-2} + \cdots + (a_{n-1}-a_n)z + a_n \\
&= -a_0z^{n+1} + Q(z).
\end{aligned}
$$

The hypothesis of the Enestrom-Kakeya theorem implies that $Q(z)$ is a polynomial with nonnegative coefficients and the sum of the coefficients is clearly a_0. Hence by Corollary 1 all the zeros of $F(z)$ lie in $|z| \leqq 1$. As all the zeros of $P(z)$ are also the zeros of $F(z)$ we have proved the Enestrom-Kakeya Theorem.

THEOREM 2. *Let* r *be the modulus of the zeros of largest modulus of* $P(z)$ *and* $M'=\max_{|z|=r}|a_0z^{n-1}+a_1z^{n-2}+\cdots+a_{n-1}|$. *Then all the zeros of* $P(z)$ *lie in the ring-shaped region* $r|a_n|/M' \leqq |z| \leqq r$ *if* $|a_n| \leqq M'$.

Proof. $P(z)=a_n+a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z$ implies, by Schwarz's lemma, that if $|z| \leqq r$ then

$$
|P(z)| \geqq |a_n| - |a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z| \geqq |a_n| - \frac{M'|z|}{r}.
$$

Hence $|P(z)|>0$ if $|z|<r|a_n|/M'$ ($\leqq r$ since $|a_n| \leqq M'$ by hypothesis). Hence all the zeros of $P(z)$ lie in the region $r|a_n|/M' \leqq |z| \leqq r$.

COROLLARY 2. *If* $a_k \geqq 0$, $k=1, 2, 3, \cdots, n-1$, $a_0>0$, *and if* $|a_n| \leqq a_0r^n+a_1r^{n-1}+\cdots+a_{n-1}r$, *then all the zeros of* $P(z)$ *lie in*

$$
\frac{r|a_n|}{a_0r^n + a_1r^{n-1} + \cdots + a_{n-1}r} \leqq |z| \leqq r.
$$

With the help of Corollary 2 we can restate the Enestrom-Kakeya Theorem in the following form:

THEOREM 3. *If* $a_0 \geqq a_1 \geqq \cdots \geqq a_{n-1} \geqq a_n > 0$, *then all the zeros of* $P(z)$ *lie in the ring-shaped region* $a_n/(a_0+a_1+\cdots+a_{n-1}) \leqq |z| \leqq 1$.

The lower limit is attained by $P(z)=z+1$. The following theorem can be easily deduced from the Enestrom-Kakeya Theorem ([1], p. 106).

THEOREM a. *All the zeros of $P(z)$ having real positive coefficients lie in $|z| \leqq \rho$, where*

$$\rho = \max\left(\frac{a_1}{a_0}, \frac{a_2}{a_1}, \cdots, \frac{a_{n-1}}{a_{n-2}}, \frac{a_n}{a_{n-1}}\right). \tag{3}$$

Clearly $\rho^n \geqq a_n/a_0$ or $a_0\rho^n \geqq a_n$. Hence,

$$M' = \max_{|z|=\rho} |a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z| = a_0\rho^n + a_1\rho^{n-1} + \cdots + a_{n-1}\rho > a_n.$$

Hence, applying Corollary 2 to $P(z)$, we can restate this theorem in the following form:

THEOREM 4. *If ρ is given by (3) then all the zeros of $P(z)$ having real positive coefficients lie in the ring-shaped region*

$$\frac{\rho a_n}{a_0\rho^n + a_1\rho^{n-1} + \cdots + a_{n-1}\rho} \leqq z \leqq \rho.$$

THEOREM 5. *If $a_0 \geqq a_1 \geqq \cdots \geqq a_{n-1} \geqq a_n > 0$, then the number of zeros of $P(z)$ in $|z| \leqq \frac{1}{2}$ does not exceed*

$$1 + \frac{1}{\log 2} \log \frac{a_0}{a_n} \cdot$$

Proof. Consider

$$\begin{aligned} F(z) &= (1 - z)(a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n) \\ &= -a_0 z^{n+1} + (a_0 - a_1) z^n + (a_1 - a_2) z^{n-1} + \cdots + (a_{n-1} - a_n) z + a_n. \end{aligned}$$

For $|z| \leqq 1$,

$$\left|\frac{F(z)}{F(0)}\right| \leqq \frac{a_0 + (a_0 - a_1) + \cdots + (a_{n-1} - a_n) + a_n}{a_n} = \frac{2a_0}{a_n} \cdot$$

If $f(z)$ is regular, $f(0) \neq 0$, and $|f(z)| \leqq M$ in $|z| \leqq 1$, then ([2], p. 171) the number of zeros of $f(z)$ in $|z| \leqq 1/2$ does not exceed $\{\log M/|f(0)|\}/\log 2$. Therefore, if $n(1/2)$ denotes the number of zeros of $F(z)$ in $|z| \leqq 1/2$ then from above

$$n(\tfrac{1}{2}) \leqq \log \frac{2a_0}{a_n} \Big/ \log 2 = 1 + \frac{1}{\log 2} \log \frac{a_0}{a_n} \cdot$$

As the number of zeros of $P(z)$ in $z \leqq 1/2$ is also equal to $n(\frac{1}{2})$ the theorem is proved.

References

1. M. Marden, The geometry of zeros, Amer. Math. Soc., Math. Surveys, No. 3, New York.
2. E. C. Titchmarsh, The theory of functions, 2nd ed., Oxford University Press, London, 1939.

ON *UC* SPACES

W. C. WATERHOUSE, Harvard University

Let M and M' be metric spaces. M is called $UC(M')$ iff every continuous function from M into M' is uniformly continuous. M is called UC iff it is $UC(R)$, where R is the real line. Levine and Saunders [**3**] began an investigation of UC spaces; this note answers some questions suggested by their article. The basic tool is Theorem A and its corollaries. Since its essentials are already known [**1**, **4**], we may state it here without proof.

DEFINITION. *A C-sequence is a sequence of pairs of points* (x_n, y_n), $x_n \neq y_n$, *such that* $d(x_n, y_n) \to 0$. *We say it has a cluster point iff the sequence* $\{x_n\}$ *(equivalently,* $\{y_n\}$*) has a cluster point.*

THEOREM A. *The following are equivalent:*

A. *M is UC.*
B. *M is $UC(M')$ for some M' containing an arc (homeomorphic image of* $[0, 1]$*).*
C. *Every two disjoint closed sets in M have positive distance.*
D. *Every C-sequence in M has a cluster point.*
E. *M is $UC(M')$ for all M'.*
F. *The metric of M induces on it the strongest uniform structure compatible with its topology.*

COROLLARY 1. *A closed subset of a UC space is UC.*

COROLLARY 2. *A UC space is complete, and hence closed as a subset of any space in which it is embedded.*

COROLLARY 3. *The derived set of a UC space is compact.*

The simplest UC spaces are thus: (1) the compact spaces, in which every sequence has a cluster point, and (2) the UD (uniformly discrete) spaces, in which there are no C-sequences (i.e., a space is UD iff there is an $\epsilon > 0$ such that $d(x, y) \geqq \epsilon$ for all $x \neq y$). This note is devoted to finding out just how far these simple spaces will take us.

If a space is the union of a compact set and a UD set, we can always make the union disjoint, because a subset of a UD set is UD. Then since a compact set and a disjoint (necessarily closed) UD set can easily be seen to have positive distance, it follows that any space which is the union of a compact set and a UD set is UC. Any such space is locally compact; this suggests the proper converse.

THEOREM 1. *A locally compact UC space can be written as the union of a compact set and a UD set.*

Proof. Let M be the space. The derived set B is compact, and therefore has a compact neighborhood, D. Then $F = M - \text{int}(D)$ is closed, so it is a UC space. If there are any C-sequences in F, they therefore must have cluster points in F;

but this is impossible, since any such cluster points would have to lie in B. Hence F is UD, and $M=D\cup F$ has the proper form.

Since a UC subspace of a space is closed, and a closed subspace of a locally compact space is locally compact, every UC subspace of a locally compact space is the union of a compact set and a UD set. This also has a good converse, but to get it we first need a bit of incidental information.

LEMMA. *Any locally totally bounded space can be embedded in a locally compact space.*

Proof. Let each point x in the space M have a totally bounded open ball $B(x, \epsilon_x)$ around it. Then it is easy to check that the union over all x in M of the corresponding open balls in the completion of M is locally compact.

THEOREM 2. *Let M be a space, every UC subspace of which is the union of a compact set and a UD set. Then M can be embedded in a locally compact space.*

Proof. We need prove only that M is locally totally bounded. Suppose on the contrary that none of the open balls $B(p, \epsilon)$ around the point p are totally bounded. Then for each there is an $\eta>0$ and a sequence of points $\{p_n\}$ in it with $d(p_i, p_j)\geqq\eta$ for $i\neq j$ and also $d(p, p_n)\geqq\eta$. Thus, in $B(p, 1)$ we get an η_1 and a sequence $\{p_{1n}\}$. Then in $B(p, \eta_1/2)$ we get η_2 and $\{p_{2n}\}$, in $B(p, \eta_2/2)$ we get η_3 and $\{p_{3n}\}$, and so on. Let N consist of p and all the points p_{kn}. A point p_{ij} is at distance $\geqq\eta_i/2$ from all other points of N; hence any C-sequence (x_n, y_n) in N must have $x_n\rightarrow p$, and N is UC. But removing any UD set from N must for some k leave the entire sequence $\{p_{kn}\}$, and hence N is not the union of a compact set and a UD set.

References

1. Masahiko Atsuji, Uniform continuity of continuous functions of metric spaces, Pacific J. Math., 8 (1958) 11–16.
2. J. L. Kelley, General Topology, Van Nostrand, New York, 1955.
3. Norman Levine and William G. Saunders, Uniformly continuous sets in metric spaces, this MONTHLY 67 (1960) 153–156.
4. John Rainwater, Spaces whose finest uniformity is metric, Pacific J. Math., 9 (1959) 567–570.

SPACES IN WHICH COMPACT SETS ARE NEVER CLOSED

NORMAN LEVINE, Ohio State University

Let P be the set of positive integers and let τ consist of $\varnothing$, P and $O_n=\{1, 2, \cdots, n\}$ for $n=1, 2, \cdots$. Then (P, τ) has the following property: the empty set is the only subset of P which is both closed and compact.

In this note we will give several characterizations of this property and list some of its implications.

REMARK 1. We shall use the following form of Zorn's Lemma: Let $\mathfrak{A}$ be a nonempty family of nonempty sets. If every chain of sets in $\mathfrak{A}$ has a lower bound,

then there is a minimal set A in $\mathfrak{A}$ (see [2]).

DEFINITION 1. *A topological space (X, τ) has property A iff $\varnothing$ is the only subset of X which is both closed and compact.*

DEFINITION 2. *A topological space (X, τ) has property B iff every nonempty closed subset of X has a proper subset which is nonempty and closed.*

LEMMA 1. *Property A implies property B.*

Proof. Suppose (X, τ) has property A and let F be closed and nonempty. Then F is not compact and there then exists an open cover $\psi = \{O\}$ of F with no finite subcover. Take $x \in F$. Then $x \in O^*$ for some O^* in ψ and $F - O^*$ is a nonempty proper closed subset of F.

DEFINITION 3. *A topological space (X, τ) has property C iff for every nonempty closed set F, there exists a chain $\Phi = \{F^*\}$ of nonempty closed proper subsets of F such that $\bigcap \{F^* \colon F^* \in \Phi\} = \varnothing$.*

LEMMA 2. *Property B implies property C.*

Proof. Let (X, τ) have property B. Let F be a nonempty closed subset of X. Let $\mathfrak{A}$ be the family of all nonempty closed proper subsets of F. Property B implies that $\mathfrak{A}$ has no minimal element. By remark 1 then, there exists a chain in $\mathfrak{A}$ which has an empty intersection. Then (X, τ) has property C.

THEOREM 1. *Properties A, B and C are equivalent.*

Proof. By Lemmas 1 and 2, it suffices to prove that property C implies property A. Let (X, τ) have property C and suppose that F is a nonempty closed set. There exists then a chain $\Gamma = \{F^*\}$ of nonempty closed proper subsets of F such that $\bigcap \{F^* \colon F^* \in \Gamma\} = \varnothing$. Since Γ, being a chain of nonempty sets, has the finite intersection property, it follows that F is not compact. Thus property A holds for (X, τ).

DEFINITION 4. *A topological space (X, τ) has property D iff for every open set $O \neq X$, there exists an open set O^* such that $O \subset O^* \neq X$, the inclusion being proper.*

REMARK 2. Property D is clearly equivalent to property B.

DEFINITION 5. *A topological space (X, τ) has property E iff for each open set $O \neq X$, there exists a chain $\Phi = \{O^*\}$ of open sets such that $O \subset O^* \neq X$, the inclusion being proper, and $X = \bigcup \{O^* \colon O^* \in \Phi\}$.*

REMARK 3. Property E is clearly equivalent to property C.

DEFINITION 6. *A topological space (X, τ) has property F iff for each $x \in X$ $c(x)$ is not compact, c denoting the closure operator.*

THEOREM 2. *Property F is equivalent to property A.*

Proof. Property A clearly implies property F. To show the converse, let Q be a nonempty closed set in X. Take $q \in Q$. If Q were compact, then $c(q)$ would be a closed subset of Q and hence $c(q)$ would be compact, contradicting property F of (X, τ).

DEFINITION 7. *A topological space (X, τ) has property G iff for each $x \in X$, $c(x)$ has a nonempty closed proper subset.*

THEOREM 3. *Property G is equivalent to property B.*

We omit the proof.

DEFINITION 8. *A topological space (X, τ) has property H at $x \in X$ iff for $x \in O \in \tau$, there exists a closed set F such that $x \in F \subset O$. (E.g., regular spaces as well as T_1 spaces have property H at every point x.)*

THEOREM 4. *Property G is equivalent to property H failing at every point X.*

Proof. Let (X, τ) have property G and suppose $x \in X$. There exists then a nonempty closed proper subset F^* of $c(x)$. Then $x \notin F^* \subset c(x)$. Then $x \in \mathcal{C}F^* \in \tau$, $\mathcal{C}$ denoting the complement operator. Now suppose there exists a closed set F such that $x \in F \subset \mathcal{C}F^*$. Then $x \in c(x) \subset F \subset \mathcal{C}F^*$ and $F^* \subset c(x) \cap \mathcal{C}c(x) = \varnothing$ which contradicts F^* being nonempty. Thus property H fails at x. Conversely, suppose that property H fails at every $x \in X$. We will show that $c(x)$ has a nonempty closed proper subset for each $x \in X$. Fix $x^* \in X$. There exists an $O^* \in \tau$ such that $c(x) \not\subset O^*$ and $x^* \in O^*$. Then $c(x^*) \cap \mathcal{C}O^*$ is a nonempty closed proper subset of $c(x^*)$.

COROLLARY 1. *If (X, τ) has property A, then (X, τ) is not regular nor is it T_1.*

THEOREM 5. *Let (X, τ) have property A and suppose that $Y \subset X$. Then* (1) *if Y is closed, then Y as a subspace of X has property A and* (2) *if Y as a subspace has property A, then Y is not compact.*

We omit the easy proof.

THEOREM 6. *Let $\{(X_\alpha, \tau_\alpha): \alpha \in \Delta\}$ be a nonempty family of nonempty topological spaces and let (X^*, τ^*) be the Cartesian product. Then (X^*, τ^*) has property A iff (X_α, τ_α) has property A for at least one α in Δ.*

Proof. Necessity: Suppose (X_α, τ_α) fails to have property A for all $\alpha \in \Delta$. Then in each X_α there exists a nonempty closed compact set F_α. For each $\alpha \in \Delta$, let P_α denote the projection mapping from X^* onto X_α. Then $\cap\{P_\alpha^{-1}F_\alpha: \alpha \in \Delta\}$ is nonempty and closed and by the Tychonoff theorem, it is also compact. This contradicts property A of (X^*, τ^*).

Sufficiency: Suppose that $(X_{\alpha^*}, \tau_{\alpha^*})$ has property A. Let $x^* \in X^*$. It suffices to show that $c^*(x^*)$ is not compact (see Theorem 2). Now

$$\begin{aligned} c^*(x^*) &= c^*(\bigcap \{\overset{-1}{P_\alpha} P_\alpha x^* : \alpha \in \Delta\}) \\ &= \bigcap \{\overset{-1}{P_\alpha} c_\alpha P_\alpha x^* : \alpha \in \Delta\} \\ &= \bigcap \{\overset{-1}{P_\alpha} c_\alpha \{x_\alpha^*\} : \alpha \in \Delta\}, \end{aligned}$$

which by the Tychonoff theorem is not compact since $c_{\alpha^*}(x_{\alpha^*}^*)$ is not compact by property A.

COROLLARY 2. *Let (X, τ) be a topological space and for each $\alpha \in \Delta$, let $(X_\alpha, \tau_\alpha) = (X, \tau)$. Let (X^*, τ^*) be the Cartesian product of this family of spaces. Then (X, τ) has property A iff (X^*, τ^*) has property A.*

References

1. John L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.
2. M. Zorn, A remark on a method in transfinite algebra, Bull. Amer. Math. Soc., 41 (1935) 667–670.

A NOTE ON MORLEY'S THEOREM

A. VANDEGHEN, Liège, Belgium

Let the internal trisectors of the angles of any triangle ABC meet in pairs to form Morley's equilateral triangle, as in Figure 1. It is known [1] that the directed angle from BC to RQ is $(B-C)/3$. Let us compare this Morley triangle of ABC with the Morley triangle of another triangle $A'B'C'$ whose sides make angles α, β, γ with those of ABC, as in Figure 2. Clearly

$$A' = A + \beta - \gamma, \qquad B' = B + \gamma - \alpha, \qquad C' = C + \alpha - \beta.$$

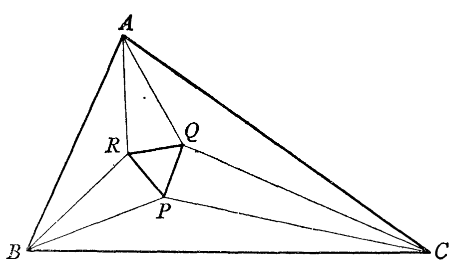


FIG. 1

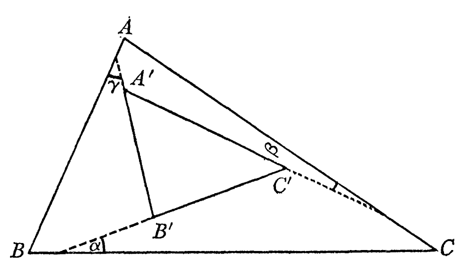


FIG. 2

A necessary and sufficient condition for the two Morley triangles to have parallel sides is

$$\frac{B-C}{3} = \alpha + \frac{B'-C'}{3},$$

which reduces to

$$\alpha + \beta + \gamma = 0.$$

It is easy to see that this condition is fulfilled for each of the following eight triangles derived from ABC:

(1) the triangle formed by the midpoints of the sides,

(2) the orthic triangle,

(3) the triangle formed by the excenters,

(4) the triangle formed by the points of contact of the sides with the incircle,

(5), (6), (7) the triangles formed by the points of contact of the sides with the three excircles,

(8) the triangle formed by the tangents at A, B, C to the circumcircle.

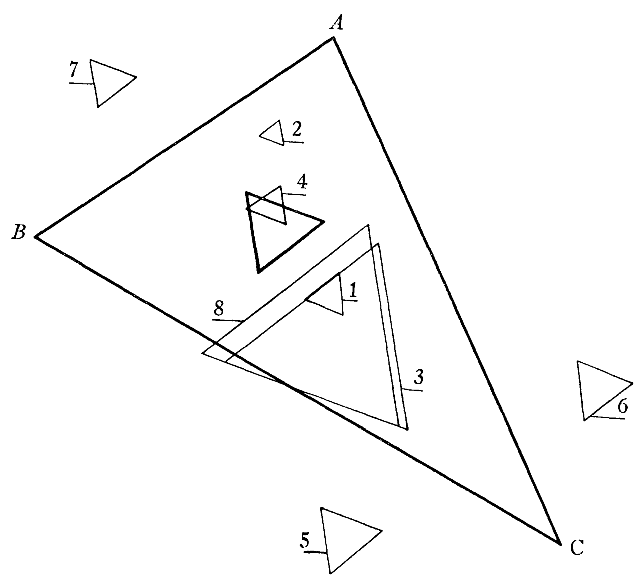


FIG. 3

The internal Morley triangles of ABC and of these eight derived triangles are numbered as above in Figure 3.

Reference

1. L. Bankoff, A simple proof of the Morley Theorem, Mathematics Magazine, 35 (1962) 223–224.

REALIZATION OF SEMI-MULTIPLIERS AS MULTIPLIERS

HARRIET FELL AND A. J. GOLDMAN, National Bureau of Standards

We deal exclusively with linear associative algebras over a common field F of scalars. Let $\mathcal{L}[A]$ be the algebra of all linear transformations of (the vector space of) an algebra A. The left regular representation of A is the homomorphism $\lambda_A: A \rightarrow \mathcal{L}[A]$ given by $\lambda_A(a) = M_a$, where M_a is defined by $M_a x = ax$ for

all $x \in A$. The elements M_a of the image $\mathfrak{M}[A] = \lambda_A(A)$ will be called (left) *multipliers* of A, and the elements of λ_A's kernel, *annih* $[A]$, are called (left) *annihilators* of A.

Let $\mathfrak{M}^*[A]$ be the subalgebra of $\mathfrak{L}[A]$ consisting of all transformations T such that, for all $x, y \in A$,

$$T(xy) = (Tx)y. \tag{$*$}$$

The associative law in A ensures that $\mathfrak{M}[A] \subset \mathfrak{M}^*[A]$, and so the members of $\mathfrak{M}^*[A]$ will be called *semi-multipliers* of A. In [1], Buck noted that $\mathfrak{M}[A] = \mathfrak{M}^*[A]$ if and only if A has a left unit.

If A is a left ideal in some algebra B, then we have a *restriction* homomor phism $\rho_B: \mathfrak{M}[B] \to \mathfrak{L}[A]$, and the associative law in B ensures that the image $\rho_B(\mathfrak{M}[B]) \subset \mathfrak{M}^*[A]$. When equality holds in this inclusion, we can say that the semi-multipliers of A have been *realized as multipliers* (of B). In [1] Buck showed how to form from A an enormous algebra B (whose dimension is at least the cardinality of $\mathfrak{M}^*[A]$) yielding this kind of realization. He asked for a construction giving a smaller and more tractable B. Our purpose in this note is to exhibit such a construction.

We take the set B to be the Cartesian product $\mathfrak{M}^*[A] \times A$. Addition and scalar multiplication are defined componentwise, and algebra multiplication is defined by

$$(T, x)(T', y) = (TT' + M_x T', Ty + xy).$$

This multiplication yields $(0, x)(0, y) = (0, xy)$ as well as

$$(T, x)(0, y) = (0, Ty + xy),$$

so that $\{(0, a) : a \in A\}$ is an imbedding of A as a left ideal in B. Furthermore every semi-multiplier T of A is realized by the multiplier $M_{(T,0)}$ of B, since

$$(T, 0)(0, y) = (0, Ty) \qquad \text{for all } y \in A.$$

Verification of the distributive and associative laws for B, with the aid of $(*)$, is routine.

The kernel of the composition $\rho_B \lambda_B$ is one natural measure of the "tightness" of the construction. In the present instance this kernel is given by

$$\text{annih } [B] = \{(M_a, -a) : a \in A\},$$

and so has the same dimension (i.e., "size") as A itself. We wish to show that this size is minimal at least for *some* algebras A, in the sense that if B is any algebra with the desired relation to such an algebra A, then

$$\dim [\ker(\rho_B \lambda_B)] \geqq \dim [A],$$

or equivalently

$$\dim [B] = \dim [\ker(\rho_B \lambda_B)] + \dim [\operatorname{im}(\rho_B \lambda_B)] \geqq \dim [A] + \dim [\mathfrak{M}^*[A]].$$

We can prove this when A is a *zero algebra* (i.e., $\lambda_A=0$). Let $\{a_j: j\in J\}$ be a basis of A and let $\{b_i: i\in I\}$ be a subset of B such that $\{\rho_B\lambda_B(b_i): i\in I\}$ is a basis of $\mathfrak{M}^*[A]$. It suffices to show that $\{a_j: j\in J\}\cup\{b_i: i\in I\}$ is a linearly independent subset of B. Suppose that

$$\sum_j \alpha_j a_j + \sum_i \beta_i b_i = 0,$$

where only finitely many of the scalars α_j, β_i are nonzero. Since $\lambda_A=0$, for all $x\in A$ we have $(\sum_i \beta_i b_i)x=0$ and thus

$$\Big\{ \sum_i \beta_i[\rho_B\lambda_B(b_i)]\Big\}(x) = 0.$$

The choice of the b_i implies that all $\beta_i=0$, and then the choice of the a_j implies that all $\alpha_j=0$, so the independence holds.

The construction is not minimal for *all* algebras A. If for example A has a left unit, one can simply take $B=A$ to achieve $\ker(\rho_B\lambda_B)=\text{annih}[A]$, which in general is lower-dimensional than A. We have not succeeded in finding a "universally minimal" construction.

Reference

1. R. C. Buck, Two remarks on the regular representations of an algebra, this MONTHLY, 67 (1960) 997.

NOTE ON A THEOREM OF STONE ON JACOBI MATRICES

T. S. CHIHARA, Seattle University

Let (a_{mn}) be a Jacobi matrix:

$$a_{mn} = 0 \quad \text{for } |m-n| > 1,$$
$$a_{n,n+1} = \bar{a}_{n+1,n} = b_n \neq 0, \qquad a_{n,n} = a_n.$$

Let $\{G_n(x)\}$ be the associated orthonormal polynomials determined by the recurrence

$$\bar{b}_{n-1}G_n(x) = (x-a_{n-1})G_{n-1}(x) - b_{n-2}G_{n-2}(x) \qquad (n = 2, 3, \cdots)$$
$$G_0(x) = 0, \qquad G_1(x) = 1,$$

and let $d\rho(x)$ be any distribution with respect to which the $G_n(x)$ are orthogonal. Finally, let P denote the support of $d\rho(x)$.

In his well-known treatise, Stone [2, Theorem 10.42] obtains the inequality, $\mathfrak{a}\leqq \text{ext}\, P\leqq 3\mathfrak{a}$, where

$$\mathfrak{a} = \sup\{\,|a_{mn}| : m, n = 1, 2, 3, \cdots\}$$
$$\text{ext}\, P = \max\{\,|\inf P|,\ |\sup P|\,\}.$$

He also shows that the equality, ext $P=3\mathfrak{a}$, is possible for a suitably chosen

Jacobi matrix and remarks that the question of a case for which $\mathfrak{a} = \text{ext } P$ remains open. We are not aware that this question has been settled and we therefore provide below an affirmative answer.

To this end, let $P_n(x) = \bar{b}_1\bar{b}_2 \cdots \bar{b}_n G_{n+1}(x)$ $(n \geqq 1)$ so that

$$(1) \qquad \begin{aligned} P_n(x) &= (x - a_n)P_{n-1}(x) - |b_{n-1}|^2 P_{n-2}(x), \quad (n = 1, 2, 3, \cdots) \\ P_{-1}(x) &= 0, \qquad P_0(x) = 1. \end{aligned}$$

Now select two positive null sequences, $\{k_n\}$ and $\{\lambda_{n+1}\}$, such that $\{\lambda_{n+1}/(k_n k_{n+1})\}$ is a chain sequence (e.g., let $0 < \lambda_{n+1} \leqq k_n k_{n+1}/4$, where $k_n > 0$ and $\lim_{n\to\infty} k_n = 0$). Then the polynomials defined by

$$\begin{aligned} Q_n(x) &= (x - k_n)Q_{n-1}(x) - \lambda_n Q_{n-2}(x) \qquad (n = 1, 2, 3, \cdots) \\ Q_{-1}(x) &= 0, \qquad Q_0(x) = 1, \end{aligned}$$

are orthogonal with respect to a distribution, $d\phi(x)$, whose support is a bounded denumerable subset of $[0, \infty)$ with 0 as its sole accumulation point [**1**, Theorems 1 and 7]. Let $[0, \beta]$ be the true interval of orthogonality for $\{Q_n(x)\}$. Then $k_n < \beta$ ([**1**], Lemma 5) and the polynomials $P_n(x) = (-1)^n Q_n(\beta - x)$ satisfy (1) with $a_n = \beta - k_n$, $\lambda_n = |b_{n-1}|^2$ and are orthogonal on $[0, \beta]$ with respect to $d\rho(x) = -d\phi(\beta - x)$. The support, P, of $d\rho(x)$ is then a denumerable subset of $[0, \beta]$ with β as its only accumulation point so that ext $P = \beta$.

But $0 < a_n < \beta$ and $\lim_{n\to\infty} a_n = \beta$, while $\lambda_{n+1} < k_n k_{n+1} < \beta^2$ (since $\{\lambda_{n+1}/(k_n k_{n+1})\}$ is a chain sequence). Thus $\mathfrak{a} = \text{ext } P$.

References

1. T. S. Chihara, Chain sequences and orthogonal polynomials, Trans. Amer. Math. Soc., 104 (1962) 1–16.

2. M. H. Stone, Linear transformations in Hilbert space, Amer. Math. Soc. Colloq. Publ., vol. XV, 1932.

ON LATTICES AND ALGEBRAS OF REAL VALUED FUNCTIONS

R. G. DOUGLAS, University of Michigan

The purpose of this note is to give an elementary proof of the following theorem about linear spaces of real valued functions.

THEOREM. *Let X be a set and $\mathfrak{A}$ a uniformly closed linear space of bounded real valued functions on X that contains the constants. Then $\mathfrak{A}$ is an algebra if, and only if, it is a lattice. (Note that in this context $\mathfrak{A}$ is a lattice if, and only if, the function $(f \vee g)(x) = \max[f(x), g(x)]$ is contained in $\mathfrak{A}$ for every pair f and g in $\mathfrak{A}$.)*

Before proving this theorem, let us remark that the theorem *can* be reduced to the case of continuous functions on a compact Hausdorff space. In this context the theorem is very closely related to the Stone-Weierstrass Theorem and, in fact, it was in this connection the theorem first occurred to the author. The proof we present, however, is quite elementary.

Proof. Assume that $\mathfrak{A}$ is an algebra. The binomial expansion for $(1-x)^{1/2}$ converges uniformly on the interval $[0, 1]$. Thus the binomial expansion for $[1-(1-f^2/\|f\|^2)]^{1/2}$ converges uniformly on X to $(f^2/\|f\|^2)^{1/2}=|f|/\|f\|$. Therefore, $|f|\in\mathfrak{A}$ for every $f\in\mathfrak{A}$. The identities

$$f\vee g = \tfrac{1}{2}\{f+g+|f-g|\} \text{ and } f\wedge g = \tfrac{1}{2}\{f+g-|f-g|\}$$

now complete the proof that $\mathfrak{A}$ is a lattice. (This is a well-known lemma that appears in the proof of the Stone-Weierstrass Theorem.)

Assume that $\mathfrak{A}$ is a lattice. The identity $fg=\frac{1}{2}\{(f+g)^2-f^2-g^2\}$ shows us that to prove $\mathfrak{A}$ is an algebra it is sufficient to show that $f^2\in\mathfrak{A}$ for every $f\in\mathfrak{A}$. We shall complete the proof of the general case after first treating a special case of functions on the interval $[0, 1]$.

Let $\{\lambda_n\}_{n=1}^{\infty}$ denote a sequence of points chosen from the open interval $(0, 1)$ so that the set $\{\lambda_n|n=1, 2, 3, \cdots\}$ is dense in $[0, 1]$. For each positive n define the function

$$k_n(x) = (1+\lambda_n)\left[(x-\lambda_n)^+ + \frac{\lambda_n^2}{1+\lambda_n}\right]$$

and the function

$$h_n(x) = k_1(x)\wedge k_2(x)\wedge\cdots\wedge k_n(x),$$

where $(x-\lambda_i)^+=(x-\lambda_i)\vee 0$. It is easy to show that the functions h_n decrease monotonically to the function x^2 and therefore converge uniformly to x^2. Thus for each $f\in\mathfrak{A}$, the functions

$$\begin{aligned} h_n(f/\|f\|) = {} & (1+\lambda_1)\left[\left(\frac{f}{\|f\|}-\lambda_1\right)^+ + \frac{\lambda_1^2}{1+\lambda_1}\right] \\ & \wedge\cdots\wedge(1+\lambda_n)\left[\left(\frac{f}{\|f\|}-\lambda_n\right)^+ + \frac{\lambda_n^2}{1+\lambda_n}\right], \end{aligned}$$

in $\mathfrak{A}$ converge uniformly to $f^2/\|f\|^2$. Therefore $f^2\in\mathfrak{A}$ for every $f\in\mathfrak{A}$ and $\mathfrak{A}$ is seen to be an algebra by our previous remark.

Let us conclude with the remark that this theorem also holds for closed linear subspaces of L_∞.

A NOTE ON NORMAL MATRICES

C. G. Cullen, University of Pittsburgh

It is well known that the product of two Hermitian matrices is itself Hermitian if and only if the two matrices commute. The generalization of this statement to normal matrices (A is normal if $A^*A=AA^*$) is not valid, although it is easy to see that if A and B are normal and commute then AB is also normal. A pair of noncommuting unitary matrices will suffice to show that the converse does

not hold. In this note we discuss briefly a sufficient condition on a normal matrix A so that for normal B we will have AB normal if and only if A and B commute. It should be observed that if A, B and AB are normal so is BA (see [2]).

A normal matrix A, being unitarily similar to a diagonal matrix, always has the spectral representation

$$A = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_k E_k, \tag{1}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the distinct eigenvalues of A and the projectors E_i are uniquely determined polynomials in A such that $E_iE_j = \delta_{ij}E_i$ and $E_1 + E_2 + \cdots + E_k = I$. The E_i are always Hermitian and B will commute with A if and only if B commutes with each of the E_i (see [1]).

For any complex matrix A we can find a unique positive semi-definite H and a unitary U (not unique if $\det(A) = 0$) such that $A = HU$. This result is a generalization of the polar representation of complex numbers and H is called the Hermitian polar matrix of A. It is not hard to see that H is the unique positive semi-definite square root of AA^*. In [2] Wiegmann proved that if A and B are normal then AB (and hence BA) is normal if and only if each factor commutes with the Hermitian polar matrix of the other.

From the spectral representation of A (1) it follows that

$$A^* = \bar{\lambda}_1 E_1 + \bar{\lambda}_2 E_2 + \cdots + \bar{\lambda}_k E_k, \tag{2}$$

$$AA^* = \lambda_1\bar{\lambda}_1 E_1 + \lambda_2\bar{\lambda}_2 E_2 + \cdots + \lambda_k\bar{\lambda}_k E_k, \tag{3}$$

and

$$H = (AA^*)^{1/2} = |\lambda_1| E_1 + |\lambda_2| E_2 + \cdots + |\lambda_k| E_k. \tag{4}$$

If (4) is the spectral representation of H then B commutes with H if and only if B commutes with each E_i, and if B commutes with each E_i then it follows from (1) that B commutes with A. Only if $|\lambda_i| = |\lambda_j|$ for $i \neq j$ could (4) fail to be the spectral representation of H.

We shall say that A has *modularly distinct eigenvalues* if unequal eigenvalues of A have unequal moduli. Our discussion yields the following result.

THEOREM. *If A and B are normal and one of the two has modularly distinct eigenvalues, then AB is normal if and only if A and B commute.*

No real matrix with a complex eigenvalue can have this property, but all positive semi-definite and all negative semi-definite hermitian matrices have modularly distinct eigenvalues; our theorem has the following consequence.

COROLLARY. *If A is positive (negative) semi-definite and B is normal then AB is normal if and only if A and B commute.*

References

1. S. Perlis, Theory of Matrices, Addison-Wesley, Reading, Mass., 1952.
2. N. A. Wiegmann, Normal products of matrices, Duke Math. J., 15 (1948) 633–638.

AFFINITIES WHICH PRESERVE LOWER DIMENSIONAL VOLUMES

WM. J. FIREY, Oregon State University

Let A be a real, $n \times n$, nonsingular matrix representing an affinity in Euclidean n-space. If the image under A of each p-dimensional simplex, $1 \leqq p \leqq n$, has the same p-dimensional volume as the original simplex, then we say A is *p-dimensional volume-preserving.* Thus the n-dimensional volume-preserving matrices make up the group of unimodular affinities and the 1-dimensional volume-preserving transformations make up the proper subgroup of orthogonal matrices.

In this note we prove the following theorem: *If A is p-dimensional volume-preserving for some $p < n$, then A is q-dimensional volume-preserving for $q = 1, 2, \cdots, n$.*

Clearly we need only prove that A is orthogonal.

Our proof depends on the isoperimetric theorem which says that, in Euclidean n-space, the sphere is the unique convex body of maximal volume for a given $(n-1)$-dimensional surface area. More precisely, we need only the weaker assertion obtained from the preceding by writing "ellipsoid" for "convex body."

First we take $p = n-1$. From the definition of the surface area of a convex body as the greatest lower bound of the surfaces of convex polyhedra containing the convex body, we conclude that A preserves the surface area of convex bodies. Let d be the absolute value of the determinant of A and let S be a sphere. Then, if S' is the image of S under A, S' is an ellipsoid of the same surface area as S. We cannot have $d > 1$ since, by the isoperimetric theorem, the volume of S', which is d times that of S, can be no more than that of S. Nor can we have $d < 1$ because A^{-1} is also $(n-1)$-dimensional volume-preserving and, from the preceding paragraph, the absolute value of the determinant of A^{-1} cannot exceed one. Hence $d = 1$. Consequently, A transforms each sphere into a sphere of the same radius and therefore preserves distances.

If $p < n-1$, we argue as follows. Let L be any linear subspace of dimension $p+1$ and let R be a rotation which sends L', the image of L under A, into L. Then RA sends L into L and preserves p-dimensional volumes. By our earlier argument, RA must preserve all distances in L. Hence, so does A. The arbitrary character of L shows that A is distance-preserving generally, i.e. orthogonal.

Editorial Note (April 18, 1965): It has been pointed out by several readers (and doubtless, if past experience is any measure, by the time this appears in print the number will have grown to astronomical size) that the theorem proved by A. J. Insel in his "A Note on the Hausdorff Separation Property in First Countable Spaces," this MONTHLY, 72(1965) 289–290, was proved more neatly in a Classroom Note, "A Note on Hausdorff Separation," by Edwin Halfar, 68(1961) 164. Dr. D. R. Andrew also called attention to the theorem in Problem 5299, this MONTHLY, 72(1965) 194, (and undoubtedly others will also).

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

GENERATION OF IRREDUCIBLE POLYNOMIALS OVER A FINITE FIELD

D. E. DAYKIN, Reading University, England

1. Introduction. In this note we extend the account given by A. A. Albert ([1], pp. 142–3) of the generation of irreducible polynomials over a finite field. Such polynomials have been found useful in electronic circuits (cf. [2]) and the new results given here will aid in tabulating them. In particular we show how, from a given irreducible polynomial of period e, all irreducible polynomials whose periods divide e may be obtained.

Let F_q be a finite field of $q=p^\nu$ elements, where p is a prime. Throughout this note all polynomials will be monic with coefficients in F_q, and by irreducible we will mean irreducible over F_q. Let $f(x)$ be a given irreducible polynomial of degree n and period e. If C is the companion matrix of $f(x)$, for each positive integer t we will write $f_t(x)$ for the characteristic function of the matrix C^t. We now state the principal results.

THEOREM 1. *If $\omega_1, \omega_2, \cdots, \omega_t$ are the t-th roots of unity, then*

$$f_t(x^t) = (-1)^{n(t+1)} \prod_{j=1}^{t} f(\omega_j x).$$

THEOREM 2. *The polynomial $f_t(x)$ is a power $[g_t(x)]^r$ of an irreducible polynomial $g_t(x)$ of degree $m=n/r$ and period $b=e/c$, where c is the highest common factor (e, t) and m is the least positive integer for which b divides q^m-1.*

THEOREM 3. *The rational canonical form of C^t is the direct sum of r companion matrices D_t of $g_t(x)$. Moreover D_t is in the first m rows and columns of the matrix HC^tH^{-1}, where the successive first m rows of H are the top rows of the matrices $I, C^t, C^{2t}, \cdots, C^{mt-t}$, and the last $n-m$ rows of H are chosen from F_q in any way which makes H nonsingular.*

Next we define a sequence of integers $\{t_i\}$. We put $t_1 = 1$ and, when $t_1, t_2, \cdots, t_{i-1}$ have been chosen, we let t_i be the least positive integer such that $t_i \not\equiv t_j q^\beta \pmod e$ for $1 \leqq j < i$, $0 \leqq \beta < m = m(t_i)$. Clearly the sequence $\{t_i\}$ is finite with $1 \leqq t_i \leqq e$.

THEOREM 4. *Each irreducible polynomial whose period divides e appears exactly once in the sequence $\{g_{t_i}(x)\}$.*

THEOREM 5. *If $n>1$ and $f(x)$ is symmetric, that is if $f(x)=x^n f(x^{-1})$, then n is even and every irreducible polynomial of degree $d>1$ whose period divides e is symmetric.*

2. Proofs of the theorems. Let $\theta_1, \theta_2, \cdots, \theta_n$ be the roots of $f(x)$. Then

$$f_t(x^t) = \prod_{i=1}^{n} (x^t - \theta_i^t) = \prod_{i=1}^{n} (-1)^{t+1} \prod_{j=1}^{t} (\omega_j x - \theta_i) = (-1)^{n(t+1)} \prod_{j=1}^{t} f(\omega_j x),$$

which is the first theorem.

The roots θ_i of $f(x)$ all have order e over F_q, and so the roots θ_i^t of $f_t(x)$ all have order b over F_q. Hence any irreducible factor of $f_t(x)$ must have period b and degree m. Since the characteristic function of θ_i^t is the same for all the roots θ_i of $f(x)$ the second theorem follows.

The period of a matrix is the period of its minimal function, and the period of C^t is b. The polynomial $g_t(x)$ is irreducible, and so if $\mu \geqq 2$ the period of $g_t^\mu(x)$ is at least p times the period b of $g_t(x)$. Hence the canonical form of C^t cannot contain a companion matrix of $g_t^\mu(x)$ with $\mu \geqq 2$. Since $f_t(x) = [g_t(x)]^r$, this proves the first part of Theorem 3. The second part is an application of Theorem II ([**3**], p. 49). As we have just shown the canonical form of C^t is a direct sum of r matrices D_t. The characteristic function $g_t(x)$ of D_t is irreducible, and so every nonzero $n\times 1$ vector u has grade m with respect to C^t. The result now follows by taking for u the vector $(1, 0, 0, \cdots, 0)$.

To prove Theorem 4 we fix b and vary t. The polynomial $g_t(x)$ has period b if and only if $(e, t)=c$, and if $t_1, t_2, \cdots, t_a$ are the distinct integers t in $1\leqq t\leqq e$ with $(e, t)=c$ then $a=\phi(b)$, where ϕ denotes Euler's function. Also the number of irreducible polynomials of period b is $\phi(b)/m$. Hence if $m>1$ some of the polynomials $g_t(x)$ must be the same. Now $g_t(x)$ has exactly m roots. Also for $1\leqq i\leqq n$ the numbers θ_i^j, $1\leqq j\leqq e$ are all distinct. Hence at most m of the polynomials $g_{t_j}(x)$ can be the same. Thus we have shown that every irreducible polynomial of period b appears m times in the sequence $\{g_t(x)\}$, $1\leqq t\leqq e$.

If $t\equiv sq^\alpha \pmod e$ then

$$f_s(x^{q^\alpha}) = \{f_s(x)\}^{q^\alpha} = \prod_{i=1}^{n} (x - \theta_i^s)^{q^\alpha} = \prod_{i=1}^{n} (x^{q^\alpha} - \theta_i^t) = f_t(x^{q^\alpha}),$$

and so $g_s(x)=g_t(x)$. This result also follows from the facts that $C^e=I$ and $C^{tq^\alpha}=C^t$. (Albert asserts ([**1**], p. 143, line 2) that if $f(x)$ is primitive and $t=sp^\alpha$, where s is prime to p, then $f_t(x)=f_s(x)$. On the contrary, if $p=2$, $q=4$, and $e=15$ then the 4 irreducible polynomials of period 15 appear in the sequence $\{g_t(x)\}$ with $g_1(x)=g_4(x)$, $g_2(x)=g_8(x)$, $g_7(x)=g_{13}(x)$, $g_{11}(x)=g_{14}(x)$, but $g_2(x)\neq g_1(x)$ and $g_{14}(x)\neq g_7(x)$.) (*Editorial note*: Dr. Albert says that the sentence in the middle of line 2 should read, "But then $f_t(x)=[f_s(x)]S^\alpha$ has known coefficients c_iS^α when the coefficients of $f_s(x)$ are known.")

Now if $t_j=sq^\alpha$ and $sq^\beta\equiv sq^\gamma \pmod e$ then since $(e, q)=1$ we have $(e, s)=(e, t_j)=c$; $q^\beta\equiv q^\gamma \pmod b$ and $\beta\equiv\gamma \pmod m$. Hence the residues modulo e of the num-

bers $s, sq, \cdots, sq^{m-1}$ yield all the numbers t in $1 \leqq t \leqq e$ with $g_t(x) = g_{t_j}(x)$, and Theorem 4 follows. Notice that $f_{e-t}(x) = x^n f_t(x^{-1})$ for $1 \leqq t < e$, because $\theta_i^t = (\theta_i^{e-t})^{-1}$ for $1 \leqq i \leqq n$.

Finally, Theorem 5 is a corollary to Theorem 4, because if $n > 1$ and $f(x)$ is symmetric then the roots of $f(x)$ and $g_t(x)$ occur in pairs like θ_i, θ_i^{-1} and θ_i^t, θ_i^{-t}.

References

1. A. A. Albert, Fundamental concepts of higher algebra, University of Chicago Press, 1956.

2. B. Elspas, The theory of autonomous linear sequential networks, IRE Trans. on Circuit theory, vol. C.T.6, No. 1, pp. 45–60.

3. H. W. Turnbull and A. C. Aitken, An introduction to the theory of canonical matrices Blackie, London, 1952.

PROBLEMS ABOUT PROBLEMS

I. D. MACDONALD, University of Newcastle, N.S.W., Australia

The desire to construct elementary numerical problems sometimes gives rise to what may be called second-level problems which have a completely different character. For instance Sholander [1], wishing to find three mutually orthogonal 3-vectors α, β, γ with integral components and the same length D, other than the stock text-book trio

$$\alpha = [1, 2, 2], \qquad \beta = [2, 1, -2], \qquad \gamma = [2, -2, 1],$$

was led to the Diophantine equations

$$\alpha_3^2 + \beta_3^2 + \gamma_3^2 = D^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2. \tag{1}$$

An appeal to Dickson [3] produced the vectors

$$\begin{aligned} \alpha &= [a^2 + b^2 - c^2 - d^2,\ 2(ac + bd),\ 2(ad - bc)], \\ \beta &= [2(ac - bd),\ b^2 + c^2 - a^2 - d^2,\ 2(ab + cd)], \\ \gamma &= [2(ad + bc),\ 2(cd - ab),\ b^2 + d^2 - a^2 - c^2], \end{aligned}$$

which Reiersöl [2] showed by matrix methods to be essentially unique.

The point in which we are interested here is that this procedure yields a window through which some fragments of advanced mathematics in the theory of numbers may be seen. It may therefore be worthwhile to indicate a few more such problems with their second-level problems in the hope of strengthening this bond between elementary and advanced topics.

Let a conic in some coordinate system be represented by the equation

$$ax^2 + 2hxy + by^2 = k, \tag{2}$$

which is to be reduced to a standard form by rotating the axes through some angle θ. If

$$x = x_1 \cos\theta - y_1 \sin\theta,$$
$$y = x_1 \sin\theta + y_1 \cos\theta,$$

as usual, substitution gives the equation $Ax_1^2+2Hx_1y_1+By_1^2=k$, where

$$A = \tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b)\cos 2\theta + h\sin 2\theta,$$
$$H = h\cos 2\theta - \tfrac{1}{2}(a-b)\sin 2\theta,$$
$$B = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)\cos 2\theta - h\sin 2\theta.$$

When θ is chosen so that $H=0$ the equation becomes

$$(a+b-\sqrt{(a-b)^2+(2h)^2})\,x_1^2 + (a+b+\sqrt{(a-b)^2+(2h)^2})y_1^2 = 2k.$$

Now in numerical problems free from troublesome surds we essentially want a, h, b, k and c to be integers, where

$$(a-b)^2 + (2h)^2 = c^2.$$

The integral solutions to this equation are well known (cf. [3], p. 165). If we take

$$a - b = (\alpha^2-\beta^2)\gamma, \qquad 2h = 2\alpha\beta\gamma, \qquad c = (\alpha^2+\beta^2)\gamma,$$

with α, β, γ integers, and put $\delta=a-\alpha^2\gamma$, (2) becomes

$$\delta(x^2+y^2) + \gamma(\alpha x+\beta y)^2 = k. \tag{3}$$

The other possibility, namely $a-b=2\alpha\beta\gamma$, $2h=(\alpha^2-\beta^2)\gamma$, $c=(\alpha^2+\beta^2)\gamma$ with $\delta=a-\frac{1}{2}(\alpha+\beta)^2\gamma$, produces essentially the same equation as (3):

$$\delta(x^2+y^2) + \tfrac{1}{2}\gamma((\alpha+\beta)x + (\alpha-\beta)y)^2 = k.$$

Thus if we want numerical examples of (1) to reduce by rotation we will do well to give numerical values to α, β, γ, δ in (3) and so avoid problems with awkward surds.

Next, suppose that we are interested in the bisectors of the angles between the straight lines

$$ax + by + p = 0, \qquad cx + dy + q = 0.$$

As the text-books prove, the equations of the bisectors are

$$\frac{ax+by+p}{\sqrt{(a^2+b^2)}} = \pm\frac{cx+dy+q}{\sqrt{(c^2+d^2)}}.$$

We shall have rational coefficients here provided a, b, p, c, d, q were rational and in addition $\sqrt{(a^2+b^2)}/\sqrt{(c^2+d^2)}$ is rational so that surds cancel out. Thus we

are led to the Diophantine equation

$$a^2 + b^2 = c^2 + d^2. \tag{4}$$

Now there is a formula which gives some solutions of this, namely

$$(\alpha\beta + \gamma\delta)^2 + (\alpha\gamma - \beta\delta)^2 = (\alpha\beta - \gamma\delta)^2 + (\alpha\gamma + \beta\delta)^2,$$

and according to Dickson (p. 254) the general solution is similar:

$$2a = \alpha\beta + \gamma\delta, \qquad 2b = \alpha\gamma - \beta\delta, \qquad 2c = \alpha\beta - \gamma\delta, \qquad 2d = \alpha\gamma + \beta\delta,$$

where the integers α, β, γ, δ are such that α and δ are even, or β and γ are even, or all four are odd.

The same equation (4) arises from problems in a quite different part of geometry. It may be proved, as it is in all the text-books, that the two tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

from the point (c, d) are perpendicular if and only if (4) holds. On the other hand tangents to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

give rise to

$$a^2 = b^2 + c^2 + d^2, \tag{5}$$

an equation similar to that treated by Sholander.

The following problem occurs on p. 131 of [4]: "A rectangular sheet of metal has four equal square portions of metal removed at the corners, and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is a maximum, the depth will be

$$\tfrac{1}{6}\{a + b - \sqrt{(a^2 - ab + b^2)}\},$$

where a, b are the sides of the original rectangle." Thus to make neat numerical examples out of this we want solutions to the Diophantine equation

$$a^2 - ab + b^2 = c^2. \tag{6}$$

There is a geometrical problem which presents a very similar equation. If (2) with $k=0$ represents a pair of lines and if t is the tangent of an angle between them, then it can be proved that

$$t = \frac{2\sqrt{(h^2 - ab)}}{a + b},$$

from which simple algebra gives

$$t^2a^2 + (2t^2 + 4)ab + t^2b^2 = (2h)^2. \tag{7}$$

The most interesting case of (7) has $t=1$. This and (6) and indeed the equation $a^2+b^2=c^2$ are all contained in the general case

$$u^2a^2 + vab + u^2b^2 = c^2, \tag{8}$$

in which u and v are given integers. We show how to solve this by a method of Dickson (p. 406). We have

$$a(u^2a + vb) = (c + ub)(c - ub),$$

so if there is a solution there are coprime integers λ, μ for which

$$\lambda a = \mu(c + ub), \qquad \mu(u^2a + vb) = \lambda(c - ub).$$

Elimination of c gives

$$(\lambda^2 - u^2\mu^2)a = (2u\lambda\mu + v\mu^2)b. \tag{9}$$

Now suppose that λ and μ are any coprime integers with $\lambda^2-u^2\mu^2\neq 0$ and $2u\lambda\mu+v\mu^2\neq 0$—we ignore the solutions arising when one of these numbers is 0. Then we can find coprime integers α and β for which

$$(\lambda^2 - u^2\mu^2)\alpha = (2u\lambda\mu + v\mu^2)\beta,$$

and we take $a=\alpha\gamma$, $b=\beta\gamma$, where γ is arbitrary. Substitution for c in $\lambda a=\mu(c+ub)$ gives

$$c = \frac{\lambda}{\mu}a - ub,$$

and we note that μ divides a. It may be verified that we now have a solution of (8).

Finally we point out a problem stated by Dickson (p. 502) to be unsolved. This is to find a parallelepiped with integral sides, face diagonals and long diagonal; in number-theoretic terms, to find integers a, b, c for which each of the following is a square:

$$b^2 + c^2,\ c^2 + a^2,\ a^2 + b^2,\ a^2 + b^2 + c^2.$$

References

1. Marlow Sholander, Rational Orthogonal Matrices, this MONTHLY, 68 (1961) 350.
2. Olav Reiersöl, Rational Orthogonal Matrices, this MONTHLY, 70 (1963) 63–65.
3. L. E. Dickson, History of the Theory of Numbers, vol. II, Washington, 1919.
4. Sir H. Lamb, An Elementary Course of Infinitesimal Calculus, Cambridge, 1919.
5. A. Sutcliffe, Complete solution of the ladder problem in integers, Mathem. Gazette, 47 (1963) 133–36.
6. Problem E 1633, this MONTHLY, 70 (1963) 1005 and 71 (1964) 795–796.

THE LIPSCHITZ CONDITION AND UNIQUENESS

GEORG AUMANN, Technische Hochschule, Munich, Germany

The standard interpretation of the Lipschitz condition for a differential equation $y'=f(x, y)$, namely, that the difference quotient of f with respect to y is bounded, does not directly show up the connection with the uniqueness of the solution to the initial value problem. But there is a simple consequence of the Lipschitz condition which reveals its influence on the uniqueness.

Let us assume that f is defined and continuous in the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$. If $a=x_0<x_1< \cdots <x_n=b$ is a subdivision of the x-interval and $x_{n+1}>b$, then by the *family of approximate solutions* of the differential equation $y'=f(x, y)$ *belonging to this subdivision* we mean the family of curves $y=y_\eta(x)$, $c \leqq \eta \leqq d$, defined by

$$y_\eta(x) = y_i + (x - x_i)f(x_i, y_i) \quad \text{for } x_i < x \leqq x_{i+1}, \qquad i = 0, \cdots, n,$$

with

$$y_0 = \eta, \; y_i = y_{i-1} + (x_i - x_{i-1})f(x_{i-1}, y_{i-1}), \qquad i = 1, \cdots n$$

and $y_\eta(x)$ defined in $x_0 \leqq x \leqq x_k$, k being the smallest natural number such that $(x_k, y_\eta(x_i))$ is outside the rectangle.

PROPOSITION 1. *If f satisfies an L-condition*

$$|f(x, y) - f(x, \bar{y})| \leqq M|y - \bar{y}|$$

for all x, y, $\bar{y}$ with a positive constant M, and if the subdivision $x_0, \cdots, x_{n+1}$ is so fine that $x_i-x_{i-1}<1/M$ for all i, then the family of approximate solutions belonging to this subdivision contains no intersections, thus covering the covered region simply.

It is illustrative to check the situation in the example $f=-y$ (using the subtangent property of the exponential curve $y=e^{-x}$ with an L-condition present) and in the examples $f=y^{1/3}$ and $f=-y^{1/3}$ without L-condition.

We get a converse of Proposition 1 by using not only the approximate solutions defined above which for distinctiveness we now call "progressive," but also the "regressive" approximate solutions $y=y_\eta^*(x)$, by introducing a number $x_{-1}<x_0$ and defining

$$y_\eta^*(x) = y_i + (x - x_i)f(x_i, y_i) \quad \text{for } x_{i-1} \leqq x < x_i, \qquad i = n, \cdots, 1, 0,$$

with $y_n=\eta$, $y_{i-1}=y_i+(x_{i-1}-x_i)f(x_i, y_i)$, $i=n, \cdots, 0$, and $c \leqq \eta \leqq d$, stopping the construction in a similar way as for the progressive case.

PROPOSITION 2. *If there is a positive constant r such that for every subdivision with steps $x_i-x_{i-1}<r$ the family of progressive approximate solutions as well as the family of regressive approximate solutions of the differential equation form simple coverings, then f satisfies an L-condition with a constant $M=1/r$ in any region which is covered by each of all these families.*

The proof of the propositions is based on the following property, verified by means of elementary analytic geometry: if $M>0$, then $|f(x_0, y_1)-f(x_0, y_2)| \leqq M|y_1-y_2|$ if and only if the lines L_j through the points (x_0, y_j) with slope $m_j=f(x_0, y_j)$, $j=1, 2$, do not intersect for x satisfying $|x-x_0|<1/M$.

REMARK. The approximate solutions considered in Proposition 1 define a differential equation "to the right"

$$y'_+ = \bar{f}(x, y), \tag{*}$$

arbitrarily close to the original equation $y'=f(x, y)$, and possessing the uniqueness property if the subdivision is fine enough.

Though $\bar{f}$ is discontinuous, equation (*) has proved to be very useful for investigations of continuity properties of the solutions of $y'=f(x, y)$.

ON SUBADDITIVE AND SUPERADDITIVE FUNCTIONS

S. P. S. RATHORE, Lucknow University, India

A real-valued function f is said to be superadditive on an interval $I=(0, a)$ if it satisfies the inequality

$$f(x+y) \geqq f(x) + f(y) \tag{1}$$

whenever x, y and $x+y$ are in I. If the inequality (1) is reversed, $f(x)$ is called subadditive. In this paper we prove two theorems. The first theorem gives a sufficient condition for $f(x)$ to be superadditive. The second theorem establishes the monotonicity of the derivative of a subadditive function under certain conditions. In what follows, I_+ refers to the interval $(0, \infty)$ and I_0 refers to the interval $(-\infty, \infty)$.

LEMMA 1. *If $f'(x)$ exists on $I=(a, \infty)$, $a\geqq 0$, then $f(x)/x$ is decreasing on I if and only if $f'(x)<f(x)/x$ on I.*

Proof. If $f'(x)<f(x)/x$, then

$$\frac{xf'(x)-f(x)}{x} < 0.$$

Hence,

$$\frac{d}{dx}\left[\frac{f(x)}{x}\right] = \frac{xf'(x)-f(x)}{x^2} < 0.$$

The converse can be established by reversing the steps of the argument.

THEOREM 1. *If $f'(x)$ exists on $I=(a, \infty)$, $a\geqq 0$, then $f(x)$ is subadditive if $f'(x)<f(x)/x$ on I, and $f(x)$ is superadditive if the inequality is reversed.*

The theorem follows from Lemma 1 and the elementary result that $f(x)$ is subadditive if $f(x)/x$ is decreasing ([1], p. 239, Theorem 7.2.4).

Examples. 1. $f(x)=x^2$ defined on $(0, \infty)$. This is superadditive.

2. $f(x)=-x^3$ defined on $(0, \infty)$. This is subadditive.

THEOREM 2. *Let $f(x)$ be subadditive on I_0 and let $f'(x)$ exist on $I=(a, \infty)$, $a \geqq 0$. If $f(x)+f(-x) \leqq 0$ for $x \in I$, then $f'(x)$ is monotone nonincreasing on I.*

Proof. Let x_1, x_2 be any two points of I. Then, for $h>0$, there exists $\epsilon>0$ such that

$$f'(x_1+x_2)-\epsilon<\frac{f(x_1+x_2+h)-f(x_1+x_2)}{h},$$

where $\lim_{h\to 0}\epsilon=0$. Hence,

$$\begin{aligned} f'(x_1+x_2)-\epsilon &< \frac{1}{h}\left[f(x_1+x_2+x_1+h-x_1)-f(x_1+x_2)\right] \\ &\leqq \frac{1}{h}\left[f(x_1+x_2)+f(x_1+h)+f(-x_1)-f(x_1+x_2)\right] \\ &\leqq \frac{1}{h}\left[f(x_1+h)-f(x_1)\right]. \end{aligned}$$

Therefore, $f'(x_1+x_2) \leqq f'(x_1)$.

Reference

1. E. Hille and R. Phillips, Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ., vol. 31.

AN EXTENSION OF OLIVIER'S THEOREM

FRANTIŠEK ŠTĚPÁNEK, Caroline University, Prague, Czechoslovakia

It is easily established that for any convergent series $\sum_{n=1}^{\infty}a_n$ of real numbers,

$$\liminf na_n \leqq 0 \leqq \limsup na_n.$$

This means that to prove $\lim na_n=0$, it is sufficient to show that the sequence $\langle na_n\rangle$ is convergent. We make use of this result to obtain a theorem which is an extension of a very old result of L. Olivier [1]. Our proof makes no appeal to the Cauchy criterion for convergence. Olivier's theorem is obtained from ours by setting $\beta=0$.

THEOREM. *Let $\sum_{n=1}^{\infty}a_n$ be a convergent series of real numbers, and suppose that there exists a constant β such that one of the following conditions holds for every nonnegative integer n (we define $a_0=0$):*

(a) $na_n \geqq (n+\beta)a_{n-1}$ (b) $na_n \leqq (n+\beta)a_{n-1}$.

Then $\lim na_n=0$.

Proof. The two conditions are equivalent as can be seen most easily by multiplying either one of them by -1. Thus it is sufficient to prove the theorem in the case that (a) holds. We define the following symbols:

$$r_1 = s_1 = 0, \qquad r_n = na_n, \qquad s_n = -(1+\beta)\sum_{i=1}^{n-1} a_i, \qquad \text{for } n \geqq 2,$$

$h_n = r_n + s_n$, for all n.

Condition (a) may then be written in the form

$$na_n \geqq (n-1)a_{n-1} + (1+\beta)a_{n-1} \qquad \text{for } n \geqq 2,$$

and this reduces at once to $h_n \geqq h_{n-1}$. Thus the sequence $\langle h_n \rangle$ is nondecreasing. But, by our convergence assumption, the sequence $\langle s_n \rangle$ has a finite limit. Accordingly, since $r_n = h_n - s_n$ for each n, it follows at once that $\langle r_n \rangle$ is convergent. This completes the proof of the theorem.

The author has benefited from discussions with Professor J. Marík. He also acknowledges gratefully the referee's help for improving the exposition of this note.

References

1. L. Olivier, Remarques sur les séries infinies et leur convergence, Crelle's J. Reine u. Angew. Math., 2 (1827) 31–44.

2. Konrad Knopp, Theory and Application of Infinite Series, Stechert-Hafner, New York, 1948.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor. 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

AN APPROACH TO IMPROVING THE TEACHING OF ELEMENTARY SCHOOL MATHEMATICS

J. H. ZANT, Oklahoma State University

Is there anything new in the teaching of elementary school mathematics or arithmetic? There may be, in spite of all the volumes which have been written about it!

For a long time psychologists and measurement specialists have told us that objectives must be stated in terms of performance-based behavioral patterns. Has this been done for elementary school mathematics? Almost always the answer is, "No." We do find objectives listed in courses of study and teachers' manuals on "Content material," such as, "Understanding addition and subtraction with two-, three- and four-digit numerals, with and without regroup-

ing," "Learning what the number 7 means," and the like. Many textbooks say nothing, but plunge into a series of activities for the children, which are performanced-based, and by which they apparently expect the child to "learn what the number 7 means," or some other vaguely implied objective. They do want and expect a modification of behavior, but it is difficult to guess exactly what modification is expected.

Furthermore, the psychologists or learning theorists suggest a definite procedure for designing and creating blocks or units of learning material for elementary school mathematics, or for any other subject matter. This may be summarized as follows:

> Specify the objectives of the materials in terms of behaviors which can be measured.
>
> When this has been done, materials should provide for individual differences so that students of varying abilities can achieve the objectives.
>
> Materials should be tested and revised, if necessary, until data can be provided to insure that the objectives are attained with the intended audience.

It is important, of course, that the objectives of elementary school mathematics be considered very carefully, and stating them in terms of behaviors which can be measured is difficult and time consuming. This task will take the best efforts of all those interested in improving the teaching of elementary school mathematics.

Given a sound set of objectives, a program or set of materials for teaching any segment of mathematics is considered successful if the children's behavior is modified in terms of the stated objectives. It is considered unsuccessful if behavior is not so modified.

Is such a procedure for constructing teaching material acceptable to mathematicians, teachers and mathematics educators? It was considered acceptable to a widely representative group of 36 such people in a recent meeting [1]. Though the project which sponsored this conference was organized to study the use of new educational media in the improvement of the teaching of elementary school mathematics, it recognized that it must first establish objectives, a way of stating them and some procedure for developing the teaching material to be presented by some media.

The term "new educational media" is not specifically defined and limited in the National Defense Education Act of 1958. For the purposes of this project, however, it is considered as limited to media of communication, and to include television, motion pictures, single concept films, film strips, projectuals, programmed instruction, and the like.

The conference also considered the most effective use of various ways of presenting mathematics in the elementary school classroom. The group was unwilling to accept the use of a single medium or mode of presentation continuously, even through a single class period, as the most effective teaching procedure. Of course, good teachers came to the same conclusion long ago. In

the use of new media then, it makes sense to use whatever is most effective for different things the teacher must do. Among the things used will be his own presentations at appropriate times.

It is expected, therefore, that if teaching material tested for effectiveness of student learning is available, the classroom teacher can make intelligent choices as to what to use during the time a student or a class is studying a particular lesson or topic. The things the teacher uses might be television, films (long or short ones), programmed instruction, film strips, overhead projectuals, and the like. This may be called an *instructional system*, and it is sometimes referred to as the *systems approach* to teaching. The systems approach to teaching allows the flexibility which every teacher needs to provide for the specific problems of his own class. We refer here to such things as the background of the students, individual differences, their abilities and capabilities, needs of the local community and the like.

A systems approach also implies that there will be available a number of elements and media dealing with various parts of the topic or lesson and so designed and tested that the teacher can make use of those elements which will produce learning most effectively. Such material does not exist to any great extent at present, and its development must be based on much wisdom, experimentation and research. This project agrees to this scheme as a working procedure, however, and accepts the responsibility of exploring specific ways to accomplish the task.

Programmed instruction itself is one element which has proved useful in producing learning in the classroom. It does represent a new concept of commitment and responsibility for learning. It is based on a careful statement of the objectives desired, and if the pupil does not learn from the program presented, it is not considered that this is his fault, but that the fault lies with the program or the programmer. Hence, the program is revised or it is rearranged; it may even be discarded and a new one written.

This does not mean that all programmed instruction which now exists is good or even that this will be so in ten years. The technology exists, however, and good programs can be produced if objectives are stated clearly in behavioral terms, and if the needed development, experimentation and research are carefully planned and executed.

The careful research, development and experimentation that have been done in recent years have produced a technology of programmed instruction which provides a model or a basis for developing effective learning programs through other media as well. It has been said that "the flurry of activity during the first decade of teaching machines and programmed learning has obscured the central issue: the impact of the experimental analysis of behavior on education."

Since programmed instruction is based on experimental analysis, then what we may call *programmed instruction based* media become a way of developing effective learning programs through any media.

In this manner, good teaching programs using media or other teaching materials can be produced. This project also accepts the responsibility of making this procedure explicit. It will take the best efforts of mathematicians, teachers, learning and measurement specialists, and media specialists, working as a team, to develop these materials, but we are convinced that it can be done.

Reference

1. First Positional Conference, Chicago, January 14, 15 and 16, 1965. *Project for the Improved Use of Newer Educational Media in Elementary School Mathematics*, James H. Zant, Project Director. Sponsored by the National Council of Teachers of Mathematics and supported by a grant from the Office of Education of the U. S. Department of Health, Education and Welfare.

AN UNTAPPED SOURCE FOR MATHEMATICS TEACHERS?

NORMAN LOCKSLEY, Alma College

When the need for college teachers in mathematics is projected, say, to 1970, all reports agree on a great increase over current demand. One forecast predicted a rise from 11,100 mathematics teachers in universities, colleges and junior colleges in 1961 to a total *requirement* for 24,950 by 1970. Additionally, the production of mathematics Ph.D.'s is expected to be 694 for the year 1969–70; short some 756 of the total requirement for that year alone [1]. Who will fill the gap?

Obviously, the non-Ph. D. The purpose of this paper is to describe one source of such mathematics teachers. As G. S. Young points out, "Is it not time for the mathematical community to face up to the fact that for a long time most undergraduate teaching will be done by non-Ph.D.'s, and to begin a study of means of identifying competence among such persons?" [2].

Recognizing this need seven years ago, Duke University began a special mathematics teaching program in 1958, designed for retired officers of the armed forces. The project has been supported since that date by the National Science Foundation. Since I completed the course in 1964, my views may be helpful to others as an assessment from within by one who has tried the program, and then put the results to work.

The object is to prepare selected retired officers of the armed services to teach mathematics in high school, or mathematics through calculus in college. By "selected" I mean selected from among those who apply. The basic entrance requirement is a bachelor's degree, with mathematics through calculus. Each participant is expected to earn the degree of Master of Arts in Teaching (Mathematics).

As a member of the 1963–64 group, I was one of 14 who began in July, 1963. Thirteen of us finished exactly a year later, July 15, 1964. The summer program gave us a quick review, a heavy dose of fundamentals in analysis, a little set theory, some finite mathematics, and a lot of calculus, emphasizing theorems. These topics were covered in two courses for a total of nine credit hours: Mathematics for Teachers, and Fundamental Concepts in Algebra, Analysis and Geometry.

With the fall semester, we were each assigned a freshman section in either college algebra or calculus, scheduled for three class hours a week. By means of a weekly class with the Director of Freshman Studies, we reinforced the topics to be discussed before our own classes met. There was no difference in the material covered or conduct of the class from those taught by other instructors. We used a mimeographed page-and-problem outline in order to be sure that all freshmen received about the same emphasis on the topics in our text. Occasional visits were made to our classes by the Director and other professors, and a common examination was given to all freshmen in the non-honors classes. The results of these examinations were quite comparable, indicating that our guidance and supervision were effective.

In the fall term we also studied Intermediate Analysis, Theory of Equations, Teaching of Mathematics and Secondary Education Principles. Then, in the spring, those of us who had started with an algebra class moved into Calculus I with their freshmen; the rest of us continued into Calculus II for our teaching. Our own courses were Differential Equations, Intermediate Analysis and Educational Psychology. I substituted Digital Computers for the latter offering.

All of our courses were staffed with top professors who gave us the benefit of their experience and study. For example, the Chairman of the Mathematics Department and originator of the program, Professor J. J. Gergen, handled all the initial summer teaching himself. In general, most of the mathematics courses were taught in "our" classroom and to the retired officer group. There were some exceptions, but not a great many.

Our part-time instructing ended in May, while we continued our own studies into the first summer school term. A split was made between those planning to teach in secondary schools and those headed for college appointments. The secondary school contingent took two courses in education, while the others studied a course in solid analytic geometry and another in modern algebra.

The requirements for the degree of M.A.T. (Mathematics) may be summarized:

	Secondary School	College
Mathematics	21 hours	27 hours
Education	15 hours	9 hours
Freshman Teaching	3 hours	3 hours

Although individual programs vary slightly from this pattern, the overall design of the course has remained relatively fixed. The current group, for example, is taking six hours of modern algebra and may elect two courses from these four: Digital Computer, Statistics, Elementary Differential Equations, and Solid Analytic Geometry.

What are the results?

Seventy-three retired officers have completed the program since 1959. Of these, 63 were teaching during the 1964–65 academic year—47 in colleges and universities and 16 in secondary schools and technical institutes. Another 13 retired officers are currently enrolled in the 1964–65 course.

From all reports, the results have been very satisfactory. Although one year, however crammed, is a relatively short time to prepare for mathematics teaching, most of the participants have had some previous preparation along the way. This ranged, for my own group, from full-time college courses and secondary school mathematics teaching to night classes and correspondence study. Afterwards, many graduates continue the road of self-study, summer institutes and further university work.

All of the graduates with whom I have discussed the program agree on its two most valuable aspects. These are felt to be first, the teaching experience and, next, the grouping of the retired officers into a single class.

I might point out that the term "retirement" is somewhat broader in military usage than its usual connotation—of a man who has completed long and honorable service and, at about 70 or so, lets go of the reins to enjoy his hobbies. In the armed forces, one *may* retire after 20 years of service and *must* retire after various periods—20 to 35 years—depending upon the rank attained. Thus, when I speak of the retired officers group, I refer to a group in their 40's and early 50's, mostly, with a median age of about 50.

With a decade or two of service expected, what are the advantages in preparing this kind of a group for college teaching?

1. Through careful selection, one can be assured of above-average intelligence, aptitude and stamina.

2. Since these men are quite mature, their interest in mathematics teaching is likely to be quite stable. The fact that fewer options are open to older people would also add to this stability! 86% of these men going into teaching is a considerably higher proportion than the 40% of mathematics Ph.D.'s from a much-younger age group [3]. Even 64% into college teaching represents a better "take."

3. Because of their varied experience, these men are able to contribute in other ways. Teaching itself is not new to them; most officers have done a lot of it. Many have attended graduate school, too, so that all their other experience may be quite valuable to smaller colleges where contact between students and faculty plays an important rôle.

There is only one major disadvantage—age. Is two decades or more of productive work worth the expense, in time and money and skill, of their education? I don't think there is any easy answer to this question—it would depend upon an individual assessment for each person. There certainly seems to be a need for competent non-Ph.D. mathematics teachers. Even if programs for older people don't produce a Weierstrass, we are assured of a steady demand for "journeyman" mathematicians. These men can contribute, therefore, if they provide good teaching at the lower levels while leaving upper level and graduate courses for the smaller number of Ph.D.'s.

It is a neat problem in operations research. How long a "half-life" is required to equate the value of greater commitment to teaching of an older group in

comparison with a smaller percentage of teachers obtained from a younger group? Since there are too many built-in uncertainties, we might leave it for the time being.

We are certain that the decade of the 60's represents a peak in the availability of officers retiring from the armed forces. This results, of course, from the fact that so many officers now in the forces entered around the World War II period. At least several thousand of these have the interest, ability and background to become effective mathematics teachers. Their availability coincides with a decrease in the number of trained mathematics teachers available for expanding classes. Yet, it appears that Duke University offers the only operating program directed specifically at this particular group of prospective instructors.

The predicted and actual shortage of mathematics teachers is so great that any *qualified* source should be investigated. In view of the comparatively early retirement age, high level of motivation and results achieved to date, the armed forces would seem to offer a promising pool for mathematical manpower.

References

1. Conference Board of Mathematical Sciences, Report of a Conference on Manpower Problems in Training Mathematicians, July 1963.

2. G. S. Young, the Ph.D. Class of 1951, this MONTHLY, 71 (1964) 787–790.

3. Committee on the Undergraduate Program in Mathematics, Report on Production of Mathematics Ph.D.'s in United States, April 1964, revised.

COMMENTS ON MASANI'S "INDIAN MATHEMATICAL MISEDUCATION"

M. M. RAO, Carnegie Institute of Technology

1. In a recent issue of the MONTHLY, an interesting article on the present and recent mathematical (mis-) education in the Indian universities has been published by P. Masani, [3]. The treatment focuses on various defects existing in the system and it also contains some generalizations together with some recommendations. Here I shall point out a few other (related) problems and correct some conclusions of Mr. Masani's article on the subject, lest some American schools may think that it gives the whole story.

As in many other countries, the Indian educational system is complex and does not admit of any quick or easy solutions. It has several of the defects described in [3]. It should also be stated that among the educators and administrators who oppose the changes and who frustrate the younger generations (discussed at length in [3]), perhaps a majority of them are the "foreign-returned" whose research talents appear to produce little more than their theses. Therefore, the (common) teachers alone should not be blamed.

2. Now I shall comment briefly on other aspects of the problem. To understand the workability and shortcomings of the system it would be desirable to make, from time to time, a statistical (or objective) evaluation of the perfor-

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (other than proposers') should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to A. E. Livingston, Dept. of Math., University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before October 31, 1965.

E 1795. *Proposed by N. D. Kazarinoff, University of Michigan*

Let $ABCDEF$ be a convex hexagon such that the perimeters of the triangles ABF, BCD, DEF, and BDF are the same. Show that the hexagon must be a triangle, that is, it must have three 180° angles. Compare 4964, [1962, 672].

E 1796. *Proposed by Louis Comtet, Boulogne, France*

Show that, as $n \to \infty$, $s(n) \equiv 1^n + 2^n + 3^n + \cdots + n^n$ is asymptotic to $en^n/(e-1)$.

E 1797. *Proposed by W. J. Blundon, Memorial University of Newfoundland*

Find all solutions in integers of the equation

$$y^2 + y = x^4 + x^3 + x^2 + x.$$

E 1798. *Proposed by Pete Said, Fresno State College, California*

A chain has n links and each link weighs one unit. p cuts are to be made in the chain in such a manner that the resulting subchains, together with the p cut links and a balance scale, can be used to weigh every multiple of the unit weight from 1 to n. Find p in terms of n.

E 1799. *Proposed by Joseph Langr, Prague, Czechoslovakia*

On the sides BC, CA, AB of the triangle $ABC \equiv T$ are chosen points A_1, A_2; B_1, B_2; C_1, C_2 in such a way that $AC_1 = AB_2 = BC$, $BA_1 = BC_2 = CA$, $CB_1 = CA_2 = AB$. (1) Show that the orthocenters H_1, H_2, H_3, H of the triangles AB_2C_1, BC_2A_1, CA_2B_1 and T lie on the Euler line of the triangle formed by the lines B_2C_1, C_2A_1, A_2B_1. (This Euler line is parallel to the line joining the incenter and circumcenter of T.) (2) Show also that $HH_1 : HH_2 : HH_3 = \cos A : \cos B : \cos C$.

E 1800. *Proposed by Joseph Langr, Prague, Czechoslovakia*

Given a circle (O), a point C on it, and two points A, B not on the circle. Find a point S on circle (O) such that SC bisects the angle ASB.

E 1801. *Proposed by Ralph Greenberg, University of Pennsylvania*

Prove that

$$\sum_{k=1}^{\infty} \zeta(2k)/2^{2k}(2k^2+k) = \log(\pi/e),$$

where $\zeta(n)=\sum_{j=1}^{\infty} j^{-n}$, the Riemann zeta function.

E 1802. *Proposed by Dov Avishalom, Tel-Aviv, Israel*

For functions of class C_n prove that

$$f^{(n)}(a) = \lim_{h\to 0}\left\{\frac{1}{h^n}\sum_{k=0}^{n}\left[(-1)^{n-k}\binom{n}{k}f(a+kh)\right]\right\}.$$

E 1803. *Proposed by F. P. Callahan, Blue Bell, Penna.*

Find the most general form of $f(x)$, defined for all real x, such that

$$\frac{f(v)-f(u)}{v-u} = f'(pu+qv)$$

holds for all u, v; where p, q are positive with $p+q=1$.

E 1804. *Proposed by Carmen Artino, John Carroll University, Cleveland, Ohio*

Let $\mathcal{V}$ be the vector space (over the real field) of all polynomials of degree less than n. Let $\mathbf{D}$ denote the differentiation operator on $\mathcal{V}$ and $\mathbf{I}$ the identity transform on $\mathcal{V}$; i.e., $f(x)\in\mathcal{V}$, $f(x)\mathbf{D}=f'(x)$ and $f(x)\mathbf{I}=f(x)$. Let $\mathbf{T}$ be a linear transformation on $\mathcal{V}$ given by $f(x)\mathbf{T}=f(x+1)$. Show that

$$\mathbf{T} = \mathbf{I} + \frac{\mathbf{D}}{1!} + \frac{\mathbf{D}^2}{2!} + \cdots + \frac{\mathbf{D}^{n-1}}{(n-1)!}.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

E 1663 [1964, 204]. *Proposed by H. D. Ruderman, Hunter College High School*

What is the maximum number of regions into which n spheres can partition space?

Comment by C. Stanley Ogilvy, Hamilton College, Clinton, New York. I believe it is possible that the solution to problem E-1663 submitted by J. D. E. Konhauser and published as II in this MONTHLY 72 (1965), p. 79, may have been misinterpreted by the editor.

A careful reading indicates to me that Mr. Konhauser intends to say that inequality for P_n cannot be sharpened, and that the maximum number of regions is therefore in fact $n(n^2-3n+8)/3$. If this is so, the word *Partial* could be deleted from the first line of page 79, loc. cit.

Incidentally the above solution agrees with that derived in a somewhat different fashion on pp. 105–6, problem 45b, *Challenging Mathematical Problems with Elementary Solutions*, Yaglom and Yaglom, Holden-Day, Inc., 1964.

Editor's Comment. Professor Ogilvy's phrasing of the first paragraph above is too kind! To set the record straight, the Editor erred in his refereeing of Mr. Konhauser's submitted solution.

F. G. Schmitt also calls attention to the Editor's mistake here.

A Triangle Inscribed in a Parabola

E 1701 [1964, 679]. *Proposed by R. F. Jackson, University of Toledo*

Prove that for any three points on a parabola with vertical axis,

$$m_1 = m_{12} + m_{13} - m_{23},$$

where m_1 is the slope of the tangent at the first point and m_{ij} is the slope of the chord through the corresponding pair of points. (This is an occasionally useful formula for numerical differentiation with irregularly spaced points.)

Solution by Sister M. Stephanie, Georgian Court College, Lakewood, N. J. A parabola with vertical axis has an equation $(x-h)^2=4a(y-k)$ with $a\neq 0$. Denoting the given points by $P_i(x_i, y_i)$, we see that $m_1=(x_1-h)/(2a)$ and $m_{ij}=(x_i+x_j-2h)/(4a)$, and the required relationship follows immediately.

Also solved by Jack Abad, H. D. Abramson, A. N. Aheart, R. W. Aldrich, Brother U. Alfred, D. P. Ambrose, J. E. Archer, Winifred Asprey, Raymond Balbes, Leon Bankoff, Merrill Barnebey, Sister Marion Beiter, E. D. Bender, Len Bertain, M. T. L. Bizley, Walter Bluger, W. R. Boland, A. J. Bridgman, J. A. Burslem, Jim Campbell, Leonard Caners, R. E. Chandler, M. M. Chawla, P. L. Chessin, Allan Chuck and Peter Goldstein (jointly), D. I. A. Cohen, Jack Dix, O. O. Dixon and R. H. Fletcher (jointly), C. L. Dotton, Neil Driscoll, Ragnar Dybvik, E. S. Eby, F. M. Eccles, J. H. Edmonston, H. K. Fallin, Jr., M. S. Fineman, David Forthoffer, H. T. Freitag, Richard Fritsche, J. A. Fuchs, Michael Goldberg, K. K. Gorowara, Nicholas Grant, R. M. Grassl, S. H. Greene, George Gruber, B. Gupta, D. M. Hancasky, J. R. Hanna, Ned Harrell, C. T. Haskell, H. A. Heckart, M. H. Hayamizu, Stephen Hoffman, J. E. Homer, Jr., R. A. Jacobson, P. W. M. John, Ken Johnson, M. Jones, J. P. Jordan, Erwin Just and Norman Schaumberger (jointly), Roman Kaluzniacki, B. W. King, B. G. Klein, E. F. Knapp, Kenneth Kramer and Steven Minsker (jointly), E. S. Langford, J. C. Lazzara, Harry Lewis, Murray Lieb, Graham Links, W. R. McEwen, R. E. Maas, D. C. B. Marsh, Gus Mavrigian (two solutions), M. R. Meck, R. J. Miller, E. J. Miranda, David Moe, T. M. Morrisette, H. L. Mourer, M. G. Murdeshwar, S. A. Naimpally, Amos Nannini, W. I. Nissen, Jr., T. L. O'Brien, Margaret Olmsted, Robert Patenaude, P. R. Pennock, Wayne Pfeiffer, Stanton Philipp, G. L. N. Rao, L. A. Ringenberg, Charles Rockland, V. K. Rohatgi, Evelyn B. Rosenthal, Alan Saleski, Chris Samelson, Horvath Sandor, M. S. R. K. Sastry, P. A. Scheinok, Camilo Schmidt, G. R. Schubert, Robin Sibson, Jr., R. P. Soni, Sidney Spital, Junior Stein, D. V. Strimling, J. S. Sullivan, M. N. S. Swamy, P. D. Thomas, Lee Thoresen, Michael Tiller, Roseanna Torretto, Andy Vince, Daniel Warner, William Wernick, John Wessner,

Brother T. C. Wesselkamper, Charles Wexler, D. A. Wood, Jeffrey Woolf, Hazel S. Wilson, K. P. Yanosko, K. L. Yocom, Brother L. F. Zirkel, and the proposer.

Just and Schaumberger observe that if $P_1, P_2, \cdots, P_n$ are distinct points on a parabola and $P_{n+1}=P_1$, then $\sum_{k=1}^{n}(-1)^{k+1}$ slope P_kP_{k+1} is the slope of the tangent to the parabola at P_1 if n is odd and is zero if n is even (provided of course that none of the lines is vertical). As a consequence of this result, if m_k is the slope of the tangent to the parabola at P_k, then $\sum_{k=1}^{n} m_k=\sum_{k=1}^{n}$ slope P_kP_{k+1} (Wernick).

Wilson's Theorem Thinly Disguised

E 1702 [1964, 680]. *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that an integer p divides $\sum_{j=1}^{p-3} j(j!)$ if and only if p is a prime.

Solution by Kenneth Kramer, Columbia College, and Steven Minsker, Brooklyn College. Since $j(j!)=(j+1)!-j!$ for $j=1, 2, 3, \cdots$, we are required to show that p divides $(p-2)!-1$ if and only if p is a prime. But $(p-1)!=(p-1)(p-2)! \equiv -1 \equiv p-1 \pmod{p}$ if and only if p is a prime (by Wilson's Theorem), and the result follows upon noting that $(p, p-1)=1$.

Also solved by Jack Abad, Dean Adams and Michael Haley (jointly), A. N. Aheart, Shair Ahmad, Brother U. Alfred, Joseph Arkin, Winifred Asprey, Joe Bechely, Gyula Bereznai, M. T. L. Bizley, Bob Blum and Steve Imbeau (jointly), W. J. Blundon, J. A. Burslem, Leonard Carlitz, P. L. Chessin, John Christopher, Allan Chuck, Allan Chuck and Peter Goldstein (jointly), D. I. A. Cohen, M. J. De Leon, G. C. Dodds, Ragnar Dybvik, L. S. Evans, N. J. Fine, N. H. Fisher, David Forthoffer, Michael Fried, J. A. Fuchs, Michael Goldberg, Jerry Goodman, Nicholas Grant, R. M. Grassl, S. H. Greene, B. Gupta, James Hale, D. M. Hancasky, M. H. Hayamizu, D. R. Hayes, Sidney Hendricksen, Stephen Hoffman, R. F. Jackson, Hajna Janos, J. P. Jordan, Roman Kaluzniacki, T. M. Little, Andrzej Makowski, D. C. B. Marsh, M. R. Meck, Eric Mendelsohn, Steven Muchnick, A. A. Mullin, W. I. Nissen, Jr., Michael Olinick, C. B. A. Peck, Stanton Philipp, Donald Quiring, Simeon Reich, S. W. Reyner, Henry Ricardo, V. K. Rohatgi, D. P. Roselle, Alan Saleski, J. P. Samson, M. S. R. K. Sastry, Ralph Schreiber, C. P. Seguin, Robin Sibson, Jr., D. L. Silverman, Henry Snyder, J. S. Sullivan, M. B. Suryanarayana and M. V. Tamhankar (jointly), G. C. Thompson, A. M. Vaidya, Emanuel Vegh, Andy Vince, Brother J. C. Wesselkamper, Charles Wexler, K. L. Yocom, and the proposer.

The Plank Problem

E 1703 [1964, 680]. *Proposed by Gyárfás András, Eötvös University, Budapest, Hungary*

A circle of radius R is completely covered with strips of scotch tape of various widths. Prove that the sum of the widths of the strips is not smaller than $2R$.

Solution by Roy O. Davies, The University, Leicester, United Kingdom. Regard the circle as the cross-section by a plane of a sphere S of radius R, and regard each strip as the cross-section of a slab whose faces are perpendicular to the strip. By an elementary theorem (area cut out on sphere $\leqq$ area cut out on circumscribing cylinder), a slab of width w cuts out an area not exceeding $2\pi Rw$ on the surface of S; and together the slabs completely cover S. Therefore $\sum 2\pi Rw \geqq 4\pi R^2$, that is, $\sum w \geqq 2R$.

Also solved by H. D. Abramson and P. R. Pennock (jointly), James Archer, Roy O. Davies (second solution), Michael Goldberg, H. V. Kronk, and the proposer.

Davies and Goldberg call attention to H. Moese, Parametr, 2 (1932); A. Tarski, *Uwagi o stopniu równowazności wielokatów*, Parametr, 2 (1932) 310–314; Th. Bang, *On covering by parallel strips*, Mat. Tidskrift B (1950) 49–53; Th. Bang, *A solution of the "plank problem,"* Proc. Amer. Math. Soc., 2 (1951) 990–993; W. Fenchel, *On Th. Bang's solution of the plank problem*, Mat. Tidskrift B (1951) 49–51; and M. Bognar, *On Fenchel's solution of the plank problem*, Acta Math., Budapest, 12 (1961), 269–270.

Abramson and Pennock point out that "circle" in the problem must be interpreted as "disk," the result being false for (geometric) circles.

Number of Unit Squares Covering a Diagonal of a Rectangle

E 1704 [1964, 680]. *Proposed by Stephen Hoffman, Trinity College, and R. B. Killgrove, San Diego State College*

If R is an $m\times n$ rectangle formed from unit squares, find the number of squares containing a segment of one diagonal of R.

Solution by C. B. A. Peck, State College, Pennsylvania. Let $S(m, n)$ be the desired number. If the diagonal passes through an interior corner, then, by similar rectangles, $k\equiv(m, n)>1$. For $k=1$, then, the diagonal passes through the sides but not the interior corners of the unit squares. There are $m-1+n-1$ of these to be crossed, the abutting sides of $m+n-1$ unit squares, so that $S(m, n)=m+n-1$ if $k=1$. By similar rectangles,

$$S(m, n) = kS(m/k, n/k) = k\left(\frac{m}{k}+\frac{n}{k}-1\right) = m+n-(m, n).$$

Also solved by Jack Abad, Brother U. Alfred, James Archer, Raymond Balbes, R. E. Ball, Jr., Merrill Barnebey, M. T. L. Bizley, Walter Bluger, Maxey Brooke, J. A. Burslem, Allan Chuck and Peter Goldstein (jointly), D. I. A. Cohen, Jack Dix, G. C. Dodds, E. S. Eby, R. B. Eggleton, N. J. Fine, David Finkel, E. T. Frankel, Michael Goldberg, Cornelius Groenewoud, B. Gupta, S. B. Guthrey, Michael Haley, D. M. Hancasky, Ned Harrell, M. H. Hayamizu, J. A. H. Hunter, R. F. Jackson, Roman Kaluzniacki, E. T. Knapp, John Koelzer, J. L. Lebbert, Harry Lewis, R. A. Little, T. M. Little, W. R. McEwen, D. C. B. Marsh, P. E. Merriman, Norman Miller, Gary Moore, T. M. Morrisette, Amos Nannini, Margaret Olmsted, Robert Patenaude, W. E. Patten, C. B. A. Peck (second solution), Wayne Pfeiffer, W. E. Philpott, S. W. Reyner, L. A. Ringenberg, V. K. Rohatgi, Charles Rockland, J. C. Roper and W. M. Self (jointly), Alan Saleski, Camilo Schmidt, J. J. Schneider, C. P. Seguin, D. L. Silverman, T. R. Smith, Sidney Spital, John Stout, J. S. Sullivan, M. B. Suryanarayana and M. V. Tamhankar (jointly), Vis Upatisringa, Andy Vince, P. S. Voigt (two solutions), Sandra White, J. C. Williams, P. O. Wood, Jr., and the proposers.

Marsh found $\sum_{k=1}^{m}(m-k+1)(n-k+1)-2\sum_{k=1}^{m}\sum_{j=0}^{m-k}\max\{0, [jn/m]-k+1\}$ for the number of k by k squares meeting the diagonal in a segment. Peck obtained $m+n+p-(m, n)-(m, p)-(n, p)+(m, n, p)$ for the corresponding problem with an m by n by p rectangular parallelepiped, a result which Brooke, Hunter, and Philpott point out, appears in Recreational Math. Mag., 2 (1961) 38–39.

Skewness of a Triangle

E 1705 [1964, 680]. *Proposed by David and Geralda Singmaster, Berkeley, California*

Given a triangle with sides a, b, c, with $a\leqq b\leqq c$. Define the *skewness* of the triangle to be

$$S = \max(a/b,\, b/c,\, c/a) \min(a/b,\, b/c,\, c/a).$$

Find the maximum and minimum skewness of a triangle. What triangles achieve the maximum and minimum?

Solution by Norman Miller, Queen's University. For an isosceles triangle $S=1$. Suppose now that $a<b<c$. Then S is the smaller of $(c/a)(a/b)=c/b$ and $(c/a)(b/c)=b/a$. Hence, for a scalene triangle, $S>1$. Given a and c, the value of b that makes S greatest is the one for which $c/b=b/a$. Then a, b, c are in geometric progression and can be written a, ar, ar^2, where $r>1$. Then $S=r^2r^{-1}=r$. The triangle inequality requires that $1+r>r^2$ or $r^2-r-1<0$. Hence $r<(1+\sqrt{5})/2 \approx 1.618$. The required range for S is $1 \leqq S<(1+\sqrt{5})/2$. S attains its minimum for an isosceles triangle; it has no maximum, but the upper bound $(1+\sqrt{5})/2$ is approached arbitrarily closely by a thin triangle whose sides are in geometric progression and whose greatest side is arbitrarily close to the sum of the other two.

Also solved by R. A. Adams, Merrill Barnebey and Dean Phelps (jointly), Sister Marion Beiter, Walter Bluger, A. J. Bridgman, J. A. Burslem, Allan Chuck and Peter Goldstein (jointly), D. I. A. Cohen, Gil de Bartolo, M. J. De Leon, F. M. Eccles, N. J. Fine, Michael Fried, Philip Fung, Michael Goldberg, S. H. Greene, D. M. Hancasky, Stephen Hoffman, R. F. Jackson, Hajna Janos and Horvath Sandor (jointly), E. S. Langford, Roman Kaluzniacki, E. F. Knapp, Kenneth Kramer and Steven Minsker (jointly), Harry Lewis, D. C. B. Marsh, T. M. Morrisette, M. G. Murdeshwar and V. K. Rohatgi (jointly), W. I. Nissen, Jr., Robert Patenaude, C. B. A. Peck, Stanton Philipp, S. W. Reyner, L. A. Ringenberg, Alan Saleski, Robin Sibson, Jr., D. L. Silverman, Harry Snyder, R. P. Soni, Sidney Spital, M. B. Suryanarayana and M. V. Tamhankar (jointly), Michael Sydlaske, Andy Vince, K. P. Yanosko, and the proposers.

Not all of these solutions were complete.

Convex Subarea of a Convex n-gon

E 1706 [1964, 680]. *Proposed by Michael Fried, University of Michigan*

The *convex subarea* of a convex n-gon is the area in the plane in which a point may be placed such that this point plus the vertices of the n-gon form a convex $(n+1)$-gon. If S represents the perimeter of a convex n-gon, what is the minimum possible value for its convex subarea?

Solution by the proposer. The convex subarea of an n-gon is the union of the "triangles" formed by a side of the n-gon and the two adjacent sides produced. For those convex n-gons with finite subarea and perimeter S, the convex subarea is

$$A = \frac{1}{2} \sum_{i=1}^{n} S_i^2 \frac{(\tan \alpha_i) \tan \alpha_{i+1}}{\tan \alpha_i + \tan \alpha_{i+1}},$$

where S_i is the length of the side at whose ends the exterior angles are α_i and $\alpha_{i+1} (\alpha_{n+1}=\alpha_1)$ and, hence, $S=\sum_{i=1}^{n} S_i$ and $2\pi=\sum_{i=1}^{n} \alpha_i$. We extremalize A by the method of Lagrange multipliers and find that $S_1= \cdots =S_n=S/n$ and

$\alpha_1 = \cdots = \alpha_n = 2\pi/n$. There clearly being no maximum subarea, this gives $A = S^2 \tan(2\pi/n)/(4n)$ as the minimum. (Of course, triangles and convex quadrilaterals do not have finite subarea.)

Also solved by J. A. Burslem, Allan Chuck and Peter Goldstein (jointly), R. F. Jackson, and D. C. B. Marsh.

All these solutions arrived at the regularity of the minimizing n-gon by unexplained "symmetry considerations" which escape the Editor.

A Vandermonde-like Determinant

E 1707 [1964, 680]. *Proposed by D. P. Roselle, Duke University*

Prove that the nth order determinant $|A_{rs}(x)|$, where $A_{rs}(x) = x^{(r-1)(r-s)}$, has the value $\prod_{j=1}^{n-1}(1-x^j)^{n-j}$.

Solution by Sidney Spital, California State Polytechnic College, Pomona, California. The assertion is clear for $x=0$. For $x \neq 0$, extraction of the common factor $x^{(1-r)(n-r)}$ from the rth row gives

$$| A_{rs}(x) | = x^{-\sigma} V_n(x^{n-1}, x^{n-2}, \cdots, x, 1)$$

with $\sigma = \sum_{r=1}^{n}(r-1)(n-r)$ and $V_n(x_1, x_2, \cdots, x_n)$ the Vandermonde determinant $|x_j^{i-1}|$ of order n. Thus

$$\begin{aligned} V_n(x^{n-1}, x^{n-2}, \cdots, x, 1) &= \prod_{s>t}(x^{n-s} - x^{n-t}) = \prod_{s>t} x^{n-s}(1 - x^{s-t}) \\ &= x^{\rho} \prod_{t=1}^{n-1}\prod_{j=1}^{n-t}(1 - x^j) = x^{\rho}\prod_{j=1}^{n-1}(1 - x^j)^{n-j}, \end{aligned}$$

where $\rho = \sum_{t=1}^{n}\sum_{s=t+1}^{n}(n-s) = \sum_{s=1}^{n}\sum_{t=1}^{s-1}(n-s) = \sigma$, and the result follows.

Also solved by A. N. Aheart, Gyula Bereznai, W. R. Boland, J. A. Burslem, M. M. Chawla, C. F. Evans, N. J. Fine, Nicholas Grant, S. H. Greene, Stephen Hoffman, R. F. Jackson, Roman Kaluzniacki, P. G. Kirmser, E. S. Langford, D. C. B. Marsh, Gus Mavrigian, Norman Miller, David Moe, M. G. Murdeshwar, Stanton Philipp, V. K. Rohatgi, M. S. R. K. Sastry, P. A. Scheinok, R. P. Soni, M. B. Suryanarayana, Elias Toubassi, Van de Vyle, and the proposer.

Inverse of an Integral Matrix

E 1708 [1964, 680]. *Proposed by J. F. Ramaley, University of California at Berkeley*

Call a matrix M *integral* if and only if the entries of the matrix are all integers. Given n integers $a_1, \cdots, a_n$ show that there exists an $n\times n$ integral matrix M whose first row consists of the n given integers and whose inverse is also integral if and only if g.c.d. $(a_1, \cdots, a_n) = 1$.

Solution by R. F. Jackson, University of Toledo. The determinants of the integral matrices M and M^{-1} are reciprocal integers and therefore ± 1. This is incompatible with any common factor, other than unity, of the elements of the given row.

A matrix with determinant ± 1 can be constructed in the following manner: Reduce in magnitude all elements of the given row which are definitely larger than the smallest, adding or subtracting the smallest as needed. Continue the reduction until one element becomes the g.c.d. of the original set, i.e., ± 1, according to hypothesis. Move this element to the first position, and complete a matrix with ± 1 in each diagonal position and zero elsewhere. The desired M is constructed by reversal of the previous steps, now applied to complete columns. This restores the original first row while preserving the magnitude of the determinant. The elements of M^{-1} are, except for sign, the corresponding minors of det M and hence are integral.

Also solved by D. I. A. Cohen, C. F. Evans, N. J. Fine, Michael Fried, D. C. B. Marsh, J. B. Muskat, Stanton Philipp, V. K. Rohatgi, D. J. Samuelson, C. P. Seguin, D. A. Smith, Sidney Spital, M. B. Suryanarayana and M. V. Tamhankar (jointly), and the proposer.

Philipp found a problem equivalent to E1708 as Exercise 24 at the end of Chapter 1 in L. Mirsky, *An Introduction to Linear Algebra.* Smith and Seguin found the sufficiency on page 770 of Eugene Schenkman, *The basis theorem for finitely generated abelian groups,* this MONTHLY 67 (1960) 770–771. Suryanarayana and Tamhankar point out that the problem is a corollary of the THEOREM: *If G is a lattice with base* $\alpha_1, \cdots, \alpha_n$ *and* $\beta = g_1\alpha_1 + \cdots + g_n\alpha_n \in G$, *then* β *can be completed to a base if and only if g.c.d.* $(g_1, \cdots, g_n) = 1$. (C. L. Siegel, *Lectures on Quadratic Forms,* Tata Institute of Fundamental Research, Bombay, (1955–56; reissued 1963), p. 12.)

Consecutive Non Square Free-Integers

E 1709 [1964, 680]. *Proposed by Jack Winter, System Development Corporation, Santa Monica, California*

Show that, for any positive integer n, there exists a sequence of a least n consecutive integers each of which contains a squared factor.

Solution by Andrzej Makowski, Powsińska 9c, Warsaw, Poland. A simple proof follows from the Chinese remainder theorem: The system of congruences $x+k\equiv 0 \pmod{p_k^2}$ $(k=1, 2, \cdots, n)$ has a solution (p_k are consecutive prime numbers). Another solution was presented by W. Sierpinski in his note *On the products of different primes* (in Polish, Wiad. Mat., 2 (1959) 204–206). The proof is by induction on n. For $n=1$ the theorem is trivial. Suppose that there exist n consecutive integers $1+k$ $(k=1, \cdots, n)$ each having a squared factor: $m_k^2 \mid 1+k (m_k>1)$. Put $m=m_1m_2\cdots m_n$. We have $m_k^2 \mid m^2(m^2-2)(1+n+1)+1+k$ $(k=1, \cdots, n)$ and $(m+1)^2 \mid m^2(m^2-2)(1+n+1)+1+n+1 = m^2(m^2-2)+1(1+n+1) = (m-1)^2(m+1)^2(1+n+1)$. Thus there exist $n+1$ such numbers: $m^2(m^2-2)(1+n+1)+1+k$ $(k=1, 2, \cdots, n, n+1)$.

Of course, either argument can be modified for arbitrary powers.

Also solved by Jack Abad, Paul Aizley, Brother U. Alfred, R. W. Ball, S. F. Becker and P. K. Subramanian (jointly), W. J. Blundon, J. A. Burslem, Leonard Carlitz, John Christopher, Allan Chuck and Peter Goldstein (jointly), D. I. A. Cohen, R. H. De Vore, R. B. Eggleton, N. J. Fine, Gerard Finley, David Forthoffer, Michael Fried, P. R. Halmos, D. M. Hancasky, R. F. Jackson, Bernard Jacobson, Ray Jurgensen, Erwin Just and Norman Schaumberger, Roman Kaluzniacki, D. C. B. Marsh, T. M. Morrisette, J. B. Muskat, Robert Patenaude, Stanton Philipp, D. A. Prener, Donald Quiring, J. P. Samson, C. P. Seguin, Robin Sibson, Jr., D. L. Silverman, G. C.

Thompson, Roseanna Torretto, A. M. Vaidya, Emanuel Vegh, Andy Vince, P. O. Wood, Jr., and the proposer.

Aizley, Jacobson, and Seguin found E1709 as Exercise 9, page 32, in Niven and Zuckerman, *An Introduction to the Theory of Numbers*, Wiley (1960), and the Editor recalls its having been assigned as an exercise in Niven's number theory class of 1951 or thereabouts. Kaluzniacki points out a similar problem in the William Lowell Putnam Mathematical Competition of March, 1955.

Powers in $\{an+b\}_{n=1}^{\infty}$

E 1710 [1964, 681]. *Proposed by D. I. A. Cohen, Princeton University, and Ralph Greenberg, University of Pennsylvania*

If a and b are relatively prime integers, prove there are infinitely many perfect powers of the form $an+b$.

Solution by D. A. Klarner, University of Alberta, Edmonton, Alberta. An obvious necessary and sufficient condition that the arithmetic progression $\{an+b: n=0, 1, \cdots\}$ contains infinitely many perfect pth powers is that b be a pth power residue of a. The necessity follows on definition of "pth power residue of a," and if an integer A exists such that $A^p \equiv b \pmod a$ then $(A+an)^p \equiv b \pmod a$ for every integer n, so that the sufficiency of the condition is equally trivial. Since we have $(a, b)=1$, $b^{\phi(a)} \equiv 1 \pmod a$ and hence $b^{p\phi(a)+1} \equiv b \pmod a$; thus $\{an+b; n=0, 1, \cdots\}$ contains infinitely many perfect $p\phi(a)+1$ powers.

Since $\{4n+3: n=0, 1, \cdots\}$ contains no squares, some arithmetic progressions are free of certain classes of perfect powers.

Also solved by Jack Abad, Shair Ahmad and K. L. Yocom (jointly), Brother U. Alfred, W. J. Blundon, Leonard Carlitz, Charles Chouteau, Allan Chuck and Peter Goldstein (jointly), R. B. Eggleton, N. J. Fine, Michael Fried, Michael Goldberg, Jerry Goodman, D. M. Hancasky, R. F. Jackson, Erwin Just and Norman Schaumberger (jointly), Andrzej Makowski, D. C. B. Marsh, T. M. Morrisette, S. S. Muchnick, J. B. Muskat, B. J. Parshall, Stanton Philipp, Donald Quiring, C. P. Seguin, W. M. Self, Robin Sibson, Jr., J. S. Sullivan, G. C. Thompson, Emanuel Vegh, Andy Vince, Brother T. C. Wesselkamper, Charles Wexler, and the proposers.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before December 31, 1965.

5300. *Proposed by A. M. Vaidya, Pennsylvania State University*

Prove that every positive integer can be expressed as a difference of two square-free integers. Compare E 1627 [1964, 686].

5301. *Proposed by Hermann Simon, McGill University*

Let H be a subgroup of the finite group G. Let q be a prime which divides the order of G but does not divide the index of H in G. Show that H is an invariant subgroup if and only if the index of the normalizer of H does not exceed q and, for every $x \in G$, $H \cap (x^{-1}Hx)$ is either $\{1\}$ or H.

5302. *Proposed by Pat Garrett, New Mexico State University*

Show that the null manifold of a compact linear operator on a Banach space need not be complemented, i.e., need not be the image of a continuous projection on the domain.

5303. *Proposed by H. S. Shapiro, University of Michigan*

Let ρ be a finite measure on the Borel sets of E_k (Euclidean k-space) such that $\rho(E_k)=1$, and which is rotation-invariant: $\rho(E)=\rho(OE)$ for every Borel set E and every orthogonal transformation O. Show that, if $u(x)$ is harmonic in E_k and $\int|u|\,|d\rho|<\infty$, then $u(0)=\int u(x)d\rho$.

5304. *Proposed by William Kolakoski, Carnegie Institute of Technology*

Describe a simple rule for constructing the sequence

$$1\ 2\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 2\ 1\ 2\ 2\ 1\ 1\ \cdots.$$

What is the nth term? Is the sequence periodic?

5305. *Proposed by Frank Dapkus, Seton Hall University, South Orange, N. J.*

Is it true that of all surfaces of revolution only spheres and circular cylinders have the property that areas of zones of equal thickness are equal?

5306. *Proposed by Fred Gross, National Bureau of Standards*

Let $\phi(x)$ and $\psi(x)$ be any two entire functions with $\phi(x)$ nonconstant. If $\psi(x)$ is of finite order and in particular of order less than k whenever $\phi(x)$ is a polynomial of degree k, then $\sin\phi(x)=\psi(x)$ must have infinitely many solutions. (Compare 5102 [1964, 564].)

5307. *Proposed by Martin J. Cohen, Beverly Hills, California*

Let $\{p_n\}$ be the sequence of primes in ascending order, $p_1=2$. Let F be any polynomial of degree $\geqq 2$ with integer coefficients. Let a_n be the integer defined by $F(n)\equiv a_n \pmod{p_n}$, $0\leqq a_n\leqq p_n-1$. Prove (or disprove) that the sequence $\{a_n\}$ is unbounded.

5308. *Proposed by Louis Comtet, Boulogne, France*

Let $A=\{a_1, a_2, \cdots\}$, $B=\{b_1, b_2, \cdots\}$ be two sequences of real numbers with the a_i all different and the $b_i\neq 0$. Define the function $f(x)$ by $f(x)=b_n$ if $x=a_n$ and $f(x)=0$ if $x\notin A$. (1) Show that $f(x)$ is of class $\leqq 2$ (of Baire). (2) Is it true that the two following conditions are equivalent?

(i) $\lim_{n\to\infty} b_n=0$,

(ii) $f(x)$ belongs to class 1.

5309. *Proposed by E. H. Feller, University of Wisconsin, Milwaukee*

Let M be a finitely generated unitary right R module, where R satisfies the

ascending chain condition for right ideals. Find an example to show that the collection of all R-homomorphisms from M to M does not satisfy the ascending chain condition for right ideals. [The elements of $\mathrm{Hom}_R(M, M)$ are written on the right.]

SOLUTIONS OF ADVANCED PROBLEMS

Density of $m+n\xi$, $(m, n)=1$

5151 [1963, 1106]. *Proposed by L. Ehrenpreis, New York University, and D. J. Newman, Yeshiva University*

If ξ is an irrational number and x any number, it is well known that integers m and n can be found such that $m+n\xi$ is arbitrarily close to x. Show, further, that the m, n can be chosen to be relatively prime.

Solution by Harvey Friedman, Highland Park High School, Illinois. We prove the

THEOREM. *Let ξ be an irrational number. Then the set of all numbers of the form $a+b\xi$, $(a, b)=1$, is dense in R.*

Proof. Select integers c, d so that $0\leqq c+d\xi<\epsilon$ and write $\gamma=c/(c, d)$, $\delta=d/(c, d)$; then $(\gamma, \delta)=1$. Let x', y' be integers selected so that $\gamma y'-\delta x'=1$. From the relation $-\delta(x'+t\gamma)+\gamma(y'+t\delta)=1$, it follows that, for an arbitrary integer t, the numbers $x'+t\gamma$, $y'+t\delta$ are relatively prime.

Given an arbitrary real number x, determine the integer t by the relations $x-(x'+y'\xi)=t(\gamma+\delta\xi)+r$, $0\leqq r<\gamma+\delta\xi$. Thus $0\leqq x-[x'+t\gamma+(y'+t\delta)\xi]<\gamma+\delta\xi$, and the result now follows from the fact that $0<\gamma+\delta\xi\leqq c+d\xi<\epsilon$.

Also solved by the proposers. Paul Erdös points out that his paper with J. H. H. Chalk also contains estimates for $|m+n\xi-x|$, $(m, n)=1$. See Erdös and Chalk, *On the distribution of primitive lattice points*, Canadian Math. Bull., v. 2, p. 91.

The Convergence of $\sum(u_n/S_n)^\alpha$, $\alpha>1$

5167 [1964, 98]. *Proposed by P. D. Barry and W. K. Hayman, University of London, England*

Let $0<u_1\leqq u_2\leqq\cdots$, $u_n\leqq n$, $S_n=\sum_{i=1}^n u_i$. Show that $\sum(u_n/S_n)^\alpha$ converges for all $\alpha>1$.

Solution by D. Borwein, University of Western Ontario. More generally, it will be shown that if $xf(x)$ is positive and nonincreasing for $x\geqq a$ and $\int_a^\infty f(x)dx<\infty$, then

$$\sum_{n=p}^{\infty} f(S_n/u_n) < \infty, \tag{1}$$

p being a sufficiently large positive integer.

We first prove that if $d_n\geqq 0$ and $D_n=\sum_{r=1}^n d_r\to\infty$, then

$$\sum_{n=p}^{\infty} d_n f(D_n) < \infty \qquad (D_p \geqq a). \tag{2}$$

In fact, (2) holds when the first hypothesis of f is replaced by the weaker one that $f(x)$ is positive and nonincreasing for $x \geqq a$, for then

$$d_n f(D_n) \leqq \int_{D_{n-1}}^{D_n} f(x)dx \qquad (D_{n-1} \geqq a)$$

and (2) follows.

Suppose first that $\lim u_n < \infty$. Then $S_n/u_n > \frac{1}{2}n$ for all large n and (1) is a consequence of (2) with $d_n = \frac{1}{2}$.

Suppose now that $\lim u_n = \infty$. Let

$$d_1 = u_1; \qquad d_n = \frac{S_n}{n} - \frac{S_{n-1}}{n-1} = \frac{nu_n - S_n}{n(n-1)} \geqq 0 \qquad (n \geqq 2).$$

Then $D_n = \sum_{r=1}^{n} d_r = S_n/n \to \infty$ and

$$d_n f(D_n) \geqq \left(\frac{u_n}{S_n} - \frac{1}{n}\right)\frac{S_n}{n} f\left(\frac{S_n}{n}\right).$$

Consequently, by (2),

$$\sum_{n=p}^{\infty} \left(\frac{u_n}{S_n} - \frac{1}{n}\right)\frac{S_n}{u_n} f\left(\frac{S_n}{u_n}\right) < \infty, \qquad \left(\frac{S_p}{p} \geqq a\right). \tag{3}$$

Let N_1 and N_2 be the sets of positive integers $n \geqq p$ such that

$$\frac{S_n}{u_n} \geqq \frac{n}{2} \quad \text{when} \quad n \in N_1; \qquad \frac{S_n}{u_n} < \frac{n}{2} \quad \text{when} \quad n \in N_2.$$

Then

$$\sum_{N_1} \frac{S_n}{nu_n} f\left(\frac{S_n}{u_n}\right) \leqq \frac{1}{2} \sum_{N_1} f\left(\frac{n}{2}\right) < \infty, \tag{4}$$

and, by (3), since $u_n/S_n - 1/n > 1/n$ when $n \in N_2$,

$$\sum_{N_2} \frac{S_n}{nu_n} f\left(\frac{S_n}{u_n}\right) < \infty. \tag{5}$$

(1) follows from (3), (4) and (5), and the proposition follows upon taking $f(x) = 1/x^\alpha, \alpha > 1$.

Also solved by George Bergman, Robert Breusch, N. G. de Bruijn (Netherlands), Paul Erdös (Hungary), and J. E. Littlewood (England).

Notes. De Bruijn proves the convergence of (1) above with the hypothesis of $f(x)$ decreasing to zero, but requiring $\int f(x) \log x dx < \infty$.

Borwein's theorem above also includes the conjecture submitted by Erdös that $\sum u_n/S_n\psi(S_n/u_n) < \infty$ if $\psi(x)$ is an increasing function for which $\psi(x)/\log x \to \infty$ and $\sum 1/n\psi(n) < \infty$. Specifically we would also have the convergence of $\sum u_n/S_n(\log S_n/u_n)^{1+\epsilon}$.

Bergman shows that the result of the problem is still true if u_n/S_n is replaced by u_{n+k}/S_n, but he shows by example that the result becomes false if the condition $u_n \leqq n$ is replaced by $u_n \leqq n^{1+1/k}$ for a fixed positive integer k.

There remains the unresolved question raised by Littlewood as to the consequences of replacing $u_n \leqq n$ by $u_n < n\omega_n$, where $\omega_n \to \infty$ sufficiently slowly.

Inverses of Matrices with Restricted Entries

5209 [1964, 689]. *Proposed by Bezalel Peleg, Hebrew University, Jerusalem*

For each matrix B whose entries are rational numbers, let $m(B)$ be the least common denominator of the entries of B. Show that, for each n, there is a nonsingular $n \times n$ matrix A all of whose entries are 0, 1 or -1, such that

$$m(A^{-1}) \geqq \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^{n-1}\sqrt{5}} - 1 \sim 0.9(1.6)^n.$$

Hadamard's formula yields $m(A^{-1}) \leqq n^{n/2}$. Can either estimate be improved?

Solution by the proposer. Let G be the constant sum homogeneous n-person game whose weights $w_1, \cdots, w_n$ are given by: $w_1 = w_2 = 1$, $w_i = w_{i-1} + w_{i-2}$, $i = 3, \cdots, n-2$, $w_{n-1} = w_{n-2}$, and $w_n = w_{n-1} + w_{n-2}$ (this game appears in Isbell, *A class of majority games*, Quar. Jour. Math., 1956, 183–7). G has exactly n minimal winning sets, $S_1, \cdots, S_n$. Denote by e_i the characteristic function of S_i, and let $e_0 = (1, \cdots, 1)$. The matrix A whose rows $a_1, \cdots, a_n$ are given by $a_i = e_{i+1} - e_1$, $i = 1, \cdots, n-1$, and $a_n = e_0$, satisfies

$$m(A^{-1}) \geqq \sum_{i=1}^{n} w_i = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^{n-1}\sqrt{5}} - 1,$$

since the only solution of $Ax = b$, $b = (0, 0, \cdots, 1)'$, is the vector

$$x = k(w_1, w_2, \cdots, w_n), \qquad k = \left[\sum_{i=1}^{n} w_i\right]^{-1}.$$

Distributivity and $(-1)x = -x$

5210 [1964, 689]. *Proposed by T. J. Kaczynski, Evergreen Park, Illinois*

Let K be an algebraic system with two binary operations (one written additively, the other multiplicatively), satisfying

(1) K is an abelian group under addition,

(2) $K - \{0\}$ is a group under multiplication, and

(3) $x(y+z) = xy + xz$ for all $x, y, z \in K$.

Suppose that for some n, $0 = 1 + 1 + \cdots + 1$ (n times). Prove that, for all $x \in K$, $(-1)x = -x$.

Solution by R. G. Bilyeu, North Texas State University. The last part of the hypothesis is unnecessary. If z denotes -1, then $z + z + zz = z(1+1+z) = z$, so $zz = 1$. Now $z(x + zx) = zx + x = x + zx$, so either $x + zx = 0$ or $z = 1$. In either case $zx = -x$.

Also solved by Carol Avelsgaard, Richard Bourgin, Robert Bowen, Joel Brawley, Jr., F. P. Callahan, M. M. Chawla (India), R. A. Cunninghame-Green (England), M. J. DeLeon, M. Edelstein, N. J. Fine, Harvey Friedman, Anton Glaser, M. G. Greening (Australia), A. G. Heinicke, Sidney Heller, G. A. Heuer, Stephen Hoffman, K. G. Johnson, A. J. Karson, Max Klicker, Kwangil Koh, C. C. Lindner, C. R. MacCluer, H. F. Mattson, C. J. Maxson, R. V. Moddy, José Morgado (Brazil), W. L. Owen, Jr., P. R. Parthasarathy (India), Harsh Pittie, Kenneth Rogers, Tôru Saitô (Japan), Camilio Schmidt, Leonard Shapiro, Frank A. Smith, George Van Zwalenberg, W. C. Waterhouse, Kenneth Yanosko, and the proposer.

Evaluating u when $\sum u_{ij}^2 \leqq 1$

5211 [1964, 689]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

A function $u \in C^2$ attains a minimum of value 0 at an interior point of a region B. If the second partial derivatives satisfy $\sum u_{ij}^2 \leqq 1$, and if every pair of points of B can be joined by a string of length $\sqrt{2}$ lying wholly in B, then $u<1$ throughout B.

Solution by W. C. Waterhouse, Harvard University. Let $x=(x_1, \cdots, x_n)$ be the minimum point; then $\partial u/\partial x_i \equiv u_i(x)=0$ for all i. Let y be any other point in B. Since x is an interior point, the hypothesis implies that there is a smooth curve Γ of length $a<\sqrt{2}$ joining x to y in B. Let Γ be $\{\phi_i(t)\}$, where t is arc length, so that $\sum_{i=1}^n \phi_i'(t)^2=1$. We have

$$u_i(\phi(t)) = \int_0^t \sum_j u_{ij}(\phi(s))\phi_j'(s)ds,$$

and

$$\begin{aligned} u(y) &= \int_0^a \sum_i \left(\int_0^t \sum_j u_{ij}(\phi(s))\phi_j'(s)ds \right) \phi_i'(t)dt \\ &\leqq \int_0^a \int_0^t \sum_{i,j} \left| u_{ij}(\phi(s))\phi_j'(s)\phi_i'(t) \right| dsdt \\ &\leqq \int_0^a \int_0^t \left[\sum_{i,j} u_{ij}(\phi(s))^2 \right]^{1/2} \left[\sum_{i,j} \phi_j'(s)^2 \phi_i'(t)^2 \right]^{1/2} dsdt \\ &\leqq \int_0^a \int_0^t dsdt = a^2/2 < 1. \end{aligned}$$

Also solved by M. Edelstein, M. D. Mavinkurve (India), Stanton Philipp, and the proposer.

The Legendre Symbol in Summation

5212 [1964, 689]. *Proposed by L. Carlitz, Duke University*

Let p be prime and let $\psi(a)=(a/p)$, the Legendre symbol. Show that, if $abcd \not\equiv 0 \pmod{p}$, then

$$S = \sum_{x,y,z=0}^{p-1} \psi(ayz+bzx+cxy+dxyz) = -[\psi(-abc)]p.$$

Solution by J. B. Muskat, University of Pittsburgh. If p is an odd prime, then $\sum_{u=0}^{p-1}\psi(ru+s)=0$, provided $r\not\equiv 0 \pmod p$. Thus

$$S=\sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\sum_{z=0}^{p-1}\psi(z[ay+bx+dxy]+cxy)=p\sum_{ay+bx+dxy\equiv 0 \,(\mathrm{mod}\ p)}\psi(cxy).$$

For each y there exists a unique $x \pmod p$ satisfying $ay+bx+dxy\equiv 0 \pmod p$, unless $b+dy\equiv 0 \pmod p$, and $x\equiv -ay(b+dy)^{-1} \pmod p$. Thus

$$S=p\sum_{\substack{y=0\\ b+dy\not\equiv 0\,(\mathrm{mod}\ p)}}^{p-1}\psi(c(-ay)(b+dy)^{-1}y)=p\sum_{y=0}^{p-1}\psi(-acy^2(b+dy)),$$

since at $b+dy\equiv 0 \pmod p$, $\psi(b+dy)=0$. Then finally

$$\begin{aligned}S&=p\sum_{y=1}^{p-1}\psi(-acy^2(b+dy))=p\sum_{y=1}^{p-1}\psi(-ac(b+dy))\\&=p\sum_{y=0}^{p-1}\psi(-ac(b+dy))-p\psi(-acb)=0-p\psi(-abc).\end{aligned}$$

Also solved by N. J. Fine, Harley Flanders, Irving Gerst, M. G. Greening (Australia), D. R. Hayes, J. B. Kelly, Kenneth Rogers, K. S. Williams, and the proposer.

Proximity of Points in Banach Space

5213 [1964, 689]. *Proposed by T. I. Seidman, The Boeing Co., Seattle*

Let X be a Banach space, S a closed convex set in X, $x_0\in(X-S)$, $x_1\in S$. Let $d=\inf\{\|x_0-x\|: x\in S\}$ and let $x'\in S$ be such that $\|x_0-x'\|<d+\epsilon$. Prove or disprove that $\|x_1-x'\|\leqq\|x_1-x_0\|+\epsilon$.

I. *Solution by J. P. Altgeld, University of Illinois.* The proposed inequality fails. Example: Let $X=R^2$ with the norm $\|(x, y)\|=\max\{|x|, |y|\}$ and let S be the convex set $\{(1, y): |y|\leqq 1\}$. Every point in S has distance 1 from (0, 0) but the distance between $(1, -1)$ and $(1, +1)$ is 2.

II. *Solution by J. M. Shaw, National Institute of Health.* The inequality is incorrect. For, let X be Euclidean 2-space, S the closed lower half-plane, $x_0=(0, 10)$, $x_1=(100, 0)$, $x'=(-4, 0)$, $\epsilon=1$. Then $d=10$, $\|x'-x_0\|=\sqrt{116}<11=d+\epsilon$, but

$$\|x-x'\|=104>101+1>\|x_1-x_0\|+\epsilon.$$

Also solved by Bruce Bailey and Bertha Thompson, Duane W. Bailey, R. G. Bilyeu, Richard Bourgin, Melvyn Ciment, J. N. Deverman, M. Edelstein, J. R. Isbell, Ben Klem, W. C. Waterhouse, Tudor Zamfirescu (Roumania), and the proposer.

Discrete Topological Subgroups

5215 [1964, 690]. *Proposed by A. Wilansky, Lehigh University*

Prove that a topological group (with more than one element) has the discrete topology if and only if it has a compact open subset which includes no

right translate of itself.

Solution by Duane W. Bailey, Amherst College. If e is the identity of a discrete group, then $\{e\}$ has the required properties. Conversely, suppose V is such a subset. We may assume that V contains e. For each $x \in V$ there are open subsets V_x and U_x of V containing x and e, respectively, and such that $V_x U_x \subset V$. Since the sets V_x, $x \in V$, cover V there are points $x_1, x_2, \cdots, x_n$ in V such that $V = \bigcup_{i=1}^n V_{x_i}$. Let $U = \bigcap_{i=1}^n U_{x_i}$. Then U is open and $VU \subset V$. Since $Va \not\subset V$ for $a \neq e$, $U = \{e\}$. Consequently, the group has the discrete topology.

Also solved by J. E. Arnold, Jr., Robert Bowen, P. R. Chernoff, R. G. Douglas and C. R. MacCluer, M. Edelstein, Murray Eisenberg, H. B. Keynes, B. G. Klein, M. D. Mavinkurve (India), John Rainwater, P. S. Schnare, Saïd Sidki, Bertram Walsh, W. C. Waterhouse, and the proposer.

Residues of Fibonacci Numbers

5216 [1964, 690]. *Proposed by Oswald Wyler, University of New Mexico*

Let p be a prime and let F_n be the nth Fibonacci number ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). Show that

(a) $F_{p-1} \equiv 0 \pmod{p}$, $F_p \equiv 1 \pmod{p}$, if $p \equiv \pm 1 \pmod{5}$.

(b) $F_p \equiv -1 \pmod{p}$, $F_{p+1} \equiv 0 \pmod{p}$, if $p \equiv \pm 2 \pmod{5}$.

Solution by D. W. Robinson, Brigham Young University. The result is well known (see Hardy and Wright, *An introduction to the theory of numbers*, 4th ed., London, 1960, p. 150) and has been extended to other quadratic recurrent sequences (see Dickson, *History of the theory of numbers*, vol. I, p. 394).

More generally, however, we may prove the following: Let $u_0, u_1, \cdots, u_n, \cdots$ be a sequence of integers satisfying the linear recurrence $u_{n+2} = tu_{n+1} - du_n$ for $n \geq 0$ where u_0, u_1, t and d are given integers. If p is a prime such that $p \nmid dt$ and $D = t^2 - 4d$, then

(1) $u_{p-1} \equiv u_0,\ u_p \equiv u_1 \pmod{p}$ if $(D/p) = 1$,

(2) $u_p \equiv (t/2)u_0,\ u_{p+1} \equiv (t/2)u_1 \pmod{p}$ if $(D/p) = 0$,

(3) $u_{p+1} \equiv du_0,\ u_{p+2} \equiv du_1 \pmod{p}$ if $(D/p) = -1$,

where (D/p) is the usual Legendre symbol. (In the last case it is clear from the recurrence that $u_p \equiv tu_0 - u_1 \pmod{p}$.)

To prove these results let

$$A = \begin{pmatrix} 0 & -d \\ 1 & t \end{pmatrix}.$$

Clearly $(u_n, u_{n+1}) = (u_0, u_1)A^n$. It may be shown by some elementary graph theory (see D. W. Robinson, *The Fibonacci matrix modulo m*, The Fibonacci Quarterly, 1 (1963) 33–34) that

$$A^{p-(D/p)} \equiv sI \pmod{p},$$

where s is an integer depending upon p. Also, since $A - tI + dA^{-1} \equiv 0$, it follows

that $A^p - tI + dA^{-p} \equiv 0$ and tr $A^p - 2t + d(1/d)$ tr $A^p \equiv 0 \pmod{p}$. Thus, tr $A^p \equiv t \pmod{p \neq 2}$. Furthermore, we note by direct calculation that this last congruence is also valid for $p = 2$.

Consequently, if $(D/p) = 1$, then $A^p \equiv sA$ and tr $A^p \equiv t \equiv st \pmod{p}$. Since $p \nmid t$, $s \equiv 1$ and $A^{p-1} \equiv I \pmod{p}$, which implies (1).

If $(D/p) = 0$, then $A^p \equiv sI$ and tr $A^p \equiv t \equiv 2s \pmod{p}$. Since $p \nmid t$ but $p \mid D$, it follows $p \neq 2$ and $s \equiv t/2 \pmod{p}$. That is, $A^p \equiv (t/2)I \pmod{p}$, which yields (2).

Finally, if $(D/p) = -1$, then $A^p \equiv sA^{-1}$ and tr $A^p \equiv t \equiv (s/d)t \pmod{p}$. Thus $s \equiv d$ and $A^{p+1} \equiv dA \pmod{p}$, which implies (3).

Also solved by Joseph Arkin, G. Di Antonio, Brother Alfred, J. H. E. Cohn (England), A. J. Bosch (Netherlands), D. I. A. Cohen, Harley Flanders, Sylvan Greene, M. G. Greening (Australia), A. W. Johnson, Jr., A. J. Keeping, Douglas Lind, H. F. Mattson, S. S. Muchnick, M. G. Murdeshwar, S. Parameswaran (India), Stanton Philipp, Simeon Reich (Israel), Henry Ricardo, M. N. S. Swamy, Van De Vyle, Emanuel Vegh, W. C. Waterhouse, and the proposer.

See also D. D. Wall, *Fibonacci series mod m*, this MONTHLY, 67 (1960) 525–532.

Uniform Polynomial Approximation

5217 [1964, 690]. *Proposed by Otomar Hajek, Prague, Czechoslovakia*

Given a real-valued function f on a compact interval $J \subset E^1$ of class Lip_A^1 [i.e., $|f(x) - f(y)| \leq A|x - y|$ for $x, y \in J$], prove that there exist polynomials p_n with $p_n \to f$ uniformly on J, p_n in the same class Lip_A^1 on J.

Using this, one may show that for Lip_A^1 maps from a compact parallelepiped of E^p to E^q, there exist uniform polynomial approximations in $\mathrm{Lip}_{A\sqrt{q}}^1$ (in the Euclidean norm). Can this be sharpened to Lip_A^1?

Solution by the proposer. Taking $J = [0, 1]$, the Bernstein polynomials have the required property. If

$$B_n(x) = \sum_{k=0}^{n} C_n^k f\left(\frac{k}{n}\right) x^k (1 - x)^{n-k},$$

then

$$B_n'(x) = n \sum_{k=0}^{n-1} C_{n-1}^k \left(f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right) x^k (1 - x)^{n-1-k}$$

and $B_n(x) \to f(x)$ uniformly (see I. P. Natanson, *Theory of Functions of a Real Variable*, p. 108). The Lipschitz assumption on f implies that

$$\max |B_n'| \leq n \cdot A \frac{1}{n} \sum_{k=0}^{n-1} C_{n-1}^k x^k (1 - x)^{n-1-k} = A,$$

which provides the required Lipschitz condition for $B_n(x)$.

No contributions were received on the final sharpened result sought for the higher dimensional case.

Theory of Numbers, 2nd ed. By B. M. Stewart. Macmillan, New York, 1964. 383 pp. $8.50.

The first edition of this book appeared in 1952 and was reviewed in this MONTHLY, 60 (1953) 274. In this second edition, much new material on modern algebra has been added, and the resulting union has been reorganized into two parts. Part I, labelled *An intuitive approach*, is longer than Part II, *An axiomatic approach*. In reality, Part I represents a traditional course in elementary number theory while Part II consists mainly of an introduction to abstract algebra.

The arrangement of the number theoretic material in this edition is more natural and represents a strengthening over the first edition. The presentation is not rushed, the author providing, in general, detailed and full arguments. Considering the type of student to whom the text is directed, the problems are well chosen, ranging from the routine to the challenging. Certain arguments and proofs could be simplified. For example, the proof given for the Möbius inversion formula is quite clumsy. On balance, however, Part I provides a good text for a one-semester undergraduate course in number theory, especially useful for students who are unable to read an abbreviated style of writing.

The subject matter of the second part ranges over such topics as groups, matrices, rings, fields, unique factorization, etc. The connection with the strictly number-theoretic portion of the text is mainly in providing motivation and examples. Nonetheless, Part II may not appeal to many as a follow-up to Part I. Also, the presentation of the material in the second part is not as successful as in the first. Example: One chapter is entitled "Modular Fields," but no definition of a modular field is given. The author appears not to have in mind a field of characteristic $\neq 0$.

Considered as a whole, the abundance of interesting material and the relatively complete explanations make the book a useful undergraduate text.

GERALD FREILICH, College of the City of New York

Vector Geometry. By Gilbert de B. Robinson. Allyn and Bacon, Boston, 1962. 176 pp. $6.95.

The geometric objects discussed in this book are, for the most part, the lines, planes, conics and quadric surfaces of Euclidean two and three space. Some of the treatment is in terms of vectors; most is not. Projective, affine, elliptic and hyperbolic geometries are mentioned briefly—so briefly, in some instances, as to render the text almost meaningless to a reader who does not already know the subject. The topics from Eulidean geometry receive a more detailed exposition, but the result is not very different from the chapters on analytic geometry in dozens of American calculus textbooks which make a nod in the direction of vectors. The book also contains a chapter on groups and linear transformations, which does not seem to tie in with the rest of the book.

HOWARD LEVI, Hunter College, City University of New York

Fourier Analysis on Groups. By Walter Rudin. Interscience, New York, 1962. ix+285 pp. $9.50.

This is a compact book. In its short span it covers the main body of knowledge about harmonic analysis of the group algebras $L^1(G)$ and $M(G)$ over the locally compact abelian (LCA) group G. $M(G)$ denotes the algebra of bounded regular measures on G, and $L^1(G)$ is the ideal in $M(G)$ of measures absolutely continuous with respect to Haar measure (or functions summable with respect to Haar measure). Under the influence of Beurling, Bochner, and Zygmund some ten analysts, including the author, have accomplished the central core of results.

Chapter 1 outlines Fourier analysis in the setting of LCA groups. Chapter 2 is devoted to the principal structure theorem for LCA groups, the algebraic-topological connections between dual groups, and Fourier relations between subgroups and annihilators.

The problem of finding all idempotents in the algebra $M(G)$, with Cohen's solution, appears in Chapter 3. This leads to the description in Chapter 4 of all homomorphisms from $L^1(G_1)$ to $M(G_2)$, with related results like extensions of homomorphisms and the invariance of $L^1(G)$ under all automorphisms of $M(G)$.

Chapters 5 and 7 give the most famous "negative" results of the theory. Chapter 5 explores the possible behavior of measures concentrated on sparse subsets of G: Kronecker sets, Helson sets, and Sidon sets in discrete groups. Sample theorem: In every nondiscrete LCA group, there exists a Cantor set whose elements are algebraically independent. This sets the stage for Williamson's theorem that $M(G)$ is a self-adjoint algebra if and only if G is discrete.

If f is a Fourier transform and g is a function whose domain contains the range of f, when is $g \circ f$ a Fourier transform? The complete solution of this problem (for transforms of functions) is given in Chapter 6. Chapter 7 centers around the problem of spectral synthesis, viz., the problem of injectivity of the correspondence between closed ideals in $L^1(G)$ and closed sets in the dual group. Wiener's Tauberian Theorem can be regarded as a special positive result (only $L^1(G)$ corresponds to the empty set). The chapter includes Schwartz's example of the impossibility of spectral synthesis in R^3 and Malliavin's general theorem for all noncompact groups.

Chapter 8 is concerned with groups whose duals can be ordered, e.g., any compact connected abelian group. This gives rise to functions and measures of "analytic" type, whose Fourier transforms vanish on the negative part of the dual group. The theorem of F. and M. Riesz on the circle is an archetype, but the general theory roams considerably.

The final chapter faces the problem of closed subalgebras of $L^1(G)$, and results are still of an exploratory nature (though Theorem 9.2.3 has one of the nicest proofs in the book). The last section determines those groups G for which the Gelfand representation of $L^1(G)$ is an algebra with the Stone-Weierstrass property.

There seem to be very few mistakes. One occurs in the proof of the Pontrjagni duality theory (part (b)). In the Appendix, where the abstract analytical tools are assembled, a few inexact statements occur, like the omission of hypotheses from the Radon-Nikodym and Fubini theorems. In the proof of Cohen's homomorphism theorem, equation (2) on page 86 seems doubtful. The proof can be sewn together by a detour.

All in all, the author has assembled a first-rate treatise. It is one of the finest examples of classical analysis in the sense that abstract methods have been used effectively for an intense investigation of important concrete algebras.

S. E. PUCKETTE, University of the South

A Course of Mathematics for Engineers and Scientists. By C. Plumpton and B. H. Chirgwin. Pergamon Press, New York, Oxford, London, Paris, 1964. Volume 4, viii+353 pp. $5.00, Volume 5, viii+202 pp. $3.75. Distributed by Macmillan, New York.

Volumes 4 and 5 are part of a seven volume series written to cover the mathematics required by science and engineering students preparing for a first degree at British and Commonwealth Universities.

Volume 4. This volume generalizes and develops the ideas and methods contained in the earlier volumes, so that the student can appreciate and use the mathematical methods required in more advanced aspects of physics and engineering. The techniques of vector analysis, matrices, and methods for finding solutions of ordinary and partial differential equations are considered. Chapter I on vector analysis is concerned with the development of differential and integral operations with vectors. The idea of a vector as an entity, defined by its transformation properties, is used. Included are: transformation of coordinates; scalar fields, gradient, vector fields; line and surface integrals, divergence and Stoke's theorems; applications; Green's theorem; surface derivatives; Green's function; variation with time; orthogonal curvilinear coordinates; Cartesian tensors. Chapter II considers Laplace's equation; series solution of ordinary differential equations, singularities; the behavior of the solution of a differential equation; Sturm-Liouville theory, eigenvalues, orthogonal functions. Chapter III considers Bessel, Legendre, Hermite and Laguerre functions, and links them with solutions of boundary and initial value problems. Methods of solution of differential equations which arise in connection with vector and scalar fields are considered in Chapter IV. Lagrange's partial differential equation and associated ordinary differential equations; field lines and level surfaces are considered. Chapter V is concerned with matrices. The definitions given are motivated well. Included are: matrix algebra, rank of matrix, reciprocal matrix, partitioned matrices, solution of linear equations, vector spaces, orthogonal matrices, eigenvalues and eigenvectors, quadratic forms, canonical forms, hermitian and unitary matrices. The treatment would be improved by the inclusion of better definitions for the *product* of two matrices, *equivalent matrices*, and *elementary operations* on matrices.

Introduction to Linear Algebra. By Frank M. Stewart. Van Nostrand, Princeton, N. J., 1963. 281 pp. $7.50.

This is a thorough, intuitive, elementary introduction to linear algebra. While the points are taken from n dimensional space, each idea is discussed in detail as it appears in three dimensional geometry. There are many diagrams. Definitions and proofs are not only clearly given, but the intuitive reasons behind them are delightfully spelled out.

Because of the care with which the basic ideas are introduced and explained, some more advanced ideas are left for a later course. The author discusses linear transformations, matrices, two approaches to determinants (one based on linear algebra, the second the traditional approach via minors) and ends with inner product spaces, eigenvalues, and the spectral theorem for symmetric transformations. Such topics as similarity and congruence of matrices, arbitrary fields and various canonical forms of matrices are not discussed.

There are lots of problems. Many difficult ideas are keyed to more complete explanations in the appendix. The author writes as though he wants the student to understand the ideas, and not merely be impressed with how much can be crowded into a single symbol.

Calculus is not a prerequisite. This text should prove very useful for students ranging from good high school seniors to college juniors. It could also be used for in-service institutes for high school teachers.

P. B. JOHNSON, University of California, Los Angeles

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

The following mathematicians have been elected to membership in the National Academy of Sciences: Professors R. H. Bing, University of Wisconsin; D. H. Blackwell, University of California, Berkeley; Mark Kac, Rockefeller Institute; and Alfred Tarski, University of California, Berkeley.

Professor Rufus Oldenburger, Director of the Automatic Control Center at Purdue University, has been elected an honorary member of the Japan Society of Mechanical Engineers.

Dr. Warren Weaver, Alfred P. Sloan Foundation, has been selected as the 13th winner of the international Kalinga Prize of 1,000 pounds sterling for the popularization of science, according to an announcement by UNESCO.

Professor James N. Eastham, Queensborough Community College, represented the

Association at the inauguration of Jacob J. Hartstein as President of Kingsborough Community College on March 25, 1965.

Professor Emeritus Lester R. Ford, Illinois Institute of Technology, represented the Association at the inauguration of Wayne F. Geisert as President of Bridgewater College on April 3, 1965.

Professor W. L. Graves, Drury College, represented the Association at the inauguration of Arthur L. Mallory as President of Southwest Missouri State College on March 26, 1965.

Professor L. H. Lange, San Jose State College, represented the Association at the inauguration of Frederic W. Ness as President of Fresno State College on April 30, 1965.

Professor H. D. Perry, Texas A & M University, represented the Association at the inauguration of Arleigh B. Templeton as President of Sam Houston State Teachers College on April 5 and 6, 1965.

Professor Irving Sussman, University of Santa Clara, represented the Association at the inauguration of Robert B. Clark as President of San Jose State College on May 4, 1965.

Professor W. L. Williams, University of South Carolina, represented the Association at the inauguration of Gordon W. Blackwell as President of Furman University on April 20, 1965.

Associate Professor Brian Abrahamson, University of Toronto, has been appointed Professor of Mathematics in the School of Physical Sciences at the University of Adelaide, Australia.

Mr. D. R. Beuerman, University of Kansas, has been appointed Instructor at Purdue University.

Associate Professor John Dyer-Bennet, Carleton College, has been promoted to Professor.

Mr. B. L. Fincher, Texas Woman's University, has been promoted to Assistant Professor.

Dr. B. E. Goodwin, University of Maryland, has been appointed Assistant Professor at the University of Delaware.

Dr. R. T. Herbst has been promoted to Director of the Mathematical Analysis and Military Apparatus Design Laboratory of Bell Telephone Laboratories in Winston-Salem, North Carolina.

Assistant Professor Donald Koontz, Elizabethtown College, has been promoted to Associate Professor.

Professor John M. Perry, Clarkson College of Technology, has been appointed Chairman of the Department of Mathematics at Wells College.

Professor Hazel Schoonmaker Wilson, Jacksonville University, retired at the end of the 1963–1964 academic year with the title of Professor Emeritus.

THE THREE COLLEGE CONTEST

Each year for the past five years, mathematics students from Antioch College (Yellow Springs, Ohio), Denison University (Granville, Ohio), and Ohio Wesleyan University (Delaware, Ohio) have participated in a two and a half hour written exam constructed and graded by a referee chosen by the mutual consent of the participating institutions. The exam generally takes place on some Saturday morning in April and is followed by a lunch and a general get-together of the participants, with the institutions rotating the role of host from year to year.

This year's referee was Professor Nathan J. Fine of Pennsylvania State University. The contest took place at Antioch College on April 17.

Referees in previous years have been Morton L. Curtis, L. M. Kelly, E. J. Mickle, and Leo Moser.

SOUTHEASTERN CONFERENCE THEORETICAL AND APPLIED MECHANICS

The Third Southeastern Conference on Theoretical and Applied Mechanics (SECTAM) will be held on March 31 and April 1, 1966, at the University of South Carolina. All research workers in the field of mechanics are invited to submit papers for consideration. All papers accepted will be presented at the conference and will be published, in full, in the proceedings volume.

The deadline for submission of papers has been tentatively set as November 15, 1965. Prospective authors should contact the Executive Chairman: Prof. J. D. Waugh, University of South Carolina, Columbia, South Carolina, or the Editorial Chairman: Prof. W. A. Shaw, Auburn University, Auburn, Alabama.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The 38th annual meeting of the Allegheny Mountain Section of the MAA was held at Lazear Hall, Washington and Jefferson College, Washington, Pennsylvania, on Saturday, May 2, 1964. In the absence of President Allen Strehler of Carnegie Institute of Technology, Wray G. Brady, Secretary, presided in the morning session. In the afternoon session, Evan Johnson, of Pennsylvania State University, presided. There were 92 persons in attendance of which 67 were members of the Association. The listing from the Employment Register was available to the persons attending the conference. A tea was held by the ladies of Washington and Jefferson College following the meeting.

At the business meeting a report from the president, Allen Strehler, was read by the acting Chairman, Wray G. Brady. Reports were heard from Lawrence Romboski and I. D. Peters to the effect that the high school mathematics contest continues to grow in both Western Pennsylvania and West Virginia, their respective areas. A call for charter members of the Association to rise was answered by Otto F. H. Burt, Professor Emeritus, of Washington and Jefferson College.

The program was as follows:

1. *Minimizing polynomials by geometric means*, by R. J. Duffin, Carnegie Insitute of Technology.

In optimizing engineering designs it often is required to minimize a generalized polynomial whose variables are the design parameters. Moreover, the minimization is to be carried out subject to inequality constraints on functions which are also generalized polynomials. By use of the inequality relating the arithmetic mean to the geometric mean, it is possible to transform this minimization problem to a related maximization problem termed the dual. The dual problem has several advantages both from an algebraic and a numerical point of view. For instance, the constraints of the dual problem are linear rather than nonlinear. The method of Geometric Programming was developed in collaboration with Dr. Clarence Zener and Dr. Elmore Peterson of the Westinghouse Research Laboratories.

2. *Characterization of exponential-type distribution in terms of conditional moments*, by E. M. Bolger and W. L. Harkness, Pennsylvania State University.

Let $f(x;\delta,\lambda)$ be a two-parameter family of exponential-type densities such that the distribution of the sum of n independent random variables $X_1, \cdots, X_n$ is reproductive in one parameter and

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 7

CONTENTS

AUGUST-SEPTEMBER 1965

THE AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Frederick A. Ficken, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. Ficken, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Executive Director, H. M. Gehman, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

General Office: SUNY at Buffalo, Buffalo, N. Y. 14214. *Executive Director:* H. M. Gehman

---

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00. For nonmembers the subscription price is \$10.00.

Published by the Association at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

PRINTED IN THE UNITED STATES OF AMERICA

THE TWO-POINT BOUNDARY PROBLEM

R. H. COLE, University of Western Ontario

1. Introduction. This paper contains a new treatment of the classical two-point boundary problem associated with a linear differential equation of order n. The significant features include a parametric definition of the adjoint boundary conditions, a definition of Green's function in terms of matrices, and an explicit development of the relationship between the scalar equation and its equivalent matrix equation.

Parametric adjoint boundary relations were initially formulated by Langer [**5**] for a system on a complex domain. Haltiner [**3**] adapted these to the real two-point problem without using matrix notation. The present use of parametric adjoint relations is suggested by results [**1, 2**] obtained for general boundary conditions. The definition of Green's matrix given here is a special case of the definition for the general problem, and it is used to suggest a convenient form of Green's function for the scalar system.

Matrix notation is used throughout the discussion, but no matrix theory beyond the elementary algebra of matrices is involved. It is thought, therefore, that these results should make this portion of the theory of differential equations available for elementary courses.

2. Notation. Matrices will be represented by German capital letters and their components by the corresponding letters in lower case italics. Matrices of one column will be called column vectors and will be represented by lower case German letters. The transpose of a matrix $\mathfrak{A}$ will be represented by $\tilde{\mathfrak{A}}$, and in particular, a row vector will be designated as the transpose of a column vector.

Let $u_1(x), u_2(x), \cdots, u_n(x)$ be any set of n functions, each having $n-1$ derivatives on a fundamental interval $[a, b]$. The vector having these functions as components will be represented by $\mathfrak{u}(x)$ and the *Wronskian matrix* of the functions by $\mathfrak{K}(\mathfrak{u}(x))$ or $\mathfrak{K}(\mathfrak{u})$. That is,

$$\mathfrak{u}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \\ \cdot \\ \vdots \\ u_n(x) \end{bmatrix} \quad \text{and} \quad \mathfrak{K}(\mathfrak{u}(x)) = \begin{bmatrix} u_1(x) & u_2(x) & \cdots & u_n(x) \\ u_1'(x) & u_2'(x) & \cdots & u_n'(x) \\ \vdots & \vdots & & \vdots \\ u_1^{(n-1)}(x) & u_2^{(n-1)}(x) & \cdots & u_n^{(n-1)}(x) \end{bmatrix}.$$

Also, if $u(x)$ is any function with $n-1$ derivatives on $[a, b]$, the vector $\mathfrak{k}(u(x))$ or $\mathfrak{k}(u)$ is defined by $\tilde{\mathfrak{k}}(u(x)) = (u(x), u'(x), \cdots, u^{(n-1)}(x))$. If $u(x) = \tilde{\mathfrak{u}}(x)\mathfrak{c}$, where $\mathfrak{c}$ is any constant vector, it is clear that

$$\mathfrak{k}(u(x)) = \mathfrak{K}(\mathfrak{u}(x))\mathfrak{c}.$$

The identity matrix will be represented by $\mathfrak{J}$. The vectors $\mathfrak{d}_1, \mathfrak{d}_2, \cdots, \mathfrak{d}_n$ and the matrix $\mathfrak{H}$ are defined by

$$\mathfrak{d}_1 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad \mathfrak{d}_2 = \begin{bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \cdots, \mathfrak{d}_n = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix}, \quad \mathfrak{H} = \begin{bmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \cdot & \cdot & \cdot \quad\quad \cdot \\ \cdot & \cdot & \cdot \quad\quad \cdot \\ 0 & 0 & 0 \cdots 1 \\ 0 & 0 & 0 \cdots 0 \end{bmatrix}.$$

It may be noted that, if $\mathfrak{H}$ is used as a left multiplier of any matrix, it has the effect of raising each of its rows one stage and replacing the last row by a row of zeros. Similarly, as a right multiplier, $\mathfrak{H}$ shifts the columns of any matrix one stage to the right. The corresponding effect of $\tilde{\mathfrak{H}}$ is easily inferred.

The differential form $L(u)$ is defined by

$$L(u) = u^{(n)} + p_1 u^{(n-1)} + \cdots + p_n u, \tag{2.1}$$

where $p_j \in C^{n-j}$ on $[a, b]$. The vectors $\mathfrak{p}$ and $\mathfrak{q}$ are defined by

$$\tilde{\mathfrak{p}} = (p_n, p_{n-1}, \cdots, p_1) \quad \text{and} \quad \tilde{\mathfrak{q}} = (p_{n-1}, p_{n-2}, \cdots, p_1, 1).$$

The first of these relations permits the representation,

$$L(u) = u^{(n)} + \mathfrak{p}\mathfrak{k}(u), \tag{2.2}$$

and the second is convenient for subsequent developments.

3. The differential equation and its adjoint. The differential equation to be considered is

$$L(u) = 0, \tag{3.1}$$

where $L(u)$ is defined by (2.1). Any vector whose components form a fundamental set of solutions of (3.1) will be called a *fundamental vector* for that equation. The vector equation

$$\mathfrak{y}' = \mathfrak{A}\mathfrak{y}, \tag{3.2}$$

where the components of the $n \times n$ matrix $\mathfrak{A}$ are continuous functions of x on $[a, b]$, is equivalent in an obvious sense to a system of n first order linear equations. Any matrix $\mathfrak{Y}(x)$ whose columns, considered as vectors, form a linearly independent set of solutions of (3.2) will be called a *fundamental matrix* for (3.2). A variety of first order systems are equivalent to the nth order equation (3.1), but one of these may be considered as a canonical system. It is represented by (3.2) with $\mathfrak{A}$ given by

$$\mathfrak{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -p_n & -p_{n-1} & -p_{n-2} & \cdots & -p_1 \end{bmatrix}. \tag{3.3}$$

It will henceforth be assumed that the matrix $\mathfrak{A}$ in (3.2) is the one defined by (3.3). The distinction of (3.2), in relation to (3.1), arises from the fact that, if $\mathfrak{u}(x)$ is a fundamental vector for (3.1), then $\mathfrak{K}(\mathfrak{u}(x))$ is a fundamental matrix for (3.2). On the other hand, the equations which follow from (3.2), $y_{i+1}=y_i'$, $i=1, 2, \cdots, n-1$, and the structure of $\mathfrak{A}$, make any fundamental matrix for (3.2) a Wronskian matrix and insure that its first row forms a fundamental vector for (3.1).

The obvious identity,

$$\mathfrak{z}\mathfrak{y}' + \mathfrak{z}'\mathfrak{y} = \frac{d}{dx}(\mathfrak{z}\mathfrak{y}), \tag{3.4}$$

or its equivalent

$$\mathfrak{z}(\mathfrak{y}' - \mathfrak{A}\mathfrak{y}) + (\mathfrak{z}' + \mathfrak{z}\mathfrak{A})\mathfrak{y} = \frac{d}{dx}(\mathfrak{z}\mathfrak{y}),$$

may be called the Lagrange identity associated with (3.2). The latter relation suggests the definition of the equation

$$\mathfrak{z}' = -\mathfrak{z}\mathfrak{A} \tag{3.5}$$

as the adjoint of (3.2). A fundamental matrix for an equation of this form is, by definition, one for which the rows form an independent set of solutions of the equation. From (3.4), it is clear that if $\mathfrak{Y}(x)$ is a fundamental matrix for (3.2), its inverse, $\mathfrak{Y}^{-1}(x)$, is a fundamental matrix for (3.5). It is reasonable to suspect that the adjoint of (3.1) will be equivalent, in some sense, to the vector equation (3.5). It is, in fact, known [6] that the last component in any vector solution of (3.5) is a solution of the familiar adjoint of (3.1). The following derivation of this result yields, as a by-product, the useful relation (3.11).

THEOREM 3.1. *The last component of any vector solution of* (3.5) *is a solution of the equation adjoint to* (3.1).

Proof. The scalar relations corresponding to (3.5) are

$$z_1' = p_n z_n \tag{3.6}$$

and

$$z_i' = -z_{i-1} + p_{n+1-i}z_n, \qquad i = 2, 3, \cdots, n. \tag{3.7}$$

From (3.7), we obtain

$$(-1)^{i-1}z_i^{(i)} = (-1)^i z_{i-1}^{(i-1)} + (-1)^{i-1}(p_{n+1-i}z_n)^{(i-1)}$$

whose sum is

$$-z_1' + p_n z_n = L^*(z_n), \tag{3.8}$$

where $L^*(v)$, the familiar adjoint of $L(u)$, is defined by

$$L^*(v) = (-1)^n v^{(n)} + (-1)^{n-1}(p_1 v)^{(n-1)} + \cdots + p_n v.$$

In view of (3.6), we infer that $L^*(z_n) = 0$. This proves the theorem.

The relations (3.7), with $0 = -z_n + z_n$ adjoined, may be written as

$$\mathfrak{H}\mathfrak{z}' = -\mathfrak{z} + z_n\mathfrak{q}. \tag{3.9}$$

From this, we get

$$(-1)^i\mathfrak{H}^{i+1}\mathfrak{z}^{(i+1)} = (-1)^{i+1}\mathfrak{H}^i\mathfrak{z}^{(i)} + (-1)^i\mathfrak{H}^i(z_n\mathfrak{q})^{(i)}, \qquad i = 0, 1, \cdots, n-1.$$

The sum of these relations is

$$\mathfrak{z} = \sum_{i=0}^{n-1} (-1)^i\mathfrak{H}^i(z_n\mathfrak{q})^{(i)}. \tag{3.10}$$

Since (3.10) exhibits the components of $\mathfrak{z}$ as linear combinations of z_n and its first $n-1$ derivatives, it may be written as

$$\mathfrak{z} = \tilde{\mathfrak{P}}\mathfrak{k}(z_n). \tag{3.11}$$

The matrix $\mathfrak{P}$ will be called the *concomitant matrix*. It can be verified that its components are given by the following relations where p_0 is defined to be 1.

$$p_{ij} = \begin{cases} \displaystyle\sum_{k=i-1}^{n-j} (-1)^k \binom{k}{i-1} p_{n-k-j}^{(k-i+1)}, & j \leqq n-i+1 \\ 0, & j > n-i+1. \end{cases}$$

All the components below the secondary diagonal are zero and the components on that diagonal are alternately $+1$ and -1 with $+1$ in the upper right hand corner. The matrix $\mathfrak{P}$ is, therefore, nonsingular.

If $\tilde{\mathfrak{z}}$ is a solution of (3.5) we have, by transposing (3.11),

$$\tilde{\mathfrak{z}} = \tilde{\mathfrak{k}}(z_n)\mathfrak{P}.$$

Substitution in (3.5) yields the fact that $\tilde{\mathfrak{k}}(z_n)$ is a solution of

$$\tilde{\mathfrak{w}}' = \tilde{\mathfrak{w}}\mathfrak{B}, \tag{3.12}$$

where

$$\mathfrak{B} = -\mathfrak{P}\mathfrak{A}\mathfrak{P}^{-1} - \mathfrak{P}'\mathfrak{P}^{-1}.$$

Further, if $\mathfrak{Z}$ is a fundamental matrix for (3.5) and $\mathfrak{v}$ is the vector whose components form the last column of $\mathfrak{Z}$, then

$$\mathfrak{Z} = \tilde{\mathfrak{K}}(\mathfrak{v})\mathfrak{P}. \tag{3.13}$$

The matrix $\tilde{\mathfrak{K}}(\mathfrak{v})$ is a fundamental matrix for (3.12) and, by Theorem 3.1, $\mathfrak{v}$ is a fundamental vector for $L^*(v) = 0$. Since $\mathfrak{B}$ is uniquely determined by $\tilde{\mathfrak{K}}(\mathfrak{v})$,

it is clear that the system corresponding to (3.12) is the canonical system associated with $L^*(v)=0$. These results suggest the following theorem.

THEOREM 3.2. *If* $\mathfrak{u}$ *is a fundamental vector for* $L(u)=0$, *there exists a fundamental vector* $\mathfrak{v}$ *for* $L^*(v)=0$ *such that*

$$\tilde{\mathfrak{K}}(\mathfrak{v})\mathfrak{P}\mathfrak{K}(\mathfrak{u}) = \mathfrak{J}. \tag{3.14}$$

Proof. Let $\mathfrak{u}$ be any fundamental vector for $L(u)=0$. The matrix $\mathfrak{K}^{-1}(\mathfrak{u})$ is a fundamental matrix for (3.5) and, therefore, may be substituted for $\mathfrak{Z}$ in relation (3.13) to yield (3.14). This is the desired result, and it is pertinent to note that the components of the fundamental vector $\mathfrak{v}$ are the components of the last column of $\mathfrak{K}^{-1}(\mathfrak{u})$.

It is of some interest to note that the Lagrange identity for the scalar case can be obtained from the identity (3.4). Let u and v be any two functions with continuous derivatives of order n, and let the vectors $\mathfrak{y}$ and $\mathfrak{z}$ be defined by

$$\mathfrak{y} = \mathfrak{k}(u)$$

$$\tilde{\mathfrak{z}} = \tilde{\mathfrak{k}}(v)\mathfrak{P}.$$

The second of these relations identifies z_n with v and is equivalent to relation (3.9). Also, $\mathfrak{k}'(u)=\mathfrak{H}\mathfrak{k}(u)+u^{(n)}\mathfrak{d}_n$. Hence we may write

$$\tilde{\mathfrak{z}}\mathfrak{y}' + \tilde{\mathfrak{z}}'\mathfrak{y} = \{-\tilde{\mathfrak{z}}'\tilde{\mathfrak{H}} + z_n\tilde{\mathfrak{q}}\}\{\mathfrak{H}\mathfrak{k}(u) + u^{(n)}\mathfrak{d}_n\} + \tilde{\mathfrak{z}}'\mathfrak{k}(u). \tag{3.15}$$

Since $\tilde{\mathfrak{z}}'\tilde{\mathfrak{H}}\mathfrak{H}\mathfrak{k}(u)=\tilde{\mathfrak{z}}'\mathfrak{k}(u)-z_1'u$ and $\tilde{\mathfrak{z}}'\tilde{\mathfrak{H}}\mathfrak{d}_n=0$, the right side of the above relation reduces to

$$z_1'u + z_n\tilde{\mathfrak{q}}\mathfrak{H}\mathfrak{k}(u) + z_n\tilde{\mathfrak{q}}\mathfrak{d}_n u^{(n)}.$$

If $p_n z_n u$ and its negative are added to this expression, we get for (3.15)

$$\tilde{\mathfrak{z}}\mathfrak{y}' + \tilde{\mathfrak{z}}'\mathfrak{y} = z_n\{u^{(n)} + \tilde{\mathfrak{p}}\mathfrak{k}(u)\} + u\{z_1' - p_n z_n\}.$$

Substituting in (3.4), we have, in view of (2.2) and (3.8),

$$vL(u) - uL^*(v) = \frac{d}{dx}\{\tilde{\mathfrak{k}}(v)\mathfrak{P}\mathfrak{k}(u)\}.$$

This is the Lagrange identity for the scalar differential forms.

4. Two-point boundary conditions. The classical two-point boundary problem arises when the solutions of the equation,

$$L(u) = 0, \tag{4.1a}$$

are required to satisfy the relations

$$\sum_{j=1}^{n} w_{ij}^{(a)} u^{(j-1)}(a) + \sum_{j=1}^{n} w_{ij}^{(b)} u^{(j-1)}(b) = 0, \qquad i = 1, 2, \cdots, n.$$

These relations can be written as

$$\mathfrak{W}^{(a)}\mathfrak{k}(u(a)) + \mathfrak{W}^{(b)}\mathfrak{k}(u(b)) = \mathfrak{o}, \tag{4.1b}$$

where $\mathfrak{W}^{(a)} = [w_{ij}^{(a)}]$ and $\mathfrak{W}^{(b)} = [w_{ij}^{(b)}]$.

The adjoint equation,

$$L^*(v) = 0, \tag{4.2a}$$

was defined in section 3. We shall define the boundary relations adjoint to (4.1b) by

$$\begin{aligned} \tilde{\mathfrak{k}}(v(a))\mathfrak{P}(a) &= \tilde{\mathfrak{t}}\mathfrak{W}^{(a)} \\ \tilde{\mathfrak{k}}(v(b))\mathfrak{P}(b) &= -\tilde{\mathfrak{t}}\mathfrak{W}^{(b)}. \end{aligned} \tag{4.2b}$$

A function $v(x)$ will be said to satisfy these relations if any nontrivial parametric vector $\mathfrak{t}$ exists for which the two relations are simultaneously satisfied. This definition of the adjoint boundary conditions is a special case of the definitions given in references [1] and [2]. It assumes the form of (4.2b) because of the relation (3.11).

THEOREM 4.1. *The system* (4.1) *and its adjoint system* (4.2) *have the same index of compatibility.*

Proof. If $\mathfrak{u}(x)$ is a fundamental vector for (4.1a), the general solution of (4.1a) is $\tilde{\mathfrak{u}}(x)\mathfrak{c}$, where $\mathfrak{c}$ is an arbitrary constant vector. This solution satisfies (4.1b) if

$$\mathfrak{D}\mathfrak{c} = \mathfrak{o},$$

where the characteristic matrix $\mathfrak{D}$ is defined by

$$\mathfrak{D} = \mathfrak{W}^{(a)}\mathfrak{K}(\mathfrak{u}(a)) + \mathfrak{W}^{(b)}\mathfrak{K}(\mathfrak{u}(b)).$$

Thus, the system is r-ply compatible (has r independent solutions), if $\mathfrak{D}$ is of rank $n-r$.

If $\mathfrak{v}(x)$ is a fundamental vector for the equation (4.2a) and $\mathfrak{K}(\mathfrak{v}(x))$ is its Wronskian matrix we may, by Theorem 3.2, assume that $\mathfrak{v}(x)$ has been chosen so that

$$\tilde{\mathfrak{K}}(\mathfrak{v}(x))\mathfrak{P}(x)\mathfrak{K}(\mathfrak{u}(x)) = \mathfrak{J}. \tag{4.3}$$

If the general solution, $\mathfrak{v}(x)\mathfrak{c}$, of (4.2a) is substituted in (4.2b), we have

$$\begin{aligned} \tilde{\mathfrak{c}}\tilde{\mathfrak{K}}(\mathfrak{v}(a))\mathfrak{P}(a) &= \tilde{\mathfrak{t}}\mathfrak{W}^{(a)} \\ \mathfrak{c}\tilde{\mathfrak{K}}(\mathfrak{v}(b))\mathfrak{P}(b) &= -\tilde{\mathfrak{t}}\mathfrak{W}^{(b)}. \end{aligned} \tag{4.4}$$

Multiplying the first of these on the right by $\mathfrak{K}(\mathfrak{u}(a))$ and the second by $\mathfrak{K}(\mathfrak{u}(b))$, we obtain by subtraction, in view of (4.3),

$$\tilde{\mathfrak{t}}\mathfrak{D} = \tilde{\mathfrak{o}}.$$

This relation is valid for a nontrivial $\mathfrak{t}$, only if $\mathfrak{D}$ is singular. Conversely, if $\mathfrak{D}$ is of rank $n-r$, there exist r independent vectors orthogonal to it. These, through either of the relations (4.4), will yield r independent determinations of $\mathfrak{c}$ which, in turn, will provide r independent solutions of the adjoint boundary system. This proves the theorem.

The familiar Green's formula,

$$\int_a^b \{vL(u) - uL^*(v)\}\,dx = \tilde{\mathfrak{k}}(v(b))\mathfrak{P}(b)\mathfrak{k}(u(b)) - \tilde{\mathfrak{k}}(v(a))\mathfrak{P}(a)\mathfrak{k}(u(a)),$$

is obtained by integrating the Lagrange identity.

THEOREM 4.2. *If u and v are any two functions with continuous derivatives of order n on* $[a, b]$, *such that u satisfies* (4.1b) *and v satisfies* (4.2b), *then*

$$\int_a^b \{vL(u) - uL^*(v)\}\,dx = 0.$$

Proof. Since v satisfies (4.2b), the right side of Green's formula reduces to

$$-\mathfrak{t}\{\mathfrak{W}^{(a)}\mathfrak{k}(u(a)) + \mathfrak{W}^{(b)}\mathfrak{k}(u(b))\}.$$

Since u satisfies (4.1b), the latter expression is zero, and the result follows.

5. Green's function. The nonhomogeneous system,

$$L(u) = r(x) \tag{5.1a}$$

$$\mathfrak{W}^{(a)}\mathfrak{k}(u(a)) + \mathfrak{W}^{(b)}\mathfrak{k}(u(b)) = \mathfrak{o}, \tag{5.1b}$$

corresponds to the matrix system,

$$\mathfrak{y}' = \mathfrak{A}\mathfrak{y} + r(x)\mathfrak{d}_n \tag{5.2a}$$

$$\mathfrak{W}^{(a)}\mathfrak{y}(a) + \mathfrak{W}^{(b)}\mathfrak{y}(b) = \mathfrak{o}. \tag{5.2b}$$

Under the assumption that the homogeneous system corresponding to (5.1) is incompatible, the Green's function, $G(x, s)$, may be derived by obtaining a solution of (5.1) in the integral form,

$$u(x) = \int_a^b G(x, s)r(s)ds. \tag{5.3}$$

The Green's matrix, $\mathfrak{G}(x, s)$, associated with (5.2) can be similarly derived. Both derivations can be achieved by using the method of variation of parameters as set forth in reference [2]. The derivation of Green's function will be given here for the sake of completeness.

Let a solution, $w(x)$, of (5.1a) be written in the form

$$w(x) = \tilde{\mathfrak{u}}(x)\mathfrak{h}(x), \tag{5.4}$$

where $\mathfrak{u}(x)$ is a fundamental vector for $L(u)=0$. Then,

$$w^{(j)} = \tilde{\mathfrak{u}}^{(j)}\mathfrak{h},\ j = 1, 2, \cdots, n-1 \quad \text{if} \quad \tilde{\mathfrak{u}}^{(j-1)}\mathfrak{h}' = 0,$$
$$w^{(n)} = \tilde{\mathfrak{u}}^{(n)}\mathfrak{h} + r(x) \qquad\qquad \text{if} \quad \tilde{\mathfrak{u}}^{(n-1)}\mathfrak{h}' = r(x).$$

Hence,

$$w^{(n)} + \tilde{\mathfrak{p}}\mathfrak{k}(w) = r(x) \quad \text{if} \quad \mathfrak{K}(\mathfrak{u})\mathfrak{h}' = \mathfrak{d}_n r(x).$$

The latter relation is valid, if

$$\mathfrak{h}(x) = \int_a^x \mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n r(s)ds.$$

It may be verified that, if this evaluation of $\mathfrak{h}(x)$ is used in (5.4), $w(x)$ is a particular solution of (5.1a). Hence, the general solution of that equation is

$$u(x) = \tilde{\mathfrak{u}}(x)\mathfrak{c} + \int_a^x \tilde{\mathfrak{u}}(x)\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n r(s)ds.$$

This solution satisfies (5.1b), if

$$\mathfrak{c} = -\mathfrak{D}^{-1}\mathfrak{W}^{(b)}\int_a^b \mathfrak{K}(\mathfrak{u}(b))\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n r(s)ds.$$

Thus, $u(x)$ is a solution of (5.1), if

$$u(x) = -\int_a^b \tilde{\mathfrak{u}}(x)\mathfrak{D}^{-1}\mathfrak{W}^{(b)}\mathfrak{K}(\mathfrak{u}(b))\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n r(s)ds + \int_a^x \tilde{\mathfrak{u}}(x)\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n r(s)ds.$$

By inserting $\mathfrak{D}^{-1}\{\mathfrak{W}^{(a)}\mathfrak{K}(\mathfrak{u}(a))+\mathfrak{W}^{(b)}\mathfrak{K}(\mathfrak{u}(b))\}$, which is equal to the identity matrix, between $\tilde{\mathfrak{u}}(x)$ and $\mathfrak{K}^{-1}(\mathfrak{u}(s))$ in the second integral and simplifying, we get

$$u(x) = \int_a^x \tilde{\mathfrak{u}}(x)\mathfrak{D}^{-1}\mathfrak{W}^{(a)}\mathfrak{K}(\mathfrak{u}(a))\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n r(s)ds$$
$$- \int_x^b \tilde{\mathfrak{u}}(x)\mathfrak{D}^{-1}\mathfrak{W}^{(b)}\mathfrak{K}(\mathfrak{u}(b))\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n r(s)ds.$$

In the interest of symmetry, $\tilde{\mathfrak{u}}(x)$ may be replaced by $\tilde{\mathfrak{d}}_1\mathfrak{K}(\mathfrak{u}(x))$. Hence, the solution is given by (5.3) with Green's function defined by

$$G(x, s) = \begin{cases} \tilde{\mathfrak{d}}_1\mathfrak{K}(\mathfrak{u}(x))\mathfrak{D}^{-1}\mathfrak{W}^{(a)}\mathfrak{K}(\mathfrak{u}(a))\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n, & s < x \\ -\tilde{\mathfrak{d}}_1\mathfrak{K}(\mathfrak{u}(x))\mathfrak{D}^{-1}\mathfrak{W}^{(b)}\mathfrak{K}(\mathfrak{u}(b))\mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n, & s > x. \end{cases}$$

Green's matrix for system (5.2) is given by (see [1], p. 465)

$$\mathfrak{G}(x, s) = \begin{cases} \mathfrak{Y}(x)\mathfrak{D}^{-1}\mathfrak{W}^{(a)}\mathfrak{Y}(a)\mathfrak{Y}^{-1}(s), & s < x \\ -\mathfrak{Y}(x)\mathfrak{D}^{-1}\mathfrak{W}^{(b)}\mathfrak{Y}(b)\mathfrak{Y}^{-1}(s), & s > x, \end{cases}$$

where $\mathfrak{Y}(x)$ is any fundamental matrix for $\mathfrak{y}' = \mathfrak{A}\mathfrak{y}$.

Green's function is independent of the specific choice of the fundamental vector $\mathfrak{u}(x)$. This can be seen by replacing $\mathfrak{u}(x)$ by $\mathfrak{C}\mathfrak{u}(x)$ in the formula for $G(x, s)$. Since $\mathfrak{K}(\mathfrak{u}(x))$ becomes $\mathfrak{K}(\mathfrak{u}(x))\tilde{\mathfrak{C}}$ and $\mathfrak{D}^{-1}$ becomes $\tilde{\mathfrak{C}}^{-1}\mathfrak{D}^{-1}$, the $\mathfrak{C}$'s cancel out leaving $G(x, s)$ unchanged. Green's matrix is similarly independent of the choice of $\mathfrak{Y}(x)$. It is of particular interest to note that, since $\mathfrak{K}(\mathfrak{u}(x))$ is a fundamental matrix for $\mathfrak{y}' = \mathfrak{A}\mathfrak{y}$, the formulas show that

$$G(x, s) = \tilde{\mathfrak{d}}_1 \mathfrak{G}(x, s)\mathfrak{d}_n.$$

Therefore, Green's function is the upper right hand corner component of Green's matrix.

The representation of $G(x, s)$ in terms of matrices has substantial advantages over the classical representation in terms of determinants (see Ince [4], p. 259). For example, the basic properties [4, p. 254], which are definitive for Green's function, are established easily in the following theorem.

THEOREM 5.1. (a) *Regarded as a function of x, Green's function is a formal solution of the boundary system* (4.1). *It fails to be a true solution because its $(n-1)$-st derivative has an upward unit jump at $x = s$.*

(b) *Regarded as a function of s, Green's function is a formal solution of the adjoint boundary system* (4.2).

Proof. Since $\tilde{\mathfrak{d}}_1 \mathfrak{K}(\mathfrak{u}(x)) = \tilde{\mathfrak{u}}(x)$, the formula for Green's function is

$$G(x, s) = \begin{cases} \tilde{\mathfrak{u}}(x)\mathfrak{a}_+, & s < x \\ \tilde{\mathfrak{u}}(x)\mathfrak{a}_-, & s > x, \end{cases} \tag{5.5}$$

where the vectors $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are independent of x. It follows that

$$\mathfrak{k}_x(G(x, s)) = \begin{cases} \mathfrak{K}(\mathfrak{u}(x))\mathfrak{a}_+, & s < x \\ \mathfrak{K}(\mathfrak{u}(x))\mathfrak{a}_-, & s > x, \end{cases} \tag{5.6}$$

where the subscript x on $\mathfrak{k}$ indicates differentiation with respect to x. Therefore, since $\mathfrak{a}_+ - \mathfrak{a}_- = \mathfrak{K}^{-1}(\mathfrak{u}(s))\mathfrak{d}_n$,

$$\mathfrak{k}_x(G(s^+, s)) - \mathfrak{k}_x(G(s^-, s)) = \mathfrak{d}_n.$$

Thus, $G(x, s)$ and each of its first $n-2$ derivatives has a removable discontinuity at $x = s$, and the $(n-1)$-st derivative has an upward unit jump at that point. It is obvious that $G(x, s)$ is a solution of $L(u) = 0$ for any value of x except s. Employing (5.6), an expression for $\mathfrak{W}^{(a)}\mathfrak{k}_x(G(a, s)) + \mathfrak{W}^{(b)}\mathfrak{k}_x(G(b, s))$ may be written down and reduced to zero. This completes the demonstration that $G(x, s)$ is a formal solution of the homogeneous boundary system.

To prove part (b) of the theorem, let the fundamental vector $\mathfrak{v}(x)$ for $L^*(v) = 0$ be chosen, as in Theorem 3.2, so that

$$\tilde{\mathfrak{K}}(\mathfrak{v})\mathfrak{P}\mathfrak{K}(\mathfrak{u}) = \mathfrak{J}. \tag{5.7}$$

We may therefore, by observing that $\mathfrak{P}\mathfrak{d}_n = \mathfrak{d}_1$, rewrite the formula for $G(x, s)$ in the form

$$G(x, s) = \begin{cases} \tilde{\mathfrak{d}}_1 \mathfrak{K}(\mathfrak{u}(x))\mathfrak{D}^{-1}\mathfrak{W}^{(a)}\mathfrak{K}(\mathfrak{u}(a))\tilde{\mathfrak{K}}(\mathfrak{v}(s))\mathfrak{d}_1, & s < x \\ -\tilde{\mathfrak{d}}_1 \mathfrak{K}(\mathfrak{u}(x))\mathfrak{D}^{-1}\mathfrak{W}^{(b)}\mathfrak{K}(\mathfrak{u}(b))\tilde{\mathfrak{K}}(\mathfrak{v}(s))\mathfrak{d}_1, & s > x. \end{cases}$$

As in part (a), this may be written as

$$G(x, s) = \begin{cases} \tilde{\mathfrak{b}}_- \mathfrak{v}(s), & s < x \\ \tilde{\mathfrak{b}}_+ \mathfrak{v}(s), & s > x, \end{cases}$$

where $\mathfrak{b}_-$ and $\mathfrak{b}_+$ are independent of s. Thus,

$$\tilde{\mathfrak{k}}_s(G(x, s)) = \begin{cases} \tilde{\mathfrak{b}}_- \tilde{\mathfrak{K}}(\mathfrak{v}(s)), & s < x \\ \tilde{\mathfrak{b}}_+ \tilde{\mathfrak{K}}(\mathfrak{v}(s)), & s > x. \end{cases} \tag{5.8}$$

Since $\tilde{\mathfrak{b}}_+ - \tilde{\mathfrak{b}}_- = -\tilde{\mathfrak{d}}_1 \mathfrak{K}(\mathfrak{u}(x))$, the use of (5.7) yields

$$\tilde{\mathfrak{k}}_s(G(x, x^+)) - \tilde{\mathfrak{k}}_s(G(x, x^-)) = -\tilde{\mathfrak{d}}_1 \mathfrak{P}^{-1}(x). \tag{5.9}$$

Since the first row of $\mathfrak{P}^{-1}(x)$ consists of zeros, except for the component $(-1)^{n-1}$ in the last place, the right side of (5.9) reduces to $(-1)^n \tilde{\mathfrak{d}}_n$. Hence, regarded as a function of s, $G(x, s)$ and each of its first $n-2$ derivatives has a removable discontinuity at $s = x$, and the $(n-1)$-st derivative has a unit jump at that point. It follows that Green's function is a formal solution of $L^*(v) = 0$.

From (5.8), using (5.7) again, we get

$$\tilde{\mathfrak{k}}_s(G(x, a))\mathfrak{P}(a) = \tilde{\mathfrak{t}}\mathfrak{W}^{(a)}$$
$$\tilde{\mathfrak{k}}_s(G(x, b))\mathfrak{P}(b) = -\tilde{\mathfrak{t}}\mathfrak{W}^{(b)},$$

where

$$\tilde{\mathfrak{t}} = \tilde{\mathfrak{u}}(x)\mathfrak{D}^{-1}.$$

Thus Green's function satisfies the adjoint boundary conditions, and the proof of the theorem is complete.

6. Remarks. The developments given here are easily extended to m-point or to integral boundary relations, since they are patterned after the corresponding developments for the general case [2]. On the other hand, they may be specialized readily to the one-point or initial value problem. This is achieved by imposing the restrictions, $\mathfrak{W}^{(a)} = \mathfrak{J}$ and $\mathfrak{W}^{(b)} = \mathfrak{O}$. In particular, Green's function, under these restrictions, becomes the one-sided Green's function used by Miller [6] as a prelude to the introduction of Green's function itself.

The fundamental identity, $\tilde{\mathfrak{K}}(\mathfrak{v})\mathfrak{P}\mathfrak{K}(\mathfrak{u}) = \mathfrak{J}$, shows clearly the reciprocal relation between the original equation and its adjoint. If the original boundary conditions (4.1b) are written in the parametric form,

$$\mathfrak{W}^{(a)}\mathfrak{k}(u(a)) = \mathfrak{t}, \quad \mathfrak{W}^{(b)}\mathfrak{k}(u(b)) = -\mathfrak{t},$$

the same identity reveals their relation to the adjoint boundary conditions. Heretofore, no such simple and direct exhibition of the adjoint relationship has been available.

References

1. R. H. Cole, The expansion problem with boundary conditions at a finite set of points, Canad. J. Math., 13 (1961) 462–479.

2. ———, General boundary conditions for an ordinary linear differential system, Trans. Amer. Math. Soc., 111 (1964) 521–550.

3. G. H. Haltiner, The theory of linear differential systems based upon a new definition of the adjoint, Duke Math. J., 15 (1948) 893–919.

4. E. L. Ince, Ordinary differential equations, Dover, New York, 1956.

5. R. E. Langer, The boundary problem of an ordinary linear differential system in the complex domain, Trans. Amer. Math. Soc., 46 (1939) 151–190.

6. K. S. Miller, Linear differential equations in the real domain, Norton, New York, 1963.

A NEW PROOF OF THE TYCHONOFF THEOREM

PETER A. LOEB, Stanford University and University of California, Los Angeles

1. Introduction. One of the most important theorems in general topology is the Tychonoff theorem, which states that the product of compact spaces is compact. We shall give here a relatively simple proof of this fundamental theorem, a proof which does not need the axiom of choice for some special cases such as the countable product of unit intervals.

Recall that if we are given a topological space X_ν for each element ν of an index set I, the cartesian product $\prod_{\nu\in I} X_\nu$ is defined to be the set of all functions x on I such that the value of x at ν, which we denote by x_ν, is a point in X_ν. An element O of the standard base for the product topology of $\prod_{\nu\in I} X_\nu$ is defined by taking a finite subset J of the index set I and open sets U_ν in X_ν for each ν in J, and setting O equal to the set

$$\left\{x \in \prod_{\nu\in I} X_\nu \,\middle|\, x_\nu \in U_\nu \text{ for each } \nu \in J\right\}.$$

By a product space we shall always mean a cartesian product with the product topology, and by the base for the product topology we shall always mean the standard base.

An example which is helpful to keep in mind when working with product spaces is given by the set of all nonnegative real valued functions on the real numbers R. For this space $I=R$ and each X_ν equals the set $\{r\in R \mid r\geqq 0\}$. The product topology for this space is the topology of pointwise convergence. Each base set O for the topology is defined by open sets $U_{\nu_1}, \cdots, U_{\nu_n}$ in a finite number of spaces $X_{\nu_1}, \cdots, X_{\nu_n}$. For each i, U_{ν_i} looks like a set of vertical slits over

the real number ν_i, and O is the set of all functions which "go through" the U_{ν_i} at the points ν_i. (See Figure 1.)

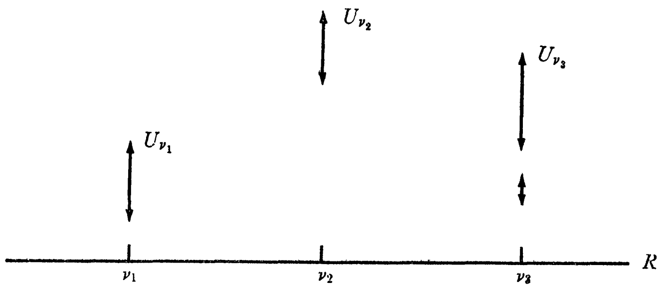


FIG. 1

2. The use of the axiom of choice. The standard proofs of the Tychonoff theorem show that the product $\prod_{\nu\in I} X_\nu$ of an arbitrary collection of compact spaces is compact by imbedding a collection of open or closed subsets of $\prod_{\nu\in I} X_\nu$ in a collection of sets which is maximal with respect to a property of finite character. The proof which we shall give shows that such a product space is compact by using a well-ordering on the index set I and a choice function on the collection of all nonempty closed subsets of all the spaces X_ν. Instead, however, of using the axiom of choice in the body of the proof to give the existence of such a well-ordering and choice function, we shall hypothesize their existence and prove the following theorem without using the axiom of choice:

THEOREM 1. *Let $\{X_\nu | \nu\in I\}$ be a family of compact spaces which is indexed by a set I on which there is a well-ordering $\geqq$. If I is an infinite set, let there also be a choice function F on the collection $\{C | C$ is closed, $C\neq\varnothing$, $C\subset X_\nu$ for some $\nu\}$. Then the product space $\prod_{\nu\in I} X_\nu$ is compact in the product topology.*

Since the axiom of choice implies that any family of compact spaces satisfies the conditions of Theorem 1, the axiom of choice and Theorem 1, together, imply that the product of an arbitrary family of compact spaces is compact. There are, however, families of compact spaces which can be shown to satisfy the conditions of Theorem 1 without using the axiom of choice; that is, families for which a well-ordering $\geqq$ and, if necessary, a choice function F can be explicitly defined. For example, if the index set I is countable, then the ordering of the natural numbers induces a well-ordering on I; if each space X_ν is a compact subset of the real numbers, then we may have F choose the least element in each closed subset of the spaces X_ν. Clearly, Theorem 1 is applicable to the product of any finite collection of compact spaces.

There are several results which make it desirable to localize the use of the axiom of choice in this way. P. J. Cohen [1] has shown that the axiom of choice is independent of the other axioms of set theory. J. L. Kelley [2] and L. E. Ward

[6] have shown that the Tychonoff theorem and even a weak Tychonoff theorem (the product of mutually homeomorphic compact spaces is compact) imply the axiom of choice. It is interesting to compare our assumption that a choice function exists on the closed subsets of the spaces X_ν with Ward's proof in which he takes a disjoint family $\mathfrak{F}$ of sets for which he is to find a choice function, puts each factor X_ν of a product space equal to the union of the sets in $\mathfrak{F}$, and then makes each element of $\mathfrak{F}$ a closed set in X_ν.

Finally, we note that the assumption that a well-ordering exists for the index set I is equivalent to the assumption that a choice function exists for the collection of nonempty subsets of I ([3], p. 273).

3. The proof. To prove that a topological space Y is compact, it is sufficient to show that any covering of Y by open sets from a base for the topology can be reduced to a finite covering. For if this is true, then any covering of Y by open sets can be replaced by a covering of base sets each of which is contained in some open set of the original covering; the new covering can be reduced to a finite covering, and each set in this finite covering can be replaced by one of the original open sets which contains it. A space Y is compact, therefore, if any collection of base sets which has no finite subcollection covering Y does not itself cover Y. With this viewpoint in mind we make the following:

DEFINITION. *A collection $\mathfrak{O}$ of open sets in a topological space Y will be called "admissible in Y," or simply "admissible," if no finite subcollection of $\mathfrak{O}$ covers Y.*

Our proof of Theorem 1 consists of taking an arbitrary admissible collection of sets from the base for the product topology and finding a point x which is in none of them. Our method is an extension of the simple proof that the product of two compact spaces is compact. A slight variation of that proof gives the following lemma for the product of topological spaces X and Y. P_X and P_Y denote the projection of $X\times Y$ onto X and Y respectively.

LEMMA 1. *Let X be compact, and $\mathfrak{O}$ be an admissible collection of base sets in $X\times Y$. There exists a point x in X such that if $\mathfrak{O}_0$ is the collection*

$$\{O \in \mathfrak{O} \mid x \in P_X(O)\},$$

then the collection

$$\{P_Y(O) \mid O \in \mathfrak{O}_0\}$$

is admissible in Y. Furthermore, the set of all such points $x\in X$ forms a closed subset of X.

Proof. The elements of $\mathfrak{O}$ are of the form $V\times W$, where V is open in X and W is open in Y, (see Figure 2). Let $\mathfrak{U}$ be the collection of all open sets $U\subset X$ for which there is a finite collection

$$\mathfrak{A}_U \subset \mathfrak{O} \quad \text{with} \quad U = \bigcap_{O\in\mathfrak{A}_U} P_X(O) \quad \text{and} \quad Y = \bigcup_{O\in\mathfrak{A}_U} P_Y(O).$$

For each U in $\mathfrak{U}$, $\mathfrak{A}_U$ covers $U \times Y$. Since no finite subcollection of $\mathfrak{O}$ covers $X \times Y$, no finite subcollection of $\mathfrak{U}$ covers X, for if X were covered by a finite collection $\{U_1, \cdots, U_n\}$ of sets in $\mathfrak{U}$, then $X \times Y$ would be equal to $\bigcup_{i=1}^n (U_i \times Y)$ and would be covered by a finite collection $\bigcup_{i=1}^n \mathfrak{A}_{U_i}$ of sets in $\mathfrak{O}$. Thus $\mathfrak{U}$ is an admissible collection of open sets in the compact space X, and consequently $\mathfrak{U}$ does not cover X. Therefore $X - \bigcup_{U \in \mathfrak{U}} U$ is a nonempty closed subset C of X. Let x be any element in C, and

$$\mathfrak{O}_0 = \{O \in \mathfrak{O} \mid x \in P_X(O)\}.$$

The collection $\{P_Y(O) \mid O \in \mathfrak{O}_0\}$ is admissible in Y, for if $\mathfrak{A}$ is a finite subcollection of $\mathfrak{O}_0$, then $\bigcap_{O \in \mathfrak{A}} P_X(O) \notin \mathfrak{U}$ since $x \in \bigcap_{O \in \mathfrak{A}} P_X(O)$, and so $Y \neq \bigcup_{O \in \mathfrak{A}} P_Y(O)$.

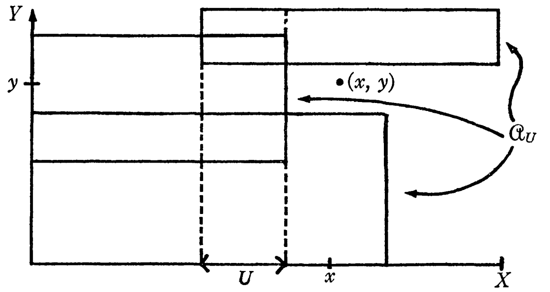


FIG. 2

Before using Lemma 1 to prove Theorem 1 for the general case, it will be helpful to see how we can use it to prove the theorem for the products of two and three compact spaces. For two compact spaces, we simply let the space Y of Lemma 1 be compact. Then, since the collection $\{P_Y(O) \mid O \in \mathfrak{O}_0\}$ is an admissible collection of open sets in the compact space Y, there is a point y in Y such that $y \notin P_Y(O)$ for any O in $\mathfrak{O}_0$. Clearly, the point (x, y) is in none of the sets O of the collection $\mathfrak{O}$. (See Figure 2.) On the other hand, if Z is the product of three compact spaces X, S, and T, then letting $Y = S \times T$, we see that Z is homeomorphic to $X \times Y$. Since we know that the product of two compact spaces is compact, we know that Y is compact and therefore that Z is compact. This method of proof, however, will not generalize to the product of an infinite number of compact spaces. To give a proof that will generalize we must again start with an arbitrary admissible collection $\mathfrak{O}$ of sets from the base for the product topology of Z. Each set O in $\mathfrak{O}$ is the product of an open set U in X and a base set $V \times W$ for the product topology of Y. Using Lemma 1, we choose a point x in X such that if $\mathfrak{O}_0$ is the set

$$\{O \in \mathfrak{O} \mid x \in P_X(O)\},$$

then the set of projections $\{P_Y(O) \mid O \in \mathfrak{O}_0\}$ is an admissible collection of base

sets in $Y=S\times T$. Again using Lemma 1 we choose a point s in S such that if $\mathcal{O}_1$ is the set

$$\{O \in \mathcal{O} \mid x \in P_X(O) \text{ and } s \in P_S(O)\}$$

then the collection of open sets $\{P_T(O) \mid O \in \mathcal{O}_1\}$ is admissible in the compact space T. Finally we choose a point t in the nonempty set

$$T - \bigcup_{O \in \mathcal{O}_1} P_T(O).$$

The point (x, s, t) is in none of the members O of the collection $\mathcal{O}$.

Clearly, in order to show that the product $\prod_{\nu\in I} X_\nu$ of an infinite collection of compact spaces X_ν is compact with the above method of proof we must assume the existence of a well-ordering $\geqq$ on the index set I. For convenience we shall also assume that there is a last index in I under the well-ordering $\geqq$. We can always make this assumption, since if $\succeq$ is a well-ordering on I for which there is no last index, then we may take the first index λ in I under the ordering $\succeq$ and define a new well-ordering $\geqq$ on I by setting $\alpha \geqq \beta$ when $\alpha \succeq \beta$ for all α and β in $I - \{\lambda\}$ and $\lambda \geqq \alpha$ for all α in I.

In proving Theorem 1, we shall use Lemma 1 as in the above examples to choose a point x_ν in a closed subset of each space X_ν. When I is an infinite set, an infinite number of choices will have to be made. It is for this reason that we must assume the existence of a choice function F on the nonempty closed subsets of the spaces X_ν.

Let $\{X_\nu \mid \nu \in I\}$ be a family of compact spaces for which such a choice function F and well-ordering $\geqq$ exist, and let Y denote the product space $\prod_{\nu\in I} X_\nu$. For each index α in I we let P_α denote the projection of Y onto X_α; i.e., $P_\alpha(x) = x_\alpha$ for all x in Y. We denote the product $\prod_{\nu \geqq \alpha} X_\nu$ of all spaces X_ν with index $\nu \geqq \alpha$ by Y_α, and the projection of Y onto Y_α, by $P_{\nu\geqq\alpha}$. If x is in Y, then $P_{\nu\geqq\alpha}(x)$ is the point x' in Y_α such that $x_\nu = x'_\nu$ for all $\nu \geqq \alpha$. Given a subset S of Y, $P_{\nu\geqq\alpha}(S)$ is the set $\{x' \in Y_\alpha \mid x' = P_{\nu\geqq\alpha}(x) \text{ for some } x \in S\}$. In particular, $P_{\nu\geqq\alpha}(Y) = Y_\alpha$. By $\alpha+1$ we shall always mean the first index greater than α in I. There is a natural homeomorphism of Y_α onto $X_\alpha \times Y_{\alpha+1}$ which takes $P_{\nu\geqq\alpha}(x)$ onto $(P_\alpha(x), P_{\nu\geqq\alpha+1}(x))$ for each x in Y.

Let $\mathcal{O}$ be a collection of base sets in Y. We shall say that $\mathcal{O}$ is *admissible on* Y_α if the set of projections $\{P_{\nu\geqq\alpha}(O) \mid O \in \mathcal{O}\}$ is admissible in Y_α. If O is an element of $\mathcal{O}$, then $P_{\nu\geqq\alpha}(O)$ is clearly a base set in Y_α. More important, however, is the fact that the homeomorphism which takes Y_α onto $X_\alpha \times Y_{\alpha+1}$ takes $P_{\nu\geqq\alpha}(O)$ onto the product of an open set $P_\alpha(O)$ in X_α and an open set $P_{\nu\geqq\alpha+1}(O)$ in $Y_{\alpha+1}$. Lemma 1 implies, therefore, that when $\mathcal{O}$ is admissible on Y_α, there is a closed subset C of X_α such that if $x_\alpha = F(C)$ and $\mathcal{O}_0 = \{O \in \mathcal{O} \mid x_\alpha \in P_\alpha(O)\}$, then $\mathcal{O}_0$ is admissible on $Y_{\alpha+1}$. In order to apply Lemma 1 in this way, we shall need to know that $\mathcal{O}$ is admissible on Y_α if $\mathcal{O}$ is admissible on $Y_{\mu+1}$ for every $\mu < \alpha$. The following lemma proves this fact for every index $\alpha > \sigma$, where σ denotes the first index in I.

LEMMA 2. *If* $\alpha > \sigma$ *and* $\mathfrak{A}$ *is a finite collection of base sets in* Y *such that*

$$Y_\alpha = \bigcup_{O \in \mathfrak{A}} P_{\nu \geqq \alpha}(O),$$

then there is an index $\gamma < \alpha$ *such that* $Y_{\gamma+1} = \bigcup_{O \in \mathfrak{A}} P_{\nu \geqq \gamma+1}(O)$.

Proof. If there is an index γ in I such that $\alpha = \gamma + 1$, then the lemma is quite trivial. However, there may be no such γ. In this case, let G be the set $\{\mu \in I \mid \mu < \alpha$ and $P_\mu(O) \neq X_\mu$ for some $O \in \mathfrak{A}\}$. There are no more than a finite number of elements in G. If $G \neq \emptyset$ let γ be the largest index μ in G; if $G = \emptyset$ let $\gamma = \sigma$. Now $P_\mu(O) = X_\mu$ for each index μ with $\gamma < \mu < \alpha$ and for each set O in $\mathfrak{A}$. For every $x \in Y$, there is an $O \in \mathfrak{A}$ such that $P_{\nu \geqq \alpha}(x)$ is in $P_{\nu \geqq \alpha}(O)$; since $P_\mu(x)$ is in $P_\mu(O)$ for every μ with $\gamma < \mu < \alpha$, $P_{\nu \geqq \gamma+1}(x)$ is in $P_{\nu \geqq \gamma+1}(O)$. Thus $Y_{\gamma+1} = \bigcup_{O \in \mathfrak{A}} P_{\nu \geqq \gamma+1}(O)$.

Using Lemmas 1 and 2, we now prove Theorem 1.

Proof of Theorem 1. Let the product space $\prod_{\nu \in I} X_\nu$ associated with the family $\{X_\nu \mid \nu \in I\}$ be denoted by Y, and let Y_α be defined as above for each α in I. Assume that I has a last index λ under the well-ordering $\geqq$. Let $\mathfrak{O}$ be an admissible collection of base sets in Y. Using induction on the well-ordered set I we shall define the coordinates x_ν of a point x which is in none of the members O of $\mathfrak{O}$. When x_ν has been defined for all ν less than an index α, we shall let $\mathfrak{O}_\alpha$ denote the set $\{O \in \mathfrak{O} \mid x_\nu \in P_\nu(O)$ for all $\nu < \alpha\}$. The coordinates x_ν are defined as follows so that for each $\alpha \in I$, $\mathfrak{O}_\alpha$ is admissible on Y_α: that is, so that no finite subset of the set of projections $\{P_{\nu \geqq \alpha}(O) \mid O \in \mathfrak{O}$ and $x_\nu \in P_\nu(O)$ for all $\nu < \alpha\}$ covers the product space $\prod_{\nu \geqq \alpha} X_\nu$:

(1) $\mathfrak{O}$ is admissible in Y_σ, where σ is the first index in I. Applying Lemma 1 to $\mathfrak{O}$ and $X_\sigma \times Y_{\sigma+1}$, we let C_σ be the closed subset of X_σ given by Lemma 1 and $x_\sigma = F(C_\sigma)$. $\mathfrak{O}_{\sigma+1}$ is admissible on $Y_{\sigma+1}$.

(2) Let α be an index in I with $\sigma < \alpha < \lambda$. Assume that we have defined points $x_\nu \in X_\nu$ for all $\nu < \alpha$ so that for each $\mu < \alpha$, $\mathfrak{O}_{\mu+1}$ is admissible on $Y_{\mu+1}$. $\mathfrak{O}_\alpha = \bigcap_{\mu < \alpha} \mathfrak{O}_{\mu+1}$. Therefore $\mathfrak{O}_\alpha$ is admissible on $Y_{\mu+1}$ for every $\mu < \alpha$. By Lemma 2, $\mathfrak{O}_\alpha$ is admissible on Y_α. Applying Lemma 1 to the collection $\{P_{\nu \geqq \alpha}(O) \mid O \in \mathfrak{O}_\alpha\}$ and $X_\alpha \times Y_{\alpha+1}$, we let C_α be the closed subset of X_α given by Lemma 1 and $x_\alpha = F(C_\alpha)$. $\mathfrak{O}_{\alpha+1}$ is admissible on $Y_{\alpha+1}$.

Since I is well-ordered (1) and (2) define x_ν for all $\nu < \lambda$ so that $\mathfrak{O}_{\mu+1}$ is admissible on $Y_{\mu+1}$ for all $\mu < \lambda$. Now

$$\mathfrak{O}_\lambda = \{O \in \mathfrak{O} \mid x_\nu \in P_\nu(O) \text{ for all } \nu < \lambda\} = \bigcap_{\mu < \lambda} \mathfrak{O}_{\mu+1}.$$

Thus by Lemma 2, $\mathfrak{O}_\lambda$ is admissible on X_λ. Since X_λ is compact,

$$X_\lambda - \bigcup_{O \in \mathfrak{O}_\lambda} P_\lambda(O)$$

is a nonempty closed subset C_λ of X_λ. Let $x_\lambda = F(C_\lambda)$. The point x in Y defined by the coordinates x_ν is in none of the sets O in $\mathfrak{O}$.

Using the axiom of choice, we have the following corollary of Theorem 1:

THEOREM 2 (TYCHONOFF). *The product of an arbitrary collection of compact spaces is compact in the product topology.*

4. The product of $\aleph$-compact spaces. A weaker notion than compactness is given by the notion of countable compactness. A topological space Y is said to be countably compact if any countable open covering of Y can be reduced to a finite covering. A space is countably compact iff each sequence has a cluster point. (See [5], pp. 138–141, and [3] pp. 162–163.) A generalization of countable compactness is given by the notion of $\aleph$-compactness, where a topological space Y is said to be $\aleph$-compact if any open covering of Y of cardinality $\leqq\aleph$ has a finite subcovering. J. Novak [4] has constructed two countably compact spaces whose product is not countably compact. On the other hand, Professor F. S. Cater of the University of Oregon has pointed out to the author that a slight modification of the above proof of the Tychonoff theorem proves the following theorem for $\aleph$-compact spaces in general and countably compact spaces in particular.

THEOREM 3. *Let $Y=\prod_{\nu\in I} X_\nu$ be the product of $\aleph$-compact spaces X_ν. Then any covering of Y of cardinality $\leqq\aleph$ by open sets in the standard base of Y has a finite subcovering.*

5. The dual proof. It is an interesting exercise to put the above proof of the Tychonoff theorem in the dual form, that is, in terms of closed instead of open sets. There is no standard terminology, however, which one can use. Instead of open base sets, one takes the Cartesian product $\prod_{\nu\in I} \mathcal{C}_\nu$, where each $\mathcal{C}_\nu = \{A \mid A \subset X_\nu \text{ and } A \text{ is closed}\}$, and works with the set of all elements C in $\prod_{\nu\in I} \mathcal{C}_\nu$ such that $C_\nu=\emptyset$ for all but a finite number of indices $\nu\in I$. If O is a base set in Y and C is given by $C_\nu=X_\nu-P_\nu(O)$ for all $\nu\in I$, then a point x in Y is in the complement of O iff x_ν is in C_ν for some $\nu\in I$. The dual form of Lemma 1 must be proved directly for $X_\alpha\times Y_{\alpha+1}$ instead of for arbitrary spaces X and Y.

I am indebted to Professors Halsey Royden and Hans Samelson for several suggestions which simplified the presentation of this proof. The material in this paper was presented to the American Mathematical Society in Pasadena on November 23, 1963.

This work was supported in part by U.S. Army Research Contract DA-31-124-ARO (D)-170 and by National Science Foundation Research Contract.

References

1. P. J. Cohen, The independence of the continuum hypothesis, Proc. Nat. Acad. Sci., Dec. 1963 and Jan. 1964.

2. J. L. Kelley, The Tychonoff product theorem implies the axiom of choice, Fund. Math., 37(1950).

3. J. L. Kelley, General Topology, Van Nostrand, Princeton, N.J., 1955.

4. J. Novak, On the Cartesian product of two compact spaces, Fund. Math., 40 (1953) 106–112.

5. H. L. Royden, Real Analysis, Macmillan, New York, 1963.

6. L. E. Ward, Jr., A weak Tychonoff theorem and the axiom of choice, Proc. Amer. Math. Soc., 13 (1962) 757–758.

GENERALIZED DROZ-FARNY TRANSVERSALS

S. T. KAO AND ARTHUR BERNHART, University of Oklahoma

1. General considerations. We consider an arbitrary triangle ABC cut by two transversals. Set up an oblique (i.e., affine) coordinate system using these transversals as axes. The x-axis intersects BC at $X_1(a_1, 0)$; CA at $X_2(a_2, 0)$; and AB at $X_3(a_3, 0)$. Similarly, the y-axis intersects the three sides of triangle ABC at $Y_1(0, b_1)$, $Y_2(0, b_2)$ and $Y_3(0, b_3)$. The directed segments X_1Y_1, X_2Y_2 and X_3Y_3 will be called "dorf" chords. The coordinates of the "dorf" points P, Q, R which divide the respective dorf chords in the ratio $r: 1-r$ are

$$P([1-r]a_1, rb_1),$$
$$Q([1-r]a_2, rb_2),$$
$$R([1-r]a_3, rb_3).$$

These points are collinear iff their determinant vanishes.

$$\begin{vmatrix} (1-r)a_1 & rb_1 & 1 \\ (1-r)a_2 & rb_2 & 1 \\ (1-r)a_3 & rb_3 & 1 \end{vmatrix} = (1-r)r \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0.$$

This dorf condition is satisfied for $r=0$ by the x-axis, and for $r=1$ by the y-axis. We demand that it be satisfied by *some* third value of r. Then

$$Q^* \equiv \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0, \tag{1}$$

and the dorf condition is satisfied for *all* values of r. We may interpret (1) as requiring that the points $J_1(a_1, b_1)$, $J_2(a_2, b_2)$ and $J_3(a_3, b_3)$ must be collinear.

This result invites an alternate approach to the configuration, (see Fig. 1).

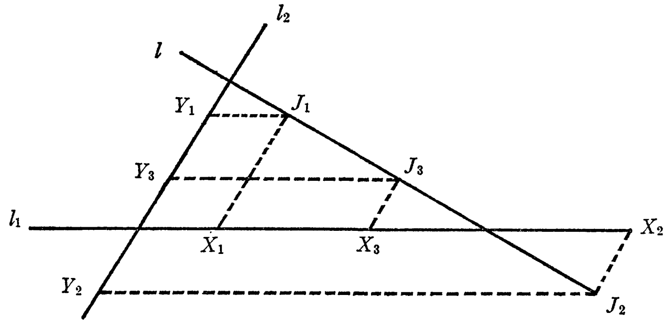


FIG. 1

Let l_1, l_2 and l be three nonconcurrent lines. On l choose arbitrarily three points J_1, J_2 and J_3. The line through J_i parallel to l_1 intercepts l_2 at Y_i; and the line through J_i parallel to l_2 intercepts l_1 at X_i. The segments X_iY_i form the sides of a triangle, unless

$$T^* \equiv \begin{vmatrix} a_1b_1 & a_1 & b_1 \\ a_2b_2 & a_2 & b_2 \\ a_3b_3 & a_3 & b_3 \end{vmatrix} = 0. \tag{2}$$

For, the intercept form of X_iY_i is $x/a_i+y/b_i=1$ and $T^*=0$ is merely the condition that the three lines $b_ix+a_iy-a_ib_i=0$ be concurrent.

An approach through projective geometry is helpful. Let range l_1 and range l_2 be connected by a projectivity which transforms X_1 into Y_1, X_2 into Y_2, and X_3 into Y_3. These ordered pairs determine the bilinear transformation

$$\begin{vmatrix} xy & x & y & 1 \\ a_1b_1 & a_1 & b_1 & 1 \\ a_2b_2 & a_2 & b_2 & 1 \\ a_3b_3 & a_3 & b_3 & 1 \end{vmatrix} = 0 \tag{3}$$

which is, in general, an equilateral hyperbola

$$Q^*xy + R^*x + S^*y + T^* = 0. \tag{3*}$$

The triangle condition $T^*\neq 0$ means that this hyperbola (3) does not pass through the origin. The dorf condition (1) means that the transformation (3) is linear, $R^*x+S^*y+T^*=0$. Moreover, the Pappus configuration of $X_1X_2X_3$ and $Y_1Y_2Y_3$ determines three cross-points $Z_1=(X_2Y_3,\ X_3Y_2)$, $Z_2=(X_3Y_1,\ X_1Y_3)$ and $Z_3=(X_1Y_2,\ X_2Y_1)$, and the points $Z_1Z_2Z_3$ are collinear. This line, called the axis of homology, or the Pascal line, has the equation

$$\begin{vmatrix} 0 & x & y & 1 \\ a_1b_1 & a_1 & b_1 & 1 \\ a_2b_2 & a_2 & b_2 & 1 \\ a_3b_3 & a_3 & b_3 & 1 \end{vmatrix} = 0 \tag{4}$$

which coincides with (3) iff condition (1) is satisfied. Consequently the requirement that P, Q, R be collinear is intimately connected with the basic concepts of projective geometry.

It is interesting to note that the collinear points J_1, J_2, J_3 lie on the Pascal line. For example, testing $J_1(a_1, b_1)$ in (4) and expanding by the first column we get $a_1b_1Q^*$, which vanishes by virtue of (1).

We may summarize these general considerations in three theorems:

THEOREM 1. *When two transversals intercept three segments on the sides of a triangle, if the points which divide these segments in a given ratio are collinear, then*

equation (1) *is satisfied. Conversely, if* (1) *is satisfied, then the points of division are collinear for every ratio.*

THEOREM 2. *Three vectors OJ_i are resolved into components OX_i, OY_i along lines l_1 and l_2 intersecting at O, so that OJ_i is the vector sum of OX_i and OY_i. The segments X_iY_i are divided by the points P_i with a common ratio. Then the points of division P_1, P_2, P_3 are collinear iff J_1, J_2, J_3 are collinear.*

COROLLARY. *Theorem* 2 *is valid for n vectors.*

THEOREM 3. *The projectivity which transforms $X_1X_2X_3$ into $Y_1Y_2Y_3$ is affine* (*i.e.*, $Q^*=0$) *iff the midpoints of X_1Y_1, X_2Y_2, X_3Y_3 are collinear.*

(Since the parallelogram $OX_iJ_iY_i$ has diagonals X_iY_i and OJ_i which bisect each other this is equivalent to saying that J_1, J_2, J_3 are collinear.)

THEOREM 4. *If the segments X_1Y_1, X_2Y_2, X_3Y_3 satisfy the dorf condition, equation* (1), *then each point (a_i, b_i) lies on the Pascal line, equation* (4), *of the projectivity $(X_1X_2X_3;\ Y_1Y_2Y_3)$.*

2. Dynamic transversals and their envelopes. By Theorem 1 the division points P, Q, R on dorf chords are collinear for each ratio, $r = X_1P:X_1Y_1 = X_2Q:X_2Y_2 = X_3R:X_3Y_3$, whenever (1) is satisfied. Let us find equations for the lines PQR and determine their envelope as the parameter r varies.

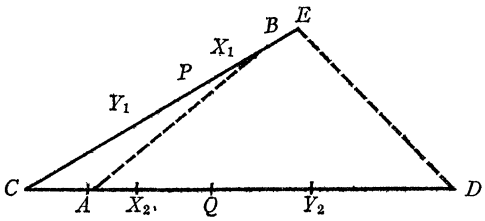


FIG. 2

For this purpose we introduce an auxiliary triangle CDE, (see Fig. 2). Choose D on CA so that X_1Y_1: $Y_1C = X_2Y_2$: Y_2D, and choose E on CB so that $EX_1:X_1Y_1 = CX_2:X_2Y_2$. This makes the position of X_1Y_1 on EC similar to that of X_2Y_2 on CD. Let $EX_1:EC = CX_2:CD = t_1$, and $EY_1:EC = CY_2:CD = t_2$. Let point P move uniformly from E at time $t=0$ to C at time $t=1$, while point Q moves uniformly from C at time $t=0$ to D at time $t=1$. The moving segment PQ has the position X_1X_2 at time $t=t_1$ and the position Y_1Y_2 at time $t=t_2$. At any time t the ratio $X_1P:PY_1$ is $(t-t_1):(t_2-t)$ which is the same as $X_2Q:QY_2$ and $X_3R:RY_3$.

Choose affine coordinates with origin $C(0, 0)$ and unit points $D(1, 0)$ and $E(0, 1)$. Then line PQR intercepts CD at $Q(t, 0)$ and CE at $P(0, 1-t)$. The intercept form of PQ becomes

$$x/t + y/(1 - t) = 1. \tag{5}$$

The dorf condition may be interpreted dynamically: As P moves uniformly along X_1Y_1 and Q moves uniformly along X_2Y_2 so also the point R, collinear with PQ, moves uniformly along X_3Y_3. Condition (1) constrains three points moving uniformly on the sides of a triangle to remain collinear at all times! Our time scale is chosen so that $t=0$ as Q passes C, and $t=1$ as P reaches C.

We exploit this dynamic viewpoint by relabelling

$$X_1 = P(t_1) = P_1, \qquad Y_1 = P(t_2) = P_2;$$

$$X_2 = Q(t_1) = Q_1, \qquad Y_2 = Q(t_2) = Q_2;$$

and

$$X_3 = R(t_1) = R_1, \qquad Y_3 = R(t_2) = R_2.$$

Point P_1 divides EC in the ratio $EP_1:P_1C=t_1:1-t_1$ and has coordinates $(0,\ 1-t_1)$ determined by the vector equation

$$P_1 = (1 - t_1)E + t_1C.$$

Similarly Q_1 divides CD in the same ratio and has coordinates $(t_1,\ 0)$ as determined by

$$Q_1 = t_1D + (1 - t_1)C.$$

The points $P_1(0, 1-t_1)$ and $Q_1(t_1, 0)$ are simply the intercepts of (5) for the parametric choice $t=t_1$.

Now consider the point W which divides P_1Q_1 in the ratio $P_1W:WQ_1=t_2:1-t_2$. We have the vector equation

$$W = t_2Q_1 + (1 - t_2)P_1,$$

whence the coordinates of W are $(t_1t_2,\ [1-t_1][1-t_2])$. As this is symmetrical in t_1 and t_2 we note that W also divides P_2Q_2 in the ratio $P_2W:WQ_2=t_1:1-t_1$.

THEOREM 5. *Any two lines P_1Q_1 and P_2Q_2 of the dynamic system* (5), *corresponding to respective times t_1 and t_2, intersect in a point W such that $P_1W:P_1Q_1=t_2$ and $P_2W:P_2Q_2=t_1$.*

Consider the point T which divides PQ in the ratio $PT:TQ=t:1-t$. The vector equation $T=tQ+(1-t)P$ gives the coordinates (x, y) of T:

$$x = t^2, \qquad y = (1 - t)^2, \tag{6t}$$

which are the parametric equations of the conic $\sqrt{x}+\sqrt{y}=1$. From $y=1-2t+t^2$ we have $x-y+1=2t$. Squaring, we obtain

$$(x - y + 1)^2 = 4x. \tag{6}$$

The quadratic terms are a perfect square, $(x-y)^2$, indicating that the conic is a parabola, with its axis of symmetry parallel to $(x-y)=0$. This is the direction of the median through C of triangle CDE. We call the T-locus given by (6) the *dynamic* parabola.

Differentiating (6t) gives $dx=2tdt$, $dy=2(t-1)dt$, indicating that the tangent line has the slope $dy:dx=(t-1):t$. Since this is also the slope of (5) we may announce:

THEOREM 6. *Every line PQ in the dynamic system* (5) *is tangent to the parabola* (6). *The point of contact T divides PQ in the ratio $PT:PQ=t$, where t is the common value of $EP:EC$ and $CQ:CD$.*

Since the T-locus is an envelope of the family of lines (5) it could also be obtained by taking its t-discriminant. Since (5) is quadratic in t, the envelope can be obtained by elementary algebra without calculus. Thus $(1-t)x+ty=t(1-t)$ can be written $t^2-(x-y-1)t+x=0$, for which (6) expresses the quadratic discriminant relation.

When the dorf condition (1) is satisfied, the third side AB of triangle ABC is also a member of the dynamic system (5). For, three transversals corresponding to times t_1, t_2 and $t=\frac{1}{2}(t_1+t_2)$ mark off dorf segments X_iY_i with their midpoints M_i. The dynamic system (5) includes an element PQ which has the same slope as BA. Let the transversals X_1X_2, Y_1Y_2 and M_1M_2, which intercept AB at X_3, Y_3 and M_3, also intercept PQ at X, Y and M. (See Figure 3.) By Theorem 5, point M is the midpoint of PQ, just as M_3 is the midpoint of X_3Y_3. We show that X_3Y_3 must coincide with PQ.

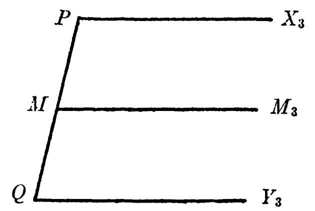


FIG. 3

Otherwise, X_3Y_3 and PQ are parallel. If these segments are also equal, they form a parallelogram, so that X_1X_2 and Y_1Y_2, as well as M_1M_2, would be parallel. Such parallel lines would intercept any transversal proportionally, so the dorf condition would be trivially satisfied. (In introducing dorf segments we excluded this contingency by postulating that the two transversals l_1 and l_2 intersected, and could be used as coordinate axes.)

Suppose that X_3Y_3 and PQ are parallel but not equal. Then PX_3 and QY_3 intersect, say at N, and MN is a median of triangle PQN. Applying the law of sines to PMN and QMN we obtain the cevian lemma,

$$PM/MQ = (PN/NQ)(\sin PNM/\sin MNQ).$$

For the median case, $PM=MQ$, the lemma becomes

$$\sin MNQ : \sin PNM = NP : NQ.$$

Since NM intersects X_1Y_1 at M_1, and NM_1 is also a median in triangle X_1Y_1N, therefore the median form of the lemma requires that $NX_1:NY_1=NP:NQ$. Accordingly X_1Y_1 is parallel to the base PQ. (The same argument applies to X_2Y_2.) But this is contrary to our assumption that the dorf chords lay along the sides of a triangle.

The foregoing contradiction shows that, indeed, X_3Y_3 coincides with PQ, and BA is tangent to parabola (6).

THEOREM 7. *Under the dorf conditions* (1) *the sides of triangle ABC and the transversals $X_1X_2X_3$ and $Y_1Y_2Y_3$ are each tangent to the dynamic parabola* (6). *Conversely, the dynamic parabola is determined by these five tangent lines.*

COROLLARY 7.1. *Let five lines be tangent to a parabola. If we treat any pair of lines as transversals of the triangle formed by the other three, the segments cut off satisfy the dorf condition.*

COROLLARY 7.2. *Let five lines be tangent to a conic. Then a dorf configuration* (*as in* 7.1) *may be formed in ten different ways iff the conic is a parabola.*

With respect to the original triangle ABC, the T-locus (6) is the parabola of Artzt inscribed in angle C. The locus of points T' isogonally conjugate in ABC to T is a *straight line* tangent to the circumcircle of ABC. This result is given by W. Fuhrmann in his *Synthetische Beweise*, (p. 159).

To locate the focus of the dynamic parabola we need Euclidean geometry. If two directly similar figures are constructed on two arbitrary segments AB and CD the center of similitude S is such that triangles SAB and SCD are similar.

For a graphical determination of S note that if AB and CD intersect at S', then circles ACS' and BDS' intersect again at S. For an analytic treatment we will employ the same letter for a point and for its Gaussian coordinate. Then

$$S = \begin{Bmatrix} A & B \\ C & D \end{Bmatrix} = G\{A, B; C, D\}, \tag{7}$$

where the symbol in braces indicates the quotient of the determinant $(AD-BC)$ by the sum of its cofactors $(A+D-B-C)$. The symbol

$$G\{A, B; C, D\} = (AD - BC)/(A + D - B - C)$$

represents the critical value of the corresponding 2×2 game. A one-dimensional projectivity $(X_1X_2X_3;\ Y_1Y_2Y_3)$ is an involution iff

$$G\{X_1, X_2; Y_1, Y_2\} = G\{X_2, X_3; Y_2, Y_3\}.$$

In the notation of (3^*) this means $R^*=S^*$.

In our current application we seek $S=G\{X_1, Y_1; X_2, Y_2\}=G\{E, C; C, D\}$. Choose S as the origin of Gaussian coordinates. Then $E:C=C:D$, therefore $C^2=DE$.

Since triangles ECS and CDS are similar their altitudes from S are proportional to the bases EC and CD, (see Fig. 4). Therefore CS lies along the symmedian of triangle CDE, and bisects angle DSE, which is twice angle DCE. Since both S and the circumcenter O of triangle CDE subtend the same angle with the chord DE, the points $DOSE$ are cyclic.

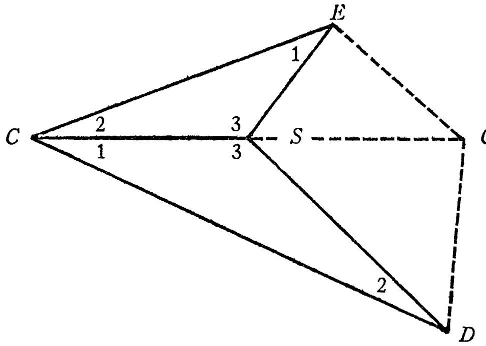


FIG. 4

But $C' = -C$ also satisfies the proportion $E:C' = C':D$. So triangles $EC'S$ and $C'DS$ are similar, $C'S$ is a symmedian of triangle $C'DE$. The quadrilateral $CDC'E$ is cyclic, for the angles at C and C' are supplementary. The line SO is the perpendicular bisector of chord CC'. Since the products of opposite sides are equal, $CD:C'E = CE\cdot C'D$, the quadrilateral is harmonic with symmedian point at the intersection of CC' and DE. (See "Polygons of Pursuit," *Scripta Mathematica*, vol. 24, p. 35.)

The point S has been defined, by (7), with respect to the auxiliary triangle CDE, formed by two particular tangents to the dynamic parabola. We next show that any pair of tangents will suffice.

We observe that $G\{E, C; C, D\} = S = G\{E, P; C, Q\}$. This follows geometrically since the center of similitude S is determined by any corresponding pairs. It also follows algebraically from (7) since $P = tC + (1-t)E$ and $Q = tD + (1-t)C$. If rows and columns are interchanged, $S = G\{E, C; P, Q\}$, which, by a repetition of the foregoing argument, reduces to $S = G\{E, P; P, R\}$. The tangent at D has been replaced by the tangent at R.

$$G\{E, C; C, D\} = S = G\{E, P; P, R\}. \tag{8}$$

Replacing first one tangent and then the other we can make the point S depend on any two tangents of the dynamic system. It remains but to show that S is the focus of the dynamic parabola. But this is evident by selecting the tangents at the ends of the *latus rectum*.

Finally, triangle SPC is similar to SQD so that angle SPC is supplementary to SQC. Accordingly $SPCQ$ is a cyclic quadrilateral, and the circumcircle of triangle PCQ passes through S.

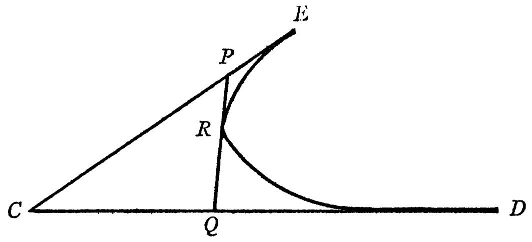


FIG. 5

THEOREM 8. *The focus S of the dynamic parabola* (6) *is the center of similitude* $G\{E, C; C, D\}$ *for the segments EC and CD formed by two tangents in the dynamic system* (5). *Also S lies on the symmedian from C of the auxiliary triangle CDE, and on the circumcircle of every triangle, such as ABC, formed by any three dynamic transversals.*

COROLLARY 8.1. *Three directed segments* X_1Y_1, X_2Y_2 *and* X_3Y_3 *satisfy the dorf condition iff they have a common center of similitude*:

$$G\{X_1, Y_1; X_2, Y_2\} = G\{X_2, Y_2; X_3, Y_3\}.$$

These cumulative considerations form a beautiful generalization of the instance considered by Droz-Farny [*Mathesis*, (1899) 162].

ON THE DISTANCE SET OF THE CANTOR MIDDLE THIRD SET, III

N. C. BOSE MAJUMDER, Calcutta University, India

Introduction. Steinhaus [1] was the first to prove that the Cantor middle third set $C \subset [0, 1]$ contains all the distances from -1 to 1, i.e. if $d \in [-1, 1]$, then the set

$$\Delta_d = \{(x, y) \mid x \in C,\ y \in C, \text{ and } y - x = d\}, \tag{1}$$

is not empty. Alternative proofs of this theorem have subsequently been given by Randolph [2] and Utz [3]. For other properties of Δ_d see [4], [5], [6], and [7].

From the results found in the papers [4] and [5], we are in a position to conjecture that Δ_d is finite for some d, and has the power c of the continuum for the other d's, but there is no d for which Δ_d is countably infinite.

In this paper we show that the conjecture is true and give a direct proof of this property of Δ_d. Another proof of the Steinhaus' theorem is an incidental result.

We shall use $\overline{\overline{A}}$ to represent the power (Cardinal number) of a set A. Hence, if $\aleph_0$ = the power of the rational set, then we shall show that, either, $1 \leqq \overline{\overline{\Delta}}_d < \aleph_0$, or $\overline{\overline{\Delta}}_d = 2^{\aleph_0} = c$, (for proof of $2^{\aleph_0} = c$, see [8]).

We first prove

THEOREM 1. *Let $d \in [-1, 1]$. Then d is such that*

$$\frac{d}{2} = \sum_{i=1}^{\infty} \frac{\gamma_i}{3^i}, \qquad \gamma_i = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}. \tag{2}$$

Also, if there is another such representation then one of them ends in only -1's, the other in only $+1$'s, one of them has a single zero more than the other, and there is no other such representation.

Proof. We use the ternary representation of

$$0 \leqq \tfrac{1}{2}(d+1) = \lambda = \sum_{i=1}^{\infty} \frac{\delta_i}{3^i}, \qquad \delta_i = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}. \tag{3}$$

Then

$$\frac{d}{2} = \lambda - \frac{1}{2} = \sum_{i=1}^{\infty} \frac{\delta_i}{3^i} - \sum_{i=1}^{\infty} \frac{1}{3^i} = \sum_{i=1}^{\infty} \frac{\delta_i - 1}{3^i} = \sum_{i=1}^{\infty} \frac{\gamma_i}{3^i}, \qquad \gamma_i = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}.$$

The first part of the theorem is thus proved.

If λ has another ternary representation,

$$\lambda = \sum_{i=1}^{\infty} \frac{\delta_i'}{3^i}, \qquad \delta_i' = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix},$$

let k be the first value of i for which $\delta_i' \neq \delta_i$, so that, $\delta_i' = \delta_i$, $i = 1, 2, \cdots, k-1$, but $\delta_k' \neq \delta_k$. Then

$$\lambda = \sum_{i=1}^{k-1} \frac{\delta_i'}{3^i} + \frac{\delta_k'}{3^k} + \sum_{i=k+1}^{\infty} \frac{\delta_i'}{3^i} = \sum_{i=1}^{k-1} \frac{\delta_i}{3^i} + \frac{\delta_k}{3^k} + \sum_{i=k+1}^{\infty} \frac{\delta_i}{3^i},$$

and hence

$$\delta_k' + \sum_{j=1}^{\infty} \frac{\delta_{k+j}'}{3^j} = \delta_k + \sum_{j=1}^{\infty} \frac{\delta_{k+j}}{3^j}. \tag{4}$$

As $\delta_k' \neq \delta_k$, we consider $\delta_k' < \delta_k$ without loss of generality, so that $\delta_k' + 1 \leqq \delta_k$.

With L the left side and R the right side of (4), we have

$$L \leqq \delta_k' + \sum_{j=1}^{\infty} \frac{2}{3^j} = \delta_k' + 1 \leqq \delta_k = \delta_k + \sum_{j=1}^{\infty} \frac{0}{3^j} \leqq R = L.$$

Therefore all equalities hold so that $\delta_k' + 1 = \delta_k$ and $\delta_i' = 2$, $\delta_i = 0$, for $i = k+1$, $k+2, \cdots$. Hence in ternary notation, either

$$\lambda = \begin{cases} \cdot\delta_1\delta_2 \cdots \delta_{k-1}\ 0\ 2\ 2\ 2 \cdots \\ \cdot\delta_1\delta_2 \cdots \delta_{k-1}\ 1\ 0\ 0\ 0 \cdots, \end{cases}$$

or else

$$\lambda = \begin{cases} \cdot\delta_1\delta_2 \cdots \delta_{k-1}\ 1\ 2\ 2\ 2 \cdots \\ \cdot\delta_1\delta_2 \cdots \delta_{k-1}\ 2\ 0\ 0\ 0 \cdots. \end{cases}$$

Consequently $d/2 = \lambda - \frac{1}{2}$ can have at most two representations of the form (2) and the further conclusions of the theorem are seen to hold.

THEOREM 2. *If* $d \in [-1, 1]$, *then the set* Δ_d *defined in* (1):

(i) *is nonempty* (*Steinhaus' Theorem*), *and*

(ii) *is either finite or has the power c of the continuum* (*i.e. there is no d for which* Δ_d *is countably infinite*).

Proof. Express d, from Theorem 1, as

$$d = \sum_{i=1}^{\infty} \frac{2\gamma_i}{3^i}, \qquad \gamma_i = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}. \tag{5}$$

With each γ_i thus determined, set

$$\begin{aligned} \alpha_i &= 1, \quad \beta_i = 0, \quad \text{if} \quad \gamma_i = -1, \\ \alpha_i &= 0, \quad \beta_i = 1, \quad \text{if} \quad \gamma_i = 1 \end{aligned}$$

and

$$\text{either} \quad \begin{Bmatrix} \alpha_i = 0 \\ \beta_i = 0 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \alpha_i = 1 \\ \beta_i = 1 \end{Bmatrix}, \quad \text{if} \quad \gamma_i = 0. \tag{6}$$

For any of the choices in (6), then

$$x = \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i} \in C, \qquad y = \sum_{i=1}^{\infty} \frac{2\beta_i}{3^i} \in C,$$

are such that

$$d = \sum_{i=1}^{\infty} \frac{2\gamma_i}{3^i} = \sum_{i=1}^{\infty} \frac{2(\beta_i - \alpha_i)}{3^i} = \sum_{i=1}^{\infty} \frac{2\beta_i}{3^i} - \sum_{i=1}^{\infty} \frac{2\alpha_i}{3^i} = y - x,$$

and (i) is established. Conversely if

$$x' = \sum_{i=1}^{\infty} \frac{2\alpha_i'}{3^i}, \quad \alpha_i' = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \text{and} \quad y' = \sum_{i=1}^{\infty} \frac{2\beta_i'}{3^i}, \quad \beta_i' = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix},$$

are such that $y'-x'=d$, then $x'\in C$, $y'\in C$ and

$$d = \sum_{i=1}^{\infty} \frac{2(\beta_i' - \alpha_i')}{3^i} = \sum_{i=1}^{\infty} \frac{2\gamma_i'}{3^i}, \qquad \gamma_i' = \beta_i' - \alpha_i' = \begin{Bmatrix} -1 \\ 0 \\ 1 \end{Bmatrix}. \tag{7}$$

Hence either $\gamma_i' = \gamma_i$ and (x', y') is one of the pairs (x, y) obtained above, or else (7) is the alternate representation of d and all other pairs (x, y) satisfying $x\in C$, $y\in C$, $y-x=d$ are obtainable by using γ_i' instead of γ_i in the recipe involving (6).

If in (5) $\gamma_i=0$ for an infinity ($\aleph_0$) of i, then (5) is the unique representation of d in this form, there are $2^{\aleph_0}=c$ choices satisfying (6) and thus $\overline{\overline{\Delta}}_d=c$.

If (5) is the unique representation of d in this form and if also (5) has $\gamma_i=0$ for a finite number (say m) of i, then $\overline{\overline{\Delta}}_d=2^m$.

Finally, if (5) and (7) are different representations of d then one of them has m (finite) and the other $m+1$ zeros and

$$\overline{\overline{\Delta}}_d = 2^m + 2^{m+1} = 3.2^m.$$

Hence the theorem is completely proved.

Another property of the Cantor set C is that each point $\lambda\in[0, 1]$ is midway between two points of C (see [2]). Incidentally we have in the following an alternative proof of this property and we also prove the

COROLLARY. *With* $0\leqq\lambda\leqq 1$, *the set*

$$\Gamma_\lambda = \left\{(x, y) \mid x\in C,\ y\in C,\ \frac{x+y}{2} = \lambda\right\},$$

is either finite (nonempty) or has the power c.

Proof. As C is symmetric in $[0, 1]$, then $(1-x)\in C$ if and only if $x\in C$. Thus

$$\Gamma_\lambda = \{(1 - x, y) \mid (1 - x) \in C,\ y \in C \text{ and } y - (1 - x) = 2\lambda - 1\}.$$

Since $-1\leqq 2\lambda-1\leqq 1$, this set is (by Theorem 2) either finite (and nonempty) or has the power c.

I am thankful to the referee for his very helpful suggestions.

References

1. H. Steinhaus, Nowa vlasnośc mnogości G. Cantora, Wektor, 1917, p. 105.
2. J. F. Randolph, Distances between points of the Cantor set, this MONTHLY, 47 (1940) 549.
3. W. R. Utz, The distance set for the Cantor discontinuum, this MONTHLY, 58 (1951) 407.
4. N. C. Bose Majumder, On the distance set of the Cantor Middle third set, Bull. Cal. Math. Soc., 51 (1959) 93.

5. N. C. Bose Majumder, Some new results on the distance set of the Cantor set, Bull. Cal. Math. Soc., 52 (1960) 1.

6. ———, Properties of the Cantor set and sets of similar type, this MONTHLY, 68 (1961) 444.

7. R. P. Boas, Jr., The distance set of the Cantor set, Bull. Cal. Math. Soc., 54 (1962) 103

8. K. Kuratowski, Introduction to Set Theory and Topology, Pergamon Press, p. 78.

A CLASSICAL THEOREM IN COMPLEX VARIABLE

E. H. CONNELL, Rice University

This is an expository paper designed to illustrate some known techniques in topological analysis. The goal of the paper is to show that if f is a function of a complex variable such that f' exists, then f' is continuous and, in fact, f'' exists. Although complete proofs are not included, the basic outline is complete and essentially self-contained. Most of the paper is taken from [1].

Let R be a region (a connected open set) in the complex plane C. The statement that $f\colon R\to C$ is differentiable means that if $z_0\in R$ then

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists. When f is differentiable at z_0, the symbol

$$\frac{f(z)-f(z_0)}{z-z_0} \text{ represents the function } h(z)=\frac{f(z)-f(z_0)}{z-z_0}$$

when $z\neq z_0$ and $h(z)=f'(z_0)$ when $z=z_0$. The function h will be continuous at z_0. If S is a circle in C, then $I(S)$ represents the interior of S.

Let $p\in C$, let S be a circle, and let $f\colon S\to C-p$ be a continuous function. Let S_1 be a circle with center p and of the same radius as S. Let g_1 be the natural radial projection of $C-p$ onto S_1 and $g_2\colon S_1\to S$ be the function which translates S_1 onto S. Then $g_2g_1f\colon S\to S$ is continuous and its homotopy class is represented by an integer n, called the index or winding number of f and written $n=\mu_S(f,p)$. If $g\colon S\to C-p$ is homotopic to f and the homotopy misses p, then $\mu_S(f,p)=\mu_S(g,p)$. If $q\in C-f(S)$ and there exists an arc from p to q in $C-f(S)$, then $\mu_S(f,p)=\mu_S(f,q)$.

LEMMA 1. *Suppose S is a circle, $f\colon S\cup I(S)\to C$ is continuous, and $p\in C-f(S)$. If $\mu_S(f,p)\neq 0$ and W is the component of the complement of $C-f(S)$ containing p, then $f(I(S))\supset W$.*

Proof. Suppose $q\in W$ and $q\notin f(I(S))$. Since there exists an arc from p to q in W, $\mu_S(f,q)\neq 0$. This is a contradiction because S may be shrunk to a point in $S\cup I(S)$. This defines a homotopy from $f|S$ to a constant map, which has index 0.

LEMMA 2. *Let $R \subset C$ be a region, $z_0 \in R$, $f: R \rightarrow C$ be continuous and let $f'(z_0)$ exist with $f'(z_0) \neq 0$. Then there exists $\delta > 0$ such that if S is a circle with center z_0 and radius less than δ then $\mu_S(f, f(z_0)) = 1$.*

Proof. (p. 72 of [1]). Let $h(z) = f(z_0) + f'(z_0)(z - z_0)$. Then $\mu_S(h, f(z_0)) = 1$. When z is close to z_0, f is close to h and thus has the same index as h.

LEMMA 3. *Let $p \in C$, S be a circle and $S_1, S_2, \cdots, S_k$ be a collection of circles in $I(S)$ such that they and their interiors are pair-wise disjoint. If*

$$f: S \cup I(S) - (I(S_1) \cup I(S_2) \cdots \cup I(S_k)) = \emptyset, \textit{ then}$$

$$\mu_S(f, p) = \sum_{i=1}^{k} \mu_{S_i}(f, p).$$

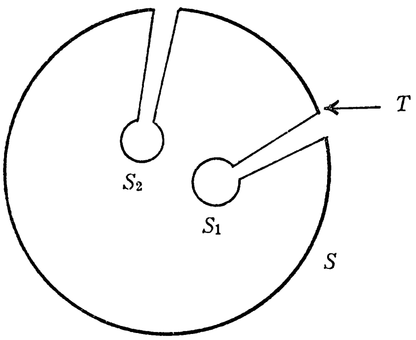


FIG. 1.

Proof. (p. 68 of [1]). Let T be the simple closed curve of Figure 1. Then $\mu_T(f, p) = 0$. (This requires the definition of index to be extended to simple closed curves, but this is not difficult.) This is true because T bounds a closed 2-cell D such that $p \notin f(D)$. Now traversing T is essentially the same as traversing S and then traversing $S_1, S_2, \cdots, S_k$ in the opposite direction. Thus

$$\mu_S(f, p) - \sum_{i=1}^{k} \mu_{S_i}(f, p) = \mu_T(f, p) = 0.$$

LEMMA 4. *Suppose $0 \subset C$ is open, $E \subset 0$, $f: 0 \rightarrow C$ is continuous on 0 and for each $z_0 \in E$, $f'(z_0)$ exists and $f'(z_0) = 0$. Then $f(E)$ has measure 0.*

Proof. (p. 72 of [1]). Note that $f(z) - f(z_0)$ is approximated by $f'(z_0)(z - z_0) = 0$ when z is close to z_0. Thus if Y is any small set containing z_0, the *ratio* of the diameter of $f(Y)$ to the diameter of Y can be made as small as desired. The proof is constructed by covering E with a collection of small sets such that the sum of the diameters of their images is $< \epsilon$.

THEOREM 1. *Suppose that S is a circle and $P\subset I(S)$ is a finite set. If h: $S\cup I(S)\to C$ is continuous and $h|I(S)-P$ is differentiable, then $|h(z_0)|\leq\max_{z\in S}|h(z)|$ for any $z_0\in I(S)$.*

Proof. (p. 73 of [1]). Suppose $z_0\in I(S)$. If $h(z_0)\in h(S)$ then $|h(z_0)|\leq\max_{z\in S}|h(z)|$ and the proof is immediate. Suppose that $h(z_0)\notin h(S)$. Let W be the component of $C-f(S)$ which contains $h(z_0)$. Since h cannot be constant on $h^{-1}(W)$, there exists a point $z_1\in I(S)$ such that $h'(z_1)$ exists, $h'(z_1)\neq 0$, and $h(z_1)\in W$. By Lemma 2 there exists a small circle S_1 with center z_1 and $\mu_{S_1}(h, h(z_1))=1\neq 0$. Let V be the component of $C-h(S_1)$ containing $h(z_1)$. By Lemma 1, $h(I(S_1))\supset V$. By Lemma 4, there exists a point $p\in V\cap W$ such that if $z\in h^{-1}(p)$, $h'(z)$ exists and $h'(z)\neq 0$. If $z\in h^{-1}(p)$, z is not a limit point of $h^{-1}(p)$, else $h'(z)$ would be 0. Since $h^{-1}(p)$ is compact, $h^{-1}(p)$ is finite, having k elements where k is a positive integer. Combining Lemma 2 and Lemma 3, $\mu_S(h, p)=k\neq 0$. Now from Lemma 1, $h(I(S))\supset W$ and the theorem follows immediately.

Hypothesis H: Let S_1 be a circle with radius $2r$ and center z_0. Let S be a circle with radius r and center z_0. Let f: $S_1\cup I(S_1)\to C$ be continuous with $f|I(S_1)$ differentiable.

THEOREM 2. *Assume Hypothesis H. If $\epsilon>0$, then there exists a $\delta>0$ such that*

$$\left|f'(z)-\frac{f(x)-f(z)}{x-z}\right|<\epsilon \quad \textit{when} \quad z\in S\cup I(S), \quad \textit{and} \quad |z-x|<\delta.$$

(Thus the difference quotient uniformly approximates the derivative on compact sets.)

Proof. There exists a $\delta>0$ such that

$$\left|\frac{f(z)-f(y)}{z-y}-\frac{f(x)-f(y)}{x-y}\right|<\epsilon \quad \begin{array}{l}\text{when } y\in S_1,\ x \text{ and } z\in S\cup I(S),\\ \text{and } |x-z|<\delta.\end{array}$$

This can easily be shown in the same manner as continuous functions on compact sets are shown to be uniformly continuous. Now for any fixed x, $z\in S\cup I(S)$ with $|x-z|<\delta$, define $h(y)$: $S_1\cup I(S_1)\to C$ by

$$h(y)=\frac{f(z)-f(y)}{z-y}-\frac{f(x)-f(y)}{x-y}.$$

Let P be the set composed of x and z. Then h is differentiable on $I(S_1)-P$ and thus by Theorem 1, $|h(y)|<\epsilon$ for all y. The proof is completed by letting $y=z$.

THEOREM 3. *Assume Hypothesis H. Then $f''(z_0)$ exists.*

Proof. Suppose that a_n is a sequence of complex numbers converging to zero with $0<|a_n|<r$ for $n=1, 2, \cdots$. Define $g_n(z)$: $S\cup I(S)\to C$ by

$$g_n(z)=\frac{f(z+a_n)-f(z)}{a_n}.$$

The proof will be completed by showing that

$$g_n'(z_0) = \frac{f'(z_0 + a_n) - f'(z_0)}{a_n}$$

is a Cauchy sequence and thus converges. Suppose that $\epsilon > 0$. From Theorem 2 it follows that there exists N such that $|g_n(z) - g_m(z)| < r\epsilon/2$ for any $z \in S \cup I(S)$ whenever $n, m > N$. Thus

$$\max_{z \in S} \left| \frac{g_n(z_0) - g_n(z)}{z_0 - z} - \frac{g_m(z_0) - g_m(z)}{z_0 - z} \right| < \epsilon \quad \text{when} \quad n, m > N.$$

By applying Theorem 1 and letting $z = z_0$, it follows that $|g_n'(z_0) = g_m'(z_0)| < \epsilon$ when $n, m > N$ and the proof is complete.

References

1. G. T. Whyburn, Topological Analysis, Princeton University Press, 1958.
2. R. L. Plunkett, Openness of the derivative of a complex function, Proc. Amer. Math. Soc., 11 (1960) 671–675.
3. S. Stoilow, Principes Topologiques de la Théorie des Fonctions Analytiques, Gauthier-Villars, Paris, 1938.
4. P. Porcelli and E. Connell, A proof of the power series expansion without Cauchy's formula, Bull. Amer. Math. Soc., 67 (1961) 177–181.
5. E. Connell, On properties of analytic functions, Duke Math. J., 28 (1961) 73–82.
6. A. Read, Higher derivatives of analytic functions from the standpoint of topological analysis, J. London Math. Soc., 36 (1961) 345–352.
7. H. Eggleston and H. Ursel, On the lightness and strong interiority of analytic functions, J. London Math. Soc., 27(1952) 260–271.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the twenty-fifth William Lowell Putnam Mathematical Competition held on December 5, 1964, have been determined in accordance with the regulations governing the Competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of California Institute of Technology, Pasadena, California. The members of the team were Norman Hyde Camien, Kenneth Kunen and Vern Sheridan Poythress; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massa-

chusetts. The members of the team were Bruce Appleby, Michael R. Rolle and Joel Spencer; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Lawrence G. Brown, Gregory W. Brumfield and Daniel M. Fendel; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Case Institute of Technology, Cleveland, Ohio. The members of the team were W. Stuart Clary, Philip J. Erdelsky and Lawrence S. Evans; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of the University of California, Berkeley, California. The members of the team were Robert Bowen, Tom Pittman and Michael Vandeman; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Robert Bowen, University of California at Berkeley; Roger Howe, Harvard University; Barry MacKichan, Harvard University; Vern Sheridan Poythress, California Institute of Technology; and Fred William Roush, University of North Carolina. To each of these a prize of seventy-five dollars is awarded. The William Lowell Putnam Prize Scholarship at Harvard has been awarded to Mr. Roger Howe, of Harvard University. The value of this scholarship is $2500 plus tuition at Harvard.

The six persons ranking second highest in the examination, named in alphabetical order, are Robert Lawrence Elgin, Pomona College; Robert Heil, Yale University; Peter Michael Kohn, Knox College; Kenneth Kunen, California Institute of Technology; Robert MacPherson, Swarthmore College; and R. L. Penner, University of Toronto.

The following teams, named in alphabetical order, won honorable mention: Michigan State University, East Lansing, Michigan, the members of the team being Stephen Elliott Crick, Jr., William A. Webb and James R. Whitney; Princeton University, Princeton, New Jersey, the members of the team being James K. Bauer, Daniel I. A. Cohen and Leopoldo Toralballa; University of California, Davis, California, the members of the team being David H. DuBois, Robert D. Hall and Larry A. Ryan; University of Pennsylvania, Philadelphia, Pennsylvania, the members of the team being Leslie Cohn, Larry Goldstein and Ralph Greenberg; and the University of Waterloo, Waterloo, Ontario, the members of the team being Roger Alfred Kingsley, Rodney Clare Wilton and Charles Robert Zarnke.

Honorable mention is given to the following twenty-four individuals, named in alphabetical order: William Ackerman, Massachusetts Institute of Technology; Bruce Appleby, Massachusetts Institute of Technology; Robert J. Bobrow, Massachusetts Institute of Technology; Fred L. Bookstein, University of Michigan; Charles Brenner, University of California at Los Angeles; Sylvian Cappell, Columbia University; Peter Van Zandt Cobb, Pomona College; Daniel I. A. Cohen, Princeton University; Victor Wayne Daniel, University of North Carolina; Philip J. Erdelsky, Case Institute of Technology; Lawrence Evans, Case Institute of Technology; Daniel M. Fendel, Harvard University; Roger Alfred Kingsley, University of Waterloo; Robert E. Maas, University of Santa Clara; T. D. MacLulich, University of Toronto; Hugh L. Montgomery, University of Illinois; Julius Z. Nadas, Case Institute of Technology; Carl Pomerance, Brown University; Michael R. Rolle, Massachusetts Institute of Technology; Milton Rosenberg, Illinois

Institute of Technology; Helen Skala, Mundelein College; Joel Spencer, Massachusetts Institute of Technology; William A. Webb, Michigan State University; and R. J. Weber, Princeton University.

A total of nineteen hundred fifteen contestants from two hundred twenty-five colleges and universities entered the Competition. Fourteen hundred thirty-nine contestants from two hundred nineteen colleges and universities participated in the examination on December 5, 1964.

The individual rankings of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

Those participating in the competition were given the following problems to solve:

Part I

1. Given a set of 6 points in the plane, prove that the ratio of the longest distance between any pair to the shortest is at least $\sqrt{3}$.
2. Find all continuous positive functions $f(x)$, for $0 \leqq x \leqq 1$, such that

$$\int_0^1 f(x)dx = 1$$

$$\int_0^1 f(x)xdx = \alpha$$

$$\int_0^1 f(x)x^2dx = \alpha^2$$

where α is a given real number.
3. Let $P_1, P_2, \cdots$ be a sequence of distinct points which is dense in the interval $(0, 1)$. The points $P_1, P_2, \cdots, P_{n-1}$ decompose the interval into n parts, and P_n decomposes one of these into two parts. Let a_n and b_n be the lengths of these two intervals. Prove that

$$\sum_{n=1}^{\infty} a_n b_n(a_n + b_n) = 1/3.$$

(A sequence of points in an interval is said to be dense when every subinterval contains at least one point of the sequence.)
4. Let p_n $(n=1, 2, \cdots)$ be a bounded sequence of integers which satisfies the recursion

$$p_n = \frac{p_{n-1} + p_{n-2} + p_{n-3}p_{n-4}}{p_{n-1}p_{n-2} + p_{n-3} + p_{n-4}}.$$

Show that the sequence eventually becomes periodic.
5. Prove that there is a constant K such that the following inequality holds for any sequence of positive numbers $a_1, a_2, a_3, \cdots$:

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + a_2 + \cdots + a_n} \leqq K \sum_{n=1}^{\infty} \frac{1}{a_n}.$$

6. Let S be a finite subset of a straight line. Say that S has the repeated distance property when every value of the distance between pairs of points of S (except for the longest) occurs at least twice. Show that if S has the repeated distance property then the ratio of any two distances between two points of S is a rational number.

Part II

1. Let u_k $(k=1, 2, \cdots)$ be a sequence of integers, and let V_n be the number of those which are less than or equal to n. Show that if

$$\sum_{k=1}^{\infty} 1/u_k < \infty,$$

then

$$\lim_{n\to\infty} V_n/n = 0.$$

2. Let S be a set of $n>0$ elements, and let $A_1, A_2, \cdots, A_k$ be a family of distinct subsets, with the property that any two of these subsets meet. Assume that no other subset of S meets all of the A_i. Prove that $k=2^{n-1}$.
3. Let $f(x)$ be a real continuous function defined for all real x. Assume that for every $\epsilon>0$

$$\lim_{n\to\infty} f(n\epsilon) = 0, \quad \text{(where } n \text{ is a positive integer).}$$

Prove that

$$\lim_{x\to\infty} f(x) = 0.$$

4. Into how many regions do n great circles (no three concurrent) decompose the surface of the sphere on which they lie?
5. Let u_n $(n=1, 2, 3, \cdots)$ denote the least common multiple of the first n terms of a strictly increasing sequence of positive integers (for example, the sequence 1, 2, 3, 4, 5, 6, 10, 12, $\cdots$). Prove that the series

$$\sum_{n=1}^{\infty} 1/u_n$$

is convergent.
6. Show that the unit disk in the plane cannot be partitioned into two disjoint congruent subsets.

Solutions. Part I

1. If three of the points are collinear, the ratio is at least 2. If not, the convex hull of the six points is either (a) a hexagon, (b) a pentagon, (c) a quadrangle, or (d) a triangle. In case (a), the convex hexagon, having angle-sum 720°, has average angle 120°.

In the other three cases, the six points include four consisting of a triangle with an interior point at which the angle sum is 360° and therefore the average angle is 120°. For, in case (b), the pentagon can be dissected into three triangles with a common vertex, and the remaining point is inside at least one of these. In case (c), either diagonal dissects the quadrangle into two triangles, at least one of which contains at least one of the remaining points. Case (d) is obvious.

In every case three of the points form a triangle with one angle at least 120°. The longest and shortest sides of this triangle have the desired ratio.

2. Multiplying the first equation by α^2, the second by -2α and the third by 1 and summing we obtain

$$\int_0^1 f(x)[\alpha - x]^2 dx = 0.$$

Thus there are no such functions $f(x)$ for any α.

3. The identity to be proved can be rewritten in the equivalent form

$$-(a_1+b_1)^3+\sum_{n=1}^{\infty}\left[(a_n+b_n)^3-a_n^3-b_n^3\right]=0.$$

Consider the partial sum

$$-a_1^3-b_1^3+(a_2+b_2)^3-a_2^3-b_2^3+(a_3+b_3)^3-a_3^3-b_3^3+\cdots$$
$$+(a_n+b_n)^3-a_n^3-b_n^3.$$

Each of the positive terms cancels one of the preceding negative terms. Because the sequence $P_0, P_1, P_2, \cdots$ is dense, every negative term eventually gets cancelled out. If we choose n sufficiently large, the only terms that are not cancelled out in the above sum are the cubes of the lengths of disjoint intervals, each of which is of length ϵ at most. The sum of the terms that are not cancelled out is therefore majorized by ϵ. It follows that the above sum is at most ϵ in absolute value for n large. Now let $n\to\infty$.

4. The same could be said about any bounded sequence of integers satisfying any recurrence relation whatever

$$p_n=f(p_{n-1}, p_{n-2}, \cdots, p_{n-N}).$$

For, if $|p_n|\leqq M$, then in a block of

$$N(2M+1)^N+N$$

consecutive p_n's there must be at least two blocks of length N that have identical entries.

A solution (perfectly legitimate) can also be obtained by showing that in our particular case all sequences p_n's satisfying the equation must end up with a string of ones.

5. Assume first that the sequence a_n is nondecreasing. We may assume that $a_n\to\infty$, otherwise the inequality is trivial. The left side can be split into the sums corresponding to even and odd n's:

$$\sum_{k=1}^{\infty}\frac{2k}{a_1+\cdots+a_{2k}}+\sum_{i=1}^{\infty}\frac{2i+1}{a_1+\cdots+a_{2i+1}}=\sum_{n=1}^{\infty}\frac{n}{a_1+\cdots+a_n}.$$

Now, if a_n is nondecreasing, then

$$\frac{2k}{a_1+\cdots+a_k+\cdots+a_{2k}}\leqq\frac{2k}{a_1+a_2+\cdots+a_k+ka_k}\leqq\frac{2k}{ka_k}=\frac{2}{a_k},$$

and similarly,

$$\frac{2i+1}{a_1+\cdots+a_i+a_{i+1}+\cdots+a_{2i+1}} \leqq \frac{2i+1}{a_1+\cdots+a_i+(i+1)a_i}$$
$$\leqq \frac{2i+1}{(i+1)a_i} < \frac{3}{a_i}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{2k}{a_1+\cdots+a_{2k}} + \sum_{i=1}^{\infty} \frac{2i+1}{a_1+\cdots+a_{2i+1}} \leqq \sum_{k=1}^{\infty} \frac{2}{a_k} + \sum_{i=1}^{\infty} \frac{3}{a_i} < 5\sum_{n=1}^{\infty} \frac{1}{a_i},$$

as was to be shown.

Now drop the assumption that the sequence a_n is nondecreasing, but $a_n \to \infty$. Let a_n^* be the nondecreasing rearrangement of the sequence a_n. Clearly $\sum_{n=1}^{\infty} 1/a_n^* = \sum_{n=1}^{\infty} 1/a_n$, and moreover, $a_1^*+a_2^*+\cdots+a_n^* \leqq a_1+a_2+\cdots+a_n$ as is easily verified. Hence, using the above,

$$\sum_{n=1}^{\infty} \frac{n}{a_1+\cdots+a_n} \leqq \sum_{n=1}^{\infty} \frac{n}{a_1^*+\cdots+a_n^*} \leqq 5\sum_{n=1}^{\infty} \frac{1}{a_n^*} = 5\sum_{n=1}^{\infty} \frac{1}{a_n}.$$

6. We can assume that the largest distance is unity. Suppose that all distances between points of S can be expressed in the form

$$(1) \qquad r_0 + r_1\xi_1 + \cdots + r_{v-1}\xi_{v-1} + r_v\xi_v$$

where $r_0, r_1, \cdots, r_v$ are rationals and $\xi_1, \xi_2, \cdots, \xi_v$ are some of the distances between points of S. For convenience denote by R any expression of the form $r_0+r_1\xi_1+\cdots+r_{v-1}\xi_{v-1}$. Assume that, if possible, the numbers $1, \xi_1, \cdots, \xi_v$ are independent over the rationals. Let m be the largest of the absolute values of the r_v's which appear in expressions of the form (1) representing distances between points of S. Let A and B be two points of S whose distance is of the form

$$\pm m\xi_v + R$$

but is also the largest possible. Let C and D denote another couple having the same distance, and assume that our labeling is such that the segment AD contains all four points $ABCD$. Thus if we set $\overline{AB}=\overline{CD}=\mu\xi_v+R$, (with $|\mu|=m$) $\overline{AD}=\alpha\xi_v+R$, $\overline{BC}=\beta\xi_v+R$, we get

(a) Either $\overline{AD}=\overline{AB}+\overline{BC}+\overline{CD}$ and thus $\alpha=2\mu+\beta$,

(b) Or $\overline{AD}=\overline{AB}+\overline{CD}-\overline{BC}$ and then $\alpha=2\mu-\beta$.

However, since we have $|\alpha|<m$, we see that neither (a) nor (b) can occur. And thus we reach a contradiction.

Part II

1. For any $N<n$ we have the estimate

$$\frac{V_n - N}{n} \leqq \sum_{k=N+1}^{V_n} 1/u_k.$$

Thus we obtain

$$\limsup_{n\to\infty} V_n/n \leqq \sum_{k=N+1}^{\infty} 1/u_k.$$

This implies the assertion.

2. Since any system of more than 2^{n-1} subsets would include two complementary sets, $k \leqq 2^{n-1}$. Suppose, if possible, $k < 2^{n-1}$. The remaining subsets, being more than 2^{n-1}, include at least one pair of complements, say A and A'. Either A or A' has a nonempty intersection with all of $A_1, \cdots, A_k$; for if A has nothing in common with A_1, then A' contains A_1, and each of $A_2, \cdots, A_k$ has a nonempty intersection with A_1 and so also with A'.

3. We prefix the following

REMARK. Let (a, b) be any interval on the real line, where $a, b > 0$. Then the set

$$\bigcup_{n=1}^{\infty} (na, nb), \qquad n = 1, 2, 3, \cdots$$

always contains a whole half line $[A, \infty)$.

Proof. If n is chosen so large that $a/b < n/(n+1)$, then $(n+1)a < nb$, and therefore the intervals (na, nb) and $((n+1)a, (n+1)b)$ overlap.

Now to the proof of the statement. Suppose the assertion is false. Then we can find an $\epsilon > 0$ and a sequence of intervals (a_i, b_i), where $a_i \to \infty$, with the property that $f(x) \geqq \epsilon$ for $a_i < x < b_i$. We now define a sequence of closed intervals $C_1, C_2, \cdots$ as follows. Let $C_1 = [a_1, b_1]$. By the Remark, we can find a closed subinterval $C_2 \subseteq C_1$ and integers n_2 and i_2 such that, for all $x \in C_2$, we have $n_2 x \in (a_{i_2}, b_{i_2})$. Proceeding in this way, we can inductively define a sequence of closed intervals C_k, where $C_k \subseteq C_{k-1}$, and integers n_k and i_k, such that $i_k > i_{k-1}$ and $n_k > n_{k-1}$, with the property that $n_k x \in (a_{i_k}, b_{i_k})$ for all $x \in C_k$. The intersection of all the closed intervals C_k is nonempty, and for $x \in \bigcap_{k=1}^{\infty} C_k$ we have $n_k x \in (a_{i_k}, b_{i_k})$ for all k. It follows that $f(n_k x) \geqq \epsilon$ for all k. But this contradicts the hypothesis of the theorem.

4. Let $f(n)$ be the number. Clearly $f(1) = 2$ and $f(2) = 4$. After $n-1$ of the circles have been drawn, yielding $f(n-1)$ regions (occurring in antipodal pairs), the nth circle, being cut twice by each of the $n-1$, is dissected into $2(n-1)$ arcs, each lying in one of the $f(n-1)$ regions. Thus $2(n-1)$ of the $f(n-1)$ regions are dissected while the rest of them are untouched. Hence

$$\begin{aligned} f(n) &= f(n-1) + 2(n-1), \\ f(n-1) &= f(n-2) + 2(n-2), \\ &\cdots\cdots\cdots\cdots \\ f(2) &= f(1) + 2. \end{aligned}$$

Adding and cancelling, we get

$$\begin{aligned} f(n) &= f(1) + 2(1 + 2 + \cdots + n - 1) \\ &= 2 + n(n - 1) \\ &= n^2 - n + 2. \end{aligned}$$

5. Taking the hint first, any divisor of a positive integer u is either of the form v, where $v \leqq \sqrt{u}$ or of the form u/v where $v \leqq \sqrt{u}$. Therefore the number of divisors of u is at most $2\sqrt{u}$. This number is an integer only if u is a square, and then $\sqrt{u}$ has been counted twice. Therefore the number of divisors of a is strictly less than $2\sqrt{u}$.

Now let $a_1, a_2, \cdots$ be the given sequence, so that $0 < a_1 < a_2 < \cdots$ and u_n is the L.C.M. of $a_1, a_2, \cdots, a_n$. Clearly, all of $a_1, a_2, \cdots, a_n$ are distinct divisors of u_n. Therefore $n < 2\sqrt{u_n}$, and $u_n > n^2/4$. The convergence of $\sum 1/u_n$ follows by comparison with $4\sum 1/n^2$.

6. Say $C = A \cup B$, where C is the unit disk and A, B are disjoint congruent subsets. Let $p \rightarrow p^*$ be the congruence of A onto B, and say the center O belongs to A, so that $O^* \in B$. Let $\overline{pq}$ be the diameter perpendicular to the line $\overline{OO^*}$, where the points p and q lie on the circumference. Denote by $|ab|$ the length of a segment $\overline{ab}$. Clearly, $|pO^*| > 1$ and $|qO^*| > 1$. But all points of A lie at a distance at most one from the center O, hence all points of B lie at a distance at most one from O^*. It follows that the points p and q belong to A. Since $|pq| = 2$, we have $|p^*q^*| = 2$, and this implies that the points p^* and q^* lie on the circumference. Therefore, either $|p^*O^*|$ or $|q^*O^*|$ is greater than one, and this contradicts $|pO| = 1 = |qO|$, completing the proof.

Erratum. This MONTHLY, 72(1965) 477–478. Mr. Samuel Klein should have been included with the persons who have been in the highest group of winners of the Putnam Competition three times instead of two times as given.

CORRECTIONS FOR "CROSS-EXAMINING PROPOSITIONAL CALCULUS AND SET OPERATIONS"

JOHN F. RANDOLPH, University of Rochester

Volume 72, No. 2, February 1965, pages 117–127.

1. In Figure 5b. the dot in the bottom quadrant was all but lost in printing.
2. Page 126 change

$$\overline{\bar{A}BC \cup (A\bar{B} \cup \bar{A}B)\bar{C}} \quad \text{to} \quad \overline{ABC \cup (A\bar{B} \cup \bar{A}B)\bar{C}}\,.$$

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

RATIONAL POWERS OF POWER SERIES

R. E. VON HOLDT, Lawrence Radiation Laboratory, University of California, Livermore

1. THEOREM. *If, for n a rational number,*

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n \equiv \sum_{k=0}^{\infty} c_k(n) x^k, \qquad a_0 > 0, \tag{1.1}$$

and

$$S(m, n) \equiv \sum_{k=0}^{m} [(n+1)k - m] a_k c_{m-k}(n), \qquad m \geqq 0, \tag{1.2}$$

then the $c_k(n)$ satisfy

$$c_0(n) = a_0^n, \tag{1.3}$$

$$S(m, n) = 0, \qquad m \geqq 1. \tag{1.4}$$

The special case of this theorem for n a positive integer and $a_0 \neq 0$, appears in Ryshik-Gradstein, "Tables of Series, Products, and Integrals," Deutscher Verlag der Wissenschaften, Berlin, 1957.

2. Lemmas.

LEMMA 1. *If $\{s_k\}_{k=1}^{\infty}$ is a sequence such that each subsequence of the form $\{s_k\}_{k=1}^{r}$ satisfies a linear dependence relation with nonzero coefficient of s_r, then $s_k = 0$ for all k.*

The lemma is obvious by mathematical induction.

LEMMA 2. *If $a(i, j) = -a(j, i)$ then*

$$\sum_{i=0}^{m} \sum_{j=0}^{m-i} a(i, j) = 0.$$

Interchanging the order of summation and interchanging the dummy indices i and j makes the sum equal to its negative and therefore equal to zero.

3. Verification of the theorem for $n = 0$.

From (1.1)

$$c_0(0) = 1, \tag{3.1}$$

$$c_m(0) = 0, \qquad m \geqq 1. \tag{3.2}$$

From (1.3)

$$c_0(0) = \overset{0}{a_0} = 1. \tag{3.3}$$

From (1.4) with (1.2)

$$\sum_{k=0}^{m-1} (k - m)a_k c_{m-k}(0) = 0, \qquad m \geqq 1. \tag{3.4}$$

(3.1) and (3.3) verify (1.3) while (3.2) and (3.4) with Lemma 1 verify (1.4).

4. Proof of the theorem for n an integer. From (1.1)

$$\sum_{k=0}^{\infty} c_k(r+1)x^k \equiv \sum_{l=0}^{\infty} a_l x^l \sum_{p=0}^{\infty} c_p(r)x^p.$$

Reordering the double sum so that the outer sum is over powers of x, we have

$$\sum_{k=0}^{\infty} c_k(r+1)x^k \equiv \sum_{k=0}^{\infty} x^k \sum_{l=0}^{k} a_l c_{k-l}(r)$$

or

$$c_k(r+1) = \sum_{l=0}^{k} a_l c_{k-l}(r) \tag{4.1}$$

and, in particular,

$$c_0(r+1) = a_0 c_0(r). \tag{4.2}$$

From (1.2) and (4.1)

$$S(m, r+1) = \sum_{k=0}^{m} [(r+2)k - m]a_k \sum_{l=0}^{m-k} a_l c_{m-k-l}(r).$$

Interchanging the order of summation, we have

$$S(m, r+1) = \sum_{l=0}^{m} a_l \sum_{k=0}^{m-l} \left\{[(r+1)k - (m-l)] + (k-l)\right\} a_k c_{m-k-l}(r).$$

From (1.2)

$$S(m, r+1) = \sum_{l=0}^{m} a_l S(m-l, r) + \sum_{l=0}^{m} \sum_{k=0}^{m-l} (k-l)a_k a_l c_{m-k-l}(r).$$

By Lemma 2 the double sum is zero, hence

$$S(m, r+1) = \sum_{l=0}^{m} a_l S(m-l, r). \tag{4.3}$$

From (4.2) and (4.3) with Lemma 1 we see that the theorem is true for $n=r+1$ if and only if the theorem is true for $n=r$. Since the theorem is true for $n=0$, by mathematical induction the theorem holds for n an integer.

5. Verification of the theorem for n a rational number. From (1.1)

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{p/q} \equiv \sum_{k=0}^{\infty} c_k(p/q)x^k, \qquad q \neq 0$$

for p and q integers. Since the power p/q will not be needed in the argument we will replace $c_k(p/q)$ by c_k. Taking the qth power of both sides of the above relation we have

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^{p} \equiv \left(\sum_{k=0}^{\infty} c_k x^k\right)^{q} \equiv \sum_{k=0}^{\infty} b_k x^k,$$

where:

$$b_0 = a_0^p = c_0^q, \tag{5.1}$$

$$\sum_{k=0}^{l} [(p+1)k - l]a_k b_{l-k} = 0, \tag{5.2}$$

$$\sum_{k=0}^{l} [(q+1)k - l]c_k b_{l-k} = 0. \tag{5.3}$$

(5.1) verifies (1.3).

Let $T(l, m)$ be the difference between c_m times (5.2) and a_m times (5.3), then

$$0 = T(l, m) = \sum_{k=0}^{l} (k - l)(a_k c_m - a_m c_k)b_{l-k} + \sum_{k=0}^{l} k(pa_k c_m - qa_m c_k)b_{l-k}.$$

From this we have

$$0 = \sum_{s=0}^{l} T(l-s, s) = \sum_{s=0}^{l}\sum_{k=0}^{l-s} (k - l + s)(a_k c_s - a_s c_k)b_{l-s-k}$$
$$+ \sum_{s=0}^{l}\sum_{k=0}^{l-s} k(pa_k c_s - qa_s c_k)b_{l-s-k}.$$

By Lemma 2 the first double sum is zero. Reordering the second double sum such that the outer sum is on $m=s+k$, we have

$$0 = \sum_{m=0}^{l} b_{l-m}\left\{\sum_{k=0}^{m} kpa_k c_{m-k} - \sum_{s=0}^{m} (m-s)qa_s c_{m-s}\right\}.$$

Replacing the dummy index s by k, we have

$$0 = \sum_{m=0}^{l} b_{l-m}\sum_{k=0}^{m} [(p+q)k - mq]a_k c_{m-k},$$

or

$$0=q\sum_{m=0}^{l} b_{l-m}S(m,\ p/q).$$

From (1.2) $S(0,\ n)\equiv 0$; hence our last relation is equivalent to

$$0=\sum_{m=1}^{l} b_{l-m}S(m,\ p/q),\qquad l\geqq 1.$$

Using this with Lemma 1 we have $S(m,\ p/q)=0,\ m\geqq 1$; thus (1.4) is verified.

This work was performed under the auspices of the U. S. Atomic Energy Commission.

AN INTEGRAL INVOLVING TURAN'S EXPRESSION FOR BESSEL POLYNOMIALS

S. K. CHATTERJEA, Bangabasi College, Calcutta, India

1. Krall and Frink [1] introduced the polynomial

$$y_n(x)=\sum_{r=0}^{n}\frac{(n+r)!}{(n-r)!r!}\left(\frac{x}{2}\right)^r; \tag{1.1}$$

it satisfies the differential equation

$$x^2y''+2(x+1)y'-n(n+1)y=0, \tag{1.2}$$

and they established among other properties:

$$y_{n+1}(x)=(2n+1)xy_n(x)+y_{n-1}(x), \tag{1.3}$$

$$\frac{1}{2\pi i}\int_c y_m(x)y_n(x)e^{-2/x}dx=\frac{(-1)^{n+1}2\delta_{mn}}{2n+1}, \tag{1.4}$$

where c is the unit circle or any contour surrounding $x=0$ and $\delta_{mn}=0,\ 1$ according as $m\neq n,\ m=n$.

Burchnall [2] found additional properties of these polynomials with the definition:

$$\theta_n(x)=x^ny_n(1/x). \tag{1.5}$$

In the present note we derive a recurrence relation for the Turan expression

$$\Delta_n=y_n(x)y_{n+2}(x)-y_{n+1}^2(x), \tag{1.6}$$

which readily yields the explicit evaluation of Δ_n, and finally we evaluate the integral

$$\frac{1}{2\pi i}\int_c\left(\frac{\Delta_n}{x^2}-\frac{1}{x}\right)e^{-2/x}dx.$$

2. From (1.3) we easily derive

$$\Delta_n + \Delta_{n-1} = 2xy_n(x)y_{n+1}(x). \tag{2.1}$$

Again,

$$\Delta_{n-1} + \Delta_{n-2} = 2xy_{n-1}(x)y_n(x). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} \Delta_n &= \Delta_{n-2} + 2xy_n(x)\left[y_{n+1}(x) - y_{n-1}(x)\right] \\ &= \Delta_{n-2} + 2(2n+1)x^2y_n^2(x). \end{aligned} \tag{2.3}$$

Now, putting $n=2m$, and applying (2.3) repeatedly, we obtain

$$\Delta_{2m} = x + 2x^2 \sum_{i=0}^{m} (4i+1)y_{2i}^2(x). \tag{2.4}$$

Similarly, putting $n=2m-1$, we derive

$$\Delta_{2m-1} = x + 2x^2 \sum_{i=1}^{m} (4i-1)y_{2i-1}^2(x). \tag{2.5}$$

It may be noted that (2.4) and (2.5) can be combined into a single formula

$$\Delta_n = x + 2x^2 \sum_{i=0}^{[n/2]} (2n+1-4i)y_{n-2i}^2(x), \tag{2.6}$$

which, when written in Burchnall's notation, reads thus (cf. [3])

$$\theta_n(x)\theta_{n+2}(x) - \theta_{n+1}^2(x) = x^{2n+1} + 2 \sum_{j=0}^{[n/2]} (2n+1-4j)x^{4j}\theta_{n-2j}^2(x). \tag{2.7}$$

Carlitz [4] obtained (2.7) in a different manner in order to prove that

$$\theta_n(x)\theta_{n+2}(x) - \theta_{n+1}^2(x) \geqq 0, \qquad x \geqq 0. \tag{2.8}$$

Here we remark that our method furnishes a shorter proof of (2.8). Szegö [5] initiated the study of the Turan inequality. It is interesting to note from (2.6) that

$$\Delta_n - x \geqq 0, \qquad \text{for all real } x. \tag{2.9}$$

Lastly by means of the orthogonality relation (1.4) and the explicit evaluation (2.6), we obtain the integral relation

$$\frac{1}{8\pi i}\int_c \left(\frac{\Delta_n}{x^2} - \frac{1}{x}\right)e^{-2/x}dx = (-1)^{n+1}\{[n/2]+1\}. \tag{2.10}$$

References

1. H. L. Krall and O. Frink, A new class of orthogonal polynomials: the Bessel polynomials, Trans. Amer. Math. Soc., 65 (1949) 100–115.
2. J. L. Burchnall, The Bessel polynomials, Canad. J. Math., 3 (1951) 62–68.
3. S. K. Chatterjea, On Turan's expression for the Bessel polynomials, to appear in Science and Culture (Indian Sci. News Assoc.)
4. L. Carlitz, A note on the Bessel polynomials, Duke Math. J., 24 (1957) 151–162.
5. G. Szegö, On an inequality of P. Turan, concerning Legendre polynomials, Bull. Amer. Math. Soc., 54 (1948) 401–405.

A PROPERTY OF THE τ-FUNCTION

N. C. SCHOLOMITI, University of Illinois, Chicago

The function $\tau(x)$ in number theory represents the number of positive integral divisors or factors of the positive integer x. The word divisor is used here in the usual sense according to which the statement that "the positive integer b is a divisor of the integer a" means that a/b is an integer or $a\equiv 0 \bmod b$.

We shall prove here by mathematical induction a property of $\tau(x)$ which was first derived and given by M. Lerch, in the year 1887, who made use of generating functions defined by power series. For an account of work relating to Lerch's formula the interested reader may consult L. E. Dickson's *A History of the Theory of Numbers*, vol. I, p. 307, and succeeding pages.

This property (Lerch's formula) can be stated as follows:

$$\tau(n) = n - \sum_{k=1}^{n-1} t(n-k, k), \tag{1}$$

where t is given in definition (1) below.

DEFINITIONS. (1) *Let $t(n-k, k)$ be the number of divisors of $n-k$, each of which is greater than k, n and k integers, $n>k\geqq 0$. Specifically $t(n, 0)=\tau(n)$.*

(2) *Let $E(n, x)=1$ if $n\equiv 0 \bmod x$, and $E(n, x)=0$ if $n\not\equiv 0 \bmod x$; n and x positive integers.*

An immediate consequence of these definitions is the property: $\sum_{u=1}^{n+1} E(n+1, u)=\tau(n+1)=t(n+1, 0)$ and thence follows:

$$\sum_{u=1}^{n} E(n+1, u) = \left(\sum_{u=1}^{n+1} E(n+1, u)\right) - E(n+1, n+1),$$

or

$$\sum_{u=1}^{n} E(n+1, u) = t(n+1, 0) - 1. \tag{2}$$

Another property is the following:

$$n \equiv n' \bmod x \quad \text{implies} \quad E(n, x) = E(n', x). \tag{3}$$

Next observe that

$$t(n-k,k) = E(n-k,k+1) + t(n-k,k+1). \tag{4}$$

This is an immediate consequence of the definitions (1) and (2) given above. (4) may be written thus:

$$t(n-k,k) = E(n-k,k+1) + t(n+1-(k+1),k+1)$$

and since $n-k \equiv n+1 \bmod (k+1)$ we have, using (3):

$$t(n-k,k) = E(n+1,k+1) + t(n+1-(k+1),k+1). \tag{5}$$

We state now the theorem:

$$n = \sum_{k=0}^{n-1} t(n-k,k). \tag{6}$$

The proof by mathematical induction is easy. (6) holds when $n=1$, because then $k=0$ since $n>k\geqq 0$: $1=t(1,0)$.

Next we prove that

$$n = \sum_{k=0}^{n-1} t(n-k,k) \quad \text{implies} \quad n+1 = \sum_{k=0}^{n} t(n+1-k,k).$$

This is the inductive step. To prove it we effect a summation over (5):

$$\sum_{k=0}^{n-1} t(n-k,k) = \sum_{k=0}^{n-1} E(n+1,k+1) + \sum_{k=0}^{n-1} t(n+1-(k+1),k+1).$$

The left hand member is n by (6). On the right we shift indices, putting $k+1=u$:

$$n = \sum_{u=1}^{n} E(n+1,u) + \sum_{u=1}^{n} t(n+1-u,u).$$

Substituting from (2): $n=t(n+1,0)-1+\sum_{u=1}^{n} t(n+1-u,u)$, or

$$n+1 = \sum_{u=0}^{n} t(n+1-u,u).$$

This completes the proof of the theorem.

Equation (6) may be written thus:

$$n = \sum_{k=0}^{n-1} t(n-k,k) = t(n,0) + \sum_{k=1}^{n-1} t(n-k,k)$$

or $t(n,0)=\tau(n)=n-\sum_{k=1}^{n-1} t(n-k,k)$. But this is (1) and our result is proved.

As an example we have:

$$\tau(17) = 17 - (t(16,1) + t(15,2) + t(14,3) + \cdots + t(9,8))$$

$$\text{or } \tau(17) = 17 - (4+3+2+1+2+1+1+1)$$

or $\tau(17) = 17 - 15 = 2$.

Observe that

$$E(n, x) = 1 + \left[\frac{n}{x}\right] + \left[-\frac{n}{x}\right],$$

where $[u]$ means, as usual, the greatest integer not greater than u. Further

$$\tau(n) = \sum_{k=1}^{n} E(n, k) = n + \sum_{x=1}^{n}\left(\left[\frac{n}{x}\right] + \left[-\frac{n}{x}\right]\right).$$

TRANSFORM METHODS FOR DIFFERENCE EQUATIONS

C. A. Grimm and W. Walther, South Dakota School of Mines and Technology

The Dirichlet series transform, which has been discussed at some length in [1] and [2], may be written in the form:

$$D\{f(n)\} = \sum_{n=0}^{\infty} a^{-sn} f(n), \qquad a > 1, s > 1.$$

Index values, 0 and ∞, and the function a^{-sn} have clearly been chosen by analogy to the Laplace transform. The generating function transform,

$$G\{f(n)\} = \sum_{n=0}^{\infty} s^n f(n),$$

is also well known (see [3]), as is the Laurent series transform (see [4]),

$$L\{f(n)\} = \sum_{n=1}^{\infty} z^{-n} f(n), \qquad |z| > R.$$

The intent of this paper is to examine transform methods for difference equations subject only to the initial requirement that the transform of $f(n+1)$ should be expressible in terms of the transform of $f(n)$. The goal will be to determine a "most suitable" transform.

We let

$$T\{f(n)\} = \sum_{n=p}^{q} g(s, n) f(n) = \phi(s) \tag{1}$$

in which n, p and q are nonnegative integers, s is a real variable, $g(s, n)$ is defined for all s on some range R and for $n \geqq p$. In addition, $g(s, p) \neq 0$ for any s on R. Then

$$T\{f(n+1)\} = \sum_{n=p}^{q} g(s, n) f(n+1) = \theta(s). \tag{2}$$

The sum (1) does not contain $f(q+1)$, which appears in the sum (2), but does contain $f(p)$, which does not appear in (2). Taking into account the coefficient function which would multiply $f(q+1)$ if (1) were extended to one more term we are led to consider the expression

$$\phi(s) + g(s, q+1)f(q+1) - g(s, p)f(p),$$

and we have the following theorem:

THEOREM I. *A necessary and sufficient condition that* $T\{f(n+1)\} = \zeta(s)[T\{f(n)\} + g(s, q+1)f(q+1) - g(s, p)f(p)]$ *for every* s *on* R *and all* $n \geqq p$ *and every function of* n *defined for* $n \geqq p$ *is that* $\zeta(s)g(s, n+1) = g(s, n)$.

Proof. Sufficiency. Let $\zeta(s)g(s, n+1) = g(s, n)$.

$$\begin{aligned} T\{f(n+1)\} &= \sum_{n=p}^{q} g(s, n)f(n+1) = \sum_{n=p}^{q} \zeta(s)g(s, n+1)f(n+1) \\ &= \zeta(s)[T\{f(n)\} + g(s, q+1)f(q+1) - g(s, p)f(p)]. \end{aligned}$$

Necessity. We now have

$$\begin{aligned} \sum_{n=p}^{q} g(s, n)f(n+1) &= \zeta(s)\left[\sum_{n=p}^{q} g(s, n)f(n) + g(s, q+1)f(q+1) - g(s, p)f(p)\right] \\ &= \zeta(s)\sum_{n=p}^{q} g(s, n+1)f(n+1). \end{aligned}$$

Hence $\sum_{n=p}^{q} [\zeta(s)g(s, n+1) - g(s, n)]f(n+1) = 0$ for each $f(n)$, $n \geqq p$ and $q \geqq p$. Thus for $f(n) = 1$ and all $q \geqq p$,

$$\sum_{n=p}^{q} [\zeta(s)g(s, n+1) - g(s, n)] = 0. \tag{3}$$

In (3), with $q = p$, we have $\zeta(s)g(s, p+1) - g(s, p) = 0$. Further, since (3) is to be zero for every value of $q \geqq p$, it follows for $q > p$ that

$$\begin{aligned} 0 &= \sum_{n=p}^{q-1} [\zeta(s)g(s, n+1) - g(s, n)] + [\zeta(s)g(s, q+1) - g(s, q)] \\ 0 &= 0 + \zeta(s)g(s, q+1) - g(s, q). \end{aligned}$$

Hence, for every $n \geqq p$,

$$\zeta(s)g(s, n+1) = g(s, n). \tag{4}$$

We consider next the type of kernel functions available under the condition (4). Since we have assumed $g(s, p) \neq 0$ for all s on R it follows that $\zeta(s)g(s, n+1) \neq 0$ for all s on R and further that $\zeta(s) \neq 0$ and $g(s, n) \neq 0$ for s on R and all integers $n \geqq p$. If we let $g_1(s, n)$ and $g_2(s, n)$ be any two solutions of (4), then

together with the above remarks we may write $g_1(s, n)/g_2(s, n) = g_1(s, n+1)/g_2(s,n+1)$ for all s on R and $n \geqq p$. Thus,

$$g_1(s, p)/g_2(s, p) = g_1(s, p + h)/g_2(s, p + h) \tag{5}$$

for all integers $h \geqq 1$. But (5) shows that the ratio $g_1(s, n)/g_2(s, n)$ is independent of n, for $n \geqq p$, and hence may be regarded as a function of s alone. Thus it is seen that

$$g_1(s, n) = \alpha(s) g_2(s, n). \tag{6}$$

Any two solutions of (4) are then related by a functional multiple as shown by (6). To find a solution of (4) we let $g(s, n) = A^n$ where A is a function of s. Hence $A^n = \zeta(s)A^{n+1}$ and therefore $A = 1/\zeta(s)$. By (6), the general solution of (4) is $g(s, n) = \alpha(s)\zeta(s)^{-n}$, $\alpha(s)$ arbitrary. Thus we are led to consider transforms of the type,

$$T\{f(n)\} = \alpha(s) \sum_{n=p}^{q} \zeta(s)^{-n} f(n). \tag{7}$$

From the proof of Theorem I we have

$$T\{f(n + 1)\} = \zeta(s)[T\{f(n)\} + \alpha(s)\zeta(s)^{-(q+1)}f(q + 1) - \alpha(s)\zeta(s)^{-p}f(p)]. \tag{8}$$

To simplify (8) it seems desirable to take $p = 1$, rather than zero, and $\alpha(s) \equiv 1$. A further simplification is obtained if $\zeta(s)^{-(q+1)}f(q+1) = 0$. Since this cannot be expected to happen in general for any finite q we impose the additional requirement that $\sum_{n=1}^{\infty} \zeta(s)^{-n}f(n)$ converge, and hence that $\lim_{n\to\infty} \zeta(s)^{-n}f(n) = 0$. With these considerations, the transform given by (7) has been specialized to

$$T\{f(n)\} = \sum_{n=1}^{\infty} \zeta(s)^{-n} f(n). \tag{9}$$

The Dirichlet series transform, with the exception of the lower limit, results if we take $\zeta(s) = a^s$; the generating function transform is obtained by taking $\zeta(s) = 1/s$. The more natural course, however, is to take $\zeta(s) = s$, the most simple nontrivial function of s. Thus we have the Laurent series transform as our "most suitable" transform and we have further specialized (9) to

$$T\{f(n)\} = \sum_{n=1}^{\infty} s^{-n} f(n). \tag{10}$$

Although we apparently lost analogy to the Laplace transform in the choice of the lower limit, 1, in actuality we have gained analogy immensely, in some cases, as can be seen by an examination of the listed results (a) through (j), given at the end of the paper.

From the ratio test, if $\lim_{n\to\infty} |f(n+1)/f(n)| = L$, then (10) converges absolutely and uniformly for all $s \geqq r > L$. Thus

$$\lim_{n\to\infty} |f(n + 1)/f(n)| = L < \infty \tag{11}$$

is a sufficient condition for the existence of a transform. Another condition for convergence is the usual exponential order condition. That is, if there is an $S>0$ such that for all $s \geqq S>0$

$$s^{-n}|f(n)| < M \quad \text{for } n > N, \tag{12}$$

where M is a constant, then (10) converges. Condition (11) can be shown to imply condition (12). The converse is not true as can be seen by considering $f(n)=\cos(n\pi/4)$ for which (12) holds but (11) does not.

We conclude by listing some results obtainable for this transform:

(a) $\lim_{s\to\infty} T\{f(n)\}=0$,
(b) $\lim_{s\to\infty} sT\{f(n)\}=f(1)$,
(c) if $T\{f(n)\}=T\{g(n)\}$, then $f(n)=g(n)$ for every $n\geqq 1$,
(d) $T\{Ef(n)\}=T\{f(n+1)\}=sT\{f(n)\}-f(1)$,
(e) $T\{E^{-1}f(n)\}=T\{f(n-1)\}=s^{-1}T\{f(n)\}+s^{-1}f(0)$,
(f) $T\{r^n f(n)\}=T\{f(n)\}\big|_{s\to s/r}$,
(g) $T\{nf(n)\}=-s(d/ds)T\{f(n)\}$,
(h) $T\{f(n)/n\}=\int_s^\infty s^{-1}T\{f(n)\}ds$,
(i) $T\{f(n)\}T\{g(n)\}=s^{-1}T\{\sum_{k=1}^n f(k)g(n-k+1)\}$,
(j) if $T\{f(n)\}=\phi(s)$, then $f(n)=d^nh(s)/ds^n\big|_{s=0}/n!$ where $h(s)=\phi(1/s)$.

It is a simple matter to construct a useful table of transforms. Only a few basic functions need to be transformed and then the more elaborate transforms are then obtained by (f), (g), and (h).

References

1. Tomlinson Fort, Linear difference equations and the Dirichlet series transform, this MONTHLY, 62 (1955) 641–645.
2. J. Ray Hanna, The Dirichlet series transform, this MONTHLY, 64 (1957) 576–581.
3. Charles Jordan, Calculus of finite differences, Chelsea, New York, 1947.
4. D. F. Lawden, On the solution of linear difference equations, Math. Gazette, 36 (1952) 193–196.

BOUNDED LINEAR FUNCTIONALS ON L_q

W. FULKS, University of Colorado

There is an elegant argument due to F. Riesz (see, e.g. F. Riesz and B. Sz.-Nagy, Functional Analysis, New York, 1955) which establishes the representation theorem for bounded linear functionals on the space L_q of measurable functions f whose qth powers, $|f|^q$, are integrable with respect to Lebesgue measure on the real line. It seems not to have been observed that the same argument applies to the space L_q of an arbitrary totally σ-finite measure space (X, S, μ). This note is a transcription of Riesz's argument to the more general situation.

We follow the notation and nomenclature of Halmos (Measure Theory, Princeton, 1950). Since we are concerned with a fixed measure μ, all references to measurability and integrability refer to μ. In particular any statement quali-

fied by the remark, $[\mu]$, means that the statement holds pointwise except on a set of measure zero (e.g. $f_1=f_2[\mu]$).

We discuss the case where $1<q<\infty$, the case $q=1$ being handled analogously. Thus p and q are assumed without further comment to be conjugate indices: they are positive numbers for which $1/p+1/q=1$.

LEMMA. *Let (X, S, μ) be a totally σ-finite measure space and let ν be a set function defined on measurable sets of finite measure which defines on each such set a totally finite signed measure. A necessary and sufficient condition that ν be the indefinite integral of a function $f\in L_p(X, S, \mu)$ is that every finite sum of the form*

$$\sum_i \frac{|\nu(E_i)|^p}{[\mu(E_i)]^{p-1}}$$

where the E_i's are disjoint sets of finite nonzero measure, be uniformly bounded. Furthermore, if B^p is the least upper bound of these sums, then $B=\|f\|_p$.

Proof. Necessity. If $f\in L_p$, then by Hölder's inequality

$$|\nu(E_i)|^p = \left|\int_{E_i} fd\mu\right|^p \leqq \int_{E_i} |f|^p d\mu[\mu(E_i)]^{p-1}.$$

Then

$$\sum_i \frac{|\nu(E_i)|^p}{[\mu(E_i)]^{p-1}} \leqq \sum_i \int_{E_i} |f|^p d\mu \leqq \int_X |f|^p d\mu = \|f\|_p^p$$

so that the sums are uniformly bounded and $B\leqq\|f\|_p$.

Sufficiency. For $\mu(F)<\infty$ we have $|\nu(E)|^p\leqq B^p[\mu(E)]^{p-1}$ for every measurable set $E\subset F$ if $\mu(E)>0$. Thus $\nu(E)\to 0$ as $\mu(E)\to 0$ so that ν is absolutely continuous with respect to μ. By the Radon-Nikodym theorem there is a function f, integrable on F, for which $\nu(E)=\int_E fd\mu$ for every measurable $E\subset F$. If F_1 and F_2 are two sets of finite measure and f_1 and f_2 are the corresponding Radon-Nikodym derivatives, then $f_1=f_2[\mu]$ on $F_1\cap F_2$, since for every measurable $E\subset F_1\cap F_2$ we have $\nu(E)=\int_E f_1 d\mu=\int_E f_2 d\mu$. Now X can be represented as the limit of an increasing sequence of measurable sets F_i of finite measure, so that we can define f uniquely $[\mu]$ on X with the property that $\nu(E)=\int_E fd\mu$ for every measurable E of finite measure.

Now let F be any set of finite measure and define $E_{k,n}=\{x\in F: k/2^n\leqq f(x)<(k+1)/2^n\}$, $k=-n2^n, \cdots, -1, 0, 1, 2, \cdots, n2^n-1$

$$E_{-n2^n-1,n}=\{x\in F: f(x)<-n\}, \qquad E_{n2^n,n}=\{x\in F: f(x)\geqq n\},$$

and

$$f_n=\sum_k \frac{\nu(E_{k,n})}{\mu(E_{k,n})}\chi_{E_{k,n}}$$

where χ_A is the characteristic function of A, and where the sum is extended over

all k from $-n2^n-1$ to $n2^n$ except those for which $\mu(E_{k,n})=0$. Clearly $|f_n(x)-f(x)|\leqq 1/2^n$ for all admissible k (i.e. $\mu(E_{k,n})\neq 0$) except possibly when $k=-n2^n-1$ or $n2^n$. For these terminal values of k we have $\mu(E_{k,n})\to 0$ as $n\to\infty$ since $|f| = \infty$ on at most a set of measure zero. Thus $f_n\to f[\mu]$ in F, consequently $|f_n|^p\to|f|^p[\mu]$ in F. But

$$\int_F |f_n|^p d\mu = \sum_k \frac{|\nu(E_{k,n})|^p}{[\mu(E_{k,n})]^{p-1}} \leqq B^p,$$

and by Fatou's lemma $\int_F|f|^p d\mu\leqq B^p$, which holds for every F of finite measure.

If $\{F_i\}$ is an increasing sequence of sets of finite measure for which $\lim F_i = X$, we set $\phi_i=|f|^p\chi_{F_i}$, and apply the monotone convergence theorem to conclude $|f|^p$ integrable and

$$\int_X |f|^p d\mu \leqq B^p.$$

This completes the proof of the lemma.

THEOREM. *Let (X, S, μ) be a totally σ-finite measure space and let A be a bounded linear functional on $L_q(X, S, \mu)$. Then there is a function $f\in L_p(X, S, \mu)$ such that for every g in $L_q(X, S, \mu)$ we have*

$$Ag = \int_X fg d\mu,$$

where $\|A\| = \|f\|_p$.

Proof. If E has finite measure, then $\chi_E\in L_q$ by Hölder's inequality, so that A acts on χ_E. The set function so defined, i.e. $\nu(E)=A\chi_E$, is clearly a totally finite signed measure on the measurable subsets of any set F of finite measure.

Let $\{E_i\}$ be a finite collection of disjoint measurable sets of nonzero finite measure, and define ϕ by

$$\phi = \sum_i \frac{|\nu(E_i)|^{p-1}\operatorname{sgn}\nu(E_i)}{[\mu(E_i)]^{p-1}}\chi_{E_i}.$$

Clearly $\phi\in L_q$ and

$$\sum_i \frac{|\nu(E_i)|^p}{[\mu(E_i)]^{p-1}} = A\phi \leqq \|A\|\,\|\phi\|_q = \|A\|\left[\sum_i \frac{|\nu(E_i)|^p}{[\mu(E_i)]^{p-1}}\right]^{1-(1/p)},$$

or

$$\sum_i \frac{|\nu(E_i)|^p}{[\mu(E_i)]^{p-1}} \leqq \|A\|^p.$$

Thus by the lemma, ν is the indefinite integral of a function $f\in L_p$ and $\|f\|_p \leqq\|A\|$.

If g is a simple function, $g=\sum_i \alpha_i\chi_{E_i}$ where $\{E_i\}$ is a finite collection of sets of finite measure, then

$$Ag = \sum_i \alpha_i\nu(E_i) = \int_X g d\nu = \int_X gfd\mu,$$

the last step being by the lemma.

If $g\in L_q$ then there is a sequence $\{g_n\}$ of simple functions converging to g in L_q: $\|g_n-g\|_q\to 0$. Thus

$$|Ag_n - Ag| = |A(g_n - g)| \leqq \|A\|\,\|g_n - g\|_q \to 0,$$

and

$$\left|\int_X g_nfd\mu - \int_X gfd\mu\right| = \left|\int_X (g_n - g)fd\mu\right| \leqq \|g_n - g\|_q\|f\|_p \to 0.$$

We conclude $Ag=\int_X gfd\mu$. And $|Ag| = |\int_X fgd\mu| \leqq \|f\|_p\|g\|_q$, so that $\|A\|\leqq\|f\|_p$, which completes the proof of the theorem.

This paper was prepared at Oregon State University under an NSF grant.

TWO DIMENSIONAL SPACES ARE QUADRILATERAL SPACES

L. M. KELLY, D. M. SMILEY, AND M. F. SMILEY; Michigan State University, Riverside, California, and the University of California, Riverside.

In [2] some of us conjectured that if X is a real normed linear space and x, y, z are linearly dependent in X, then

$$(1)\qquad |x| + |y| + |z| + |x+y+z| \geqq |x+y| + |y+z| + |z+x|.$$

Joram Lindenstrauss has observed that (1) is a minor consequence of his result [1] that every normed linear space of dimension two is isometric to a subspace of $L_1[0, 1]$. Two of us have given a geometric proof of (1) based on the convexity of the unit ball of X (unpublished). Here we give a very simple algebraic proof of (1).

Since x, y, z are linearly dependent we have $\rho x+\sigma y+\tau z=0$ for real numbers ρ, σ, τ not all zero.

Case 1. ρ, σ, τ are all nonnegative (or all nonpositive). We may choose our notation so that $\rho, \sigma, \tau\geqq 0$, $\rho\geqq\sigma, \tau$ and $\tau>0$. For, if $\sigma=\tau=0$ then $x=0$ and (1) is trivial. We may then write $z=\alpha x+\beta y$ where $-\alpha= -\rho\tau^{-1}\leqq -1, \beta= -\sigma\tau^{-1}\leqq 0$ and $\alpha\leqq\beta$. With $k=(\alpha-\beta)\alpha^{-1}$ and $l=\alpha^{-1}$ we have

$$0\leqq k\leqq 1, \qquad 0\leqq l\leqq 1, \qquad ky - lz = x + y$$

as well as $(1-k)y-(1-l)z= -(z+x)$. Then

$$|x+y| \leqq k|y| + l|z|, \qquad |z+x| \leqq (1-k)|y| + (1-l)|z|$$

and we get $|x+y|+|z+x| \leqq |y|+|z|$. But we also have $|y+z| \leqq |x| + |x+y+z|$ and (1) follows by addition.

Case 2. One of ρ, σ, τ is negative and the other two are nonnegative (or one of ρ, σ, τ is positive and the other two are nonpositive). We may choose our notation so that $\tau<0$, ρ, $\sigma \geqq 0$ and $\sigma \geqq \rho$. Here we have $z=\alpha x+\beta y$ with $\alpha, \beta \geqq 0$ and $\alpha \leqq \beta$. With $k=(\beta-\alpha)(1+\beta)^{-1}$, $l=(1+\beta)^{-1}$ we verify that

$$0 \leqq k \leqq 1, \quad 0 \leqq l \leqq 1, \quad kx + l(x+y+z) = x+y,$$
$$(1-k)x + (1-l)(x+y+z) = x+z.$$

Manipulations like those used for Case 1 then prove that (1) holds.

M. F. Smiley's research was partially supported by NSF grant GP-1447.

References

1. Joram Lindenstrauss, On the extension of operators with a finite-dimensional range, Ill. J. Math., 8 (1964) 488–499.

2. D. M. Smiley and M. F. Smiley, The polygonal inequalities, this MONTHLY, 71 (1964) 755–760.

Editorial Note. The problem raised by the Smileys was also solved by Dr. Roy O. Davies in a letter to the Smileys dated Dec. 2, 1964.

THE HOLOMORPHS OF FREE ABELIAN GROUPS OF FINITE RANK

N. C. HSU, State University of New York at Buffalo

1. Introduction. A group is called complete if it has no outer automorphism and has no central element except for the trivial one. The automorphisms of the holomorphs of certain groups have been studied by several authors (see the references), and it is known that the holomorph of the infinite cyclic group, i.e., the free abelian group of rank 1 is not complete (see [2] page 1010). In this note we prove

THEOREM. *The holomorph of a non-cyclic free abelian group of finite rank is complete.*

Hence, a non-cyclic free abelian group of finite rank is an example of an abelian group, in which $x \rightarrow x^2$ is not an automorphism, that has a complete holomorph. (See [6] page 609.)

2. The proof of the theorem. It suffices to show that the holomorph under consideration has no outer automorphism. Let F_n be the free abelian group of rank n, and U_n be the automorphism group of F_n. The elements of F_n are n-tuples of rational integers and the elements of U_n are n by n matrices of rational integral coefficients whose determinants are ± 1. The image of an element $(c_1, c_2, \cdots, c_n)$ of F_n under the automorphism (x_{ij}) is

$$(c_1, c_2, \cdots, c_n)(x_{ij}) = \left(\sum_i c_i x_{i1}, \sum_i c_i x_{i2}, \cdots, \sum_i c_i x_{in} \right).$$

The holomorph G_n of the group F_n is defined as follows: The elements of G_n are the pairs (ξ, x) with $\xi \in U_n$, $x \in F_n$; the multiplication of elements of G_n is given by the rule $(\xi_1, x_1)(\xi_2, x_2) = (\xi_1\xi_2, x_1\xi_2 + x_2)$, where $\xi_1\xi_2$, $x_1\xi_2 + x_2$ are ordinary matrix operations. We imbed F_n into G_n in an obvious way. The first cohomology group $H^1(U_n, F_n)$ of U_n over F_n is the factor group of the subgroup of the principal characters in the group of the crossed characters of U_n into F_n, where a crossed character is a mapping u of U_n into F_n subject to the condition $u(\xi_1\xi_2) = u(\xi_1)\xi_2 + u(\xi_2)$ and where a principal character is a mapping v of U_n into F_n defined by $v(\xi) = a - a\xi$ for some fixed element a of F_n.

LEMMA 1. *For $n \geqq 2$, $H^1(U_n, F_n)$ vanishes.*

Proof. Let

$$\alpha = \begin{bmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \cdots 0 \\ \cdot & \cdot & \cdot \cdot \cdot \cdot \\ 0 & 0 & 0 \cdots 1 \\ -1 & 0 & 0 \cdots 0 \end{bmatrix}.$$

Then $\alpha^n = -\epsilon$, where ϵ is the n by n unit matrix. For any crossed character u and for any element ξ of U_n, $u(\xi)$ is determined by $u(\alpha)$, for ξ and α^n commute. In particular, α^n and

$$\beta = \begin{bmatrix} 1 & 1 & 0 \cdots 0 & 0 \\ 0 & 1 & 0 \cdots 0 & 0 \\ \cdot & \cdot & \cdot \cdot \cdot \cdot & \cdot \\ 0 & 0 & 0 \cdots 1 & 0 \\ 0 & 0 & 0 \cdots 0 & 1 \end{bmatrix}$$

commute, and therefore $u(\alpha) = (\sum_{i=2}^n a_i - 2b_2, a_2, a_3, \cdots, a_n)$ and $u(\beta) = (0, b_2, 0, \cdots, 0)$ for some integers $b_2, a_2, a_3, \cdots, a_n$. If v is the principal character induced by

$$(-b_2, a_2 - b_2, a_2 + a_3 - b_2, \cdots, \sum_{i=2}^n a_i - b_2),$$

then $u(\alpha) = v(\alpha)$ and therefore $u = v$ by a previous remark. This shows that every crossed character is a principal character, and the proof is complete.

COROLLARY 1. *Every automorphism of the holomorph G_n of the free abelian group F_n of rank $n \geqq 2$ which maps $F_n \subset G_n$ onto itself is an inner automorphism.*

Proof. The first cohomology group $H^1(U_n, F_n)$ is isomorphic to the factor group B/I, where B is the group of all automorphisms of G_n that map $F_n \subset G_n$ onto itself and where I is the group of all inner automorphisms of G_n (see, e.g. [5] page 3). Hence the corollary follows.

LEMMA 2. *For any n, F_n is a characteristic subgroup of G_n.*

Proof. It is shown by W. H. Mills [4, p. 390] that if A is a finitely generated abelian group which does not contain any element of order 2, then A is the only normal maximal abelian subgroup of its holomorph isomorphic to A. This proves the statement.

Lemma 2 together with Corollary 1 implies that the holomorph under consideration has no outer automorphism and we have the theorem.

References

1. Yu. A. Gol'fand, On the group of automorphisms of the holomorph of a group, Rec. Math. (Mat. Sbornik) N. S. 27 (69) (1950) 333–350.
2. N. C. Hsu, The group of automorphisms of the holomorph of a group, Pacific J. Math., 11 (1961) 999–1012.
3. G. A. Miller, On the multiple holomorph of a group, Math. Ann., 66 (1908) 133–142.
4. W. H. Mills, Multiple holomorphs of finitely generated abelian groups, Trans. Amer. Math. Soc., 71 (1951) 379–392.
5. ———, The automorphisms of the holomorph of a finite abelian group, Trans. Amer. Math. Soc., 85 (1957) 1–34.
6. W. Peremans, Completeness of holomorphs, Nederl. Akad. Wetensch. Proc. Ser. A, 60 (1957) 608–619.
7. W. Specht, Gruppentheorie, Berlin, 1956.

ABSOLUTELY INDEPENDENT GROUP AXIOMS

R. A. JACOBSON* AND K. L. YOCOM, South Dakota State University

1. Introduction. In a recent paper [1], Harary utilized the notion of axioms that "never" hold in order to develop the concepts of "very" independent and "absolutely" independent axiom systems. It was pointed out that the usual group axioms are not very independent; and, indeed, one is quite fortunate to find any very independent axiom systems. It is the purpose of this paper to present a set of, not only "very," but "absolutely" independent axioms both for a group and for a semi-group with right identity.

It is well known that both a group and a semi-group with a right identity consist of a set S of elements and a binary operation, $*$, that satisfy certain axioms. We shall assume that closure is an inherent part of the definition of binary operation. In the following, all elements e, x, y, z are in S.

2. Semi-group with right identity. We use the axioms:

(1) *there exists an element e such that $x * e = x$ for all x;*

(2) $x * (y * z) = (x * y) * z$ *for all x, y, z and $z \neq e$.*

---

* Presently at Houghton College.

The existence of a right identity element implies that $(x * y) * e = x * (y * e)$ for all x, y; hence the requirement that $z = e$ is excluded in axiom (2). The simple contradiction of the axioms would read:

$\sim$(1) for every element y there exists an x such that $x * y \neq x$;

$\sim$(2) there exist elements x, y, z, $z \neq e$, such that $x * (y * z) \neq (x * y) * z$.

The requirements that the axioms "never" hold are:

($\bar{1}$) for all x, y, then $x * y \neq x$

($\bar{2}$) for all x, y, z, and $z \neq e$, then $x * (y * z) \neq (x * y) * z$.

The following examples, in which x, y are contained in the set S of positive integers, indicate that the axioms are "absolutely" independent.

BINARY OPERATION	IDENTITY	AXIOMS
$x * y = x + y - 1$	$e = 1$	1, 2
$x * y = x + y$	none	$\bar{1}$, 2
$x * y = \begin{cases} x, & y = 1 \\ y + 1, & y \neq 1 \end{cases}$	$e = 1$	1, $\bar{2}$
$x * y = x + 1$	none	$\bar{1}$, $\bar{2}$

3. Group. The following axioms define a group:

(i) *there exists an element e such that $x * e = x$ for all x, $x \neq e$;*

(ii) *for all x, y, z, $z \neq e$; $x * (y * z) = (x * y) * z$;*

(iii) *$x * w = y$ has a unique solution w for all x, y, $x \neq y$.*

At first glance, it appears we have omitted these requirements: $e * e = e$, $x * (e * e) = (x * e) * e$, and $e * w = e$ has a solution. It is also apparent that these are satisfied if we can prove that $e * e = e$. Employing a proof by contradiction, we assume that $e * e = a$, $a \neq e$. Since $a \neq e$, $a * w = e$ has a unique solution b. If $b = e$, we have the contradiction $e = a * b = a * e = a$. If $b \neq e$, let $c = e * b$. Then $e * c = e * (e * b) = (e * e) * b = a * b = e$. Furthermore, $a * c = a * (e * b) = (a * e) * b = a * b = e$; hence, from the uniqueness condition of (iii), $c = b$. Since $e * b = c$ and $e * c = e$, this asserts that $e * b = b$ and $e * b = e$, or finally $b = e$, which is invalid. Hence $e * e = e$ and axioms (i), (ii), (iii) define a group.

The simple contradictions of the axioms are:

$\sim$(i) for all y there exists an x, $x \neq y$, such that $x * y \neq x$;

$\sim$(ii) there exist x, y, z, $z \neq e$, such that $x * (y * z) \neq (x * y) * z$;

$\sim$(iii) there exist x, y, $x \neq y$, such that $x * w = y$ does not have a unique solution.

We assert that the axioms "never" hold if:

($\bar{\text{i}}$) for all x, y, if $x \neq y$, then $x * y \neq x$;

($\overline{\text{ii}}$) for all x, y, z, if $z \neq e$, then $x * (y * z) \neq (x * y) * z$;

($\overline{\text{iii}}$) for all x, y, if $x \neq y$, then there is no unique solution for $x * w = y$.

The following examples, in which x, y are contained in the set S of all integers, indicate that the preceding group axioms are "absolutely" independent.

Binary Operation	Identity	Axioms
$x*y=x+y$	$e=0$	(i), (ii), (iii)
$x*y=y$	none	$\overline{\text{(i)}}$, (ii), (iii)
$x*y=x-y$	$e=0$	(i), $\overline{\text{(ii)}}$, (iii)
$x*y=x$	$e=0$	(i), (ii), $\overline{\text{(iii)}}$
$x*y=\begin{cases} \lvert y\rvert+1, & xy\neq 0 \\ x, & x\neq 0,\ y=0 \\ -\lvert y\rvert-1, & x=0,\ y\neq 0 \\ -2, & x=y=0 \end{cases}$	$e=0$	(i), $\overline{\text{(ii)}}$, $\overline{\text{(iii)}}$
$x*y=\begin{cases} \lvert x\rvert+\lvert y\rvert, & xy\neq 0 \\ \lvert x\rvert+1, & x\neq 0,\ y=0 \\ 1+\lvert y\rvert, & x=0,\ y\neq 0 \\ 2, & x=y=0 \end{cases}$	none	$\overline{\text{(i)}}$, (ii), $\overline{\text{(iii)}}$
$x*y=\begin{cases} y+2, & x-y>2 \\ y+3, & x-y\leqq 2 \end{cases}$	none	$\overline{\text{(i)}}$, $\overline{\text{(ii)}}$, (iii)
$x*y=x+1$	none	$\overline{\text{(i)}}$, $\overline{\text{(ii)}}$, $\overline{\text{(iii)}}$

4. Remarks. It is quite apparent that if S contains but one or two elements, the axioms can be neither "very" nor "absolutely" independent. In other words, the degree of independence of the axioms is associated with the order of S. We thus introduce the terminology; a set of axioms are " . . . " independent (mod n), where n is the number of elements in S. In particular, the preceding axiom systems are both "absolutely" independent (mod $\aleph_0$).

A tantalizing further problem is to find an x such that the above axiom systems are either "very" or "absolutely" independent (mod x). Furthermore, if x cannot be finite for the above axiom systems, perhaps a change in axioms will yield a finite x. Finally, having found group axioms that are absolutely independent (mod $\aleph_0$), we are immediately intrigued by the axioms for abelian groups, rings, etc.

Reference

1. F. Harary, A very independent axiom system, this Monthly, 68 (1961) 159–162.

A geometrician named Klein
Thought the Moebius Band asinine:
"Though its out side is in,
Still it's ugly as Sin;
It ain't Round, like that Bottle o'mine!"*

* The first two lines may be either sung or declaimed; the last three lines are to be sung, to any appropriate tune, in the key of E flat.

M. M. H. Coffee, and J. J. Zeltmacher, Jr., The University of Illinois

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Md. 20740.

ON THE FIXED POINTS OF $w=(a\bar{z}+b)/(c\bar{z}+d)$

NATHAN ELJOSEPH, University of Tel-Aviv, Israel

1. Introduction. It is known that the number of fixed points of

$$(1) \qquad w = \frac{az+b}{cz+d} \qquad ad-bc \neq 0$$

is 1 or 2, except in the case $w=z$. This is not true for

$$(2) \qquad w = \frac{a\bar{z}+b}{c\bar{z}+d} \qquad ad-bc \neq 0.$$

The following cases arise for (2):

(i) no fixed points (example: $w=-1:\bar{z}$);

(ii) one fixed point (example: $w=\bar{z}+1$, the point at infinity);

(iii) two fixed points (example: $w=2\bar{z}$, $z=0$ and $z=\infty$);

(iv) an infinite number of fixed points (example: $w=\bar{z}$, all the points on the real axis). We shall subsequently see that these are all the possible cases.

The transformation (2) may be factored into $w=\bar{z}$ and (1). Hence generalized circles (i.e. Euclidean circles and straight lines) will be transformed into generalized circles, and angles will be preserved in magnitude but reversed in sense.

Some of the following results may be found in C. Caratheodory [1], proved by a different method.

2. Preliminaries. The following lemma is basic for the main discussion.

LEMMA. *If* (2) *has three fixed points, then it has an infinite number of fixed points, and all the fixed points lie on a* (*generalized*) *circle.* (This is clearly not true for (1) when $w=z$.)

Proof. Without loss of generality we assume that one of the fixed points is the point at infinity (since this may be brought about by a suitable transformation of form (1)). Then $c=0$, and (2) may be written as

$$(3) \qquad w = \lambda\bar{z}+\mu \qquad \lambda \neq 0.$$

Denote the other two fixed points by r and s, so that $r=\lambda\bar{r}+\mu$ and $s=\lambda\bar{s}+\mu$. Every point t on the line l joining r and s may be written as $t=kr+(1-k)s$,

where k is an arbitrary real number. Since r and s are fixed points, t is a fixed point.

If there is an additional fixed point f, then by the same method we see that each point on all lines joining f to the points of l is fixed, and the same is true for the points on the line through f parallel to l. Hence each point of the plane is fixed. This is impossible, for (2) takes the form (3) since $z=\infty$ is fixed, $\mu=0$ since the origin is fixed, $\lambda=1$ since $z=1$ is fixed, and hence $w=\bar{z}$. But $z=i$ is not fixed.

REMARK. From the Lemma, it follows that the cases mentioned in Section 1 exhaust all the possibilities for the number of fixed points.

3. Geometric treatment.

THEOREM 1. *If* (2) *has an infinite number of fixed points it is an inversion with respect to a* (*generalized*) *circle. This is an involutory transformation.*

Proof. We may assume, without loss of generality, that the point $z=\infty$ is fixed. Hence straight lines are transformed into straight lines. All the fixed points are on a straight line l (by lemma of Section 2). Let P be a point on l, A a point not on l, and A' its image by (2). Then $A \neq A'$ since A is not on l. The line PA is transformed into PA', and the angle between PA and l is reversed. Hence PA' is the reflection of PA with respect to l. The line AA' meets l at B which is fixed. The line BA' is the image of BA and both form the same angle (with reversed sense) with l. Thus AA' is perpendicular to l, and the triangles APB and $A'PB$ are congruent. Hence A' is the reflection of A with respect to l.

Theorem 1 shows that if (2) has an infinite number of fixed points it is involutory. We shall now determine to what extent the converse is true.

THEOREM 2. *If* (2) *is involutory and has at least one fixed point, then it has an infinite number of fixed points.*

Proof. Without loss of generality we may assume that the point at infinity is a fixed point. Then straight lines will be transformed into straight lines. Choose any point A which is not fixed (such a point exists, by the lemma), and let its image be A'. Choose a second point B which is also not fixed and not on the line AA' (by the same lemma). If its image B' is on AA', take for B any other point which is not fixed but on the line BB' (see Lemma). Hence we have found a nondegenerate quadrilateral Q whose vertices are not fixed points and form two pairs of corresponding points.

In Q, the lines AB and $A'B'$, AB' and $A'B$ are interchanged, and AA' and BB' are each fixed lines. It follows that the point of intersection of the diagonals S of Q is fixed. If in Q at least one pair of opposite sides is not parallel, these sides (extended) meet at a second finite fixed point T. Suppose that Q is a parallelogram. Since the transformation is involutory, it is possible to interchange A and A', B and B', so that $AA'BB'$ is in the positive sense, as in Figure 1. Choose any point C on AB (extended). Its image C' is on $A'B'$. The line SC is trans-

formed into SC' since S is fixed. But both have equal angles (in the reversed sense) with AB and $A'B'$. Hence CC' passes through S. The triangles ASC and $B'SC'$ are congruent, and $AC=B'C'$. Thus $AC\neq A'C'$ and AC' is not parallel to $A'C$. Their point of intersection is a second finite fixed point. Our theorem now follows from the lemma.

REMARK. The transformation (2) may be involutory without having any fixed points, for example $w=-1:\bar{z}$.

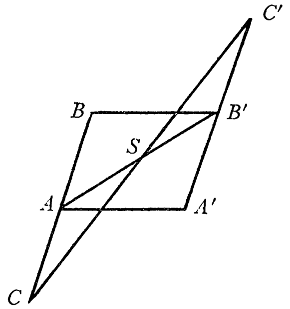


FIG. 1

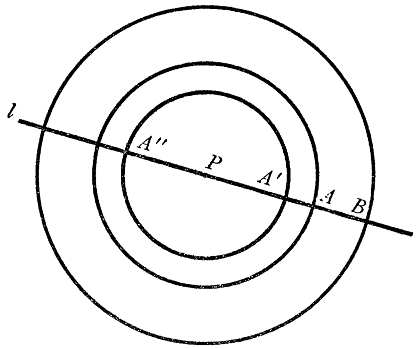


FIG. 2

THEOREM 3. *If* (2) *is involutory and has no fixed points it is an imaginary inversion* (*i.e. the product of an inversion and a rotation through* 180° *about the center of the circle of inversion*).

Proof. Let P be the image of the point at infinity. Then P is finite and P and ∞ are images of one another. The image of any straight line l through P is another straight line l' through P. Let α be the angle between l and l' at P. The angle between l and l' at infinity is $360°-\alpha$, and therefore, since (2) reverses the sense of the angles, the angle between l and l' at P is α. Thus $360°-\alpha=\alpha$ so that $\alpha=180°$, and l is fixed. Hence every line through P is fixed. (I am indebted to the referee for this short proof.)

Draw a circle with center P. The circle is orthogonal to the pencil of lines through P, and its image will also be a circle with center P since the transformation is isogonal. Hence the image of A is either A' or A'' as in Figure 2. Denote by r the radius of the original circle and by r' that of its image. Let $r>r'$. Draw another circle with center P and radius $R>r$. We shall show that the radius of the image of this circle $R'<r'$. It is sufficient to prove that B', the image of B which is on l, is inside the inner circle of the figure. As (2) is the product of a translation, rotation, homothety, inversion, and reflection with respect to a straight line, separation properties on l are invariant since l is a fixed line. If the image of A is A', the order of points is $(PA'AB\infty)$. The image is $(\infty AA'B'P)$ so that B' is between A' and P. A similar argument applies if the image of A is A''.

If we write $r'=f(r)$, then if $r'>r$, $f(r)$ is a decreasing function of r. Moreover $r=f(r')$. Because of the continuity of (2), there exists an r for which $r=f(r)$. Hence there is a circle m with center P which is fixed. As there is no fixed point, and each line through P is fixed, the image of each point of m is its antipode. Rotating the plane through 180° about P, we find that each point of m is fixed. The theorem follows by Theorem 1.

4. Analytic treatment. We shall now treat the problem analytically, rediscovering Theorems 1–3 as well as calculating the equations of the circles of real or imaginary inversion.

(a) Suppose that (2) has an infinite number of fixed points.

(a_1) Let $c=0$. Then (2) may be written in the form (3). For every fixed point we have $z=\lambda\bar{z}+\mu$, and hence $\bar{z}=\bar{\lambda}z+\bar{\mu}$. By substitution, $z(1-\lambda\bar{\lambda})=\lambda\bar{\mu}+\mu$. Since there is more than one finite fixed point, $|\lambda|=1$ and $\lambda\bar{\mu}+\mu=0$.

If $\mu=0$, write $\lambda=e^{i\phi}$. Then (3) becomes $w=e^{i\phi}\bar{z}$. Each point of the line $z=re^{i\phi/2}$ is fixed $(-\infty<r<\infty)$, and the transformation is a reflection with respect to this line. If $\mu\neq 0$, then $\lambda=-\mu/\bar{\mu}$, and (3) takes the form $w/\mu+\bar{z}/\bar{\mu}=1$. The fixed points are found by solving $z/\mu+\bar{z}/\bar{\mu}=1$. Put $z/\mu=a+bi$, $\mu=\mu_1+\mu_2 i$. Substituting, we find that $a=1/2$, $x=\mu_1/2-b\mu_2$, $y=\mu_2/2+b\mu_1$. Hence $\mu_1x+\mu_2y=|\mu|^2/2$, and (3) is a reflection with respect to this line.

(a_2) Let $c\neq 0$. We can write (2) in the form

$$w=\frac{\alpha\bar{z}+\beta}{\bar{z}+\delta},\qquad \beta\neq\alpha\delta. \tag{4}$$

If z is a fixed point, then $z=(\alpha\bar{z}+\beta):(\bar{z}+\delta)$, or $\bar{z}=(\bar{\alpha}z+\bar{\beta}):(z+\bar{\delta})$. On substituting we have

$$z^2(\bar{\alpha}+\delta)+z(\delta\bar{\delta}-\alpha\bar{\alpha}+\bar{\beta}-\beta)-(\alpha\bar{\beta}+\beta\bar{\delta})=0. \tag{5}$$

Here $\delta=-\bar{\alpha}$ since otherwise, the quadratic equation (5) yields at most two fixed points. Substituting, we have $z(\bar{\beta}-\beta)-(\bar{\beta}-\beta)=0$. Similarly, $\beta=\bar{\beta}$ so that β is real. From (4) we now have

$$w=\frac{\alpha\bar{z}+\beta}{\bar{z}-\bar{\alpha}}\cdot \tag{6}$$

Hence,

$$(w-\alpha)(\bar{z}-\bar{\alpha})=\beta+\alpha\bar{\alpha}. \tag{7}$$

If z is a fixed point, the right-hand side of (7) must be positive. Denoting it by R^2, we can write (7) as $w-\alpha=R^2/(\bar{z}-\bar{\alpha})$ which is an inversion.

(b) If (2) is involutory, we may assume without loss of generality that *if* there is a fixed point it is finite. Thus (2) can be written as (4). Since the transformation (2) is to coincide with its inverse, $\alpha=-\bar{\delta}$, and (4) takes the form (7), where $\beta+\alpha\bar{\alpha}\neq 0$. If there is *one* fixed point, then $\beta+\alpha\bar{\alpha}>0$ and there is an infinite

number of fixed points. Moreover $\beta=\bar{\beta}$ and β is real. For the case of no fixed points we have $\beta+\alpha\bar{\alpha}<0$.

We have thus proved Theorems 1–3 and have calculated in each case the equations of the circles of inversion.

5. Noninvolutory transformations without fixed points. An example of such a transformation is $w=i:\bar{z}$.

Since $c\neq 0$, (2) may be written in the form (4).

THEOREM 4. *A necessary and sufficient condition for* (4) *to be noninvolutory and to have no fixed points is*

$$-(\bar{\alpha}\bar{\delta}-\alpha\delta+\beta-\bar{\beta})^2>\{2(\beta+\bar{\beta})+|-\alpha+\bar{\delta}|^2\}\cdot|\alpha+\bar{\delta}|^2. \tag{8}$$

Proof. (a) If $\alpha=-\bar{\delta}$ then β cannot be real (by (b) of Section 4). If we write (4) in the form (7) we see that a necessary and sufficient condition for (2) to be noninvolutory and to have no fixed point has been found. (8) is also fulfilled.

(b) Suppose $\alpha\neq-\bar{\delta}$, then by (b) of Section 4, (4) is noninvolutory. We require that $z\neq(\alpha\bar{z}+\beta)/(\bar{z}+\delta)$ for every z, or $z\bar{z}+\delta z-\alpha\bar{z}-\beta\neq 0$. Hence, if $z=x+yi$, $-\alpha=a_1+a_2i$, $-\beta=b_1+b_2i$, $\delta=d_1+d_2i$, there must be no simultaneous real solution of the system

$$\begin{aligned} x^2+y^2+(d_1+a_1)x+(a_2-d_2)y+b_1&=0,\\ (d_2+a_2)x+(d_1-a_1)y+b_2&=0. \end{aligned} \tag{9}$$

The condition $\alpha\neq-\bar{\delta}$ implies that (9_2) represents a straight line. Equation (9_1) represents a circle, real, null or imaginary. If the transformation is to have no fixed point, the distance of the center of the circle from the line must be greater than its radius. In the case of an imaginary circle there is no fixed point, and (8) is automatically fulfilled, as will be seen in the next paragraph.

Let R be the radius of (9_1). Then

$$R^2=\left(\frac{d_1+a_1}{2}\right)^2+\left(\frac{a_2-d_2}{2}\right)^2-b_1=\left|\frac{-\alpha+\bar{\delta}}{2}\right|^2+\frac{\beta+\bar{\beta}}{2}.$$

The condition is

$$\pm\frac{(d_2+a_2)(-d_1-a_1)/2+(d_1-a_1)(d_2-a_2)/2+b_2}{|\alpha+\bar{\delta}|}>R,$$

where the sign is chosen so that the left-hand side of the inequality is nonnegative. By squaring both sides we have (8). If (9_1) is an imaginary circle, then R^2 is a negative number, and (8) is fulfilled.

REMARK. The remark at the end of Section 2 can also be deduced from (5), or from the fact that the fixed points are the points of intersection of a line and a circle.

Reference

1. C. Caratheodory, Theory of Functions of a Complex Variable, vol. 1, Chelsea, New York, 1958, sections 28, 38–44.

TWO DEFINITIONS OF LOCAL COMPACTNESS

P. S. SCHNARE, Louisiana State University in New Orleans

1. Introduction. The usual definitions of local compactness on an *arbitrary* space appear in several forms. (Some, like Bourbaki, require the space to be Hausdorff.) Two representative definitions are:

(1.1) DEFINITION. *A space X is locally compact if and only if each point in X belongs to an open set U such that* $\mathrm{Cl}(U)$ ($\mathrm{Cl}=$closure) *is compact*, [1, p. 71].

(1.2) DEFINITION. *A space X is locally compact if and only if each point in X is a member of a compact neighborhood*, [2, p. 146].

Most such definitions are readily seen to be equivalent to (1.1), and clearly (1.1) implies (1.2). This note provides counterexamples to show that (1.1) is not equivalent to (1.2) for T_1 spaces or normal spaces. (The terms normal, regular, and compact in this paper are those of Kelley [2].) It is also shown that local compactness: (1.1) is not an invariant under local homeomorphisms even if the spaces involved are T_1 or normal. Finally, a necessary and sufficient condition is given for the equivalence of the two definitions.

2. Counterexamples.

(2.1) *Example*: Let N denote the set of all natural numbers throughout this section. Let $A_n=\{n\}\times N\times\{1\}$, $n=1, 2, \cdots$; $B_1=\emptyset$, and $B_n=\{(n, 1, 2), (n, 2, 2), \cdots, (n, n-1, 2)\}$, $n=2, 3, \cdots$. Let $C_n=A_n\cup B_n$, $n=1, 2, \cdots$ and $X=\bigcup_{n=1}^{\infty} C_n$. Take as a basis for X all sets C such that for some n, $C\subset C_n$ and $C_n\setminus C$ is finite. Then X is locally compact in the sense of (1.1) and T_1. Let $Y=(N\times\{1\})\cup(N\times\{2\})$ with topology $\mathfrak{J}=\{C\subset Y: (N\times\{1\})\setminus C \text{ is finite}\}$. The space Y is a T_1 space which is not locally compact in the sense of (1.1), since the closure of every nonvoid open set is the whole space which is not compact, but Y is locally compact in the sense of (1.2).

(2.2) DEFINITION. *A function $f: X\rightarrow Y$ is a local homeomorphism from the space X onto the space Y if and only if f is a surjection i.e. $f[X]=Y$ and for each $x\in X$ there exists an open neighborhood of x: $U\subset X$ and an open neighborhood of $f(x)$: $V\subset Y$ such that $f|U: U\rightarrow V$ is a homeomorphism of U onto V.*

(2.3) *Remark*: Every local homeomorphism is a continuous open mapping.

We now define a function $f: X\rightarrow Y$, where X and Y are the spaces defined at the beginning of this section by the formula: $f(n, m, l)=(m, l)$. Then f is a local homeomorphism from X onto Y which does not preserve local compactness in the sense of (1.1).

Incidentally, in view of (2.3) this provides a stronger solution to problem 5056 [1962, 926; 1963, 1017; 1964, 562].

(2.4) *Remark*: Local compactness in the sense of (1.2) *is* an invariant under the weaker hypothesis that the function f be open and continuous.

(2.5) *Example*: Let N have for its topology $\mathfrak{I}=\{\emptyset, \{1\}, \{1, 2\}, \cdots, \{1, 2, \cdots, n\} \cdots\}$ then N is T_0, and normal (there are no disjoint closed subsets). It is locally compact: (1.2) but not locally compact: (1.1).

(2.6) *Remark*: It is easy to extend (2.5) to an example of a normal space which is "rich" in disjoint closed sets with the same disparity between (1.1) and (1.2). Also, using (2.5) as the range space, one can construct a local homeomorphism from a normal locally compact: (1.1) space.

3. Remarks. The reason for the disagreement in definitions (1.1) and (1.2) is that in an arbitrary space the closure of a compact set may fail to be compact. This difficulty does not arise for Hausdorff spaces (trivially) nor for regular spaces (Kelley [2]; Problem B (b), p. 161). Hence, the definitions are equivalent in such spaces.

(3.1) THEOREM. *Let X be locally compact*: (1.2). *Then X is locally compact*: (1.1) *if and only if $C\subset X$ and C compact implies that* Cl(C) *is compact.*

Proof. Suppose that C compact implies that Cl(C) is compact. Let x be an arbitrary element of X. Then we can find a compact neighborhood N of x and an open set U such that $x\in U\subset N$. But then $\mathrm{Cl}(U)\subset\mathrm{Cl}(N)$ and since Cl(N) is compact, Cl(U) is compact.

Conversely, suppose that X is locally compact: (1.1) and let $C\subset X$ be compact. Then $C\subset\bigcup_{k=1}^{n} U_k$ for some finite choice of open U_k's, each with compact closure. But then $\mathrm{Cl}(C)\subset\bigcup_{k=1}^{n}\mathrm{Cl}(U_k)$ which is compact. Hence, Cl(C) is compact.

References

1. J. G. Hocking and G. S. Young, Topology, Addison-Wesley, Reading, Mass., 1961.
2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.

WHEN IS A LOOP A GROUP?

F. D. PARKER, SUNY at Buffalo

When faced with the multiplication (Cayley) table for a finite system of order n with a binary operation it is easy to determine whether or not the system forms a quasi-group (the table must form a latin square). If, in addition, there is an element which serves as a right and left identity, the system is a loop. The next question is whether the system is a group, and this now means "Is the binary operation associative?" If it is necessary to test all possibilities, then $2n^3$ multiplications must be made, which is discouraging. Sometimes one can find a counter-example, or else show that the system is isomorphic to a system known

to be a group. There is sometimes no easy way out; it must be established whether or not we have a group, yet no counterexample appears and no isomorphic system seems to be at hand.

Zassenhaus [1] suggests the "rectangle rule" by which the rows and columns of the Cayley table are rearranged so that the identity is in every position on the main diagonal (the normal form). Then numbering each of the elements as though they were elements of a square matrix, Zassenhaus shows that the associative law is equivalent to the requirement that $a_{ij}a_{jk}=a_{ik}$. This is the "rectangle rule" and provides the basis for another test, which has both a decreased number of necessary operations and an algorithmic form to recommend it.

Consider a table in normal form and the elements numbered as in a matrix. We premultiply the elements in the second row by a_{12}, then post-multiply the elements in the second column by a_{21}. If the system is a group, then the rectangle rule turns the second row into $(a_{11}, a_{11}, a_{13}, a_{14}, \cdots, a_{1n})$ and the second column into $(a_{11}, a_{11}, a_{31}, a_{41}, \cdots, a_{n1})$. At this point the square matrix of order two in the upper left corner is composed only of a_{11} (the identity), the first and second rows are identical, and the first and second columns are identical. Now premultiply the third row by a_{13} and postmultiply the third column by a_{31}. If the system is a group, the third row is $(a_{11}, a_{11}, a_{11}, a_{14}, a_{15}, \cdots, a_{1n})$ and the third column is $(a_{11}, a_{11}, a_{11}, a_{41}, a_{51}, \cdots, a_{n1})$. By continuing in this fashion, the matrix is reduced completely to a matrix containing only a_{11} (the identity). If, at any stage of the reduction, the upper left matrix of the appropriate size has any entry which is not the identity, then the system is not a group. Moreover, after row k and column k have been treated as indicated, the first k rows should be identical and the first k columns should be identical. Failure in this respect at any stage again disqualifies the system as a group.

The total number of operations necessary is surprisingly small. It would seem at first that n operations are necessary in carrying out the first premultiplication, but the first product is $a_{12}a_{21}$, which is a_{11} by the structure of the normal form. The second product is $a_{12}a_{22}$, which is a_{12}; but this will be postmultiplied by a_{21} in the column multiplication, and so the result of the premultiplication and post-multiplication is the identity, again by the structure of the normal form. Consequently only $2(n-2)$ individual multiplications have been necessary. Now the first and second columns are equal as well as the first and second rows, otherwise the loop is *not* a group, and the third row is $(a_{31}, a_{31}, a_{33}, a_{34}, \cdots, a_{3n})$. Premultiplication by a_{13} makes it unnecessary to calculate the first three products, so that the third row and third column are tested by making $2(n-3)$ individual operations. Finally, if the first k rows and columns are tested for equality at each step, only $(n-2)(n-1)$ individual products need be made to complete the entire test.

Reference

1. H. Zassenhaus, The Theory of Groups, Chelsea, New York, 1959.

HARMONICITY AND HOLOMORPHIC MAPPINGS

K. V. RAJESWARA RAO, Harvard University

It is well known that, in spaces of more than one complex variable, harmonicity is not preserved under holomorphic homeomorphisms. The following example which exhibits this phenomenon may be of some interest.

Let C^2 denote the space of two complex variables. It is known (see [1], p. 45) that there exists a holomorphic one-to-one mapping $F: C^2 \to C^2$ such that F omits a neighborhood of the origin. If r denotes the distance of z from the origin, then $h(z) \equiv 1/r^2$ is a bounded harmonic function on $F(C^2)$. But, in view of Liouville's theorem, $h \circ F$ cannot be harmonic on C^2.

Reference

1. S. Bochner and W. T. Martin, Several complex variables, Princeton Univ. Press, Princeton, N. J., 1948.

A PROPERTY OF THE LOGARITHM

D. S. GREENSTEIN, Northwestern University

One important limit involving the logarithm is

$$\lim_{x \to \infty} \frac{\log x}{x} = 0.$$

Since it is becoming customary in elementary calculus courses to use the definition

$$\log x = \int_1^x t^{-1}\, dt$$

to deduce properties of the logarithm (including the logarithm of a product equalling the sum of the logarithms of its factors), the following simple demonstration of the above limit should be of interest.

From the defining integral, it is easily shown (or intuitively demonstrated by means of areas) that for $x > 1$,

$$0 < \log x < x.$$

Since $\log x = 2 \log x^{1/2}$, it necessarily follows that

$$0 < \frac{\log x}{x} < \frac{2x^{1/2}}{x} = 2x^{-1/2},$$

from which the limit is easily deduced. Finally, it should be noted that the substitution $x = t^{-1}$ also yields $\lim_{t \to 0} t \log t = 0$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITOR: JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington D. C. 20005

METHODS OF IMPROVING INSTRUCTION BY GRADUATE ASSISTANTS

JOYCE A. SHANA'A, University of Oklahoma

With increased enrollments, universities throughout the country have been forced to rely on graduate students to teach beginning courses. The importance of these courses to the students enrolled and their impact on the entire program within a college should not be minimized. In some institutions, these courses are taught by student assistants who have been handed a book, sometimes a schedule, and then waved good-bye until grades are turned in, or until adverse complaints force further supervision.

The University of Oklahoma depends on graduate assistants to teach mathematics to approximately 2600 students each semester. Our faculty of 30 full-time members has been concerned with maintaining reasonable objectives, high levels of instruction, just grading policies, and permeating each course with the spirit and philosophy inherent in sophisticated mathematics while depending upon a changing, increasing group of graduate assistants (73 this year) to furnish instruction in the elementary courses. These assistants form a heterogeneous group in terms of professional goals, personality, levels of mathematical maturity, and teaching experience. All, however, are expected to teach six hours of class per week and to maintain required scholarship standards both as teachers and as students.

This paper describes several measures which have been adopted to improve the quality of instruction given by our graduate assistants. These follow closely the recommendations of Panel 6 on the Conference on Manpower Problems in the Training of Mathematicians (report in this MONTHLY, February, 1964).

For many years certain senior staff members have been responsible for the coordination of those courses which involve multiple sections. Now that a course may have twenty or more sections taught by nearly as many people, the importance of this supervision has been re-emphasized. Each course with several sections has a coordinator who has the teaching maturity to act both as a supervisor and as a reference resource. The coordinator holds regular meetings with his group; topics of discussion include the objectives of the course, the emphases desired by the department, tests, etc. During the semester, the coordinator visits each section, often more than once, with the joint purpose of offering constructive criticism and noting excellence in instruction. At this level, few graduate assistants have problems with the mathematical concepts involved; however, many teaching practices are improved through awareness of needed areas of improvement.

The remedial algebra courses are often taught by our new graduate assistants. These people are starting their first graduate courses as well as facing their first teaching experience. Many have never planned to teach; their problems are often compounded by ignorance of teaching techniques, by communication difficulties, by necessary social adjustments, and by fear. The coordinator for this group has a time-consuming job. He needs to disperse information about the standard practices of the University. He must coordinate course objectives among a group lacking familiarity with departmental policy. He should visit the classroom frequently to observe teaching mechanics and presentation methods and should advise on testing and grading procedures. Unofficially he becomes a counselor and sympathetic listener.

This group meets weekly at first. Many problems solve themselves by presentation in free discussion. All instructors help design reasonable common midterm and final examinations which are given simultaneously to all sections. These are graded on an equivalent scale, and thus provide a useful guidestick to the beginning instructor in evaluating his classes' achievements and his own grading practices. To allow time to adequately carry out the above tasks, this coordinator has part of his teaching load reduced.

From the coordination sessions emerged the idea of exchange classroom visits among the assistants. The constructive criticisms resulting from these visits are received seriously and acted on swiftly with a minimum of resentment about departmental interference. Occasionally a pair of instructors will then collaborate on making and giving a quiz. These are effective practices with both old and new graduate assistants.

Preceding the fall semester of 1964, our twenty-seven new graduate assistants were asked to attend a four-day seminar during registration week. Thus we eliminated much of the necessary paper work of those first weeks of class, introduced them to the school and community, aided them in selection of suitable course schedules (both in level and number), and acquainted them with the department's requirements of them as students and instructors. The schedule, given below, describes our orientation program. Present plans indicate this will become a regular part of the fall enrollment.

FIRST MORNING: Get acquainted session. Welcomes by our chairman and the graduate dean. Paper work—payroll forms, keys, etc. Tour of the library.

FIRST AFTERNOON: Discussion by participants of the following topics: What is a good teacher—for the beginning student, the upperclassman, and the graduate? How much academic freedom does the new instructor have? (Excellent topic.) What is modern mathematics? What are the objectives of our algebra and analysis courses?

Assignment: Plan homework assignments to cover, in four lessons, chapter 1 of the algebra text.

SECOND MORNING: Discussion of homework assignments. Lecture by a senior staff member on sets, the real number system and applications of these in teaching elementary mathematics courses.

SECOND AFTERNOON: Panel discussion by experienced graduate students on our course offerings (level of difficulty, needed background and logical sequence), professors and their expectations, and additional degree requirements.

Assignment: Design an hour quiz over chapter 1.

THIRD MORNING: Enrollment.

THIRD AFTERNOON: Presentation of units on absolute values, inequalities, and graphing by another staff member, aided by four of the new group.

THIRD EVENING: Picnic for entire mathematics department and their families.

Assignment: Prepare a ten-minute teaching unit for presentation to the entire group.

FOURTH DAY: Return of tests with written comments on each. Individual grading of an examination (taken from files) followed by discussion of the variance in grades. Individual presentations of units with class comments on each.

Active participation in these sessions far exceeded expectations; many additional topics including the keeping of a grade book, homework—its weight and checking, and correct use of a grading curve were tabled because of lack of time. Advance planning and a concerned skilled director contribute largely to the success of this type of program.

Two departmental practices seek to improve instruction by increasing contact time with the students. Special help sessions are held in all undergraduate courses four afternoons a week. These, staffed by graduate assistants and instructors, average in attendance 75 students each day and provide free individual tutoring. In addition, each staff member posts and keeps as many office hours per week as he teaches.

As a final step in developing a unity of purpose and to allow free exchange of ideas, both the senior and the junior staffs meet monthly to discuss common problems. Our chairman and the members of the standing committees on departmental policies invite frank discussion of all problems within the department.

The quality of the instruction given in our department, specifically by the graduate assistants, has improved by application of the above practices. We recommend them to other departments blessed with large teaching staffs of graduate assistants.

A COMPARISON OF THE MATHEMATICAL COMPETENCIES OF EDUCATION AND NON-EDUCATION MAJORS ENROLLED IN A LIBERAL ARTS COLLEGE

DORA HELEN SKYPEK, Emory University

Several studies in the past twenty years purporting to compare the academic ability of undergraduates in teacher training programs to that of college students in other major fields have generally found the teacher trainees to rank below other groups. The majority of the studies were based on teachers' college students or university-related College of Education students. Furthermore, the subjects in many cases were freshmen or included freshmen.

It was assumed that students in the Upper Division of a liberal arts college—students who have survived the academic and psychological wars of the first two years of general college education and who have committed themselves to an intensive program of specialized training—might more nearly represent

tapes are transferred to film for later use by schools and other groups. The project is coordinated by the American Association for the Advancement of Science, and production costs are being underwritten by grants from the National Science Foundation and the Timken Roller Bearing Company. The Conference Board of the Mathematical Sciences is one of the cooperating scientific societies.

TWELFTH-GRADE MATHEMATICS

A report describing the fourth- and fifth-year mathematics course offerings in 66 selected high schools in 20 states has recently been released by the U. S. Office of Education. *Emerging Twelfth-Grade Mathematics Programs* is the title of the report prepared by Lauren G. Woodby of the Office of Education staff. A general lack of agreement among high school teachers on what course or courses should be offered at this level, and consequent variety in practice, is a major finding of the report. Copies may be obtained from Dr. Woodby.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before January 31, 1966.

E 1805. *Proposed by I. J. Schoenberg, U. S. Army Research Center, University of Wisconsin*

We wish to inscribe a triangle ABC, having given sides a, b, c, in the circle $\|x\|=r$ of the Minkowski (Banach) plane M_2 of points $x=(x_1, x_2)$ with the norm $\|x\|=\max(|x_1|, |x_2|)$. Since M_2 is a metric space, it is necessary for the possibility of this construction that (i) a, b, c satisfy the triangle inequality, and (ii) a, b, and c do not exceed the diameter $2r$. Show that these conditions are also sufficient.

E 1806. *Proposed by Kaidy Tan, Fukien Normal College, China*

On two sides AB, AC of triangle ABC describe equilateral triangles ABD, ACE outside the given triangle. Let G, H be the centroids of these two triangles respectively, and let M be the midpoint of BC. Prove that $AC \geqq AB$ if and only if $MG \geqq MH$.

E 1807. *Proposed by Peter Rolland, Jr., University of Illinois*

Let p be a polynomial of degree n. Prove that

$$\sum_{k=0}^{n} \frac{p^{(k)}(0)}{(k+1)!} \cdot x^{k+1} = \sum_{k=0}^{n} (-1)^k \frac{p^{(k)}(x)}{(k+1)!} x^{k+1}, \qquad p^{(0)}(x) \equiv p(x).$$

E 1808. *Proposed by H. T. Croft, Cambridge University, England*

C is a configuration of 5 points P_i $(i=1, \cdots, 5)$ in a closed domain D. Find $M = \max_C \min_{i \neq j} P_iP_j$, in the following cases:
(i) D is the unit square;
(ii) D is the unit equilateral triangle;
(iii) D is the unit circle;
(iv) D is the surface of the unit sphere.
In (iv) P_iP_j may be interpreted either as Euclidean or great-circle distances: essentially the same problem results.

E 1809. *Proposed by Pete Said, Fresno State College*

A square has sides of length 2. It is to be rolled along a curve without slipping and so that the center of the square moves in the horizontal line $\{(x, y) \mid y = 2^{1/2}\}$. What is the equation of the principal period of the curve?

E 1810. *Proposed by John Harvey and John E. Wetzel, University of Illinois*

For each natural number n let $C(n)$ be the sum of the cubes of the decimal digits of n. Show that for every n, iterating C eventually leads to one of the following repeating cycles: 1, 55–250–133, 136–244, 153, 160–217–352, 370, 371, 407, 919–1459.

E 1811. *Proposed by Irving Adler, North Bennington, Vermont*

If $a_1, \cdots, a_n$ are distinct positive integers, the positive integer s is said to be attainable through $\{a_1, \cdots, a_n\}$ if there exist nonnegative integers $x_1, \cdots, x_n$ such that $x_1a_1 + \cdots + x_na_n = s$. The critical number for a set is the least positive integer N such that every integer $n \geqq N$ is attainable through that set.

If a and d are relatively prime, prove that the critical number for $\{a, a+d, a+2d\}$ is $(m+d-1)a-d+1$, where m is the least positive integer $\geqq \frac{1}{2}(a-1)$.

Compare problem E 1637 [1964, 799]. Generalizations are welcome. See note to E 1601 [1964, 433].—Ed.

E 1812. *Proposed by Samuel Stern, State University College at Buffalo, N. Y.*

Let R be a set satisfying all ring axioms, omitting commutativity of addition. If R has a right multiplicative identity prove that R is a ring.

E 1813. *Proposed by Milton Legg, University of Minnesota, Duluth*

Let A be a row semi-magic square and let A_j denote the matrix obtained by replacing the jth column of A by a column of 1's. Show that $\det(A_j)$ is independent of j.

E 1814. *Proposed by R. C. Thompson, University of California, Santa Barbara*

Show that there exist infinitely many sets of three consecutive integers, each of which is a sum of two squares of positive integers.

SOLUTIONS OF ELEMENTARY PROBLEMS

Minimal Sum of Binomial Coefficients

E 1711 [1964, 793]. *Proposed by A. J. Goldman, National Bureau of Standards*

Let p, N, r be positive integers with $r>1$. Find a p-part partition of N into nonnegative integers n_i (*i.e.*, $\sum_{i=1}^{p} n_i=N$) which minimizes the sum of binomial coefficients $\sum_{i=1}^{p} C(n_i, r)$. (We take $C(n, r)=0$ if $r>n$.)

I. *Solution by Lawrence Yue Lin Tong, Iona College, New Rochelle, New York.* If $N=pq+r$ with $0\leqq r<p$, we shall prove that the minimizing partition consists of r $(q+1)$'s and $\overline{p-r}$ q's. We have

$$\begin{aligned} C(a+1, r) + C(b-1, r) &= C(a, r) + C(a, r-1) + C(b, r) - C(b-1, r-1) \\ &\leqq C(a, r) + C(b, r) \end{aligned}$$

if $a\leqq b-1$ (since then $C(a, r-1)\leqq C(b-1, r-1)$). If now $(n_1, n_2, \cdots, n_p)$ is a partition of N and $n_i+2\leqq n_j$ for some i and j, then $\sum_{k=1}^{p} C(n_k^*, r)\leqq \sum_{k=1}^{p} C(n_k, r)$ with $n_i^*=n_i+1$, $n_j^*=n_j-1$, and $n_k^*=n_k$ otherwise. Thus, the minimal sum occurs for those partitions in which $|n_i-n_j|\leqq 1$ for all i and j. If R of the n_i's have the value $Q+1$ and the remaining $p-R$ have the value Q, then $0\leqq R\leqq p$ and $N=R(Q+1)+(p-R)Q=pQ+R(=p(Q+1)$ if $R=p$!), as asserted earlier.

II. *Solution by E. S. Langford, U. S. Naval Postgraduate School, Monterey, California.* The minimum is found by making the x_j's "as equal as possible." For example, writing $N=pq+r$ with $p>r\geqq 0$, one can take $x_1=x_2=\cdots=x_r=q+1$ and $x_{r+1}=x_{r+2}=\cdots=x_p=q$. This is a consequence of a very general result which is sometimes referred to as the "flyaway package" or "blast-off package" problem. This general result is an obvious extension of Theorem 3 in G. G. den Broeder, Jr., R. E. Ellison, and L. Emerling, *On optimum target assignments*, Operations Res., 7 (1959) 322–326.

THEOREM. *Suppose that $f_1, f_2, \cdots, f_p$ are convex functions with domain the nonnegative integers (so that $\Delta f_j(x)$ is decreasing). A p-tuple of nonnegative integers $(x_1, x_2, \cdots, x_p)$ is said to be N-optimal if $\sum x_j = N$ and $\sum f_j(x_j) \geqq \sum f_j(x_j')$ for any p-tuple of nonnegative integers $(x_1', x_2', \cdots, x_p')$ with $\sum x_j' = N$. Suppose that $(x_1, x_2, \cdots, x_p)$ is N-optimal. Let j^* be an index which maximizes $\Delta f_j(x_j)$. Then $(x_1, x_2, \cdots, x_j^*+1, \cdots, x_p)$ is $(N+1)$-optimal.* (That is, at every step, one chooses the index with the greatest "marginal value.")

In the special case $f_1 = f_2 = \cdots = f_p = f$, an "optimal sequence" of j^*'s is seen to be $1, 2, \cdots, p, 1, 2, \cdots, p, \cdots$.

The above theorem applies also to minimization of concave functions by taking negatives. $f(x) = \binom{x}{r}$ is concave by Bernoulli's formula.

It is interesting to note that taking x_j equal to the greatest integer in $(N+j-1)/p$ also yields a minimum. An independent proof of this can be given by Dynamic Programming methods.

Also solved by J. C. Abad, M. A. Bershad, D. I. A. Cohen, Michael Goldberg, D. M. Hancasky, J. E. MacDonald, Jr., Robert Maas, D. C. B. Marsh, R. L. Syverson, and the proposer.

Probability of a Jack-Queen Contact

E 1713 [1964, 793]. *Proposed by W. A. Thompson, Jr., University of Delaware*

What is the probability that at least one Jack and Queen are adjacent in a shuffled deck of ordinary playing cards?

Solution by Jack C. Abad, Lincoln High School, San Francisco. We first determine A, the number of arrangements of 44 C's, 4 J's, and 4 Q's for which no J is adjacent to a Q. Then the probability p we seek is given by

$$p = 1 - \frac{(4!)^2(44!)A}{52!}\cdot$$

We count these A arrangements by considering the $\binom{8}{4} = 70$ permutations of 4 J's and 4 Q's and dropping C's into the appropriate gaps. In every permutation $(a_1, a_2, \cdots, a_8)$, consider the seven ordered pairs (a_k, a_{k+1}) and let i be the number of ordered pairs that are either (JQ) or (QJ). If n_i is the number of permutations having i J-Q pairs, $n_1 = n_7 = 2$, $n_2 = n_6 = 6$, and $n_3 = n_4 = n_5 = 18$. In each permutation, we drop in i C's, one in each gap between a J-Q pair. The other 44-i C's can be distributed in the 9 positions (front, seven gaps, end) in

$$\binom{44 - i + 8}{8}$$

ways. So

$$A = \sum_{i=1}^{7} n_i \binom{52 - i}{8} = 27{,}061{,}623{,}270,$$

and

$$p = 1 - \frac{27,061,623,270}{52,677,670,500} = \frac{284,622,747}{585,307,450} \approx .4863.$$

Also solved by Walter Bluger, W. W. Funkenbusch, Bernard Greenspan, F. W. Herlihy, E. S. Keeping, Donald Quiring, Sidney Spital, B. R. Toskey, and the proposer.

A Limit

E 1714 [1964, 793]. *Proposed by Stanton Philipp, Long Beach, California*

Let n and x be positive integers, and define $N_n(x)$ to be the number of positive integers d such that (1) $d|x$ and (2) $x \leqq d^2 \leqq n^2$. Prove that as $n \to \infty$,

$$\lim n^{-2} \sum_{x=1}^{n^2} N_n(x) = \frac{1}{2}\cdot$$

Solution by John Christopher, Sacramento State College, Sacramento, California. Let $S_n = N_n(1) + N_n(2) + \cdots + N_n(n^2)$. A positive integer, d, is counted in $N_n(x)$ if and only if $x = kd$ where $1 \leqq k \leqq d$. Thus d is counted in exactly d of the terms in the sum S_n. Therefore, $S_n = 1 + 2 + \cdots + n = \frac{1}{2}(n^2 + n)$, and $n^{-2}S \to \frac{1}{2}$ as $n \to \infty$.

Also solved by J. C. Abad, W. J. Blundon, Leonard Carlitz, D. I. A. Cohen, P. G. Engstrom, N. J. Fine, Jerry Goodman, M. G. Greening (Australia), D. C. B. Marsh, D. R. Shoemaker, L. Y. L. Tong, E. H. Umberger, K. S. Williams, K. L. Yocom, and the proposer.

A Countably Infinite to One Continuous Function

E 1715 [1964, 793]. *Proposed by R. F. Jolly, University of California, Riverside*

Does there exist a continuous function f on the closed interval [0, 1] which is exactly countably infinite to one (i.e., for each y in the range of f, $f^{-1}(y)$ is infinite and countable)?

Solution by G. P. Speck, Raymond College, Stockton, California. Recalling the construction of the Cantor ternary set S and its complement $C(S)$ define f as follows:

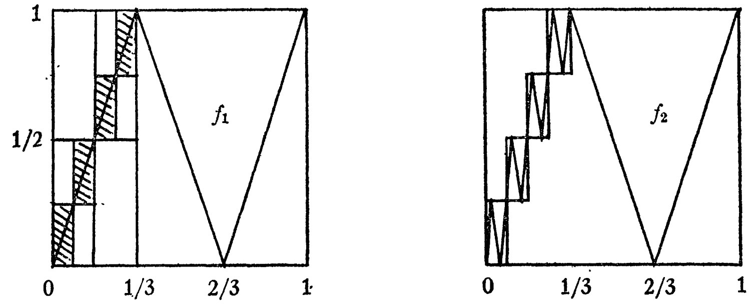
$$f(x) = \{1/2 + 1/2 \sin 6\pi x, \qquad 1/3 \leqq x \leqq 2/3.$$

$$f(x) = \begin{cases} 1/4 + 1/4 \sin 18\pi x, & 1/9 \leqq x \leqq 2/9. \\ 3/4 + 1/4 \sin 18\pi x, & 7/9 \leqq x \leqq 8/9. \end{cases}$$

$$f(x) = \begin{cases} 1/8 + 1/8 \sin 54\pi x, & 1/27 \leqq x \leqq 2/27. \\ 3/8 + 1/8 \sin 54\pi x, & 7/27 \leqq x \leqq 8/27. \\ 5/8 + 1/8 \sin 54\pi x, & 19/27 \leqq x \leqq 20/27. \\ 7/8 + 1/8 \sin 54\pi x, & 25/27 \leqq x \leqq 26/27. \end{cases}$$

Continue to define f on $C(S)$ in keeping with the procedure suggested in the preceding three steps. Finally, define f over $[0, 1]$ as the continuous extension of f as defined over $C(S)$. If E is the set of end-points of the open intervals comprising S and $0 \leqq y \leqq 1$, it is clear that $f^{-1}(y)$ is countably infinite if $(S-E) \cap f^{-1}(y) = \phi$. Since f agrees with the Cantor singular function on E, these functions agree on $\bar{E} = S$. But the usual discussion of the Cantor singular function shows it to be one to one on S, so that $(S-E) \cap f^{-1}(y)$ is at most a singleton. Thus, f is countably infinite to one.

Solution by the proposer. While solutions can be given in the form of a series expansion, the result is perhaps easier to see when described geometrically.



The only change from f_1 to f_2 is that which takes place in the shaded rectangles. These are treated exactly as the "original rectangle" $[(0, 0); (1, 0); (1, 1); (0, 1)]$. Continue the process to obtain $f_n(x) \to f(x)$ uniformly for $0 \leqq x \leqq 1$. Clearly f is continuous and countably infinite to one.

Also solved by A. H. Kruse.

Number of Residue Classes Modulo n in a Set of Sums

E 1716 [1964, 794]. *Proposed by J. D. Brillhart, University of San Francisco, and R. L. Graham, Bell Telephone Laboratories*

If $(a_1, a_2, \cdots, a_n)$ denotes some arrangement of the first n positive integers, show that at least $\sqrt{n}$ distinct residue classes modulo n occur in the set $\{\sum_{j=1}^{k} a_j : 1 \leqq k \leqq n\}$.

I. *Solution by Jack C. Abad, Lincoln High School, San Francisco.* With each arrangement $(a_1, a_2, \cdots, a_n)$ associate an n-tuple $R = (r_1, r_2, \cdots, r_n)$, where $0 < r_k < n-1$ and $r_i \equiv \sum_{j=1}^{i} a_j \pmod{n}$. These r_i are names for the residue classes that occur in $\{\sum_{j=1}^{k} a_j : 1 \leqq k \leqq n\}$. If there are m distinct integers in some R, we will show that $m \geqq \sqrt{n}$ for $n > 2$. The proposed theorem is not true for $n = 2$.

If we form the $n-1$ ordered pairs (r_k, r_{k+1}) from some R, we notice (for $n > 1$): the $n-1$ ordered pairs are distinct; there exists at most one ordered pair where $r_k = r_{k+1}$. Since there are m distinct integers in R, we can form at most $m(m-1)$ ordered pairs with different components from them. So m must satisfy the inequality $m(m-1)+1 \geqq n-1$, or $m \geqq (1+\sqrt{(4n-7)})/2$. It can be shown that

$1+\sqrt{(4n-7)} \geqq 2\sqrt{n}$ for $n \geqq 4$, with strict inequality for $n \geqq 5$. For $n=4$, the arrangement (2, 4, 3, 1) gives rise to $R=(2, 2, 1, 2)$, so that $m=2=\sqrt{4}=\sqrt{n}$. For $n=3$, $m=2$ for every arrangement and $2>\sqrt{3}$. For $n=2$, the arrangement (1, 2) has the associated pair (1, 1), and so $m=1 \ngeqq \sqrt{2}=\sqrt{n}$.

II. *Solution by Michael Goldberg, Washington, D. C.* The problem is equivalent to the following: Consider a closed-circuit railroad whose perimeter is n miles ($n>1$). Locate p stations along the circuit so that one can begin a trip at a station, travel a_1 miles to another station, travel a_2 miles in the same direction to still another station, and continue in this way until each integral number of miles from one to n has been traveled.

The number of possible trips between different stations is $p(p-1)$. Furthermore, a complete circuit can be made from any station. Therefore, if no trip length is repeated, the largest number of different trips that can be made from p stations is given by $n=p^2-p+1$. Hence, $p>\sqrt{n}$ since $n>1$. The milepost numbers correspond to a minimal set of residues. Equivalent sets can be obtained by translation. Sometimes, other sets are possible.

For some special values of p, the maximum $n=p^2-p+1$ can be attained. For example, for $p=5$, and $n=21$, the five stations can be at mileposts 0, 1, 4, 14, 16. If we begin at station zero, take single steps for trips, then double steps, etc., we obtain the trips a_i in the order 1, 3, 10, 2, 5, 4, 12, 6, 13, 7, 14, 8, 15, 9, 17, 16, 19, 11, 18, 20, 21. For $p=6$, and $n=31$, the six stations can be at mileposts 0, 1, 3, 8, 12, 18.

Also solved by Leonard Carlitz, N. J. Fine, M. G. Greening (Australia), D. R. Shoemaker, and the proposer.

Dual of Ptolemy's Theorem on a Sphere

E 1717 [1964, 794]. *Proposed by V. F. Ivanoff, San Carlos, California*

Prove the dual of Ptolemy's theorem on the spherical surface, namely: In the circumscribed spherical quadrilateral with the sides a, b, c, d touching the incircle in that order,

$$\cos \tfrac{1}{2}(a, c) \cdot \cos \tfrac{1}{2}(b, d) = \cos \tfrac{1}{2}(a, b) \cdot \cos \tfrac{1}{2}(c, d) + \cos \tfrac{1}{2}(a, d) \cdot \cos \tfrac{1}{2}(b, c).$$

(Note. This proposition holds for any plane quadrilateral whether circumscribed or not.)

Solution by Michael Goldberg, Washington, D. C. The stated problem is given on page 177 of *Spherical Trigonometry*, by W. J. McClelland and T. Preston (1890). This refers to a hint on page 33 where the analogue of Ptolemy's theorem is proved as follows: If a, b, c, d be in order the sides and e and f the diagonals of a spherical quadrilateral inscribed in a circle, then $\sin \frac{1}{2}e \sin \frac{1}{2}f = \sin \frac{1}{2}a \sin \frac{1}{2}c + \sin \frac{1}{2}b \sin \frac{1}{2}d$. For if a', b', c', d', be the chords of the arcs which form the sides of the quadrilateral and e' and f' those of the diagonals, it is clear that a', b', etc. are the sides and diagonals of a plane quadrilateral inscribed in a circle,

and therefore, $a'c'+b'd'=e'f'$. But $a'=2r \sin \frac{1}{2}a$, $b'=2r \sin \frac{1}{2}b$, etc., where r is the radius of the sphere. Therefore etc.

If the polar dual of the figure is constructed, then each side is replaced by an angle which is its supplement. The circumscribed circle is replaced by the inscribed circle. Hence, the sought theorem is demonstrated.

Also solved by Van de Vyle and the proposer.

An Application of Bertrand's Conjecture

E 1718 [1964, 794]. *Proposed by J. D. Cloud, Manhattan Beach, California*

Prove that, in every solution in positive integers of the Diophantine equation $x^n=y!$, either $n=1$ or $x=1$.

Solution by Donald G. Kanzaki, Student, Drew University. If $x=1$ or $n=1$, then $x^n=y!$ clearly has a solution in positive integers. Now suppose that x, $n>1$ and $x^n=y!$ If now p is a prime factor of x^n, p^2 is a factor of $y!$ Let p be the greatest prime factor of $y!$ By Bertrand's Conjecture [W. J. Le Veque, *Topics in Number Theory*, vol. 1, Addison-Wesley (1956) §§ 6–8], $y\geqq p>y/2$ so that $2p>y$ and p^2 is not a factor of $y!$ This contradiction proves the assertion of the problem.

Also solved by J. C. Abad, W. T. Bailey, M. A. Bershad, Walter Bluger, W. J. Blundon, M. B. Boisen, Jr., Maxey Brooke, J. L. Brown, Jr., John Christopher, Allan Chuck, D. I. A. Cohen, R. J. Eckert, R. B. Eggleton (Australia), P. K. Garlick, A. A. Gioia, Michael Goldberg, Jerry Goodman, M. G. Greening (Australia), D. M. Hancasky, Sidney Kravitz, Joel Kugelmass, E. S. Langford, C. C. Lindner, J. E. MacDonald, Jr., Andrzej Makowski (Poland), D. C. B. Marsh, T. M. Morrisette, J. B. Muskat, C. B. A. Peck, Harsh Pittie, Simeon Reich (Israel), G. P. S., Gary Sampson and David Zitarelli (jointly), P. S. Schnare, J. J. Schneider, D. L. Silverman, Sidney Spital, Dmitri Thoro, L. Y. L. Tong, B. R. Toskey, A. M. Vaidya, Thomas Vanden Eynden, Van de Vyle, R. N. Vawter, Emanuel Vegh, W. C. Waterhouse, P. J. Welsh, Jr., D. A. Wick, K. S. Williams, Dale Woods, Kenneth Yanosko, and the proposer.

Garlick, Lindner, and Sampson and Zitarelli located E 1718 as an exercise on page 172 of Niven and Zuckerman, *An Introduction to the Theory of Numbers*, Wiley (1960), and Spital found it posed in Uspensky and Heaslet, *Elementary Number Theory*, McGraw-Hill (1939), page 94. Brooke points out that W. E. Heal proved long ago that $y!$ is not a power for $y>1$. [Math. Mag., 1 (1882) 208–9].

On Reversing the Digits of an Integer

E 1719 [1964, 794]. *Proposed by D. I. A. Cohen, Princeton University*

Let R be the number formed by reversing the digits of the n-digit number N, $R>N$. Prove that $\sqrt{R}\leqq R-N$, and prove that if $10\sqrt{R}\geqq R-N$, then n is even.

Solution by C. B. A. Peck, State College, Pennsylvania. These statements are false, the smallest counterexamples being $N=78$ and $N=102$, respectively. We can rather easily describe the cases in which the first statement fails and in the process say something about the second. This will also lead to weaker statements on the lines of the first. By the familiar method for taking the square root, the number of digits in $[\sqrt{R}]$ is $[(n+1)/2]$. $R>N$ implies that at most the first

$[(n-1)/2]=a$ digits of N and R can agree. If the first digit to disagree is the mth, then $m \leqq [(n+1)/2]$ and $R-N$ has $n-m$ or $n-m+1$ digits according as the mth digits differ by 1 or by more than 1. In the *second* case, $R-N$ has at least

$$n-(a+1)+1=\left[\frac{n+1}{2}\right]+1$$

digits, hence exceeds $\sqrt{R}$, as desired for the first statement. It need not exceed $10\sqrt{R}$, and this leads to a contradiction of the second statement if n is odd. The smallest such example is $N=204$. In the *first* case, the first statement is satisfied if $m \leqq a$, but a counterexample to the second when $m \leqq a$ is furnished by $N=10002$. Again, if $m=a+1$ and n is odd, $R-N=99\cdot 10^a$ and $(R-N)^2 =9801\cdot 10^{n-3}$, which has $n+1$ digits and thus exceeds R, as desired for the first statement. It never comes up to the $n+2$ digits of $100\cdot R$, hence every such N contradicts the second statement, the smallest being $N=102$. Finally, if $m=a+1$ and n is even, $R-N=9\cdot 10^a$ and

$$(R-N)^2=81\cdot 10^{n-2},$$

which is not always $\geqq R$. Of course, no contradiction of the second statement is possible here, since n is even. The numbers N which contradict the first statement can be characterized as follows: n is even, all but the center two digits are symmetrical in the center, the center two differ by one (the larger to the right), and the last two, when reversed, form a number exceeding 80. Since, for given even n, the largest such N is $10^n-10^{n/2}-1$ (all 9's except for an 8 just left of center), the inequality $k\sqrt{R} \leqq R-N$ is satisfied for n odd if $k_n \leqq 1$ and for n even if, and only if,

$$k^2 \leqq 81\cdot 10^{n-2}/(10^n-10^{(n/2)-1}-1).$$

The greatest lower bound for the k's is thus $9/10$, so it is true that $9\sqrt{R} < 10(R-N)$ for any $n>1$.

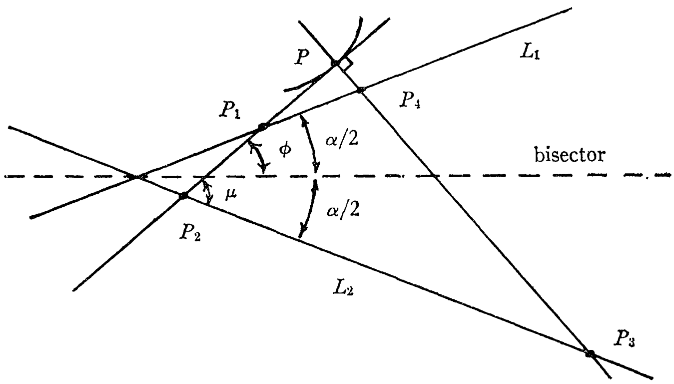
Also solved by M. G. Greening (Australia), E. S. Langford, D. E. McOwen, D. C. B. Marsh, P. R. Nolan (Ireland), Sidney Spital, and Van de Vyle. Counterexamples were supplied by J. C. Abad, R. W. Aldrich, Joseph Arkin, M. A. Bershad, Walter Bluger, Allan Chuck, P. K. Garlick, Michael Goldberg, R. A. Jacobson, J. W. Losse, Simeon Reich (Israel), G. P. S., Gail K. Smith, and B. B. Winter.

A Locus

E 1720 [1964, 794]. *Proposed by R. L. Caskey, Oklahoma State University*

Find a locus such that the tangent and the normal at each of its points intersect two given lines in four concyclic points. (Dedicated to Dr. N. A. Court.)

Solution by Sidney Spital, California State Polytechnic College, Pomona, California. We begin by assuming that the given lines L_1 and L_2 are not parallel. Referring to the figure, we see that for P_1, P_2, P_3, P_4 to be concyclic, $\mu+\lambda=\pi$. From the trigonometry of the intersections $\mu-\lambda=\alpha-\pi/2$ and $\phi=\mu-\alpha/2$.



Eliminating λ, then μ gives respectively $\mu=\frac{1}{2}(\alpha+\pi/2)$, $\phi=\pi/4$. Using the symmetry about the angle bisector, the desired locus must therefore be any one of the family of lines which intersect the bisector at an angle of $\pm\pi/4$. Since this result is independent of the coordinates of our locus it also holds for the remaining case of L_1 and L_2 parallel ($\alpha=0$).

Also solved by D. I. A. Cohen, Michael Goldberg, Ned Harrell, V. F. Ivanoff, D. C. B. Marsh, P. R. Nolan (Ireland), Daniel Pedoe, Robin Robinson, B. R. Toskey, Van de Vyle, K. S. Williams, and the proposer.

Van de Vyle gives also the following degenerate cases: Any line through the point A common to the given lines (when they meet), any circle on A as center, part of a circle on A as center and part of the four lines tangent to this circle and meeting one of the bisectors of the angles determined by the given lines, at an angle of $\pi/4$.

The proposer points out that E 1720 and a solution are given in Mathesis, 29 (1909) 190.

Oops! E 1676 Again!

E 1721 [1964, 911]. *Proposed by J. C. Van Rhijn, Vollenhove, The Netherlands*

Given an ellipse E with foci F_1 and F_2, a point P outside E, the tangents PR_1 and PR_2 from P to E, and a positive number $f(0<f<1)$. Find the locus of P if $PR_1 \cdot PR_2=f\cdot PF_1\cdot PF_2$.

Editor's comment: This problem was posed as E 1676 [1964, 317] and a solution published in this MONTHLY, 72 (1965) 188–189.

Comment by M. S. Klamkin, University of Minnesota. The result follows immediately from some results on ellipses in C. Zwikker, *Advanced Plane Geometry*, North-Holland, Amsterdam (1950), pages 98 and 112: If the parametric equation of the ellipse E is $z=a\cos\theta+ib\sin\theta$ and R_1 and R_2 are given by $\theta=u+q$ and $\theta=u-q$, respectively, then

$$-PR_1\cdot PR_2=\left(\frac{a+b}{2}\right)^2 e^{2iu}+\left(\frac{a-b}{2}\right)^2 e^{-2iu}-\frac{c^2}{2}\cos 2q \quad (c^2=a^2-b^2)$$

and $PF_1 \cdot PF_2 = -PR_1 \cdot PR_2/\cos^2 q$. If now $|PR_1 \cdot PR_2| = f|PF_1 \cdot PF_2|$, then $\cos^2 q = f$ and P is given by $(a \cos u + ib \sin u)/\cos q$; in other words, the locus of P is an ellipse confocal with E.

Following Zwikker, we then note the following generalization of the usual reflection property of ellipses: The tangents to an ellipse from a point make equal angles with the lines joining that point to the foci.

Also solved by Robert Bart, J. D. E. Konhauser, E. S. Langford, Robin Robinson, and the solvers of E1676.

A Curious Sequence

E 1722 [1964, 911]. *Proposed by Norman Schaumberger and Erwin Just, Bronx Community College, New York City*

It is known that $\sum_{j=0}^{n} \cos j\theta$ does not approach a limit as $n \to \infty$ for any θ. Show that as $n \to \infty$, $\lim \sum_{j=0}^{n} \binom{n}{j} \cos j\theta$ does not exist or is zero.

I. *Solution by W. L. Nicholson, General Electric Company, Richland, Washington.* The binomial and De Moivre theorems give

$$\sum_{j=0}^{n} \binom{n}{j} (\cos j\theta + i \sin j\theta) = \sum_{j=0}^{n} \binom{n}{j} e^{ij\theta} = (1 + e^{i\theta})^n$$
$$= (1 + \cos\theta + i \sin\theta)^n = \left(2 \cos \frac{\theta}{2}\right)^n \left(\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2}\right).$$

Equating real and imaginary parts,

$$\sum_{j=0}^{n} \binom{n}{j} \cos j\theta = \left(2 \cos \frac{\theta}{2}\right)^n \cos \frac{n\theta}{2}, \qquad \sum_{j=0}^{n} \binom{n}{j} \sin j\theta = \left(2 \cos \frac{\theta}{2}\right)^n \sin \frac{n\theta}{2} \cdot$$

Both sums tend to 0 as $n \to \infty$ if $|2 \cos \frac{1}{2}\theta| < 1$ (that is, if $2\pi/3 < \theta < 4\pi/3$ (modulo π)), while the sine sums are identically 0 if $\theta = 0$. Otherwise, both sums diverge since $\cos \frac{1}{2}n\theta$ and $\sin \frac{1}{2}n\theta$ continually oscillate.

II. *Solution by N. J. Fine, Pennsylvania State University.* Writing S_n for the sum in the problem, we have

$$S_{2n} = \cos n\theta \left(2 \cos \frac{\theta}{2}\right)^{2n} = \left(2 \cos^2 \frac{n\theta}{2} - 1\right)\left(2 \cos \frac{\theta}{2}\right)^{2n}$$
$$= 2S_n^2 - \left(2 \cos \frac{\theta}{2}\right)^{2n}.$$

If S_n converges, so does S_{2n} and therefore also $(2 \cos \frac{1}{2}\theta)^{2n}$. Hence,

$$|2 \cos \tfrac{1}{2}\theta| < 1 \quad \text{or} \quad |2 \cos \tfrac{1}{2}\theta| = 1.$$

In the first case, $S_n \to 0$. In the second, $S_{2n} = \cos n\theta$ which converges only if

$\theta \equiv 0$ (modulo π). But this implies that $|2 \cos \frac{1}{2}\theta| = 2$, contrary to $|2 \cos \frac{1}{2}\theta| = 1$. Hence, S_n can only converge to 0.

Also solved by A. N. Aheart, R. G. Albert, W. A. Al-Salam, Robert Bart, M. A. Bershad, J. L. Brown, Jr., P. L. Chessin, D. I. A. Cohen, Huseyin Demir (Turkey), M. S. Demos, N. Ersec, A. C. Fang, Stuart Goff, Michael Goldberg, D. M. Hancasky, E. R. Hansen, Stephen Hoffman, J. M. Horner, R. F. Jackson, R. A. Jacobson, M. S. Klamkin, E. S. Langford, H. H. Leavitt, D. C. B. Marsh, C. B. A. Peck, Simeon Reich (Israel), Ira Rosenbaum, G.P.S., Y. P. Sabharwal (India), P. A. Scheinok, Robin Sibson (England), Sidney Spital, R. A. Struble, L. Y. L. Tong, W. C. Waterhouse, Charles Wexler, and the proposer.

$$\sum_{j=1}^{[(k-1)/2]} \cos (2jn\pi/k)$$

E 1723 [1964, 911]. *Proposed by Franklin C. Smith, Minneapolis, Minnesota*

If k is a positive integer and $\theta = 2\pi j/k$, $j = 1, 2, \cdots, k-1$, show that

$$\sum_{n=1}^{[(k-1)/2]} \cos n\theta = -1/4\{2 + (-1)^{k+j} + (-1)^j\},$$

where $[x]$ denotes the greatest integer not exceeding x.

Solution by G. P. Speck, Raymond College, U.O.P., Stockton, California. It is well known that

$$\sum_{n=1}^{N} \cos n\theta = -\frac{1}{2} + \frac{\sin (N + \frac{1}{2})\theta}{2 \sin \frac{1}{2}\theta}$$

if $\theta \not\equiv 0$ (modulo 2π), and obviously $[(k-1)/2] + \frac{1}{2}$ is $k/2$ if k is odd and $(k-1)/2$ if k is even. The given sum is therefore $-\frac{1}{2}$ if k is odd and $-\{1 + (-1)^j\}/2$ if k is even, and the result follows.

Also solved by J. C. Abad, R. G. Albert, W. A. Al-Salam and A. A. Gioia (jointly), Ronald Alter, Robert Bart, M. A. Bershad, J. L. Brown, Jr., J. A. Burslem, Jim Campbell, Leonard Carlitz, P. L. Chessin, D. I. A. Cohen, Huseyin Demir (Turkey), Kenneth Dieter, P. K. Garlick, Michael Goldberg, D. M. Hancasky, E. R. Hansen, Stephen Hoffman, R. A. Jacobson, Erwin Just, and Norman Schaumberger (jointly), M. S. Klamkin, E. S. Langford, Steve Ligh, D. C. B. Marsh, J. G. Rau, Simeon Reich (Israel), S. M. Robinson, Y. P. Sabharwal (India), P. A. Scheinok, D. R. Shoemaker, F. C. Smith, Sidney Spital, L. Y. L. Tong, C. A. Tsonis, Van de Vyle, W. C. Waterhouse, and the proposer.

A Previously Posed Triangle Inequality

E 1724 [1964, 911]. *Proposed by H. W. Guggenheimer, University of Minnesota*

If S is the area of a triangle ABC, show that

$$abc \geqq 8S^{3/2}/3^{3/4},$$

with equality for the equilateral triangle.

I. *Solution by A. Oppenheim, University of Malaya, Kuala Lumpur, Malaya.* It is an interesting exercise to see how many triangle inequalities can be deduced from the inequality

$$(1)\qquad (\lambda a^2 + \mu b^2 + \nu c^2)^2 \geqq 16(\mu\nu + \nu\lambda + \lambda\mu)S^2$$

which is valid for all real λ, μ, ν, with equality if and only if

$$\lambda:\mu:\nu = b^2 + c^2 - a^2 : c^2 + a^2 - b^2 : a^2 + b^2 - c^2.$$

If λ, μ, ν are all positive, the familiar arithmetic-geometric mean leads from (1) to

$$(2)\qquad (\lambda a^2 + \mu b^2 + \nu c^2)^2 \geqq 48(\lambda\mu\nu)^{2/3}S^2.$$

To obtain E 1724 take $\lambda a^2 = \mu b^2 = \nu c^2$ in (2).

II. *Solution by J. S. Frame, Michigan State University.* Let A, G, and H denote respectively the arithmetic, geometric, and harmonic means of the positive quantities $x=(b+c-a)/2$, $y=(c+a-b)/2$, $z=(a+b-c)/2$. Then $A\geqq G\geqq H$, and $S^2=(x+y+z)xyz=3AG^3$.

$$\begin{aligned} abc &= (y + z)(z + x)(x + y) = (x + y + z)(yz + zx + xy) - xyz \\ &= (3A)(3G^3/H) - G^3 \geqq 8AG^3/H \geqq 8AG^3/(AG^3)^{1/4} = 8(S^2/3)^{3/4}. \end{aligned}$$

Equality holds only when $A=G=H=x=y=z$, and the triangle is equilateral.

III. *Solution by Sidney Spital, California State Polytechnic College, Pomona, California.* With $s=(a+b+c)/2$, the problem can be recast to that of minimization of the triangle circumradius $r(a, b, c) = abc/4s$ subject to the constraints of area $\sqrt{[s(s-a)(s-b)(s-c)]}=S$ (a positive constant), and nonnegative side lengths a, b, c. Since the circumradius approaches $+\infty$ as any one of the sides approaches zero (while the area S is held fixed) the boundary of the allowable region of a, b, c maximizes r. Therefore the desired minimum must be an internal one, and from the symmetry of $r(a, b, c)$ and the constraints, it must be given by the equilateral triangle $a=b=c$, whence $s=3c/2$ and $S=c^2\sqrt{3}/4$. Thus,

$$r(a, b, c) = abc/(4S) \geqq c^3/(4S) = r(c, c, c)$$

and $abc\geqq c^3 = 8S^{3/2}/3^{3/4}$.

IV. *Solution by F. Leuenberger, Feldmeilen/ZH, Switzerland.* An easy sharpening of E 1724 is $abc=4RS\geqq 8sS/3^{3/2}\geqq 8S^{3/2}/3^{3/4}$, where R is the circumradius and s is the semiperimeter. These inequalities follow from the well-known estimates $3^{3/2}R\geqq 2s$ and $s\geqq S^{1/2}3^{3/4}$ in which there is equality if and only if the triangle is equilateral.

Editor's comment: Aheart, Marsh, and Sister Stephanie point out that E 1724 was conclusion (2) in E 1454 [1961, 177], posed by Leonard Carlitz and solved by F. Leuenberger and Carlitz in

this MONTHLY 68 (1961) 805–806. The solutions which appear here differ considerably, however, from those to E 1454.

Also solved by A. N. Aheart, H. S. Al-Amiri, R. G. Albert (two solutions), J. W. Baldwin, Robert Bart, M. A. Bershad, D. A. Blaeuer, Farrell Bloch, W. J. Blundon, B. A. Bursack, Jim Campbell, Leonard Carlitz, W. J. Carpenter, Mannis Charosh, M. M. Chawla and K. M. Das (jointly) (India), N. A. Childress, R. T. Coffman, D. I. A. Cohen, T. R. Curry, Huseyin Demir (Turkey), Kenneth Dieter, Ragnar Dybvik (Norway), A. C. Fang, Neal Felsinger, N. J. Fine, Michael Goldberg, S. H. Greene, Raoul Hailpern, D. M. Hancasky, J. R. Hanna, M. Hayamizu, R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), M. S. Klamkin, Kenneth Kramer, E. S. Langford, Ruth S. Lefkowitz, R. E. Lowney, Doyle McOwen, D. C. B. Marsh, E. L. Magnuson, Norman Miller, W. O. Moser, Margaret Olmsted, J. P. Phillips, Lawrence Rafsky, G. L. N. Rao (India), Simeon Reich (Israel), Judith Richman, L. A. Ringenberg, Bernard Rosner, J. P. Ruebsamen, M. S. R. K. Sastry, P. A. Scheinok, Ralph Schreiber, D. B. Sheinson, Sister M. Stephanie, R. L. Syverson, C. A. Tsonis, Van de Vyle, Simon Vatriquant (Belgium), W. C. Waterhouse, and the proposer, and the solvers of E 1454.

$\sum_{d/n} d\phi(d)$

E 1725 [1964, 912]. *Proposed by Hwa S. Hahn, Pennsylvania State University*

If $\phi(x)$ is Euler's function and n is a positive integer with the prime decomposition $n=\prod_{p|n} p^h$, prove that

$$\sum_{d|n} d\phi(d) = \prod_{p|n} (p^{2h+1}+1)/(p+1).$$

I. *Solution by W. A. Al-Salam and A. A. Gioia, Texas Technological College, Lubbock, Texas.* If x is any complex number and k is a positive integer, we show that

$$\sum_{d|n} d^x\phi(d^k) = \prod_{d|n}\left\{1+\frac{p^{x+k-1}(p-1)[p^{h(x+k)}-1]}{p^{x+k}-1}\right\}.$$

Consider

$$(1)\qquad \sum_{d|n} d^x\phi(d^k) = n^x\sum_{d|n}\left(\frac{n}{d}\right)^x\phi(d^k).$$

Since the arithmetic functions $f(n)=n^{-x}$ and $g(n)=\phi(n^k)$ are obviously multiplicative, their Dirichlet product $\sum_{d|n}(n/d)^{-x}\phi(d^k)$ is a multiplicative function. It follows that (1) is also multiplicative and need be evaluated only at prime powers p^h.

$$\sum_{d|p^h} d^x\phi(d^k) = \sum_{i=0}^{h} p^{xi}\phi(p^{ki}) \quad = 1+\sum_{i=1}^{h} p^{xi}p^{ki-1}(p-1)$$

$$= 1+\frac{p^{x+k-1}(p-1)(p^{(x+k)h}-1)}{p^{x+k}-1}.$$

The theorem follows by taking the product of these factors over all prime power divisors of n.

II. *Solution by K. S. Williams, University of Toronto, Toronto, Ontario, Canada.* Arguing as in Solution I, we find that $\sum_{d|n} d^\alpha\{\phi(d)\}^\beta$ is

$$\prod_{p|n}\left\{1+h\left(\frac{p-1}{p}\right)^\beta\right\} \quad\text{or}\quad \prod_{p|n}\left\{1+\frac{p^\alpha(p-1)^\beta[p^{h(\alpha+\beta)}-1]}{p^{\alpha+\beta}-1}\right\}$$

according as $\alpha+\beta=0$ or $\alpha+\beta\neq 0$.

Similar arguments lead to corresponding expressions when ϕ is replaced by other familiar number theoretic functions.

Also solved by J. C. Abad, R. G. Albert, C. W. Barnes, M. A. Bershad, J. A. Burslem, John Christopher, D. I. A. Cohen, Huseyin Demir (Turkey), M. S. Demos, Bob De Vore, R. B. Eggleton (Australia), N. J. Fine, P. K. Garlick, Jerry Goodman, D. M. Hancasky, R. A. Jacobson, M. S. Klamkin, D. A. Klarner, E. S. Langford, Marilyn McLennan, D. C. B. Marsh, J. Mersky and Eric Weiss (jointly), J. B. Muskat, Harsh Pittie, D. P. Roselle, P. A. Scheinok, Frank Servas, Jr., Sidney Spital, L. Y. L. Tong, A. M. Vaidya, Simon Vatriquant (Belgium), Emanuel Vegh, Charles Wexler, and the proposer.

Langford calls attention to the formulas

$$\sum_{d|n}\phi(d)/d=\prod_{p|n}\left[1+\frac{h(p-1)}{p}\right] \quad\text{and}\quad \sum_{d|n}d/\phi(d)=\prod_{p|n}\left(1+\frac{hp}{p-1}\right)$$

which are attributed to Catalan [Dickson, *History of the Theory of Numbers*, vol. 1, page 130].

The proposer points out that $n^{-2}\sum_{d|n}d\phi(d)$ is the probability that $ax\equiv b \pmod n$ has a solution when a and b are selected randomly from $\{1, 2, \cdots, n\}$. He observes that this probability is greater than 1/2 if n has at most two distinct prime factors and is always less than 1/2 if n has at least three such factors.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate signed sheets and should be mailed before March 31, 1966.

5310. *Proposed by S. W. Golomb, University of Southern California*

Represent each of the $n!$ permutations on n letters as a product of disjoint cycles, and let T_n^k be the number of such permutations for which the longest cycle has at most k letters. Show that

$$T_{n+1}^k=\sum_{j=0}^{k-1}(n)_jT_{n-j}^k,$$

where $(n)_j=n!/(n-j)!$.

5311. *Proposed by D. Ž. Djoković, University of Belgrade, Yugoslavia*

Let $x_1<x_2<\cdots<x_n$ be real numbers and

$$f(x;x_1,\cdots,x_n)=(x-x_1)(x-x_2)\cdots(x-x_n),$$

$$M = \max_{x_1 < x < x_n} \left| f(x; x_1, \cdots, x_n) \right|,$$

$$F(x; x_1, \cdots, x_n) = \frac{1}{M} f(x; x_1, \cdots, x_n),$$

$$\phi(x_1, x_2, \cdots, x_n) = \int_{x_1}^{x_n} F(x; x_1, \cdots, x_n) dx.$$

Prove (or disprove) the inequality $(-1)^k(\partial\phi/\partial x_k) > 0$.

5312. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Prove (or disprove) the equation

$$\begin{vmatrix} 1 & 1 & \cdots 1 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial^{n-1}}{\partial x_1^{n-1}} & \dfrac{\partial^{n-1}}{\partial x_2^{n-1}} \cdots & \dfrac{\partial^{n-1}}{\partial x_n^{n-1}} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots 1 \\ x_1 & x_2 & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots x_n^{n-1} \end{vmatrix} = 1!2! \cdots n!.$$

5313. *Proposed by J. M. Quoniam, Saint-Etienne (Loire), France*

If ABC is a triangle with circumcenter O, orthocenter H, circumradius R, and area Δ, prove that

$$\cos 2A + \cos 2B + \cos 2C = \frac{OH^2}{2R^2} - \frac{3}{2}, \tag{1}$$

$$\sin 2A + \sin 2B + \sin 2C = 2\Delta/R^2. \tag{2}$$

5314. *Proposed by D. R. Brillinger, Princeton University*

Show that the following integral is absolutely convergent:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\sin \omega_1}{\omega_1} \frac{\sin \omega_2}{\omega_2} \cdots \frac{\sin \omega_k}{\omega_k} \frac{\sin(\omega_1 + \omega_2 + \cdots + \omega_k)}{(\omega_1 + \omega_2 + \cdots + \omega_k)} d\omega_1 \cdots d\omega_k.$$

5315. *Proposed by G. A. Heuer and R. G. Lee, Concordia College, Moorhead, Minn.*

If f is a function integrable on $[0, 1]$, let $Af = A^1 f$ be the function defined on $(0, 1]$ by

$$Af(x) = (1/x) \int_0^x f(x)\, dx; \qquad Af(0) = 0.$$

Let $A^{n+1}f = A(A^n f)$ for $n = 1, 2, \cdots$; obviously, if f has a limit at 0, so does Af.

(1) Find a function f_0, bounded and continuous on $(0, 1]$, such that f_0 does not have a limit at 0 but Af_0 does.

(2) Find a function f_1, bounded and continuous on $(0, 1]$, such that Af_1 does not have a limit at 0.

(3) Does there exist, for each positive integer n, a function f_n, continuous and bounded on $(0, 1]$, such that $A^n f$ has no limit at 0?

(4) Is there a continuous bounded function f on $(0, 1]$ such that for every positive integer n, $A^n f$ fails to have a limit at 0?

5316. *Proposed by J. T. Fleck and Carl Evans, Cornell Aeronautics Laboratory, Buffalo, N. Y.*

Show that if A and B are nonnegative definite Hermitian matrices of order n, the characteristic roots of AB are real and nonnegative.

5317. *Proposed by Daniel I. A. Cohen, Princeton University*

Show that there exists a number k such that there are $n-1$ consecutive composite integers less than e^{kn} for all n.

5318. *Proposed by Oystein Ore, Yale University*

Prove that there exist only a finite number of integers n for which the factorial $n!$ is the sum of two integral squares. Try to find all such n.

5319. *Proposed by V. K. Rohatgi, Michigan State University*

Let n, x_i $(i=1, 2, \cdots, k)$ be nonnegative integers. For $n \geqq 1$, define a function recursively as follows:

$$(0; 0, 0, \cdots, 0) = 1,$$

$$(n; x_1, x_2, \cdots, x_k) = \begin{cases} 0 & \text{if } \sum_{i=1}^{k} x_i > n, \\ \sum_{y_1=0}^{x_1} \cdots \sum_{y_k=0}^{x_k} (-1)^\alpha (n-1; y_1, \cdots, y_k) & \text{otherwise,} \end{cases}$$

where $\alpha = \sum_{i=1}^{k} (x_i - y_i)$.

Find $(n; x_1, \cdots, x_k)$ explicitly in terms of $n, x_1, \cdots, x_k$.

SOLUTIONS OF ADVANCED PROBLEMS

Series of Trigonometric Functions

5156 [1963, 1107; 1964, 1141]. *Proposed by Don Kirkham, Iowa State University*

For $-\frac{1}{2}\pi \leqq \theta \leqq \frac{1}{2}\pi$, prove that

$$\theta \cot \theta + \tfrac{1}{3} \cot^3 \theta(\tan \theta - \theta) - \tfrac{1}{5} \cot^5 \theta(-\tfrac{1}{3} \tan^2 \theta + \tan \theta - \theta)$$
$$+ \tfrac{1}{7} \cot^7 \theta(\tfrac{1}{5} \tan^5 \theta - \tfrac{1}{3} \tan^3 \theta + \tan \theta - \theta) - \cdots = \tfrac{1}{8}(\pi^2 - 4\theta^2).$$

II. *Solution by Roy O. Davies, The University, Leicester, England.* The earlier solution of the problem which covered the situation $|\theta| \leqq \pi/4$ may be extended to other values of θ. The following formula from the earlier solution will be used:

$$(1)\quad f(x) = \tfrac{1}{2}x^2 - \tfrac{1}{4}(1 + \tfrac{1}{3})x^4 + \tfrac{1}{6}(1 + \tfrac{1}{3} + \tfrac{1}{5})x^6 - + \cdots = \tfrac{1}{2}(\arctan x)^2.$$

So, if $x = \cot\theta$, $(n+\frac{1}{4})\pi \leqq \theta \leqq (n+\frac{3}{4})\pi$, $n = 0, \pm 1, \pm 2, \cdots$, then $f(x) = \frac{1}{2}[(n+\frac{1}{2})\pi - \theta]^2$.

Note also that the series

$$(2)\qquad \cot\theta - \tfrac{1}{3}\cot^3\theta + \tfrac{1}{5}\cot^5\theta - + \cdots = (n + \tfrac{1}{2})\pi - \theta$$

for $(n+\frac{1}{4})\pi \leqq \theta \leqq (n+\frac{3}{4})\pi$, and diverges for all other values of θ.

Now let $S(\theta)$ represent the series of the problem and let $x = \cot\theta$. Then

$$\begin{aligned} S(\theta) &= \theta(x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - + \cdots) + \tfrac{1}{3}x^2 - \tfrac{1}{5}x^2(-\tfrac{1}{3} + x^2) + \cdots \\ &= \theta((n + \tfrac{1}{2})\pi - \theta) + (1\cdot\tfrac{1}{3} + \tfrac{1}{3}\cdot\tfrac{1}{5} + \cdots)x^2 - (1\cdot\tfrac{1}{5} + \tfrac{1}{3}\cdot\tfrac{1}{7} + \cdots)x^4 + \cdots \\ &= \theta((n + \tfrac{1}{2})\pi - \theta) + f(x) \\ &= (2n + 1)^2\pi^2/8 - \tfrac{1}{2}\theta^2, \qquad (n + \tfrac{1}{4})\pi \leqq \theta \leqq (n + \tfrac{3}{4})\pi. \end{aligned}$$

This, with the earlier solution, establishes $S(\theta) = \pi^2/8 - \theta^2/2$ for $|\theta| \leqq 3\pi/4$.

We also deduce the divergence of $S(\theta)$ in the interval $(3\pi/4, 5\pi/4)$ (and corresponding intervals mod 2π) by noting the identity

$$S(\theta) = n\pi(\cot\theta - \tfrac{1}{3}\cot^3\theta + \tfrac{1}{5}\cot^5\theta - \cdots) + S(\theta - n\pi)$$

and comparing with (2).

Also solved by W. C. Waterhouse.

Editorial Note. The note at the end of the former solution [1964, 1141] referred to the proposer's verification in the case $\theta = 7\pi/6$. This should have read $\theta = 7\pi/16$.

m-convexity

5219 [1964, 801]. *Proposed by Martin Cohen, Beverly Hills, California*

Let a function f be called m-convex if, for some m, $0 \leqq m \leqq 1$, and every x_1, x_2 in the domain of f, we have $f(mx_1 + (1-m)x_2) \leqq mf(x_1) + (1-m)f(x_2)$. f is called convex if it is m-convex for all m, $0 \leqq m \leqq 1$. In problem 5088 [1964, 330] it is proved that if f is bounded and if f is $\frac{1}{2}$-convex, then f is convex.

Determine necessary and sufficient conditions on m such that all m-convex functions are convex. Give examples of functions which are m-convex but not convex.

I. *Solution by Roy O. Davies, Leicester University, England.* There are no such m. For $m = 0$, 1, any nonconvex f is a counterexample. For $0 < m < 1$, let M denote the smallest field of real numbers containing m, that is, the set of all numbers expressible in the form

$$(a_0m^r + a_1m^{r-1} + \cdots + a_r)/(b_0m^s + b_1m^{s-1} + \cdots + b_s),$$

where r, s are nonnegative integers, the a_i and b_j are integers, and the denominator is not zero. Using Zorn's lemma, select a maximal set Y of real numbers linearly independent over M, with $1 \in Y$. We note that M is countable while Y is not countable. A number

$$(1) \qquad x = \mu_1 y_1 + \cdots + \mu_n y_n \qquad (\text{distinct } y_\nu \in Y;\, \mu_1, \cdots, \mu_n \in M)$$

cannot vanish unless $\mu_1 = \cdots = \mu_n = 0$, and every real x is uniquely (except for zero terms) expressible in the form (1). It is easy to verify that the function $f(x)$ defined as the coefficient μ_i (possibly zero) of 1 in (1) is m-convex, indeed "m-linear." But $f(x) = x$ on the everywhere dense set M, while if $y^* \in Y - M$, then $f(x) = 0$ on the everywhere dense set $\{\mu y^*: \mu \in M\}$, easily implying that f is discontinuous and not convex.

In addition we see that the function $f(x) + x^2$ is also m-convex, but neither m-linear nor convex. But it is easy to prove that whenever $0 < m < 1$, m-convexity together with boundedness produces convexity.

II. *Solution by Solomon Marcus, Institutul de Matematică, Bucharest, Rumania.* The problem is completely solved by Ervin Deák in *Über konvexe und interne Funktionen, sowie eine gemeinsame Verallgemeinerung von beiden,* (Ann. Univ. Sci. Budapest. Eötvös, Sect. Math., tomus V (1962), 109–154). Theorem 19 of this paper includes higher dimensional situations and says that an m-convex function defined on an open set K in R^n is convex in K if and only if it is continuous in K.

Also solved by John B. Kelly.

Editorial Note. The problem was proposed as a generalization of 5088 and might, therefore, have included a boundedness hypothesis. As observed above, convexity would then follow. It is possible to give a proof analogous to that given for 5088 [1964: 330, 804].

Characterizing a Unique Functional of $C[0, 1]$

5221 [1964, 801]. *Proposed by Oliver Gross, the RAND Corporation, Santa Monica, California*

The notion of an integral has been characterized, extended and examined from many points of view. Many definitions embody explicitly some sort of limiting procedure. A characterization in which the limiting aspect is obscured might be of interest.

Relative to the class $C_{[0,1]}$ of real valued continuous functions on the closed unit interval, define the operator S as follows:

$$Sf(x) = \tfrac{1}{2}(f(\tfrac{1}{2}x) + f(1 - \tfrac{1}{2}x)).$$

Clearly, S takes $C_{[0,1]}$ into (in fact, onto) itself. Prove that there exists a unique function K on $C_{[0,1]}$ such that for all f, (1) $K(f) \in \text{Range } (f)$, and (2) $K(Sf) = K(f)$.

Solution by D. Ž. Djoković, University of Belgrade, Yugoslavia. The existence is clear since we can take

$$K(f) = I(f) \equiv \int_0^1 f(x)dx \in \text{Range } (f).$$

In order to prove uniqueness of K we start from the identity

$$S^n f(x) = \frac{1}{2^n}\left[\sum_{k=0}^{2^{n-1}-1} f\left(\frac{k}{2^{n-1}} + \frac{x}{2^n}\right) + \sum_{k=1}^{2^{n-1}} f\left(\frac{k}{2^{n-1}} - \frac{x}{2^n}\right)\right],$$

which can be proved by induction. It follows that $S^n f(x) \to I(f)$ as $n \to \infty$ uniformly for $x \in [0, 1]$. That is, if $\epsilon > 0$

$$I(f) - \epsilon < S^n f(x) < I(f) + \epsilon, \qquad n > n_0(\epsilon), \qquad x \in [0, 1].$$

Consequently $K(f) = K(S^n f) \in (I(f) - \epsilon, I(f) + \epsilon)$, implying that $K(f) \equiv I(f)$.

Also solved by Robert Cohen, Roy O. Davies (England), Michael Edelstein, P. G. Engstrom, A. G. Heinicke, Sim Lasher, M. D. Mavinkurve (India), A. G. P. M. Nijet (Netherlands), and Stanton Philipp.

Notes. (1) Davies observes that $S^n f$ converges uniformly for every Riemann-integrable f and may converge uniformly for non-Riemann-integrable functions—e.g., the characteristic function of the set of all rational multiples of $\sqrt{2}$, wherein $S^n f \to 0$. For f integrable-R, the functional $I(f)$ is, clearly, not necessarily in the range of f.

(2) In an obvious fashion, the same result may be obtained using

$$Sf = \lambda f(\lambda x) + (1 - \lambda) f(1 - (1 - \lambda)x), \; 0 < \lambda < 1.$$

Subnets of Convergent Nets

5222 [1964, 802]. *Proposed by M. Rajagopalan and A. Wilansky, Lehigh University*

Must every convergent net in a metric space have a subnet whose range has at most one limit point?

Solution by E. T. Ordman, Princeton University. No. The following construction gives a counterexample in the Euclidean plane.

Let D be the class of finite subsets $s = \{a_1, a_2, \cdots, a_n\}$ of the interval (0, 1). We may partially order D by letting $s \geqq t$ whenever $s \supset t$; any two elements are followed by their union, so $\geqq$ directs D. Any element of D follows at most finitely many others, and D is uncountable (since $\{a\} \in D$ for $0 < a < 1$), so any cofinal subset of D must be uncountable. We shall define a net T with domain D into the Euclidean plane which is one-to-one and which converges to zero. Then any subnet of T will include in its range images of the points of some cofinal subset of D; that is, the range of the subnet will be uncountable, and thus have an uncountable number of limit (in fact, condensation) points other than zero.

T may be constructed as follows. Let D_n be the subset of D consisting of n-element subsets of $(0, 1)$; D_n may be mapped one-to-one into $(0, 1)$ by writing each $s = \{a_1, a_2, \cdots, a_n\}$ in ascending order, writing each a_k as a decimal expansion (terminating if possible) $a_k = \cdot a_{1,k}a_{2,k}, \cdots$, and then letting $\delta(s) = \cdot a_{1,1}a_{1,2} \cdots a_{1,n}a_{2,1}a_{2,2} \cdots$. δ is then one-to-one into $(0, 1)$, in general not onto. We now introduce polar coordinates into the plane; (r, a) is the point a radians from the axis on the circle of radius r. Define $T(s) = (1/n, \delta(s))$ for $s \in D_n$; T is then one-to-one and eventually inside each circle of radius $1/n$, so T converges to zero.

Also solved by Ethan Akin, M. M. Chawla (India), Michael Edelstein, Hewitt Kenyon, J. R. Porter, and the proposers.

Editorial Note. As indicated by Chawla, the problem may also be resolved by the use of Theorem 6, p. 71 in J. L. Kelley, *General Topology.*

As posed originally, the problem asked for exactly one limit point, which admits the trivial example $\{(-1)^n\}$ whose range is $\{-1, 1\}$ and has no limit points. Happily the problem was accepted by the solvers as intended by the proposers.

The Torsion Subgroup of an Infinite Abelian Group

5223 [1964, 802]. *Proposed by C. R. MacCluer, University of Michigan*

Let L be the additive Abelian group of all ∞-tuples $(a_1, a_2, \cdots)$ where the nth entry is drawn from the integers modulo p^n, p a fixed prime, and let addition in L be coordinate-wise. Let G be the torsion subgroup and H the subgroup of all elements that have almost all zero entries. Show that H is not a direct summand of G. (This provides an example showing that purity of H does not imply that H is a direct summand even in the torsion case.)

Solution by Victor Keiser, University of Colorado. Let $\bar{g} = (a_1, a_2, \cdots) \in G$. Since $\bar{g}$ has finite order, almost all entries satisfy $(a_i, p) = p$. For all such a_i the equation $px = a_i$ has a solution, say h_i. Let $\bar{h} = (h_1, h_2, \cdots)$ where h_i is the solution of $px_i = a_i$ if it exists and $h_i = 0$ otherwise. Then $(\bar{g} - p\bar{h}) \in H$ because almost all its entries are zero. Thus $p(H + \bar{h}) = H + \bar{g}$, so we see that G/H is divisible.

Now suppose that H is a direct summand of G, say $G = H \oplus S$. Then $S \cong G/H$, so S must be divisible. But every element of p^iS has a zero in the ith position, so $p^iS \neq S$ for some i. Hence S is not divisible.

Also solved by D. Ž. Djoković (Yugoslavia), Jack R. Porter, Burnett R. Toskey, and the proposer.

Representation of a Legendre Sum

5224 [1964, 802]. *Proposed by L. Carlitz, Duke University*

Let $\psi(a) = (a/p)$, the Legendre symbol. Show that if $abcd \not\equiv 0 \pmod{p}$,

$$S = \sum_{x,y,z=0}^{p-1} \psi(ax^2 + by^2 + cz^2 - 2dxyz) = -p\{\psi(a) + \psi(b) + \psi(c) + \psi(-abc)\}.$$

Solution by J. B. Muskat, University of Pittsburgh. We use the formula

$$\text{(A)} \qquad \sum_{x=0}^{p-1} \psi(x^2+n) = p - 1 - p\psi^2(n),$$

proved by Ernst Jacobsthal (J. Reine Angew. Math., 1932 (1907) 238–245). Thus we have

$$\sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\sum_{z=0}^{p-1} \psi(ax^2+by^2+cz^2-2dxyz)$$

$$= \psi(c)\sum_{x=0}^{p-1}\sum_{y=0}^{p-1}\sum_{z=0}^{p-1} \psi([z-dxyc^{-1}]^2 - d^2x^2y^2c^{-2} + ax^2c^{-1} + by^2c^{-1})$$

$$= \psi(c)\sum_{x=0}^{p-1}\sum_{y=0}^{p-1} [p-1-p\psi^2(-d^2x^2y^2c^{-2}+ax^2c^{-1}+by^2c^{-1})] \qquad \text{(by (A))}$$

$$= \psi(c)p\left[p(p-1) - \sum_{x=0}^{p-1}\sum_{y=0}^{p-1} \psi^2(by^2c - d^2x^2y^2 + ax^2c)\right]$$

$$= \psi(c)p\left\{p(p-1) - \sum_{\substack{x=0 \\ d^2x^2\neq bc}}^{p-1} \sum_{y=0}^{p-1} \psi^2(y^2[bc-d^2x^2]+acx^2) - \sum_{\substack{x=0 \\ d^2x^2=bc}}^{p-1} \sum_{y=0}^{p-1} \psi^2(acx^2)\right\}$$

$$= \psi(c)p\left\{p(p-1) - \sum_{\substack{x=0 \\ d^2x^2\neq bc}}^{p-1} [p-1-\psi(-acx^2(bc-d^2x^2)^{-1})] - p\sum_{\substack{x=0 \\ d^2x^2=bc}}^{p-1} 1\right\}$$

$$= p\psi(c)\left\{p(p-1) - (p-1)(p-1-\psi(bc) + \sum_{\substack{x=0 \\ d^2x^2\neq bc}}^{p-1} \psi(-acx^2\psi(bc-d^2x^2) - p[1+\psi(bc)]\right\}$$

$$= p\psi(c)\left\{-1-\psi(bc)+\psi(ac)\sum_{x=1}^{p-1}\psi(d^2x^2-bc)\right\}$$

$$= p\psi(c)\left\{-1-\psi(bc)+\psi(ac)\left[-\psi(-bc)+\psi(d^2)\sum_{x=0}^{p-1}\psi(x^2-bcd^{-2})\right]\right\}$$

$$= p\{-\psi(c)-\psi(b)+\psi(a)[-\psi(-bc)+p-1-p\psi^2(-bcd^{-2})]\} \qquad \text{(by (A))}$$

$$= p\{-\psi(c)-\psi(b)-\psi(-abc)+\psi(a)[p-1-p]\}$$

$$= -p\{\psi(a)+\psi(b)+\psi(c)+\psi(-abc)\}.$$

Also solved by N. J. Fine, Irving Gerst, M. G. Greening (Australia), K. S. Williams, and the proposer.

Editorial Note. Fine in his proof first develops the interesting general formula:

$$\sum_{x_1,x_2,\cdots,x_n=0}^{p-1} \psi(a_1x_1^2+\cdots+a_nx_n^2)=\psi(a_1a_2\cdots a_n)\sum\psi(x_1^2+\cdots+x_n^2)$$

with $a_1a_2\cdots a_n\not\equiv 0 \pmod p$.

A Function with Prescribed Horizontal Tangents

5225 [1964, 802]. *Proposed by Solomon Marcus, University of Bucharest, Romania*

Let B be a given, nowhere dense, and perfect set on the real line. For any function F, with continuous derivatives on $[0, 1]$, put

$$S_1=\{t: F'(t)=0\},\qquad S_2=\{F(t): t\in S_1\}.$$

Does there exist a function F such that $S_2=B$? (See a closely related problem, no. 5114 [1964, 693].)

Solution by Roy O. Davies, the University, Leicester, England. B must also be bounded (since F is continuous) and of measure zero (intuitively obvious since F' represents the local magnification of lengths under the mapping $t\to F(t)$, and easily proved; see S. Saks, *Theory of the Integral*, p. 266). The conditions are now sufficient.

Let $c=\inf B$, $d=\sup B$, enumerate the intervals of $[c, d]\backslash B$ as $J_n=(c_n, d_n)$, and set

$$j_n=|J_n|,\qquad r_n=\sum_{\nu=n}^{\infty}j_\nu,\qquad i_n=j_n/r_1^{1/2}(r_n^{1/2}+r_{n+1}^{1/2}),$$

observing that $j_n/i_n\to 0$ and

$$\sum_{n=1}^{\infty}i_n=r_1^{-1/2}\sum_{n=1}^{\infty}(r_n-r_{n+1})/(r_n^{1/2}+r_{n+1}^{1/2})=r_1^{-1/2}\sum_{n=1}^{\infty}(r_n^{1/2}-r_{n+1}^{1/2})=1.$$

Write $m\prec n$ if $c_m<c_n$, and set

$$a_n=\sum_{m\prec n}i_m,\qquad b_n=a_n+i_n,\qquad I_n=(a_n, b_n).$$

Clearly the I_n are disjoint, with $a_m<a_n$ iff $m\prec n$, and everywhere dense in $[0, 1]$, and $[0, 1]\backslash\bigcup I_n$ is of measure zero.

On $[0, 1]$ let $d_n(x)$ denote the distance of x from $[0, 1]\backslash I_n$, and define $f_n(x)=(4j_n/i_n^2)d_n(x)$, $f(x)=\sum_{n=1}^{\infty}f_n(x)$, $F(t)=c+\int_0^t f(x)dx$. Then F has the continuous derivative f because each f_n is continuous and the series $\sum f_n$ is uniformly convergent (since $f_n(x)\neq 0$ only for $x\in I_n$, and $\sup f_n(x)=2j_n/i_n\to 0$). Also F is strictly increasing because $F'(x)>0$ on $\bigcup I_n$. We have

$$F(b_n)-F(a_n)=\int_{a_n}^{b_n}f(x)dx=\int_{a_n}^{b_n}f_n(x)dx=j_n.$$

Since $[0, 1]\backslash\bigcup I_m$ and $[c, d]\backslash\bigcup J_m$ are of measure zero, we have

$$F(a_n) = c + \int_0^{a_n} f(x)dx = c + \sum_{m\prec n} \int_{a_m}^{b_m} f(x)dx = c + \sum_{m\prec n} j_m = c_n,$$

and $F(b_n) = c_n + j_n = d_n$. Thus $F(I_n) = J_n$, and

$$S_2 = F(\{t: F'(t) = 0\}) = F(\{t: f(t) = 0\}) = F([0, 1]\backslash\bigcup I_n)$$
$$= [c, d]\backslash\bigcup J_n = B.$$

Also solved (partially) by S. W. Benton, Jr.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University

COLLABORATING EDITORS: K. O. MAY, University of California, Berkeley and E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. Programmed Materials: K. O. May, Univ. of California, Berkeley, Calif. 94704. Films: E. P. Vance, Oberlin College, Oberlin, Ohio 44074.

Fourier Series and Orthogonal Functions. By Harry F. Davis. Allyn and Bacon, Boston, 1963. xii+403 pp. $11.95.

This book is intended primarily as a text for a year-long undergraduate course to be taken immediately after first (and hence normally unrigorous) courses which include some treatment of elementary differential equations and vector concepts. The enrollment would include prospective mathematicians, engineers and physicists. It is addressed also to the private reader.

The six chapters are entitled, in order, "Linear Spaces," "Orthogonal Functions," "Fourier Series," "Legendre Polynomials and Bessel Functions," "Heat and Temperature," and "Waves and Vibrations; Harmonic Analysis." There is also an Appendix ("Functions on Groups") for those knowing some modern algebra. The chapters can be read in many permutations. There are a large number of exercises (with answers to many), illustrative examples, and diagrams. Physical and geometrical illustrations and motivations are found scattered usefully throughout the text, as well as in specialized chapters and sections. Many remarks are made illuminating the particular significance of individual hypotheses in theorems and the relationships between similar-sounding or apparently conflicting theorems.

On the other hand, in following the folksy style of many lecture platforms, the writing often becomes frothy, superficial and vague. (It is also disjointed, perhaps the result of seeking after extreme flexibility in the order of topics.) There is a genuine danger that a naive reader could come to believe that he has learned a great deal more mathematics than would be the case.

A good deal more of current mathematical terminology is used in this book than is customary in a discussion of these topics at this level. Its readers will thus derive the benefit of a larger than ordinary mathematical vocabulary. However, they may not obtain sufficiently precise, well-rounded or deep understanding. References to appropriate sources to which to turn for elaboration would be very useful; there are hardly any.

LEE LORCH, University of Alberta

Factor Analysis as a Statistical Method. By D. N. Lawley and A. E. Maxwell. Butterworth Inc., Washington, D. C., 1963. 117 pp. $4.25.

This book quite successfully describes various topics in factor analysis in a manner both intelligible and meaningful to a statistician. The authors are to be congratulated for their sophisticated treatment of the subject, although motivation is often lacking and discussion of all topics is brief.

Consistent utilization of matrix notation throughout the book provides a quick grasp of the essence of factor analysis and allows a wealth of material to be contained in a small volume. The necessary principles of matrix algebra are included in an appendix. However, a section might have been added to justify certain iterative techniques for solving matrix equations that are used in the estimation procedures.

Statisticians should be particularly pleased with the chapter on maximum likelihood estimation of factor loadings and residual variances. Also of interest are maximum likelihood estimation given prior information concerning the matrix of factor loadings, and procedures for estimating factor scores.

Approximate estimation techniques and the method of principal components are also described. Objectives of principal component analysis are distinguished from those of factor analysis, but the reader may sense a degree of bias regarding their relative merits and a certain arbitrariness in the choice between the two methods in any particular application.

The book may be considered sufficient enlightenment for a mathematician (statistician) with a casual interest in factor analysis. More appropriately it is recommended as a basis for further study, and a wide selection of references is included for this purpose.

JOHN I. THORNBY, Bucknell University

Degrees of Unsolvability. By Gerald E. Sacks. Annals of Mathematics Studies, no. 55. Princeton University Press, Princeton, N. J., 1963. 171 pp. $3.50.

This book is, in large part, a gathering together in one place of a variety of

difficult for the mathematical reader to learn some physics from it. This is most regrettable since the book deals with both a large number and a great variety of applications, and since the many drawings, tables and numerical examples indicate the intention of the authors to give a presentation of the theory which is as concrete, accessible and practical as possible.

W. MAGNUS, New York University

Fonctions de Variables Réelles, Volume I. By H. G. Garnir. Gauthier-Villars, Paris, 1963. 518 pp.

This interesting book on elementary real analysis is the first part of a two volume work on this subject by Professor Garnir. A second volume devoted to integration theory is in preparation. The level of the presentation would make this book quite accessible to students who have completed a standard course in calculus and have some acquaintance with elementary matrix algebra. The exposition is more traditional in style than that of other books that have been written recently on this subject, and much emphasis is placed on the development of manipulative technique, though adequate attention is given to such matters as the topology of n-dimensional space and the importance of linearity considerations. The book contains an unusually large and varied collection of exercises with hints provided for the difficult problems.

The first two chapters and an appendix are devoted to a brief introduction to set algebra, a fairly standard account of the topological properties of n-dimensional Euclidean space, and a discussion of the real and complex number systems, in that order. Chapters 3 through 7 deal with the continuity, differentiability, and convergence properties of real and complex valued functions defined on subsets of n-dimensional space. The sections on uniform convergence and change of variables are especially well written. The remaining chapters are devoted to a detailed discussion of the elementary functions, an exposition of the techniques of anti-differentiation, and an account of linear differential equations with constant coefficients. The book does not have an index.

In the reviewer's opinion, it is unfortunate that the author did not include a set theoretic discussion of the function concept since this could have been used to add clarity to the material on sequences, countable sets, operators, etc. The style of presentation would also seem to make it difficult for a student to recognize the material that is essentially peripheral to the main development.

ANTHONY L. PERESSINI, Oregon State University

Continued Fractions. By A. Ya. Khintchin, translated into English by Peter Wynn. Noordhoff, Groningen, 1963. 101 pp. Dfl. 16,25.

The first Russian edition was published in 1935, the second in 1949 and the third, published after the death of the author, without any alterations with the exception of a few notes of a bibliographical character, in 1961. The first and second editions are virtually the same.

This monograph is concerned with the metric theory of continued fractions, which deals with the problem of determining the measure of a set of numbers whose simple continued fraction representations satisfy some prescribed condition. For example, the measure of the set to which the number x belongs only in case the sequence of partial denominators in the simple continued fraction for x is bounded, is 0. More generally, the nth partial denominator $a_n(x)$ may be required to satisfy the inequality $a_n(x) \leqq \phi(n)$, $(n=1, 2, 3, \cdots)$, in which case the convergence or divergence of the series $\sum_{n=1}^{\infty} [1/\phi(n)]$ determines a property of the set of all such x. Again, if $q_n(x)$ is the denominator of the nth approximant of the simple continued fraction for x, then there exists a number B such that $q_n(x) < e^{Bn}$ for all sufficiently large n and almost all x.

The first two chapters of the book contain the basic properties of simple continued fractions including certain inequalities in the approximation of numbers. For the beautiful metric theory of the third chapter, it is assumed that the reader has a knowledge of the elementary theory of measure of sets of numbers.

H. S. WALL, The University of Texas

Continued Fractions. By A. Ya. Khinchin. University of Chicago Press, Chicago, 1964. 94 pp. Cloth \$5.00, paper \$1.95.

This is the translation by Scripta Technica, Inc. of the book of the preceding review. The English translation is edited by Herbert Eagle, Brown University.

H. S. WALL, The University of Texas

The Application of Continued Fractions and their Generalizations to Problems in Approximation Theory. By A. N. Khovanskii (the Russian edition was published in 1956), translated into English by Peter Wynn. Noordhoff, Groningen, 1963. 209 pp. Dfl. 30.

This book is largely a table of continued fraction expansions for various particular functions, compiled from original sources and with references included. The first chapter gives a survey of the analytic theory of continued fractions, including convergence theorems and methods for expansion of series into continued fractions. Chapter II, entitled "Continued Fraction Expansions of Certain Functions," gives a method of Lagrange for solution of certain differential equations by means of continued fractions and obtains expansions for binomial functions, the arc tangent, natural logarithm, e^x, tangent and hypergeometric functions. Chapter III contains Padé approximants for certain functions, e.g., e^x, log x, sin x, the error function and the gamma function. The last chapter contains a brief account of certain generalizations of continued fractions first given by Euler. The many formulas should be useful for computation. The bibliography contains a list of 109 references.

H. S. WALL, The University of Texas

Functions of a Complex Variable. By Gino Moretti. Prentice-Hall, Englewood Cliffs, N. J., 1964. 456 pp. $10.25.

This is a book written by an applied scientist for other applied scientists on the more practical aspects of the theory of complex variables. Along with the basic elements of the theory, the book has chapters on Jacobian elliptic functions, Bessel functions, Fourier series, and Laplace and Fourier transforms. There are also several sections on the Dirac delta function (found in the chapter entitled "Some Higher Transcendental Functions").

Although the mathematics is sometimes imprecise and on occasion questionable, there is much to recommend the book to its intended audience. The author has an informal style that is generally very readable. He strives to give the reader insight and frequently points out the "best way" to look at a particular problem or result based on his experience with the material. There are a wealth of figures (217 of them), numerous examples, and many problems ranging from the routine to the imaginative.

The point of view is always the practical. The author states in the preface that he regards mathematics as "a tool for solving physics problems." The "impractical" aspects of the theory are either ignored or just mentioned in passing—the latter done at times in an unsatisfactory way. For example, in the chapter on conformal mapping, a brief discussion on the possibility of mapping certain domains conformally onto the unit disc is essentially concluded with the statement: "Riemann proved that this could be done, but his proof is not completely satisfactory." One is left with the impression that perhaps the Riemann mapping theorem is not true.

D. R. SHERBERT, University of Illinois

BRIEF MENTION

Ne Plus Ulcers, By R. M. Winger. Fry and Smith, Phoenix, 1964. x+214 pp.

A charming collection of essays on many subjects, including mathematics, by the urbane and sensitive geometer who was for many years professor and executive officer of the mathematics department of the University of Washington.

Differential Equations and their Applications. Edited by Ivo Babuška. Academic Press, New York, 1964. 247 pp. $12.00.

Proceedings of a conference held in Prague, September 1962.

Elements of Ordinary Differential Equations, 2nd ed. By Michael Golomb and Merrill Shanks. McGraw-Hill, New York, 1964. 411 pp. $8.95.

Extensively revised.

Textbook of Algebra, vol. I and II, 7th ed. by G. Chrystal. Chelsea, New York, 1964. 584 and 630 pp. Cloth $3.95 each and paper $2.35 each.

Modern Pure Solid Geometry, 2nd ed. By Nathan Altshiller-Court. Chelsea, New York 1964, 353 pp. $6.00.

Essentially the original text, 1935, with various additions of new material.

Management and Mathematics: The Practical Techniques Available. By Allan Fletcher and Geoffrey Clarke. Gordon and Breach, New York, 1964. 235 pp. $7.50.
For the nonmathematical executive.

Mathematical Foundations of Thermodynamics. By R. Giles. Pergamon (Macmillan), New York, 1964. 228 pp. $10.00.

Einführung in die Theorie der algebraischen Zahlen und Funktionen. By Martin Eichler. Birkhäuser Verlag, Basel, 1963. 341 pp. $5.90.

Differentialoperatoren der mathematischen Physik. By Gunter Hellwig. Springer Verlag, Berlin, 1964. 253 pp. DM 36.00.

The Gamma Function. By Emil Artin. Translated by Michael Butler. Holt, Rinehart, and Winston, New York, 1964. 39 pp. $2.25.
A translation of a classic monograph which first appeared in the *Hamburger Mathematische Einzelschriften* in 1931.

Polynomial Expansions of Analytic Functions. By Ralph Boas and Creighton Buck. Academic Press, New York, 1964. 77 pp. $4.00.
Revised edition. First published in 1957.

L'Enseignement Mathématique, vol. IV (ser. II) nos. 1–2. Revue Internationale. Organe officiel de la Commission internationale de l'Enseignement Mathématique.
This number contains: (1) papers presented at the Bologna Conference of the commission (by Artin, Viola, Freudenthal, Stone, Libois, Lombardo-Radice, Morin, Cartan, Campedelli, Roghi, and Servais); (2) the Commission's report for 1959–1962; (3) the report of the commission's meeting at Saltsjobaden, 1962; (4) Professor Pienes' report on the "Education of Teachers" presented at Stockholm.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professors H. L. Garstens, University of Maryland; D. B. Goodner, Florida State University; C. E. Hardgrove, Northern Illinois University; R. W. Sloan, State University College at Oswego; and John Wagner, Michigan State University, represented the Association at the 19th National TEPS Conference in New York City on June 22–25, 1965.

Professor E. G. Begle, Stanford University, and Dr. B. E. Rhoades, CUPM, represented the Association at the summer conference of the Association of State Supervisors of Mathematics at Michigan State University on June 14–19, 1965.

Brother Louis Francis Zirkel, Archbishop Molloy High School, Jamaica, New York, represented the Association at the Solemn Blessing and Dedication of St. Augustine

Hall, St. Louise de Marillac Hall and Blessed Gabriel Perboyre Hall at St. John's University on April 27, 1965.

Professor Leon W. Cohen, University of Maryland, represented the Association at the banquet celebrating the 100th meeting of the National Science Board on May 27, 1965.

Professor R. E. Dowds, Butler University, represented the Association at the inauguration of W. N. Haines as President of Franklin College on May 14 and at the Dedication of B. F. Hamilton Memorial Library on the campus of Franklin College on May 15, 1965.

Professor J. N. Eastham, Queensborough Community College, represented the Association at the inauguration of S. B. Gould as President of the State University of New York on May 13, 1965.

Professor Harry M. Gehman, SUNY at Buffalo, represented the Association at the Governor's Library Conference, held in Albany, N. Y., on June 24–25, 1965.

Professor C. B. Stortz, Northern Michigan University, represented the Association at the inauguration of R. L. Smith as President of the Michigan Technological University on April 9, 1965.

Professor Mark Kac, The Rockefeller Institute, has succeeded Professor Gustav A. Hedlund as Chairman of the Division of Mathematics, National Academy of Sciences—National Research Council.

Professor R. W. McKelvey, University of Colorado, has been appointed Chairman of the Department of Mathematics.

Visiting Associate Professor M. R. Parameswaran, Michigan State University, has been appointed Professor and Head of the Department of Mathematics at the University of Kerala, Trivandrum, India.

Professor J. P. Russell, Professor of Mathematics and Administrative Officer of the Department, has been named Associate Dean of the Polytechnic Institute of Brooklyn at the Long Island Graduate Center at Farmingdale. He will continue also as Professor of Mathematics.

Professor W. R. Scott, University of Kansas, has been appointed Professor at the University of Utah.

Associate Professor G. F. Simmons, Colorado College, has been promoted to Professor.

Mr. J. G. Solomon, Radio Corporation of America, Moorestown, New Jersey, has been appointed Research Specialist at the High Energy Physics Laboratory of the University of Pennsylvania.

Mr. J. C. Warren, Jr., Consultant in Mathematics Education, has been appointed Assistant Professor at the College of Notre Dame and will serve as Chairman of the Department of Mathematics.

Associate Professor Abraham Weinstein, Nassau Community College, has been appointed Chairman of the Department of Mathematics.

Mr. R. O. Wells, Jr., New York University, has been appointed Assistant Professor at Rice University.

Mr. Leon Battig, University of Wisconsin, died on February 15, 1965. He was a member of the Association for 40 years.

Mr. S. A. Joffe, Mutual Life Insurance Company, New York, died on November 8, 1964. He was a charter member of the Association.

Professor Clifford Marburger, Franklin and Marshall College, died on March 15, 1965. He was a member of the Association for 32 years.

Assistant Professor John Raleigh, Temple University, died on December 26, 1964. He was a member of the Association for 7 years.

Assistant Professor J. B. Wells, Jr., University of Kentucky, died on November 12, 1964. He was a member of the Association for 8 years.

TRAVEL GRANTS TO THE INTERNATIONAL CONGRESS OF MATHEMATICIANS

Travel grants will be made to a number of qualified mathematicians for attendance at the International Congress of Mathematicians, to be held in Moscow on August 14–24, 1966. Selection of the grantees will be made by the Committee on Travel Grants (of the Division of mathematics of the National Academy of Sciences—National Research Council) enlarged to include representatives of all mathematical societies affiliated with the Division and representatives of the various governmental agencies concerned. Younger mathematicians are encouraged to apply. Special efforts will be made to support their attendance, and they will have the opportunity to supplement their applications by submitting additional information.

Applications can be obtained from the Division of Mathematics, National Academy of Sciences—National Research Council, 2101 Constitution Ave., N.W., Washington, D. C. 20418. Applications must be received on or before November 1, 1965.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for the three year term July 1, 1965, to June 30, 1968, by a mail vote of the Association in the Sections indicated:

Illinois	J. M. H. Olmsted, Southern Illinois University
Iowa	D. W. Wall, State University of Iowa
Louisiana-Mississippi	P. K. Rees, Louisiana State University
Maryland-D.C.-Virginia	Dorothy L. Bernstein, Goucher College
Michigan	K. W. Folley, Wayne State University
Minnesota	E. J. Camp, Macalester College
Philadelphia	Emil Grosswald, University of Pennsylvania
Southern California	R. C. James, Harvey Mudd College
Texas	Martin Wright, University of Houston

The highest percentage of voters was 54% in the Louisiana-Mississippi Section, followed by the Iowa Section with 45%. One candidate was elected by a majority of one vote, another by a majority of two.

RAOUL HAILPERN, *Associate Secretary*

ANNOUNCEMENT OF L. R. FORD AWARDS

At its meeting on January 27, 1965, in Denver, Colorado, the Board of Governors authorized a number of awards, to be named after Lester R. Ford, Sr., to authors of expository articles published in the MONTHLY and the MATHEMATICS MAGAZINE. A maximum of six awards will be made annually; each award is in the amount of $100. The articles are to be selected by a subcommittee of the Committee on Publications appointed for this purpose.

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 8

CONTENTS

OCTOBER 1965

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

THIS IS THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

General Office: SUNY at Buffalo, Buffalo, N. Y. 14214. *Executive Director:* H. M. GEHMAN

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are \$6.00. For nonmembers the subscription price is \$10.00.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

A GENERAL CURRICULUM IN MATHEMATICS FOR COLLEGES

W. L. DUREN, JR., Chairman, CUPM

1. A report to the Association. The Committee on the Undergraduate Program in Mathematics (CUPM) hereby presents to the Mathematical Association of America (MAA) its report on "A General Curriculum in Mathematics for Colleges" [1]. As a standing committee of the Association, CUPM requests that the Board of Governors receive this report and transmit it to the Sections of the Association for their study and comment. CUPM would like to see each Section of the Association draft a resolution, or committee report of its own, stating its evaluation of this General College Curriculum in Mathematics (CMC) which CUPM here proposes, and on the issues related to it.

CUPM has the authority under its charter from MAA to issue its own studies and recommendations; in the past we have used this authority to publish our curriculum recommendations on certain aspects of the college mathematics curriculum. This general college curriculum is so fundamental and so broad in its scope, however, that your Committee prefers to present this report to you for your study and comment before we do further work on it. We have endeavored to avoid being overly prescriptive of what colleges should do. Moreover, there are many issues involved in a general mathematics curriculum which are of such importance that CUPM should not attempt to resolve them for the college mathematics establishment. The "general curriculum" of this report consists essentially of an array of modular course units which goes as far as we feel we should at the present time towards offering a guide for curriculum construction in college mathematics. We think the proposed system is flexible enough to provide for local variation, for a variety of different individual curricula, and for enough dynamic progression in the college curriculum to permit these issues to be settled by the natural processes of work and discussion in the mathematical community.

In the beginning we set out to construct a recommended curriculum for "small" colleges but we soon found that this was not a significant specification. For one thing "small colleges" are no longer small. Moreover, it appeared that larger institutions might be able to use the same report as a minimal statement from which more advanced and complex department programs could be extended. So we changed our specification to prepare a list of course offerings few enough in number and conservative enough in content so that a staff of as few as four teachers can teach it. Thus the program we offer in this report is a minimal one which can serve as a basis for richer and more advanced programs wherever and whenever these are feasible. The various Panel recommendations [2] of CUPM represent what we have to say about richer programs in special areas of college mathematics. We hope that our basic program is so designed that the curriculum of a particular college can progress through it without the necessity of abolishing it to supplant it by an entirely new program five years from now. Indeed, a number of university and college departments already have

curricula ahead of the general curriculum proposed in this report, at least for their better prepared students. Therefore we wish to emphasize that this severely economized program of 14 semester-courses is not to be interpreted either as an ultimate goal or a limitation on undergraduate mathematics.

2. Secondary school prerequisites. The general curriculum in mathematics, which CUPM proposes, assumes well-prepared students who offer upon entrance $3\frac{1}{2}$ or 4 years of high school mathematics including: A. Geometry and intermediate algebra, B. A study of the elementary functions, i.e. polynomials, rational and algebraic functions, exponential, logarithmic and trigonometric functions, including the more elementary parts of two- and three-dimensional analytic geometry.

Recognizing that in many situations some of these prerequisites must still be taught in college we describe the following one-semester remedial course.

MATH. 0. *Elementary Functions and Coordinate Geometry* (3 semester hours). A precalculus course covering topics B above.

3. The proposed general curriculum in mathematics. For students who can meet the prerequisites CUPM proposes that the following list of lower division courses be available each semester as needed. (See the report itself for details and sample course outlines.)

MATH. 1. *Introductory Calculus* (3 or 4 semester hours). A differential and integral calculus of polynomials and other elementary functions.

MATH. 2, 4. *Multivariate Calculus, Limits and Differential Equations* (3 or 4 semester hours each). Courses which we describe in two different versions. In the preferred version Math. 2 is a multivariate calculus, while Math. 4 strengthens limits and series and provides an introduction to linear differential equations. A more conventional form is also described in which Math. 2 continues the single variable calculus, while Math. 4 takes up multivariate calculus and a brief introduction to differential equations.

MATH. 2P. *Introductory Probability*. A course of first year level which can follow Math. 1 and may be substituted for Math. 2 when it is appropriate for particular students.

MATH. 3. *Linear Algebra* (3 or 4 semester hours). Linear systems, matrices, vectors, linear transformations, unitary geometry with characteristic values. Applications to geometry.

For the upper division, not necessarily offered each semester, our report proposes the following 9 semester courses.

MATH. 5. *Advanced Multivariable Calculus*. Primarily vector calculus, Stokes' and Green's theorems and a brief introduction to boundary value problems of partial differential equations.

MATH. 6. *Algebraic Structures*. Groups, rings and fields.

MATH 7. *Probability and Statistics*. Two versions are displayed, one emphasizing statistical inference and one going more deeply into probability.

MATH. 8. *Numerical Analysis.* Numerical methods for integration, differential equations, matrix inversion, estimation of characteristic roots. Oriented towards machine computation.

MATH. 9. *Geometry.* A number of types of courses might do here. We present outlines for a Euclidean geometry for prospective teachers and a differential geometry.

MATH. 10. *Applied Mathematics.* A course to illustrate the principles and basic styles of thought in solving physical or other scientific problems by mathematical methods. Specific content will depend on the persons involved.

MATH. 11 or 11–12. *Introductory Real Variable Theory.* Standard material on the real numbers, continuity, limits, differentiation and integration. The student will learn to make the proofs.

MATH. 13. *Complex Analysis.* Complex numbers, elementary functions, analytic functions, contour integration, Taylor and Laurent Series, conformal mapping, boundary value problems, integral transforms.

Introduction to Computer Science. The computer, algorithms, programming languages, problem solving. Many mathematics departments now offer such a course at first or second year level but for the general mathematics student and for his mathematics teacher CUPM feels that the question of how to begin computation and how to relate it to mathematics is not ready to be resolved by a recommendation.

4. Some course sequences which this general curriculum provides. In the major sequences below we adopt the limit of six semester courses in the upper division program of a student, although our program in many cases would provide more.

a. Two one-year sequences for liberal arts students, or majors in social and behavioral sciences, and business administration students are: Math. 1, 2 or Math. 1, 2P.

b. An advanced placement two-year sequence for mathematics or physical science students is: Math. 2, 3, 4, 5 or 2P, or 7 as appropriate.

c. A mathematics major program for students bound toward graduate mathematics is given by: Math. 1, 2, 3, 4, 5, 6, 10, 11, 12, 13. There are additional courses available to adapt this sequence for special interests.

d. A mathematics major program for applied mathematicians might be: Math. 1, 2, 2P, 3, 4, 5, 6, 8, 10, 11, 12 or 13. This exhausts the limit we assume but more courses are available. Computer Science, or equivalent introduction to computing, is a necessary supplement.

e. A program for prospective theoretically oriented graduate students in biological, management or social sciences substantially meeting the recommendation of that Panel is: Math. 1, 2P, 2, 3, 4, 7. Computer Science should be included.

f. The mathematics content of a program for prospective career computer

scientists (not mathematics majors) is covered by: Math. 1, 2, 2P, 3, 4, 6, 7, 8. Obviously Computer Science and more in the area of computation is also needed.

g. A physics major bound for graduate work in physics would find good support in: Math. 1, 2, 3, 4, 5, 2P, 10, 13. Additional available courses: Math. 6, 7, 8, 11, 12.

h. The minimal program recommended for mathematics teachers of grades 9–12 by the Teacher Training Panel can be met by: Math. 1, 2, 3, 4, 6, 2P, 7, 9 and two semester electives, provided Math. 9 is a year-course of appropriate geometry.

On the other hand, this program does not provide for the training of elementary teachers. Colleges having this responsibility should offer the special course program recommended for this purpose by the Panel on Teacher Training [2]. The proposed program also does not provide for deep remedial instruction or for a noncalculus mathematics appreciation course to meet a liberal arts requirement.

We observe also that advanced high schools are already teaching courses for superior students covering the material of Math. 1, 2, 2P, and even 3. So advanced placement as high as Math. 4 is possible.

5. Issues in the general mathematics curriculum. We call attention to some issues which inevitably arise in planning a general curriculum in mathematics for colleges, one which will serve as many purposes as possible, and as economically as possible. College teachers generally and particularly those participating in MAA Section evaluations of this report may have ideas different from ours on the resolution of these problems. Naturally, we hope you will agree generally with the way in which we have tried to meet them. Here are some of the main ones.

1) What is the right normal starting point for college mathematics in 1965? Should Math. 0 have been eliminated? Or should we have provided for a year of precalculus mathematics as was customary some years ago?

2) What accomplishment should be expected from the pivotal Introductory Calculus, Math. 1?

3) To what extent can we count on high schools taking over the teaching of Math. 0, 1, 2, 2P, and 3? When?

4) Should Probability, Math. 2P, have been offered in noncalculus form? Should it include statistics? Combinatorial algebra?

5) Should we say that a three-hour Computer Science, which might be labelled Math. 2C, is the business and even responsibility of mathematics departments to teach? Alternatively can and should a shorter introduction to programming be adjoined to existing mathematics courses? Should mathematics courses be modified to include homework on the computer? These are hot issues. Our attitude was: Let us wait and see.

6) Do mathematics teachers share our view that multivariable calculus should begin in Math. 2 and continue with fortification from linear algebra

thereafter? Textbooks are a problem but not an insurmountable one. See the reasons for an affirmative answer in the report.

7) This report subsumes introductory differential equations into the lower division calculus sequence as some current texts now do it. Was this the right decision or should the first study of differential equations be held back for a separate course in the second or third year?

8) We wanted to include in the upper division courses an intermediate differential equations course of somewhat more theoretical nature than the conventional introduction. It was crowded out by the demand of chairmen of graduate departments for a year of real variables. Texts for such a course are now available. Should it be recognized in the general undergraduate curriculum?

9) There is a current issue on the proper place to introduce the technique of (exterior) differential forms. It appears now in some lower division calculus texts. We let it appear in our program for the first time in Math. 5, which is a third-year level advanced calculus. It also appears in the differential geometry version of Math. 9. What do mathematics teachers think about this?

10) What geometry should the general curriculum include? We believed that, within some bounds, it is better to let a teacher teach what he knows and likes instead of trying to specify some preferred geometry.

11) We omitted Number Theory from the list of 14 courses. Obviously this is a good course when there is someone who knows how to teach it and students to take it. Its omission here emphasizes the minimal character of this department program. Should number theory be included as a must in such a program?

12) Alternatively, or perhaps additionally, should the Algebraic Structures, Math. 6, have been extended to a year course where it could have included a little number theory and more advanced linear algebra than one can cover in the introductory and low-level Math. 3?

13) Should the general college curriculum include a terminal year-course in mathematics appreciation, or a cultural or philosophical course in mathematics for liberal arts students? Without recommending against it we did not include it in this report.

14) Can teacher education, including elementary teacher education, be absorbed into the general mathematics curriculum, or must it be, as we decided, essentially a special program?

15) Will the expansion in both the numbers and rate of students entering college force a return to deeper remedial instruction? Or can we safely predict that high school graduates will be better and better trained in mathematics each year?

16) Returning to advanced undergraduate mathematics, how much upper division mathematics can normally be required of a major? We used six semester courses as a norm.

17) Should there be separate *undergraduate* courses in applied mathematics? What is a course in applied mathematics? We assumed that all mathematics would be taught with applications in mind and in addition we thought that,

where it can be well done, at least one course in applied mathematics should be available.

18) Does the general college curriculum here described provide an adequate undergraduate education to serve as a foundation for undertaking contemporary graduate mathematics? If not, is it possible to do better within the limits of time normally allowed for undergraduate majors in a college? Also with the available human material, teacher and student?

19) We offered an outline for a one-semester course in real variables and another one for a year-course, as requested by chairmen of graduate departments. Is a year of real variables a really necessary offering for undergraduates or is it a luxury course which will be repeated in graduate school?

These are some of the questions which revolve around the general mathematics curriculum where a widespread study and discussion should be very fruitful for mathematics teaching. Some other perennial questions were omitted because their discussion is seldom fruitful. Some of these have to do with rigor in calculus (you cannot make yourself understood in what you say about it), with where we will get the teachers, and the textbooks.

6. How this report was constructed. CUPM has several Panels: the Panel on Mathematics for Physical Sciences and Engineering, the Panel on Pregraduate Training in Mathematics, the Panel on Teacher Training in Mathematics, the Panel on Mathematics for Biological, Management and Social Sciences. Also there is a standing Advisory Group on Communications (Library and publications). For several years the Panels have studied intensively the special needs of mathematical training in their areas of interest and have published their recommendations [2]. To prepare this general curriculum report CUPM formed an all-panel subcommittee which reported back to CUPM as a whole and now CUPM transmits it to the Association. Two members of the School Mathematics Study Group were included to insure a proper articulation with the new high school calculus course as planned by that group. This report was based not only on the panel reports of CUPM but also on the studies of leading departments of mathematics in the country.

The take-off point was the year-course on elementary functions, polynomials, rational and algebraic functions, logarithms, exponential, and trigonometric functions which have long occupied a dominant position in the mathematics curriculum in the last years of high school and first year of college. Usually it has been offered as precalculus mathematics, or with at most a smattering of calculus, under such names as "college algebra, trigonometry and analytic geometry," or "mathematical analysis—an integrated approach," or "fundamentals," or just "college mathematics." This material is recognizable in our proposed program as Math. 0, 1. What we did was to say that the latter half of it could and should become the calculus of the elementary functions, while the responsibility for teaching the first half should normally be assigned to the high schools.

Next we thought about flexibility and multiple tracks with many possible

entrance points and many suitable exits from the program to accommodate to the diversity of students that colleges now serve. This led us to the semester building blocks, our modular units for many course sequences. While many colleges will offer separate "honors" tracks parallel to the main track, and that may be best where it can be done, our minimal approach attempts to take care of the spread in achievement, and to some extent the spread in ability, of entering students by advanced placement in the General College Curriculum. That is why it is important, wherever it is possible, to offer all of the lower division courses every semester.

In the report we offer some course outlines as samples, in some cases multiple samples, of possible interpretations of the courses which we list with deliberately bare, rather generic, "college catalogue" descriptions. These outlines were not all prepared in the Subcommittee. Some were prepared in Panels and for some we had the help of mathematicians outside CUPM. We assume that mathematicians who use this report will construct their own course outlines and hence do not regard our outlines as detailed recommendations of CUPM.

References

1. *A General Curriculum in Mathematics for Colleges*, available free of charge from CUPM, P. O. Box 1024, Berkeley, California 94701.

2. The following CUPM reports are available from CUPM: *Recommendations on the Undergraduate Mathematics Program for Engineers and Physicists, Tentative Recommendations for the Undergraduate Mathematics Program for Students in the Biological, Management and Social Sciences, The Pregraduate Preparation of Research Mathematicians.* (This is an idealized program. The Panel will shortly publish *Preparation for Graduate Study in Mathematics*, on how these idealized goals may be absorbed into regular curricula.) *Recommendations for the Training of Teachers of Mathematics, Recommendations for the Undergraduate Mathematics Program for Work in Computing*, and *Basic Library List, 1965*. Another report expected soon is one on *Applied Mathematics* in the undergraduate curriculum.

ON ABSOLUTELY CONTINUOUS FUNCTIONS

DALE E. VARBERG, Hamline University, St. Paul, Minn.

1. Introduction. The purpose of this note is to discuss the concept of absolute continuity for functions of a real variable. This topic has been well explored so most of our results will not be new. What is new is our approach. We begin with an elegant inequality which the author discovered lying buried as an innocent problem in Natanson's book [4]. From this inequality, which we dignify by calling the Fundamental Lemma, many well-known and some new results follow in an almost trivial fashion.

The notion of bounded variation is intimately connected with that of absolute continuity. We assume that the reader is familiar with this notion as well as the following standard properties of functions of bounded variation.

Associated with a function f of bounded variation on $[a, b]$ are three monotonically increasing functions p, n and t (the positive, negative and total variations) which satisfy:

$$f(x) = f(a) + p(x) - n(x);\ t(x) = p(x) + n(x), \tag{1.1}$$

(1.2) *f' exists almost everywhere and is integrable,*

(1.3) $t'(x) = |f'(x)|$ *almost everywhere* (see [5] p. 15),

(1.4) *if f is continuous on $A = [a, b]$, then* $t(b) = \int_{-\infty}^{\infty} N_A(y)dy$,

where $N_A(y)$, the socalled Banach indicatrix, is defined by the statement: $N_A(y)$ is the number of solutions x of the equation $y=f(x)$ in A (see [4] p. 225).

There are several equivalent definitions for absolute continuity. We will say that f is absolutely continuous on $[a, b]$ if for any $\epsilon>0$ there exists $\delta>0$ such that for any finite number of nonoverlapping subintervals (a_k, b_k) of $[a, b]$ with $\sum(b_k-a_k)<\delta$, $\sum|f(b_k)-f(a_k)|<\epsilon$. An equivalent definition is obtained if, for example, the absolute values are omitted in the last inequality (see [5] pp. 50, 51). It is obvious that an absolutely continuous function is continuous and it is easy to show that it is also of bounded variation.

2. The fundamental lemma and a related theorem.

FUNDAMENTAL LEMMA [4, p. 241]. *Let f be a function defined on the set $[a, b]$ and let E be any subset on which f' exists and satisfies $|f'(x)| \leqq K$. Then*

$$m^*f(E) \leqq Km^*(E), \tag{2.1}$$

where m^* denotes exterior Lebesgue measure and

$$f(E) = \{y \mid y = f(x) \text{ for some } x \in E\}.$$

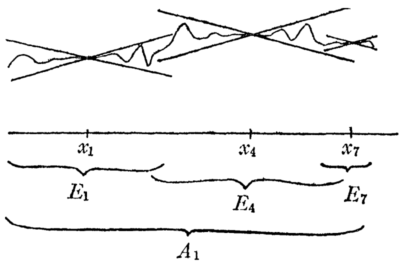
Proof. In case E is finite or denumerable, (2.1) is obvious since both sides are then zero. Suppose that E is non-denumerably infinite and let $\epsilon>0$ be given. Let A be an open set containing E and such that $m(A) \leqq m^*(E)+\epsilon$. For each $x \in E$, there exists a symmetric open interval E_x about x such that $E_x \subset A$ and

$$|f(y) - f(x)| \leqq (K + \epsilon)|y - x| \tag{2.2}$$

for all $y \in E_x$ (if a or b belongs to E, the corresponding E_a or E_b would not be symmetric but this will not affect our arguments). The collection $\{E_x\}$ forms an open covering of E which by the Lindelöf Theorem may be reduced to a denumerable covering, say $\{E_n\}$, the midpoint of E_n now being denoted by x_n.

Our first step is to demonstrate that for any finite N,

$$m^*f\left[\bigcup_{n=1}^{N} E_n\right] \leqq (K + \epsilon)m\left[\bigcup_{n=1}^{N} E_n\right]. \tag{2.3}$$



A diagram may be helpful. The wavy curve represents the graph of f, it being by (2.2) trapped on E_n between line segments having slopes $K+\epsilon$ and $-(K+\epsilon)$ which pass through $(x_n, f(x_n))$. $\bigcup_{n=1}^N E_n$ is an open set composed of a finite number of disjoint open intervals $A_1, A_2, \cdots, A_k$ each of which is a union of some of the E_n's. Now if (2.3) holds for the union of those E_n's which make up each A_j, it will clearly hold for $\bigcup_{n=1}^N E_n$. Thus we may suppose that $\bigcup_{n=1}^N E_n$ is an open interval.

On $\overline{E}_n$ (the closure of E_n), define a linear function f_n by

$$f_n(x) = (K + \epsilon)(x - x_n) + f(x_n)$$

and note that f_n attains a maximum and a minimum, say M_n and m_n, at the right and left endpoints of $\overline{E}_n$ respectively. Let $M_i = \max(M_1, M_2, \cdots, M_N)$, $m_j = \min(m_1, m_2, \cdots, m_N)$. On $\bigcup_{n=1}^N E_n$ we may be sure that f does not get above M_i nor below m_j.

Next, for $x \in \bigcup_{n=1}^N E_n$, let

$$g(x) = \begin{cases} (K + \epsilon)(x - x_j) + f(x_j) & \text{if } x_i \geqq x_j \\ -(K + \epsilon)(x - x_j) + f(x_j) & \text{if } x_i < x_j; \end{cases}$$

g is a linear function which has as big a range as f could possibly have, i.e., $f[\bigcup_{n=1}^N E_n] \subset g[\bigcup_{n=1}^N E_n]$ and hence

$$m^*f\left[\bigcup_{n=1}^N E_n\right] \leqq mg\left[\bigcup_{n=1}^N E_n\right] = (K + \epsilon)m\left[\bigcup_{n=1}^N E_n\right].$$

The proof of (2.1) now follows easily.

$$\begin{aligned} m^*f(E) &\leqq m^*f\left[\bigcup_{n=1}^\infty E_n\right] \leqq \lim_{N\to\infty} m^*f\left[\bigcup_{n=1}^N E_n\right] \\ &\leqq \lim_{N\to\infty} (K + \epsilon)m\left[\bigcup_{n=1}^N E_n\right] = (K + \epsilon)m\left[\bigcup_{n=1}^\infty E_n\right] \leqq (K + \epsilon)m(A) \\ &\leqq (K + \epsilon)[m^*(E) + \epsilon]. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have the result.

A much weaker version of our first theorem is stated in [**3**, p. 195].

THEOREM 1. *Let f be defined and measurable on the set $[a, b]$ and let E be any measurable subset on which f' exists (finite sense). Then*

$$m^*f(E) \leqq \int_E |f'(x)| \, dx. \tag{2.4}$$

Proof. Suppose first that $|f'(x)| < B$ (an integer) on E. Let

$$E_k^n = \left\{ x \in E \,\Big|\, \frac{k-1}{2^n} \leqq |f'(x)| < \frac{k}{2^n} \right\}, \qquad k = 1, 2, \cdots, B2^n, \ n = 1, 2, \cdots.$$

Then for each n,

$$\begin{aligned} m^*f(E) &= m^*f\left(\bigcup_k E_k^n \right) = m^*\left[\bigcup_k f(E_k^n) \right] \leqq \sum_k m^*f(E_k^n) \\ &\leqq \sum_k \frac{k}{2^n} m(E_k^n) \qquad \text{by the Fundamental Lemma} \\ &= \sum_k \frac{k-1}{2^n} m(E_k^n) + \frac{1}{2^n} \sum_k m(E_k^n). \end{aligned}$$

Therefore

$$m^*f(E) \leqq \lim_{n \to \infty} \left[\sum_k \frac{k-1}{2^n} m(E_k^n) + \frac{1}{2^n} \sum_k m(E_k^n) \right] = \int_E |f'(x)| \, dx.$$

To handle the case where f' is not bounded, let

$$\begin{aligned} A_k &= \{ x \in E \mid k - 1 \leqq |f'(x)| < k \}, \qquad k = 1, 2, \cdots, \\ m^*f(E) &= m^*f\left(\bigcup_k A_k \right) \leqq m^*\left[\bigcup_k f(A_k) \right] \leqq \sum_k m^*f(A_k) \\ &\leqq \sum_k \int_{A_k} |f'(x)| \, dx = \int_E |f'(x)| \, dx. \end{aligned}$$

3. Absolutely continuous functions. We shall use the term "null set" for a set of Lebesgue measure zero.

THEOREM 2. *If f is absolutely continuous on the set $[a, b]$ and if N is any null subset, then $mf(N) = 0$, i.e., an absolutely continuous function maps null sets into null sets.*

COROLLARY. *An absolutely continuous function maps measurable sets into measurable sets.*

This theorem and its corollary are well known and have standard proofs (see, for example [4], pp. 249, 50), so we omit them. We now know that absolutely continuous functions are continuous, of bounded variation and map null sets

into null sets. These three properties actually characterize such functions as we see from the following theorem.

THEOREM 3 (BANACH-ZARECKI, [4] p. 250). *If f is continuous and of bounded variation on $[a, b]$ and if f maps null sets into null sets, then f is absolutely continuous on $[a, b]$.*

Proof. Let (a_k, b_k) be nonoverlapping subintervals of $[a, b]$ and let $E_k = \{x \in [a_k, b_k] \mid f'(x) \text{ exists}\}$. Since $m\{[a_k, b_k] - E_k\} = 0$ and f maps null sets into null sets, $mf([a_k, b_k]) = mf(E_k)$ and therefore

$$\begin{aligned}
\sum_{k=1}^{n} |f(b_k) - f(a_k)| &\leqq \sum_{k=1}^{n} mf([a_k, b_k]) = \sum_{k=1}^{n} mf(E_k) \\
&\leqq \sum_{k=1}^{n} \int_{E_k} |f'(x)|\, dx \qquad \text{by Theorem 1} \\
&= \sum_{k=1}^{n} \int_{a_k}^{b_k} |f'(x)|\, dx
\end{aligned}$$

which $\to 0$ as $\sum_{k=1}^{n} (b_k - a_k) \to 0$. This last conclusion follows from the fact that f' is integrable (see (1.2) and a well-known property of the Lebesgue integral [4] p. 148).

The proofs of the next two theorems are almost identical with that above.

THEOREM 4 (cf. [3, p. 183]). *If f is continuous, if f' exists for all but a finite or denumerable set of points and if f' is integrable on $[a, b]$, then f is absolutely continuous on $[a, b]$.*

THEOREM 5. *If f is continuous on $[a, b]$, if f' exists almost everywhere and is integrable on $[a, b]$, and if f maps null sets into null sets, then f is absolutely continuous on $[a, b]$.*

The next result is an immediate consequence of Theorem 4.

THEOREM 6 (cf. [4] p. 266). *If $f'(x)$ exists for all $x \in [a, b]$ and if f' is integrable there, then f is absolutely continuous on $[a, b]$.*

We have already characterized absolutely continuous functions as functions which are continuous, of bounded variation, and map null sets into null sets. Theorems 7 and 10 provide another characterization, namely, that of being indefinite integrals.

THEOREM 7. *Let f be integrable on $[a, b]$ and let $F(x) = \int_a^x f(t)dt$. Then F is absolutely continuous on $[a, b]$.*

This theorem follows easily from the definition of absolute continuity and a well-known property of the Lebesgue integral (see [4, p. 252]). The proof of the next theorem is fairly difficult and it cannot, we think, be materially simpli-

fied by ideas that we have developed so far. Perhaps the simplest proof may be found in [5, p. 48] but see also [4, p. 253].

THEOREM 8. *Let f be integrable on $[a, b]$ and let $F(x)=\int_a^x f(t)dt$. Then $F'(x)=f(x)$ almost everywhere on $[a, b]$.*

THEOREM 9 (cf. [4, p. 246]). *If f is absolutely continuous and $f'(x)=0$ almost everywhere on $[a, b]$, then f is a constant function.*

Proof. Let $E=\{x\in[a, b] \mid f'(x)=0\}$ and $E'=[a, b]-E$. Then

$$mf([a, b]) \leqq mf(E) + mf(E').$$

Now $mf(E)$ is 0 by the Fundamental Lemma while $mf(E')$ is 0 by Theorem 2. But since f is continuous, $f([a, b])$ is a point or a closed interval. It cannot be a closed interval, because its measure is 0, and it is therefore a single point.

Our next theorem together with Theorem 8 show in what sense integration and differentiation are inverse operations.

THEOREM 10. *Let f be absolutely continuous on $[a, b]$. Then $f(x)=f(a)+\int_a^x f'(t)dt$ for all $x\in[a, b]$.*

Proof. We give a standard proof of this well-known theorem. Being absolutely continuous, f is of bounded variation and so f' is integrable. Let $g(x)=f(a)+\int_a^x f'(t)dt$. By virtue of Theorems 7 and 8, g is absolutely continuous and $g'(x)=f'(x)$ almost everywhere on $[a, b]$. It follows that $g-f$ is also absolutely continuous, and moreover, has a zero derivative almost everywhere. Hence by Theorem 9, $g(x)-f(x)$ is a constant which in turn must be 0 since $g(a)=f(a)$. The conclusion follows.

We turn our attention to three theorems for monotonically increasing functions. Corresponding results are of course valid for decreasing functions but we shall not need them. The first of these theorems is well known, but the second and third may be new.

THEOREM 11. *Let f be monotonically increasing on $[a, b]$. Then $f(x)=g(x)+h(x)$ where both g and h are monotonically increasing, g is absolutely continuous and $h'(x)=0$ almost everywhere.*

Proof. Let $g(x)=f(a)+\int_a^x f'(t)dt$ and $h(x)=f(x)-g(x)$. Clearly $h'(x)=0$ almost everywhere and g is monotonically increasing. To see that h is monotonically increasing, let x and y be fixed numbers such that $a\leqq y<x\leqq b$. Consider a new function f^* identical with f on $[a, x]$ and such that $f^*(z)=f(x)$ for $z\geqq x$. Then for $k>0$,

$$\int_y^x \frac{f^*(z+k)-f^*(z)}{k}dz = \frac{1}{k}\int_x^{x+k} f^*(z)dz - \frac{1}{k}\int_y^{y+k} f^*(z)dz$$

$$= f(x) - \frac{1}{k}\int_y^{y+k} f^*(z)dz \leqq f(x) - f(y).$$

But by Fatou's Lemma,

$$\int_y^x f'(z)dz = \int_y^x \lim_{k\to 0^+} \frac{f^*(z+k) - f^*(z)}{k} dz \leqq f(x) - f(y).$$

Hence $g(x)-g(y) \leqq f(x)-f(y)$ or $h(y) \leqq h(x)$.

THEOREM 12. *Let f be monotonically increasing on the set $[a, b]$ and let E be the subset where f' exists. Then*

$$\int_a^b f'(x)dx = m^*f(E) \leqq f(b) - f(a).$$

REMARK. The reader may wish to check the correctness of this result for the Cantor middle third function.

Proof. By Theorem 1, $m^*f(E) \leqq \int_E f'(x)dx = \int_a^b f'(x)dx$. To show the reverse inequality, we use Theorem 11 to write $f(x)=g(x)+h(x)$, where both g and h are monotonically increasing, g is absolutely continuous and $h'(x)=0$ almost everywhere.

It is intuitively clear that $m^*f(E) \geqq mg(E)$ (note that we need not write $m^*g(E)$ for E is measurable and hence, by the corollary to Theorem 2, so is $g(E)$). To see that this inequality really holds, let $\epsilon>0$ be given and let $B=\bigcup B_k$ be a union of disjoint open intervals B_k and such that $f(E) \subset B$ but $m(B)<m^*f(E)+\epsilon$. $\{f^{-1}(B_k)\}$ is a collection of disjoint measurable sets whose union contains E. In fact, $f^{-1}(B_k)$ is an interval, a point or empty. Moreover, it is clear that

$$mg[f^{-1}(B_k)] \leqq mB_k.$$

Therefore

$$\begin{aligned} mg(E) &\leqq m\{\bigcup g[f^{-1}(B_k)]\} \leqq \sum mg[f^{-1}(B_k)] \\ &\leqq \sum m(B_k) = m(B) < m^*f(E) + \epsilon \end{aligned}$$

from which the claim follows.

Thus

$$\begin{aligned} m^*f(E) &\geqq mg(E) \\ &= mg([a, b]) \quad \text{(using the fact that } g \text{ maps null sets into null sets)} \\ &= g(b) - g(a) = \int_a^b g'(x)dx \qquad \text{(by Theorem 10)} \\ &= \int_a^b f'(x)dx. \end{aligned}$$

THEOREM 13. *Let f be monotonically increasing on the set $[a, b]$, let A be a measurable subset and let E be the subset of A where f' exists. Then*

$$\int_A f'(x)dx = m^*f(E) \leqq m^*f(A). \tag{3.1}$$

If f is absolutely continuous, the inequality becomes an equality.

Proof. Suppose first that f is absolutely continuous and let C be an open subset of $[a, b]$, i.e., let $C = \bigcup_n C_n$, where $\{C_n\} = \{(a_n, b_n)\}$ is a finite or denumerable collection of non-overlapping open intervals. Then

$$\begin{aligned} mf(C) &= mf(\bigcup C_n) = m[\bigcup f(C_n)] = \sum mf(C_n) \\ &= \sum [f(b_n) - f(a_n)] = \sum \int_{a_n}^{b_n} f'(x)dx = \int_C f'(x)dx. \end{aligned}$$

To extend this result to an arbitrary measurable set A, let $\{A_n\}$ and $\{B_n\}$ be sequences of open sets such that $A \subset A_n$, $f(A) \subset B_n$, $\lim_{n\to\infty} m(A_n) = m(A)$ and $\lim_{n\to\infty} m(B_n) = m\,f(A)$. Since

$$mf(A) \leqq mf[f^{-1}B_n \cap A_n] \leqq mf[f^{-1}B_n] \leqq m(B_n),$$

and since $m(B_n) \to mf(A)$ as $n \to \infty$, we see that

$$\begin{aligned} m^*f(A) &= \lim_{n\to\infty} mf[f^{-1}B_n \cap A_n] \\ &= \lim_{n\to\infty} \int_{f^{-1}B_n \cap A_n} (f'(x)dx \qquad \text{(since } f^{-1}B_n \cap A_n \cap (a, b) \text{ is open)} \\ &= \int_A f'(x)dx. \end{aligned}$$

To complete the proof of (3.1) in the absolutely continuous case requires only that we note that $mf(E) = mf(A)$ (see Theorem 2).

Finally in the general case, we use Theorem 11 to write $f(x) = g(x) + h(x)$. Then, noting that E is measurable since A is measurable, we have

$$\begin{aligned} mf(A) \geqq m^*f(E) \geqq mg(E) &\qquad \text{(by the proof of the previous theorem)} \\ = \int_E g'(x)dx &\qquad \text{(by the derivation above)} \\ = \int_E f'(x)dx = \int_A f'(x)dx. \end{aligned}$$

On the other hand by Theorem 1, $m^*f(E) \leqq \int_E f'(x)dx = \int_A f'(x)dx$ which, together with the inequality above, implies (3.1).

The last two theorems allow us to prove two theorems which give formulas for the total variation of an absolutely continuous function.

THEOREM 14 (cf. [1] p. 209). *If f is of bounded variation on $[a, b]$, then its total variation satisfies*

$$t(b) \geqq \int_a^b |f'(t)|\, dt. \tag{3.2}$$

The equality holds if f is absolutely continuous.

Proof. Let $E = \{x \in [a, b] \mid t'(x) \text{ exists}\}$. Then

$$\begin{aligned} t(b) &= t(b) - t(a) \geqq m^*t(E) \\ &= \int_a^b t'(x)dx && \text{(by Theorem 12)} \\ &= \int_a^b |f'(x)|\, dx && \text{(by (1.3))}. \end{aligned} \tag{3.3}$$

To prove the second contention, we note that equality holds at (3.3) if f is absolutely continuous since then t is absolutely continuous, as is easily shown.

THEOREM 15 (cf. [2] p. 606). *Let f be of bounded variation on $[a, b]$, let t be its total variation and let A be a measurable subset of $[a, b]$. Then $m^*t(A) \geqq \int_A |f'(x)|\, dx$. The equality holds if f is absolutely continuous.*

Proof. Let E be the subset of A where f' exists. Then

$$\begin{aligned} m^*t(A) \geqq m^*t(E) &= \int_E t'(x)dx && \text{(by Theorem 13)} \\ &= \int_E |f'(x)|\, dx && \text{(by (1.3))} \\ &= \int_A |f'(x)|\, dx. \end{aligned} \tag{3.4}$$

Finally, we note that equality holds at (3.4) if f is absolutely continuous.

Let $E = \{x \in [a, b] \mid f'(x) = 0\}$. It is obvious that there are functions for which $f(E)$ is finite or denumerable. A recent advanced problem in this MONTHLY [1963, 5114] asks for the construction of a function f with a continuous derivative such that $f(E)$ is non-denumerable. We will not take space to construct such a function but we do wish to point out that $f(E)$ cannot be too large; in fact, it must always be a set of measure 0.

THEOREM 16. *Let f be defined on $[a, b]$ and let $E = \{x \in [a, b] \mid f'(x) = 0\}$. Then $mf(E) = 0$.*

Proof. Apply the Fundamental Lemma with $k = 0$.

This theorem allows a partial converse as we shall see in Theorems 17 and 19.

THEOREM 17. *If f is monotonically increasing on the set $[a, b]$ and if M is any measurable subset for which $mf(M)=0$, then $f'(x)=0$ almost everywhere on M.*

Proof. Let $f(x)=g(x)+h(x)$ where both g and h are monotonically increasing, g is absolutely continuous and $h'(x)=0$ almost everywhere. Then

$$\int_M f'(x)dx = \int_M g'(x)dx = mg(M) \qquad \text{(by Theorem 13)}$$

$$\leqq m^*f(M) \qquad \text{(by proof of Theorem 12)}$$

$$= 0.$$

Since $f'(x) \geqq 0$, we conclude that $f'(x)=0$ almost everywhere on M.

THEOREM 18. *Let f be continuous and of bounded variation on the set $[a, b]$ and let M be any subset for which $mf(M)=0$. Then $mt(M)=0$, $t(x)$ being the total variation of f on $[a, x]$.*

Proof. From the introduction (1.4), we know that $t(b)=\int_{-\infty}^{\infty} N_{[a,b]}(y)dy$. Hence for any subinterval (a_k, b_k) of $[a, b]$,

$$mt[(a_k, b_k)] = t(b_k) - t(a_k) = \int_{-\infty}^{\infty} [N_{[a,b_k]}(y) - N_{[a,a_k]}(y)]dy$$

$$= \int_{-\infty}^{\infty} N_{(a_k,b_k]}(y)dy = \int_{-\infty}^{\infty} N_{(a_k,b_k)}(y)dy.$$

This result is easily extended to arbitrary open sets, that is, for any open set A

$$mt(A) = \int_{-\infty}^{\infty} N_A(y)dy.$$

Next, let $\{A_n\}$ be a sequence of open sets such that $f(M) \subset A_n$ and $\lim_{n\to\infty} m(A_n) = mf(M)=0$. Then

$$m^*t(M) \leqq mt[f^{-1}A_n] = \int_{-\infty}^{\infty} N_{f^{-1}A_n}(y)dy = \int_{A_n} N_{f^{-1}A_n}(y)dy \leqq \int_{A_n} N_{[a,b]}(y)dy,$$

and the latter $\to 0$ as $n \to \infty$.

THEOREM 19. *Let f be continuous and of bounded variation on the set $[a, b]$ and let M be a measurable subset for which $mf(M)=0$. Then $f'(x)=0$ almost everywhere on M.*

Proof.

$$\int_N |f'(x)|\, dx \leqq m^*t(M) \qquad \text{(by Theorem 15)}$$

$$= 0 \qquad \text{(by Theorem 18).}$$

The conclusion follows.

Any mathematical paper ought to both answer and raise questions. Theorem 19 is well known in case f is absolutely continuous [3, p. 194]. Our proof allows us to get by by assuming only that f is continuous and of bounded variation. Question: Is Theorem 19 true if we just assume that f is of bounded variation? If the answer is yes, is it sufficient to assume that f' exists almost everywhere?

This research was partially supported by the Air Force Office of Scientific Research.

References

1. Lawrence M. Graves, The Theory of Functions of Real Variables, McGraw-Hill, New York, 1956.

2. E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Vol. 1, 3rd ed., Cambridge, 1927.

3. H. Kestelman, Modern Theories of Integration, Oxford, 1937.

4. I. P. Natanson, Theory of Functions of a Real Variable (Translated from Russian), Revised Edition, Ungar, New York, 1961.

5. Frigyes Riesz and Béla Sz.-Nagy, Functional Analysis (Translated from the second French edition), Ungar, New York, 1955.

SOME RESULTS ON LINEAR OPERATORS ON LATTICE GROUPS

E. S. LANGFORD, Autonetics, Division of North American Aviation
(Now with the U. S. Naval Postgraduate School, Monterey, Calif.)

In a preliminary section we develop, for completeness, some well-known properties of Abelian lattice groups; (see [1]).

The second section includes several results concerning linear operators on lattice groups. These are all proved within the framework of abstract lattice groups—representation theorems are not used.

I. Preliminary results. We begin with several definitions. We shall concern ourselves only with Abelian groups, although this is not necessary for much of the theory. A system $\{G; +, \geqq\}$ is a *partially-ordered group* (*po*-group: Birkhoff) if

(i) $\{G; +\}$ is an abstract group.
(ii) $\{G; \geqq\}$ is a partially-ordered set.
(iii) "$\geqq$" is invariant under "$+$". That is, if $a \geqq b$, then $a+c \geqq b+c$ for every $c \in G$.

A partially-ordered set is a *lattice* if every pair of elements in the set have both a least upper bound and a greatest lower bound. A *lattice group* (l-group: Birkhoff) is of course a partially-ordered group which is also a lattice.

G will now denote a lattice group. If $a, b \in G$, the least upper bound of a and b will be denoted by $a \vee b$ ("a cup b"). The greatest lower bound will be denoted

by $a\wedge b$ ("a cap b"). It is evident that these elements must be unique. Since $a\geqq b$ if and only if $(-b)\geqq(-a)$, we have the result that $(-a)\vee(-b)=-(a\wedge b)$, and dually, with cup and cap interchanged. This implies that in a partially-ordered group, the existence of least upper bounds implies the existence of greatest lower bounds and conversely. It is obvious that the lattice operations (viz., cap and cup) are commutative, by definition.

Examples of such systems abound. For several other interesting examples see [1].

Example 1.1. The additive group of real numbers under the usual definition of order.

Example 1.2. The multiplicative group of positive rational numbers, where $r\geqq s$ means that r is some integral multiple of s.

Example 1.3. The set of all real-valued functions on an arbitrary set S under pointwise addition and where $f\geqq g$ means that $f(s)\geqq g(s)$ for every $s\in S$.

Example 1.4. More generally, if G is any lattice group, then the direct union of an arbitrary number of copies of G is a lattice group under the same "componentwise" definitions as in Example 1.3.

Example 1.5. Consider Example 1.4. Then the subset of elements of which all but a finite number of components are zero is a lattice group.

REMARKS. A *lattice-group isomorphism* is a group isomorphism which preserves order in both directions. The group of Example 1.2 is seen to be isomorphic to a group of the type of Example 1.5, namely the group of all (countable) sequences of integers which are eventually zero. The mapping is, of course, $(n_1, n_2, \cdots, n_k, 0, \cdots)\to 2^{n_1}\cdot 3^{n_2}\cdot \ldots \cdot p_k^{n_k}$, where p_k is the kth prime.

Note that the set of all continuous functions on an interval is a lattice group, but that the set of differentiable functions on the interval is not.

For any $a\in G$ we define the *positive part* of a (denoted by a^+) by $a^+=a\vee 0$; we define the *negative part* of a (denoted by a^-) by $a^-=(-a)\vee 0=-(a\wedge 0)$. The *absolute value* of a (denoted by $|a|$) is defined by $|a|=a\vee(-a)$. Note that in Example 1.2, if $r=p/q$ is in lowest terms, then $r^+=p$ and $r^-=q$. For any $a\in G$, a^+, a^-, and $|a|$ are always nonnegative. Four well-known properties of lattice groups are

$$a\vee b+a\wedge b=a+b \quad \text{("modular law")}, \tag{1.1}$$

$$a=a^+-a^- \quad \text{("Jordan decomposition")}, \tag{1.2}$$

$$a^+\wedge a^-=0, \tag{1.3}$$

$$|a|=a^++a^-=a^+\vee a^-. \tag{1.4}$$

Let G^+ denote the set of all nonnegative elements of G. It is clear that any partially-ordered group G is essentially determined by G^+ since $a\geqq b$ if and only if $(a-b)\in G^+$. (1.2) says that G^+ is *generating*; that is, $G=G^+-G^+$. Clifford has shown [2] that in any partially-ordered group, the fact that G^+ is generating is equivalent to the Moore-Smith property, that any pair of elements has an upper

bound. Note that in Example 1.2, G^+ is the set of positive integers. (1.1) generalizes the fact that the product of two integers is the product of their l.c.m. by their g.c.d.

It is easily verified that $(a\wedge b)+c=(a+c)\wedge(b+c)$ and dually for cup. Using this and (1.1), one can show that a lattice group is always distributive with respect to the lattice operations. That is, $(a\vee b)\wedge c=(a\wedge c)\vee(b\wedge c)$ and dually, with cap and cup interchanged.

II. Linear operators on lattice groups. A *linear operator* on a lattice group G is a mapping $E: G\to G$ such that $E(a+b)=Ea+Eb$. Any collection of linear operators on G can itself be partially ordered by defining $E\geqq F$ to mean that $Ea\geqq Fa$ whenever $a\geqq 0$. This is often called the *natural* or *operator* ordering. A linear operator E is said to be *positive* if $Ea\geqq 0$ whenever $a\geqq 0$. This is equivalent to $E\geqq 0$ in the operator ordering, where 0 denotes the zero operator, $0a=0$ for every $a\in G$. It follows that positive linear operators are order-preserving; that is, if $E\geqq 0$ and $a\geqq b$, then $Ea\geqq Eb$. Let I denote the identity operator, $Ia=a$ for every $a\in G$. If E is a linear operator, and $E'=I-E$, then E' is likewise linear: it is called the *complement* of E.

A *contractor* E is a linear operator such that E and E' are both positive: evidently this is equivalent to $I\geqq E\geqq 0$. Clearly E is a contractor if and only if E' is a contractor. If E and F are contractors, then their composition EF is a contractor. Contractors have the useful property of commuting with the lattice operations. That is

$$E(a\vee b) = Ea\vee Eb, \tag{2.1}$$

$$E(a\wedge b) = Ea\wedge Eb, \tag{2.2}$$

$$E|a| = |Ea|. \tag{2.3}$$

For proofs, see Leader [3]. These are not defining properties of contractors, though. If $Ea=2a=a+a$, then E satisfies (2.1) through (2.3), although it is clearly not a contractor.

An idempotent contractor is called a *projector*. If P and Q are projectors and $a\geqq 0$, we have (Leader [3])

$$PQa = Pa\wedge Qa. \tag{2.4}$$

From this it follows that projectors always commute, since the lattice operations are commutative. Projectors can be characterized as follows.

THEOREM 2.1. *A necessary and sufficient condition that a contractor P satisfy* (2.5) *is that P be a projector.*

$$a\wedge Pb = Pa\wedge b = P(a\wedge b) \quad \text{for} \quad a, b\geqq 0. \tag{2.5}$$

Proof. Note that the second equality in (2.5) implies the first by symmetry. Suppose that P is a projector. Then $b=Pb+P'b$, where $Pb\wedge P'b=PP'b=0$.

Thus $Pa \wedge b = Pa \wedge (Pb + P'b) = Pa \wedge (Pb \vee P'b) = (Pa \wedge Pb) \vee (Pa \wedge P'b)$ by the distributive law. But $0 \leqq Pa \wedge P'b \leqq P(a \vee b) \wedge P'(a \vee b) = PP'(a \vee b) = 0$. Using (2.1) and the fact that $Pa \wedge Pb \geqq 0$ gives (2.5).

Conversely, suppose that P is a contractor which satisfies (2.5). Letting $b = Pa$ gives $Pa = Pa \wedge Pa = a \wedge P^2a = P^2a$. Since G^+ is generating, $Pa = P^2a$ for every $a \in G$; therefore P is a projector.

We remember that a partial-ordering is *simple* if for $a, b \in G$, either $a > b$ or $b \geqq a$. For a partially-ordered group, this is equivalent to $G = G^+ \cup (-G^+)$. A lattice group G is *Archimedean* if $a, b \in G$ and $na \leqq b$ for every n imply that $a \leqq 0$. (This generalizes the Archimedean property of the real numbers, which is usually stated that if $a, b > 0$, then $na > b$ for some integer n. The contrapositive form is needed for the generalization since the ordering in a lattice group need not be simple.)

Not all lattice groups are Archimedean.

Example 2.1. We consider the group W_2 of ordered pairs of rational numbers under the ordinary componentwise addition and under the lexicographic ordering: $(x_1, y_1) \geqq (x_2, y_2)$ means that either $x_1 > x_2$ or that $x_1 = x_2$ and $y_1 \geqq y_2$. Note that the ordering is simple. W_2 is not Archimedean since, for example, $0 < n(0, 1) < (1, 0)$ for every integer n. Every linear operator P on W_2 can be represented by a 2×2 matrix of rationals

$$P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

(The element $(x, y) \in W_2$ is to be considered as a column vector

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

in this representation.) Using this representation, we can completely characterize all contractors on W_2.

THEOREM 2.2. *$E = \|a_{ij}\|$ is a contractor on W_2 if and only if*

$$1 \geqq a_{11} \geqq 0, \qquad a_{12} = 0, \qquad (a_{21} \textit{ is unrestricted}), \qquad 1 \geqq a_{22} \geqq 0,$$

with the further restrictions that $a_{11} = 1$ implies that $a_{22} = 1$ and $a_{21} \leqq 0$, and $a_{11} = 0$ implies that $a_{22} = 0$ and $a_{21} \geqq 0$. Further, the only projectors on W_2 are the trivial ones, 0 *and* I.

Proof. Let $a = (x, y)$ be an arbitrary nonnegative element of W_2. That is, $x > 0$ or $x = 0$ and $y \geqq 0$. Writing $a \geqq Ea \geqq 0$ and solving for inequalities in the a_{ij} gives the theorem. The fact that there are no nontrivial projectors on W_2 is shown by writing $E^2 = E$ and equating components. (This is not a special property of W_2. It can be shown that there do not exist any nontrivial projectors on any simply-ordered lattice group.)

We have already noted that projectors always commute. The result is not true, in general, for contractors.

Example 2.2. In W_2, let

$$E = \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then $EF(1, 1) = (0, 0)$ but $FE(1, 1) = (0, \frac{1}{2})$.

It is interesting, however, to see that contractors always commute with projectors.

THEOREM 2.3. *Let P be a projector and E a contractor on a lattice group G. Then $PE = EP$.*

Proof. We need only consider $a \geqq 0$. Then $Ea \leqq a$ implies $P'Ea \leqq P'a$. Likewise since $P' \leqq I$, $P'Ea \leqq Ea$. Therefore $P'Ea \leqq P'a \wedge Ea$. Similarly, $EPa \leqq Pa \wedge Ea$. Therefore $P'Ea + EPa \leqq P'a \wedge Ea + Pa \wedge Ea$. Since Pa and $P'a$ are disjoint, so are the two elements on the right-hand side of the inequality. Using (1.1) and the distributive law, $P'Ea + EPa \leqq (Pa \vee P'a) \wedge Ea = (Pa + P'a) \wedge Ea = a \wedge Ea = Ea$. That is, $P'E + EP \leqq E$ in terms of the operator ordering. Since $P'E = E - PE$, this shows that $EP \leqq PE$. Since P' is also a projector, the same is true with P replaced by P'. Thus $EP' \leqq P'E$. That is, $E - EP \leqq E - PE$, and therefore $EP \geqq PE$.

We conjecture that two contractors on an Archimedean lattice group must always commute.

It can be verified that the set of all projectors on a lattice group forms a Boolean algebra where $P \wedge Q = PQ$ and $P \vee Q = P + Q - PQ$. The identity of the algebra is of course the identity operator I. This shows (via (2.4)) that the operator R defined by

$$\begin{aligned} Ra &= Pa \wedge Qa && \text{for } a \geqq 0 \\ Ra &= R(a^+) - R(a^-) && \text{for arbitrary } a \end{aligned}$$

is a projector and provides the greatest lower bound for P and Q in the operator ordering. Again, there is no immediate extension to contractors as we show by an example.

Example 2.3. Let E and F be contractors on a lattice group G. Then the operator H defined as above need not be linear. Consider W_2 again and let

$$E = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}.$$

Then $H(2, 4) = (1, 2)$, $H(2, -4) = (1, -5)$, but $H[(2, 4) + (2, -4)] = H(4, 0) = (2, -2)$.

Two linear operators E and F on a lattice group G are said to be *disjoint* if $|Ea| \wedge |Fa| = 0$ for every $a \in G$. If P and Q are projectors, then using (2.3) and

(2.4) one can see that P and Q are disjoint if and only if $PQ=0$. It will come as no surprise that this does not in general hold for contractors.

Example 2.4. Let E and F be as in Example 2.2. Then $EF=0$, yet

$$E(1, 1) \wedge F(1, 1) = (0, 1).$$

If G is Archimedean, however, the statement is true.

THEOREM 2.4. *Suppose that E and F are contractors on an Archimedean lattice group G. Then $EF=0$ if and only if E and F are disjoint.*

Proof. We can assume that $a \geqq 0$. Since $F \leqq I$ and $E \geqq 0$, we have that $EF \leqq E$. Similarly $EF \leqq F$, and therefore $0 \leqq EFa \leqq Ea \wedge Fa$. G need not be Archimedean to show this. Therefore the composition of two disjoint contractors is always 0.

Conversely, suppose that $EFa=0$. Let $b=Ea \wedge Fa$. Since $b \leqq Fa$, we have that $Eb \leqq EFa=0$. Since $b \geqq 0$, this shows that $Eb=0$. We show now that $b \leqq (E')^n Ea$ for $n=0, 1, 2, \cdots$, where for any operator H, H^0 is taken to be I. The case $n=0$ has been done. Suppose now that $b \leqq (E')^k Ea$. Since $Eb=0$,

$$b = b - Eb = E'b \leqq (E')^{k+1} Ea$$

which completes the induction. Adding the first n of these inequalities gives

$$\begin{aligned} nb &\leqq \sum_{k=0}^{n-1} (E')^k Ea = \sum_{k=0}^{n-1} (E')^k (I - E')a \\ &= \sum_{k=0}^{n-1} \left[(E')^k a - (E')^{k+1} a\right] = a - (E')^n a \leqq a. \end{aligned}$$

By the Archimedean property, $b=0$.

This paper contains a part of the author's doctoral dissertation, submitted to Rutgers—The State University, June, 1963. The author wishes to thank Prof. S. Leader for his aid and encouragement in writing the dissertation.

References

1. G. Birkhoff, Lattice Theory (Rev. Ed.), Amer. Math. Soc. Colloq. Pub., Vol. XXV, 1948.
2. A. H. Clifford, Annals of Math., 41 (1940) 467.
3. S. Leader, Separation and Approximation in Topological Vector Lattices, Canad. J. Math., 11 (1959) 286–296.

According to an old Professor, it is so cold in the winter in Alaska, that a special value of π is used there to compensate for the climatic conditions. This number is called Eskimo π.

ON THE FUNCTIONAL EQUATION $f(x+y)=f(x)\cdot f(y)$

S. W. DHARMADHIKARI, Indian Statistical Institute, Calcutta

1. Introduction and summary. Let R_r be the set of rational numbers. If g is a complex-valued function on R_r, satisfying $g(x+y)=g(x)+g(y)$, for all x and y in R_r, then it can be easily shown that $g(x)=cx$ for all $x\in R_r$ where $c=g(1)$; (see e.g. [1], p. 116). Sometimes we come across the functional equation

$$f(x+y)=f(x)\cdot f(y), \tag{1}$$

where f is a complex-valued function on R_r. It is easy to verify that if such a function f vanishes anywhere, then it vanishes everywhere. We will therefore exclude this case and assume that f never vanishes. The pertinent question then is whether an f satisfying (1) is necessarily of the form $f(x)=e^{cx}$ for all $x\in R_r$, where $c=\log f(1)$. If f is real-valued, the answer is in the affirmative. This is well known; but we will state it as a theorem for convenience in future reference.

THEOREM 1. *Let f on R_r be real-valued and nonzero. Let f satisfy* (1) *and let $c=\log f(1)$. Then $f(x)=e^{cx}$ for all $x\in R_r$.*

We intend to show here that this theorem fails if f is allowed to have complex values. First we express every complex-valued f satisfying (1) in a special form involving an integer-valued function α. Next, functions f of the form $f(x)=e^{cx}$ are characterised in terms of α. Further investigation of the nature of α leads to the desired result.

Notation: A rational number x will always be expressed as a ratio m/n of integers m and n, with $n>0$. The set of integers will be denoted by I, and I^+ will denote the set of positive integers. The greatest common divisor of two integers m and n is denoted by (m, n) and the notation $m|n$ means that m divides n.

2. The general form of functions satisfying the condition (1). The following theorem will first be established.

THEOREM 2. *Let f on R_r be complex-valued and nonzero. Let c be a logarithm of $f(1)$. Then f satisfies* (1) *if, and only if, for every rational $x=m/n$,*

$$f(x)=\exp\left[cx+2\pi ix\alpha(n)\right], \tag{2}$$

where $\alpha(n)\in I$ and satisfies

$$0\leqq \alpha(n)<n, \quad \textit{for } n\geqq 2; \textit{ and} \tag{3}$$

$$n\,|\,\alpha(kn)-\alpha(n), \quad \textit{for all } k \textit{ and } n \textit{ in } I^+. \tag{4}$$

Proof. (a) Suppose that f satisfies (1). Then, for $x\in R_r$ and for $m\in I$, we have

$$f(mx)=[f(x)]^m. \tag{5}$$

Therefore if $n \in I^+$, $n \geqq 2$, then $f(1/n)$ is an nth root of $f(1)=e^c$. Thus

$$f(1/n) = \exp\left[\frac{c + 2\pi i\alpha(n)}{n}\right], \tag{6}$$

where $\alpha(n)$ is an integer satisfying (3). Finally, for a rational $x=m/n$, (5) and (6) give

$$f(x) = \exp\left[x\{c + 2\pi i\alpha(n)\}\right]. \tag{7}$$

Now, let $(m, n)=1$. The rational $x=m/n$ can then also be written as km/kn, where $k \in I^+$. Therefore (7) gives

$$f(x) = \exp\left[x\{c + 2\pi i\alpha(kn)\}\right]. \tag{8}$$

The values given by (7) and (8) are the same if, and only if, $x[\alpha(kn)-\alpha(n)]$ is an integer. This is the case if, and only if, condition (4) holds. This proves the "only if" part.

(b) Let f be given by (2), where α satisfies the condition (4). Then, from the way (4) was derived from (7) and (8), it follows that $f(x)$ is independent of the way in which x is expressed as a ratio of integers. Now let $x_j=m_j/n_j$, $(j=1, 2)$, be two rationals. Then

$$f(x_1)\cdot f(x_2) = \exp\left[(x_1 + x_2)c + 2\pi i\{x_1\alpha(n_1) + x_2\alpha(n_2)\}\right] \tag{9}$$

and

$$f(x_1 + x_2) = \exp\left[(x_1 + x_2)\{c + 2\pi i\alpha(n_1n_2)\}\right]. \tag{10}$$

Thus the exponents differ by

$$2\pi i\left[m_1\frac{\alpha(n_1n_2) - \alpha(n_1)}{n_1} + m_2\frac{\alpha(n_1n_2) - \alpha(n_2)}{n_2}\right]; \tag{11}$$

(4) easily implies that the coefficient of $2\pi i$ in (11) is an integer. Thus (9) and (10) are identical. That is, f satisfies (1). This proves the "if" part and completes the proof of the theorem.

THEOREM 3. *Let f on R_r be complex-valued and nonzero. Let c be a logarithm of $f(1)$. Then, for some complex constant d and for all $x \in R_r$, $f(x)$ equals e^{dx} if, and only if, f can be expressed in the form* (2), *where $\alpha(n)$ is an integer which satisfies* (3) *and is such that, for some $\nu \in I$, not depending on n,*

$$n \mid \alpha(n) + \nu, \quad \textit{for all } n \in I^+. \tag{12}$$

Proof. Let $f(x)=e^{dx}$. Then f satisfies (1) and can therefore be expressed in the form (2). The two forms of f give the same value if, and only if, $x[(c-d)+2\pi i\alpha(n)]$ is an integral multiple of $2\pi i$. Since c and d are both logarithms of $f(1)$, we must have $(c-d)=2\pi i\nu$, for some $\nu \in I$. Taking $x=1/n$, we see that (12) must hold.

Conversely, if f is given by (2) and α satisfies (12), then $f(x)=e^{dx}$, where $d=c-2\pi iv$. This completes the proof of the theorem.

It is clear that (12) implies (4). If the reverse implication were true, then Theorem 1 could be extended to the case when f is complex-valued. We will therefore study some consequences of (3) and (4). This will throw some light on the nature of the function α.

3. The function α. The following lemma from number theory will be useful.

LEMMA 1. *Let $k\in I$, $n\in I$ and $(k, n)=1$. Let a and b run through all residues modulo k and n respectively. Then $bk-an$ runs through all residues modulo kn.*

LEMMA 2. *Under conditions* (3) *and* (4), *$\alpha(kn)$ is uniquely determined by $\alpha(k)$ and $\alpha(n)$ whenever $(k, n)=1$.*

Proof. Equation (4) shows that, for some $a\in I$,

$$\alpha(kn) = \alpha(n) + an. \tag{13}$$

(3), applied to $\alpha(kn)$ and $\alpha(n)$, shows that $0\leqq a<k$. Similarly,

$$\alpha(kn) = \alpha(k) + bk, \tag{14}$$

where $0\leqq b<n$. The values (13) and (14) are identical if, and only if,

$$bk - an = \alpha(n) - \alpha(k). \tag{15}$$

Lemma 1 shows that there is at the most one pair (a, b) which satisfies (15). We must show that there is at least one such pair (a, b). Observe that $\alpha(n)-\alpha(k)$ takes all integral values from $-(k-1)$ to $(n-1)$. So let $x\in I$ and $-k<x<n$. Lemma 1 implies that, for some a and b, the integers $bk-an$ and x have the same residue modulo kn. That is,

$$kn \mid bk - an - x.$$

But $bk-an-x\geqq-(k-1)n-(n-1)=-kn+1$ and

$$bk - an - x \leqq (n-1)k - (-k+1) = kn - 1.$$

Therefore $bk-an-x=0$. This proves the lemma.

The significance of Lemma 2 is that, under conditions (3) and (4), the function α is completely determined by its restriction to the primes and their powers. The next lemma is concerned with this restriction.

LEMMA 3. *Let p be a prime. Then, conditions* (3) *and* (4) *imply that, for every $k\in I^+$*

$$\alpha(p^k) = \sum_{j=0}^{k-1} c_j p^j, \tag{16}$$

where $c_j\in I$ and $0\leqq c_j<p$; $(j=0, 1, \cdots)$.

Proof. Put $c_0=\alpha(p)$. If $k\geqq 2$, then (4) shows that $p^{k-1}\mid\alpha(p^k)-\alpha(p^{k-1})$. Hence

$\alpha(p^k)=\alpha(p^{k-1})+c_{k-1}p^{k-1}$, where $c_{k-1}\in I$. That $0\leqq c_j<p$ follows from (3). The lemma follows.

We now want to reverse the process. For every prime p, choose $c_j\in I, j\geqq 0$, satisfying $0\leqq c_j<p$ for all j. Define $\alpha(p^k)$ for every $k\in I^+$ by (16). Write every positive integer n in the form

$$n = p_1^{k_1}\cdots p_\nu^{k_\nu}, \tag{17}$$

where the primes $p_1, \cdots, p_\nu$ are taken in the increasing order. Let $n_1=p_1^{k_1}$ and $n_2=n/n_1$. Then $n=n_1n_2$, where $(n_1, n_2)=1$. The proof of Lemma 2 shows that, if $\alpha(n_j)$, $(j=1, 2)$, are defined to satisfy (3), then there is a unique choice of $\alpha(n)$ such that $0\leqq\alpha(n)<n$ and

$$n_j \mid \alpha(n) - \alpha(n_j), \qquad (j = 1, 2). \tag{18}$$

Having defined α for powers of primes, we can thus define $\alpha(n)$ for all n in such a way that (3) holds and conditions of the form (18) are satisfied. Does this function α satisfy (4)?

We have to check whether

$$n' \mid n \Rightarrow n' \mid \alpha(n) - \alpha(n'). \tag{19}$$

We will use induction on the number ν of primes occurring in (17). Observe that, when $\nu=1$, (19) follows immediately from (16). For $\nu>1$, write n in the form $n=n_1n_2$ as described above. Let $n'\mid n$. Two cases arise.

Case (i): For some j, $n'\mid n_j$. We have

$$\alpha(n) - \alpha(n') = [\alpha(n) - \alpha(n_j)] + [\alpha(n_j) - \alpha(n')].$$

n_j, and hence n', divides the first bracket because of (18). Also, n' divides the second bracket because of the induction hypothesis. Thus $n'\mid\alpha(n)-\alpha(n')$ in this case.

Case (ii): If we are not in case (i), then $n'=n_1'\cdot n_2'$, where $n_j'\mid n_j$ for $j=1, 2$. Therefore, from case (i), $n_j'\mid\alpha(n)-\alpha(n_j')$. Writing

$$\alpha(n) - \alpha(n') = [\alpha(n) - \alpha(n_j')] + [\alpha(n_j') - \alpha(n')]$$

and using (18), we see that $n_j'\mid\alpha(n)-\alpha(n')$. Since $(n_1', n_2')=1$, it follows that $n'\mid\alpha(n)-\alpha(n')$ in this case also.

Thus (19) or, equivalently, (4) is verified and the c_j's still need to satisfy only the restriction that $0\leqq c_j<p$.

We are now in a position to prove that (12) is strictly stronger than (4). For $p=2$, let $c_0=1$ and $c_j=0$ for $j\geqq 1$. For other primes, choose $c_j=0$ for all j. This means that

$$\alpha(2^k) = 1 \quad \text{for all } k \in I^+; \text{ and} \tag{20}$$

$$\alpha(n) = 0 \quad \text{if } n \text{ is odd.} \tag{21}$$

The definition of α can now be completed as explained before. Suppose that this

choice of α satisfies (12). Then $\nu=-1$ because of (20), and $\nu=0$ because of (21). This contradiction shows that (12) cannot hold.

We have thus shown that Theorem 1 cannot be extended to the case when f is complex-valued. A complex-valued f satisfying (1) may therefore "misbehave" even on the rationals.

Acknowledgments: The author is grateful to Dr. V. Venugopal Rao and Dr. N. S. Gopalakrishnan for useful discussions. This research was supported by the Council of Scientific and Industrial Research under their scheme of "Scientists' Pool," while the author was at the University of Poona.

Reference

1. E. Kamke, Theory of Sets, Dover, New York, 1950.

A GENERALIZATION OF n-SCALE NUMBER REPRESENTATION

J. A. FRIDY, University of North Carolina and Rutgers, The State University

1. Introduction. In [1] J. L. Brown, Jr., obtained a characterization of a complete sequence of positive integers, where $\{a_p\}_1^\infty$ is complete if, for each positive integer k, there exists a sequence $\{\alpha_p\}_1^\infty$ of zeros and ones such that $k=\sum_{p=1}^\infty \alpha_p a_p$. This sum can be viewed as a generalization of the binary representation of k (cf. Corollary to Theorem 1 in [1]). In this paper the latter point of view is extended to obtain a generalization of the representation of the nonnegative integers in the scale of n: if n is an integer greater than 1 and k is a nonnegative integer, then k can be uniquely expressed by $k=\alpha_0+\alpha_1 n+\cdots+\alpha_m n^m$, where $\alpha_p=0, 1, \cdots$, or $n-1$.

In Section 3 a similar generalization is obtained for the fractional representation: if n is an integer greater than 1 and $x\in[0, 1]$, then x can be expressed uniquely by $x=\sum_{p=1}^\infty \alpha_p/n^p$, where $\alpha_p=0, 1, \cdots$, or $n-1$. In this context uniqueness tacitly implies the convention by which $.0111\cdots$ is identified with $.1000\cdots$ in binary. This agreement is used throughout the third section.

2. n-Bases for the nonnegative integers.

DEFINITION 1. *The sequence $\{a_p\}_1^\infty$ of positive integers is an n-basis for the nonnegative integers if, for each nonnegative integer k, there exists a sequence $\{\alpha_p\}_1^\infty$ such that $\alpha_p=0, 1, \cdots$, or $n-1$, and $k=\sum_{p=1}^\infty \alpha_p a_p$.*

THEOREM 1. *Let $\{a_p\}_1^\infty$ be a nondecreasing sequence of positive integers with $a_1=1$. Then $\{a_p\}_1^\infty$ is an n-basis for the nonnegative integers if and only if*

$$a_{m+1} \leqq 1+(n-1)\sum_{p=1}^{m} a_p, \quad \textit{for } m=1, 2, \cdots. \tag{1}$$

Proof. If $a_{m+1}>1+(n-1)\sum_{p=1}^{m} a_p$ for some m, then it is clear that $a_{m+1}-1$ cannot be expressed in the form $\sum_{p=1}^{\infty} \alpha_p a_p$.

Conversely, suppose that (1) is true. First note that since $a_1=1$, it is trivial to express 0, 1, $\cdots$, and $n-1$ in the desired form; that is, each nonnegative integer $k \leqq (n-1)a_1$ can be written as $k=\alpha_1 a_1$ $(0\leqq \alpha_1 \leqq n-1)$. Proceeding by induction, we assume that each $k \leqq (n-1)\sum_{p=1}^{m} a_p$ can be expressed by $k=\sum_{p=1}^{m} \alpha_p a_p$. We show that each $k \leqq (n-1)\sum_{p=1}^{m+1} a_p$ can be expressed by $k=\sum_{p=1}^{m+1} \alpha_p a_p$. Using the inductive hypothesis, we may assume that k satisfies

$$(n-1)\sum_{p=1}^{m} a_p < k \leqq (n-1)\sum_{p=1}^{m+1} a_p.$$

Let j be the least positive integer such that $k \leqq (n-1)\sum_{p=1}^{m} a_p + j a_{m+1}$. Then $j \leqq n-1$, and

$$(n-1)\sum_{p=1}^{m} a_p + (j-1)a_{m+1} < k \leqq (n-1)\sum_{p=1}^{m} a_p + j a_{m+1};$$

or

$$(n-1)\sum_{p=1}^{m} a_p - a_{m+1} < k - j a_{m+1} \leqq (n-1)\sum_{p=1}^{m} a_p.$$

Applying (1) to the left hand member,

$$0 \leqq k - j a_{m+1} \leqq (n-1)\sum_{p=1}^{m} a_p;$$

thus by the inductive hypothesis there exists a sequence $\{\alpha_p\}_1^m$ $(\alpha_p=0, \cdots$, or $n-1)$ such that $k-ja_{m+1}=\sum_{p=1}^{m}\alpha_p a_p$. Taking $\alpha_{m+1}=j$, this gives $k=\sum_{p=1}^{m+1}\alpha_p a_p$.

COROLLARY. *Let $\{a_p\}_1^\infty$ be a nondecreasing sequence of positive integers with $a_1=1$. Then the condition, $a_{m+k+1}\leqq 1+(n-1)\sum_{p=1}^{m} a_p$ $(m=1, 2, \cdots)$, is necessary and sufficient for the sequence to remain an n-basis after the deletion of k arbitrary terms.*

Proof. Straightforward using Theorem 1.

THEOREM 2. *The following are equivalent:*

(i) *the sequence $\{a_p\}_1^\infty$ provides exactly one n-basis representation of each nonnegative integer;*

(ii) *$a_1=1$, and $a_{m+1}=1+(n-1)\sum_{p=1}^{m} a_p$, for $m=1, 2, \cdots$;*

(iii) *$a_m=n^{m-1}$, for $m=1, 2, \cdots$.*

Proof. (i) implies (ii): Suppose that $\{a_p\}_1^\infty$ is an n-basis but (ii) does not hold for all m; i.e., $a_{m+1}<1+(n-1)\sum_{p=1}^{m} a_p$ for some m. Then $a_{m+1}\leqq (n-1)\sum_{p=1}^{m} a_p$, so by Theorem 1 we can get $a_{m+1}=\sum_{p=1}^{m}\alpha_p a_p$. But also $a_{m+1}=\sum_{p=1}^{m+1}\beta_p a_p$,

where $\beta_{m+1}=1$ and $\beta_i=0$ for $i\leqq m$, whence the representation of a_{m+1} by $\{a_p\}_1^\infty$ is not unique.

(ii) implies (iii): Assuming (ii), we have $a_1 = n^0$; and if $a_p = n^{p-1}$ for $p=1, \cdots, m$, then $a_{m+1}=1+(n-1)\sum_{p=1}^m n^{p-1}=1+(n-1)(n^m-1)/(n-1)=n^m$.

That (iii) implies (i) is obvious.

3. n-Bases for an interval. Throughout the following $\{r_p\}_1^\infty$ will denote a nonincreasing sequence of real numbers with limit zero, and $S=\sum_{p=1}^\infty r_p$ (finite or infinite).

DEFINITION 2. *The sequence $\{r_p\}_1^\infty$ is an n-basis for the interval $[0, (n-1)S]$ if, for each $x\in[0, (n-1)S]$, there exists a sequence $\{\alpha_p\}_1^\infty$, such that $\alpha_p=0, 1, \cdots,$ or $n-1$, and $x=\sum_{p=1}^\infty \alpha_p r_p$.*

THEOREM 3. *The sequence $\{r_p\}_1^\infty$ is an n-basis for $[0, (n-1)S]$ if and only if*

$$r_m \leqq (n-1)\sum_{p=m+1}^{\infty} r_p, \qquad \textit{for } m = 1, 2, \cdots. \tag{2}$$

Proof. Suppose that $r_m>(n-1)\sum_{p=m+1}^\infty r_p$ for some m, and consider a number x such that $(n-1)\sum_{p=m+1}^\infty r_p<x<r_m$. If $x=\sum_{p=1}^\infty \alpha_p r_p$ then we must have $\alpha_1=\cdots=\alpha_m=0$ (since $x<r_m\leqq\cdots\leqq r_1$); but then $x\leqq(n-1)\sum_{p=m+1}^\infty r_p<x$, an absurdity.

Suppose that (2) is true, and let $x\in(0, (n-1)S]$. (The conclusion is trivial for $x=0$.) Since $\{r_p\}_1^\infty$ is nonincreasing with limit 0 and $x>0$, we can let $p_1=\min[p: r_p\leqq x]$. If $r_{p_1}=x$ we're done. If $r_{p_1}<x$ and k is the greatest integer not exceeding $n-1$ such that $kr_{p_1}\leqq x$, choose $\alpha_{p_1}=k$. Again, if equality holds we're done. If not, then $x-\alpha_{p_1}r_{p_1}>0$ and we can choose p_2 so that $p_2=\min[p>p_1: r_p\leqq x-\alpha_{p_1}r_{p_1}]$. As before we can assume that $r_{p_2}<x-\alpha_{p_1}r_{p_1}$, and if k is the greatest integer not exceeding $n-1$ such that $kr_{p_2}\leqq x-\alpha_{p_1}r_{p_1}$, we choose $\alpha_{p_2}=k$. If equality holds we're finished; if not, then $x-\sum_{i=1}^2 \alpha_{p_i}r_{p_i}>0$, and we can continue the process. In general, we choose

$$p_j = \min\left[p > p_{j-1}: r_p \leqq x - \sum_{i=1}^{j-1} \alpha_{p_i}r_{p_i}\right],$$

and in case $r_{p_j}<x-\sum_{i=1}^{j-1}\alpha_{p_i}r_{p_i}$, we take

$$\alpha_{p_j} = \max\left[k: 1 \leqq k \leqq n-1 \text{ and } kr_{p_j} \leqq x - \sum_{i=1}^{j-1} \alpha_{p_i}r_{p_i}\right].$$

If at any step of the process equality holds, then we get $x=\sum_{i=1}^j \alpha_{p_i}r_{p_i}$, and we have finished.

If the process does not terminate, then we get an increasing infinite sequence of partial sums which is bounded above by x. It follows that so long as $x-\sum_{i=1}^j \alpha_{p_i}r_{p_i}\leqq(n-1)\sum_{p=p_j+1}^\infty r_p$ for each j, we can express x as the limit of this sequence; i.e., $x=\sum_{i=1}^\infty \alpha_{p_i}r_{p_i}$.

We argue by contradiction; suppose that j is the least positive integer such that

$$(n-1)\sum_{p=p_j+1}^{\infty} r_p < x - \sum_{i=1}^{j} \alpha_{p_i} r_{p_i}. \tag{3}$$

If (3) is true for some j, then $\alpha_{p_j}=n-1$; for

$$r_{p_j} \leqq (n-1)\sum_{p=p_j+1}^{\infty} r_p < x - \sum_{i=1}^{j} \alpha_{p_i} r_{p_i}$$

implies that

$$(\alpha_{p_j}+1) r_{p_j} < x - \sum_{i=1}^{j-1} \alpha_{p_i} r_{p_i};$$

by the choice of α_{p_j}, this can occur only if $\alpha_{p_j}=n-1$. Thus (3) can be written as

$$x - \sum_{i=1}^{j-1} \alpha_{p_i} r_{p_i} > (n-1)\sum_{p=p_j+1}^{\infty} r_p + (n-1) r_{p_j} = (n-1)\sum_{p=p_j}^{\infty} r_p \geqq r_{p_j-1}.$$

By the choice of p_j, the preceding line implies that $r_{p_j-1}=r_{p_{j-1}}$. But j was chosen to be the least such positive integer, so (3) must be true for $j=1$; however,

$$(n-1)\sum_{p=p_1+1}^{\infty} r_p < x - \alpha_{p_1} r_{p_1} = x - (n-1) r_{p_1}$$

implies that

$$r_{p_1-1} \leqq (n-1)\sum_{p=p_1}^{\infty} r_p < x,$$

which contradicts the choice of p_1.

EXAMPLE. Let $r_p=2^{-p}$ $(p=1, 2, \cdots)$. Then $S=\sum_{p=1}^{\infty} 2^{-p}=1$,

$$r_m = \sum_{p=m+1}^{\infty} r_p \quad (m = 1, 2, \cdots),$$

and if $x\in[0, 1]$ and $x=\sum_{p=1}^{\infty} \alpha_p r_p$ ($\alpha_p=0$ or 1), then $\alpha_1\alpha_2\alpha_3 \cdots$ is the binary fraction representation of x.

The special case $n=2$ in the preceding theorem can be stated as follows: each number $x\in[0, S]$ can be expressed as a sum of distinct elements of the sequence $\{r_p\}_1^{\infty}$ if and only if each r_p does not exceed the sum of its successors. A similar problem has been studied in [2], [3], and [4]; that being to express each positive integer as a sum of distinct terms of a given sequence. The given sequence consists, however, of reciprocals of positive integers, and each sum can contain only a finite number of terms.

The following are immediate consequences of Theorem 3.

COROLLARY 1. *If* $\{r_p\}_1^\infty$ *is an n-basis for* $[0, (n-1)S]$ *and* m *is a positive integer greater than* n, *then* $\{r_p\}_1^\infty$ *is an m-basis for* $[0, (m-1)S]$.

COROLLARY 2. *If* $\sum_{p=1}^\infty r_p$ *diverges and* n *is an integer greater than* 1, *then* $\{r_p\}_1^\infty$ *is an n-basis for* $[0, \infty)$.

COROLLARY 3. *In order that* $\{r_p\}_1^\infty$ *remain an n-basis for* $[0, (n-1)S]$ *after the deletion of k arbitrary terms, it is necessary and sufficient that*

$$r_m \leqq (n-1) \sum_{p=m+k+1}^{\infty} r_p, \quad \text{for } m = 1, 2, \cdots.$$

THEOREM 4. *The following are equivalent:*

(i) *the sequence* $\{r_p\}_1^\infty$ *provides exactly one n-basis representation of each number in* $[0, (n-1)S]$;

(ii) $r_m = \sum_{p=m+1}^\infty r_p$, *for* $m = 1, 2, \cdots$;

(iii) $r_m = r_1 n^{1-m}$, *for* $m = 1, 2, \cdots$.

Proof. (i) implies (ii): Suppose that $\{r_p\}_1^\infty$ is an n-basis for $[0, (n-1)S]$ but that (ii) is not true; i.e., for some m, $r_m < (n-1)\sum_{p=m+1}^\infty r_p$. Let x be a number such that $r_m < x < (n-1)\sum_{p=m+1}^\infty r_p$. Using the same construction as in the proof of Theorem 3, we can write $x = \sum_{p=1}^\infty \alpha_p r_p$, where not all of $\alpha_1, \alpha_2, \cdots, \alpha_m$ are zero (since $r_m < x$). On the other hand, the sequence $\{r_p\}_{m+1}^\infty$ is an n-basis for the interval $[0, (n-1)\sum_{p=m+1}^\infty r_p]$; and since $x < (n-1)\sum_{p=m+1}^\infty r_p$, we can write $x = \sum_{p=m+1}^\infty \beta_p r_p$, where not all of the β_p's equal $n-1$. Hence, the n-basis representation of x is not unique.

(ii) implies (iii): Assuming (ii), we note that

$$r_1 = (n-1)\sum_{p=2}^{\infty} r_p = (n-1)(S - r_1),$$

whence $r_1 = (n-1)n^{-1}S$. The rest is straightforward induction.

(iii) implies (i): This is immediate since, if $\{r_p\}_1^\infty$ satisfies (iii), then $\{r_p\}_1^\infty$ is merely a multiple of the geometric sequence $\{n^{-p}\}_{p=1}^\infty$, which gives the unique representation of $[0, 1]$ in negative powers of n.

References

1. J. L. Brown, Jr., Note on complete sequences of integers, this MONTHLY, 68 (1961) 557–560.

2. P. Erdös and S. Stein, Sums of distinct unit fractions, Proc. Amer. Math. Soc., 14 (1963) 127–131.

3. P. J. van Albada and J. H. van Lint, Reciprocal bases for the integers, this MONTHLY, 70 (1963) 170–174.

4. H. S. Wilf, Reciprocal bases for integers, Bull. Amer. Math. Soc., 67 (1961) 456.

ON PERIODICITY IN GENERALIZED FIBONACCI SEQUENCES

D. M. BLOOM, Brooklyn College

1. Introduction. We define a *generalized Fibonacci sequence* (abbreviated "GF-sequence") to be a sequence $S=(S_n)$ of integers ($n=0, \pm 1, \pm 2, \cdots$) satisfying the following two conditions:

$$S_n + S_{n+1} = S_{n+2} \qquad \text{(all } n\text{)} \tag{1}$$

$$\text{g.c.d. } (S_n\colon n = 0, \pm 1, \pm 2, \cdots) = 1. \tag{2}$$

Observe that we are slightly stretching the usual idea of "sequence" by allowing the subscripts n to be negative as well as positive.

Since any sequence (S_n) of integers satisfying (1) is evidently a constant multiple of a sequence satisfying both (1) and (2), we have not lost generality of treatment by requiring (2). It is clear that any GF-sequence (S_n) is uniquely determined by the values S_0 and S_1, with $(S_0, S_1)=1$.

The celebrated *Fibonacci sequence* is the GF-sequence (F_n) defined by

$$F_0 = 0; \qquad F_1 = 1. \tag{3}$$

Almost as familiar is the *Lucas sequence* (L_n), defined by

$$L_0 = 2; \qquad L_1 = 1. \tag{4}$$

Throughout this paper, F and L will denote the GF-sequences defined by (3) and (4) respectively.

Let S be a GF-sequence; let m be a positive integer. Suppose there exist integers k and r ($r>0$) such that, for all n,

$$m \mid S_n \quad \text{if and only if} \quad n \equiv k \pmod{r}.$$

Then m is said to be *periodic in S with period r*, and we write $r=P_S(m)$. Actually, if m is a factor of just one term of S, then m is necessarily periodic in S (Theorem 1 below). It is well known that every positive integer is periodic in the Fibonacci sequence (Theorem 2). On the other hand, the integer 5 is not periodic in L, whereas the integer 2 is periodic in every GF-sequence. Thus two questions naturally arise:

(A) *Which GF-sequences S have the property that every positive integer is periodic in S?*

(B) *Which positive integers are periodic in every GF-sequence?*

In what follows, we shall answer both questions. The answer to (A) is surprisingly simple (Theorem 6), and depends only on some elementary facts about periods, of which we give a brief self-contained discussion in Section 2 (Theorems 1–4). The answer to (B) is more complicated (Theorem 7), and here we need some more precise information about periods which is given without proof in Theorem 5.

2. Results on periods.

THEOREM 1. *If the positive integer m is a factor of at least one term of the GF-sequence S, then m is periodic in S.*

Proof. Assume that $m \mid S_k$. The number of ordered pairs (S_n, S_{n+1}) which are distinct modulo m cannot exceed m^2; hence there exist integers n and r $(0<r \leq m^2)$ such that $S_n \equiv S_{n+r}$, $S_{n+1} \equiv S_{n+1+r}$ (mod m). It easily follows, using (1), that $S_q \equiv S_{q+r}$ for all q. In particular, $0 \equiv S_k \equiv S_{k+r}$ (mod m), so that $m \mid S_{k+r}$. We have thus shown that m divides at least *two* terms of S. Hence there exists a least positive integer t such that, for some q, m divides both S_q and S_{q+t}. Using the known relation

$$S_{a+b} + (-1)^b S_{a-b} = S_a L_b$$

(valid for all a, b), it follows by induction on i that

$$m \mid S_{q+it} \qquad (i = 0, \pm 1, \pm 2, \cdots). \tag{5}$$

Moreover, by minimality of t, m can divide no other terms of S except the terms (5). Hence m is periodic in S with period t.

THEOREM 2. *Every positive integer is periodic in F.*

Proof. Clearly every integer divides $0 = F_0$, and thus Theorem 2 follows at once from Theorem 1. Note also that if $P_F(m) = r$, then

$$m \mid F_n \quad \text{if and only if} \quad r \mid n. \tag{6}$$

It follows easily from (6) that if a, b are relatively prime, then

$$P_F(ab) = \text{l.c.m.}\ (P_F(a), P_F(b)).$$

THEOREM 3. *If m is periodic in the GF-sequence S, then $P_S(m) = P_F(m)$.*

Proof. Let $m \mid S_k$. The relation

$$S_{k+n} \equiv S_{k+1} F_n \pmod{m} \tag{7}$$

is clearly true for $n = 0$, 1 and can then be proved for all n by induction using (1). Since $m \mid S_k$, (1) and (2) imply that $(m, S_{k+1}) = 1$; hence (7) shows that $m \mid S_{k+n}$ if and only if $m \mid F_n$, and the theorem follows.

By virtue of Theorem 3, $P_S(m)$ depends only on m and not on S; hence we shall drop the subscript S and simply write $P(m)$, the "period of m."

Suppose that m is periodic in L, and that n is the least positive integer such that $m \mid L_n$. The relation $L_{-k} = \pm L_k$ implies that $-n$ is the greatest negative integer such that $m \mid L_{-n}$. Hence $P(m) = 2n$ if $m > 2$. We have proved (cf. Theorem 1):

THEOREM 4. *If the integer $m > 2$ is a factor of a term of L, then $P(m)$ is even.*

Some more specific information about periods is collected in the following theorem:

THEOREM 5. *Let p be an odd prime with $p\neq 5$; let $\epsilon=1$ if $p\equiv\pm 2$* (mod 5), *$\epsilon=-1$ if $p\equiv\pm 1$* (mod 5). *Then*

(a) *$P(p)$ divides $p+\epsilon$.*

(b) *Writing $p+\epsilon=rP(p)$, r is odd if $p\equiv 3$* (mod 4), *even if $p\equiv 1$* (mod 4).

(c) *If p^k is the highest power of p dividing $F_{P(p)}$, then $P(p^{k+n})=p^nP(p)$ for all $n\geqq 0$.*

The proofs are not quite in keeping with the simplicity of our presentation, and are omitted. Part (a) is proved in [**1**]. Part (c) is referred to in [**2**], and can be proved by arguments similar to those used to prove that every odd prime power has a primitive root. References to part (b) would be appreciated, if any are known.

3. Answer to question A. Let S, T be GF-sequences. Suppose that there exist integers a and b such that

$$S_n = (-1)^a T_{n+b} \quad (\text{all } n);$$

i.e., S can be obtained from T "by translation" together with a possible uniform sign change. Such sequences S, T will be called *equivalent*; clearly "equivalence" is indeed an equivalence relation. By Theorem 2, a sequence equivalent to F has the property described in Question A of Section 1. Interestingly, the converse is true:

THEOREM 6. *Assume every positive integer is a factor of at least one term of the GF-sequence S. Then S is equivalent to F.*

Proof. Given a fixed n, let $r=P(S_n)$. By hypothesis, F_r is a factor of some term S_m. Since also $S_n \mid F_r$ by definition of r, it follows that $S_n \mid S_m$. This in turn implies that $m\equiv n$ (mod $P(S_n)$), i.e.,

$$m \equiv n \pmod{r}. \tag{8}$$

Since $P(F_r)=r$ and $F_r \mid S_m$, (8) implies that $F_r \mid S_n$. Since we already know that $S_n \mid F_r$, it follows that $|S_n| = |F_r|$. We have thus shown that every term of S is (up to sign) equal to a term of F.

The terms of any GF-sequence are eventually of constant sign and increasing in magnitude. Hence, replacing S by an equivalent sequence if necessary, we may assume that $0<S_1<S_2<S_3$. Hence, for suitable a, b, c,

$$S_1 = F_a, \qquad S_2 = F_b, \qquad S_3 = F_c,$$
$$0 < a < b < c.$$

Then $F_{b+1}=F_{b-1}+F_b\geqq F_a+F_b=F_c$, so that we must have $c=b+1$. This shows that S_2, S_3 are *consecutive* terms of F. It follows that S is equivalent to F.

4. Answer to question B. Let n be a fixed positive integer. We define an ***n-pair*** to be any ordered pair of consecutive terms in any GF-sequence, ***reduced modulo n***. For any sequence S, let $A_n(S)$ be the set of all n-pairs appearing in S.

LEMMA 7.1. *If S and T are GF-sequences such that $A_n(S)\cap A_n(T)\neq\varnothing$, then $A_n(S)=A_n(T)$.*

Proof. This follows immediately from (1).

LEMMA 7.2. *If S is a GF-sequence, then n is periodic in S if and only if $A_n(S)$ contains a pair whose first coordinate is zero (modulo n).*

Proof. Obvious (cf. Theorem 1).

LEMMA 7.3. *The following two conditions are equivalent:* (a) *n is periodic in every GF-sequence.* (b) *Every n-pair is in $A_n(S)$ for some GF-sequence S in which n is periodic.*

Proof. (a) implies (b) by definition. Conversely, assume (b), let S be any GF-sequence, and let (x, y) be any n-pair in $A_n(S)$. By assumption, $(x, y)\in A_n(T)$ for some GF-sequence T in which n is periodic; by Lemma 7.1, $A_n(S)=A_n(T)$. Hence, by Lemma 7.2, periodicity of n in T implies periodicity of n in S, proving (a).

LEMMA 7.4. *For fixed n, the number of n-pairs with the property of Lemma* 7.3(b) *is precisely $P(n)\phi(n)$, ϕ being the Euler function.*

Proof. Denoting reduction modulo n by brackets, each such pair must be of the form $[S_{a+k-1}, S_{a+k}]$, where $n|S_a$ and $0<k\leqq P(n)$. Such a pair is then uniquely determined by k and by the value of S_{a+1} (mod n). There are $P(n)$ choices for k, and $\phi(n)$ choices for $[S_{a+1}]$ (by (2)), which yields a total of $P(n)\phi(n)$ n-pairs of the required type. It is not hard to show these pairs are all different.

LEMMA 7.5. *The total number of n-pairs is precisely $\phi_2(n)$, the number of ordered pairs (a, b) of integers modulo n such that g.c.d.$(a, b, n)=1$.*

Proof. If $(a, b, n)=1$ and k is the product of all primes which divide b but not a, then $(kn+a, b)=1$. Hence there exists a GF-sequence S defined by $S_0=kn+a$, $S_1=b$, and then $[S_0, S_1]=[a, b]$ is an n-pair. Conversely, if $(a, b, n)\neq 1$, then $[a, b]$ cannot be an n-pair, by (2).

It can be shown that

$$\phi(n) = n \prod_p (1 - 1/p), \tag{9}$$

$$\phi_2(n) = n^2 \prod_p (1 - 1/p^2), \tag{10}$$

where p runs through the distinct prime factors of n. (9) is of course well known. To prove (10), we observe that if $(m, n)=1$ then the distinct ordered pairs modulo mn are precisely the pairs (a_1m+b_1n, a_2m+b_2n) where a_1, a_2 are taken modulo n and b_1, b_2 are taken modulo m. Here the relation

$$\text{g.c.d.}(a_1m + b_1n, a_2m + b_2n, mn) = 1$$

is equivalent to the two relations $(a_1, a_2, n)=1$ and $(b_1, b_2, m)=1$. This shows

that ϕ_2 is multiplicative, so that (10) need be proved only when $n=p^\alpha$, a prime power. In the latter case, $(a, b, p^\alpha)\neq 1$ if and only if $p|a$ and $p|b$, which gives $p^{\alpha-1}$ choices for a (modulo p^α) and the same for b. Hence $\phi_2(p^\alpha)=p^{2\alpha}-p^{2(\alpha-1)}$, which agrees with (10).

For alternate proofs of (10) see [3].

We now have the machinery necessary to answer Question B of Section 1. We say a positive integer n is *good* if it is periodic in every GF-sequence (equivalently, if it is a factor of at least one term of every GF-sequence). By Lemmas 7.3 through 7.5, n is good if and only if $\phi_2(n)=P(n)\phi(n)$, i.e., (by (9), (10)) if and only if

$$P(n) = n \prod_p (1 + 1/p), \tag{11}$$

where p runs over the prime factors of n. Thus the good integers are known as soon as the periods $P(n)$ are known. The following theorem shows, however, that only the periods $P(p)$, $P(p^2)$, p prime, need be considered.

THEOREM 7. (a) *The integers* 1, 2, 4 *are good. Every other good integer is of the form* p^k *or* $2p^k$, *where* p *is an odd prime and* $k\geqq 1$.

(b) *An odd prime* p *is good if and only if* $P(p)=p+1$. (*Here* $P(n)$ *denotes the period of* n *in any GF-sequence, cf. Theorem* 3.) *In particular, a good odd prime must be congruent to* 3 *or* 7 (mod 20).

(c) *Let* p *be a good odd prime and let* $k\geqq 2$. *Then* p^k *is good if and only if* p^2 *does not divide* F_{p+1} (*i.e., if* $P(p^2)>P(p)$).

(d) *Let* p^k *be good, where* p *is an odd prime and* $p^k>3$. *Then* $2p^k$ *is good if and only if* $p\equiv 7$ *or* 43 (mod 60). *The integer* 6 *is good.*

Proof. Let n be good. By Theorem 4, all factors >2 of n must have even periods. If n has relatively prime factors a, b both greater than 2, it follows that the GF-sequence S defined by $S_1=a$, $S_2=b$ has no term divisible by both a and b, and hence no term divisible by n; contradiction. Moreover, $P(8)=6$ (seen by inspection of the sequence F), so that 8 (and hence any higher power of 2) is not good, by (11). Part (a) of the theorem follows.

Part (b) follows readily from (11) and Theorem 5. Equation (11) also shows that if p^2 is good then $P(p^2)=p(p+1)>P(p)$; conversely, if p is good and $P(p^2)>P(p)$ then Theorem 5(c) gives $P(p^{n+1})=p^nP(p)=p^n(p+1)$, which shows by (11) that p^{n+1} is good for all $n\geqq 0$. Thus part (c) of Theorem 7 is proved.

Let p^k be good, as in (d), with $p^k>3$. Then $P(p^k)=(p+1)p^{k-1}$ by (11), and

$$P(2p^k) = \text{l.c.m.}\ (P(2),\ P(p^k)) = \text{l.c.m.}\ (3,\ (p + 1)p^{k-1}).$$

It follows from (11) that $n=2p^k$ is good if and only if 3 does not divide $(p+1)p^k$, i.e., $p\equiv 1$ (mod 3). Since also $p\equiv 3$ or 7 (mod 20) by part (b), this is equivalent to $p\equiv 7$ or 43 (mod 60) as stated in (d). Finally, (11) shows that $6=2\cdot 3$ is good, since inspection of (F_n) yields $P(6)=12$.

5. Further questions. Theorem 7(c) motivates the question: is $P(p^2)=P(p)$ possible? Any information would be welcome, as the answer seems not to be known.

Another type of question is motivated by the second half of Theorem 7(b). Let Q be the set of primes congruent to 3 or 7 (modulo 20). Of the first eighteen members of Q (i.e., all members of Q below 350), thirteen are good; of the nine members of Q between 350 and 550, all nine are good. Question: are "most" of the primes in Q good? In fact, if $f(x)=\text{card}\{p\in Q: p\leqq x\}$ and $g(x)=\text{card}\{p\in Q: p\leqq x, p \text{ good}\}$ ("card" denoting cardinality), does $\lim_{x\to\infty} g(x)/f(x)$ exist? If so, what is it? In fact, are such questions approachable? At any rate, it seems that the Fibonacci numbers still provide much material for investigation.

References

1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 3rd ed., Clarendon Press, Oxford, 1954.

2. L. E. Dickson, History of the Theory of Numbers, vol. I, Carnegie Institute of Washington, 1919, pp. 393–412. Chelsea, New York, 1952, reprint.

3. N. J. Fine, Solution of Problem E 1542, this MONTHLY, 70 (1963) 760.

EXPANSION OF ANALYTIC FUNCTIONS IN INFINITE SERIES AND INFINITE PRODUCTS WITH APPLICATION TO MULTIPLE VALUED FUNCTIONS

ALEXANDER ARCACHE, School of Nabatieh, Lebanon

Let the function $f(z)$ be analytic in a region D of the z-plane, and let $z=x$ be a point in D. Let $f(xe^h)=\phi(h)$ be expanded in a power series in h (holding x fixed). We have:

$$\phi(h)=\phi(0)+\frac{h}{1!}\phi'(0)+\frac{h^2}{2!}\phi''(0)+\cdots+\frac{h^n}{n!}\phi^{(n)}(0)+\cdots.$$

This series converges absolutely and uniformly when $(z=h)$ belongs to a closed disk whose center is the origin and whose radius ρ is less than the radius of convergence R

$$(z=h)\in(|z|\leqq\rho<R).$$

The radius of convergence R is given by Hadamard's formula ([1], 140–141)

$$R=\frac{1}{\limsup_{n\to\infty}\sqrt[n]{\left|\dfrac{1}{n!}\phi^{(n)}(0)\right|}}.$$

We obtain:

$$f(xe^h) = f(x) + \frac{h}{1!} A_1 f(x) + \frac{h^2}{2!} A_2 f(x) + \frac{h^3}{3!} A_3 f(x) + \cdots + \frac{h^n}{n!} A_n f(x) + \cdots,$$

with

$$A_1 f(x) = x\frac{df(x)}{dx} \qquad A_2 f(x) = x\frac{d}{dx}[A_1 f(x)]$$

$$A_3 f(x) = x\frac{d}{dx}[A_2 f(x)] \cdots A_{(k+1)} f(x) = x\frac{d}{dx}[A_k f(x)] \cdots$$

Expansion in infinite products. If we put $\psi(h) = \log f(xe^h)$ then (with x fixed)

$$\psi(h) = \psi(0) + \frac{h}{1!}\psi'(0) + \frac{h^2}{2!}\psi''(0) + \cdots + \frac{h^n}{n!}\psi^n(0) + \cdots,$$

valid for $|h| \leqq \rho' < R'$, where R' is given by (1). We have

$$\exp[\psi(h)] = \exp[\psi(0)] \exp\left[\psi'(0)\frac{h}{1!}\right] \exp\left[\psi''(0)\frac{h^2}{2!}\right] \cdots \exp\left[\psi^{(n)}(0)\frac{h^n}{n!}\right] \cdots.$$

This may be written in the following form:

$$f(xe^h) = f(x)[P_1 f(x)]^{h/1!}[P_2 f(x)]^{h^2/2!}[P_3 f(x)]^{h^3/3!} \cdots [P_n f(x)]^{h^n/n!} \cdots.$$

Upon examination of the coefficients we obtain

$$P_1 f(x) = \exp\left[x\frac{d}{dx}\log f(x)\right]$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$P_{(k+1)} f(x) = \exp\left[x\frac{d}{dx}\log P_k f(x)\right]$$

$$\cdots\cdots\cdots\cdots\cdots$$

Application to multiple-valued functions. If $f(z)$ is a multiple-valued function, and if a is a branch point $(z=a) \notin D$, then we can replace x by $x-a$ and put $h=2k\pi i$, where k is any integer, subject to later restrictions. The result is

$$f[(x-a)e^{2k\pi i}] = f(x-a) + \frac{2k\pi i}{1!} A_1 f(x-a) + \frac{(2k\pi i)^2}{2!} A_2 f(x-a) + \frac{(2k\pi i)^3}{3!} A_3 f(x-a) + \cdots + \frac{(2k\pi i)^n}{n!} A_n f(x-a) + \cdots,$$

where $(x \neq a)$ and $|2k\pi i| \leqq \rho < R$, whence $|k| \leqq \rho/2\pi < R/2\pi$. For the infinite products we obtain

$$f[(x-a)e^{2k\pi i}] = f(x-a)[P_1 f(x-a)]^{2k\pi i/1!}[P_2 f(x-a)]^{(2k\pi i)^2/2!}$$
$$[P_3 f(x-a)]^{(2k\pi i)^3/3!} \cdots [P_n f(x-a)]^{(2k\pi i)^n/n!} \cdots,$$

with $(x \neq a)$ and, as before, $|k| \leqq \rho'/2\pi < R'/2\pi$.

We put

$$x - a = re^{i\theta} = re^{i(\theta+2k\pi)},$$

where r is positive and $e^{2k\pi i} = 1$. The branch point $z = a$ is a singular point.

We can select the single-valued branch function corresponding to $k = 0$,

$$f_{(0)}(x-a) = f(re^{i\theta}), \qquad (r > 0, \alpha < \theta < \alpha + 2\pi).$$

Then all the branches are given by

$$f_{(k)}(x-a) = f[re^{i(\theta+2k\pi)}] = f(re^{i\theta}e^{2k\pi i}),$$

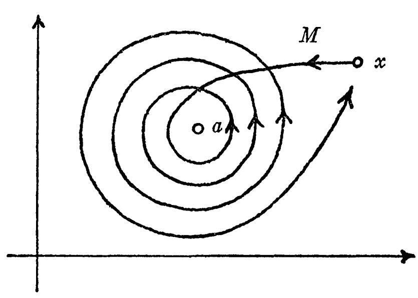
with the same restrictions on r and θ.

We can extend the range of θ in order to include $\theta = \alpha$.

The branch functions are defined but not continuous for $\theta = \alpha$ (see [2], 56–57, 80–81), but if we take these branch functions all together, $f_{(k+1)}(x-a)$ being the continuation of $f_{(k)}(x-a)$, they form the multiple-valued function $f(x-a)$ which is continuous for $\theta = \alpha$ and $r > 0$.

If a point M chosen on the branch $f_{(0)}(x-a)$ describes a path from $(z = x)$ surrounding k times the branch point $(z = a)$ and returns to its initial position $(z = x)$, the value of the function becomes $f_{(k)}(x-a)$.

We suppose that this path is inside D and that there are no singular points inside or on this path (other than $z = a$).



The expansion formulas become

$$f_{(k)}(x-a) = f_{(0)}(x-a) + \frac{2k\pi i}{1!} A_1 f_{(0)}(x-a) + \frac{(2k\pi i)^2}{2!} A_2 f_{(0)}(x-a)$$
$$+ \frac{(2k\pi i)^3}{3!} A_3 f_{(0)}(x-a) + \cdots + \frac{(2k\pi i)^n}{n!} A_n f_{(0)}(x-a) + \cdots.$$

valid for $x\neq a$ and $2\pi|k|\leqq\rho<R$, and

$$f_{(k)}(x-a)=f_{(0)}(x-a)[P_1f_{(0)}(x-a)]^{2k\pi i/1!}[P_2f_{(0)}(x-a)]^{(2k\pi i)^2/2!}$$
$$[P_3f_{(0)}(x-a)]^{(2k\pi i)^3/3!}\cdots[P_nf_{(0)}(x-a)]^{(2k\pi i)^n/n!}\cdots,$$

with $x\neq a$ and $2\pi|k|\leqq\rho'<R'$. (These branches may be distinct or not distinct.) In a similar way one may readily obtain expansion formulas for the case in which the path surrounds several branch points.

References

1. L. V. Ahlfors, Complex analysis, McGraw-Hill, New York, 1953.
2. R. V. Churchill, Complex variables and applications, 2nd ed., McGraw-Hill, New York, 1960.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146.

FOURIER TRANSFORMS OF TWO POISSON KERNELS

D. A. SPRECHER, Syracuse University

1. Introduction and notation. This paper deals with the evaluation of the Fourier integrals (2.1) and (2.2) below. Special cases of these transforms were treated by S. Bochner [2] and J. Horváth [5] (in connection with the existence proof of generalized Hilbert transforms in L^2), but neither author succeeded in evaluating these integrals without knowing in advance their inverse transforms and both needed different methods for odd and even n in (1.1).

In what follows, $\mathbf{t}=(t_1,\cdots,t_n)$ and $\mathbf{x}=(x_1,\cdots,x_n)$ will denote vectors in n-dimensional Euclidean space, R^n. The inner product of $\mathbf{t}$ and $\mathbf{x}$ will be designated by $\mathbf{t}\cdot\mathbf{x}$, and the length of $\mathbf{t}$ by $|\mathbf{t}|$. Also, $\mathbf{x}^j$ will represent the vector $\mathbf{x}$ with zero jth component; $\mathbf{x}^{jk}$ will denote the vector $\mathbf{x}$ whose jth and kth components are zero. The notation $d\mathbf{x}^j$ and $d\mathbf{x}^{jk}$ is self explanatory.

For the real parameter $\alpha>0$ and integer s we define the Poisson kernels

$$\begin{aligned} p_s(\mathbf{x};\alpha) &= (|\mathbf{x}|^2+\alpha^2)^{-(n+1)/2-s} \\ \mathbf{p}_s(\mathbf{x};\alpha) &= \mathbf{x}p_s(\mathbf{x};\alpha). \end{aligned} \tag{1.1}$$

We designate the Fourier transforms of p_s and $\mathbf{p}_s$ by P_s and $\mathbf{P}_s$, respectively. Since $\mathbf{p}_s$ is a vector, we have, in fact, $\mathbf{P}_s=(\mathbf{P}_{s1},\cdots,\mathbf{P}_{sn})$, where $\mathbf{P}_{sj}$ denotes the Fourier transform of the jth component of $\mathbf{P}_s$, namely, x_jp_s.

2. Statement of the results. We are considering the Fourier integrals

$$P_s(\mathbf{t};\alpha) = (2\pi)^{-n/2}\int_{R^n} p_s(\mathbf{x};\alpha)e^{i\mathbf{x}\cdot\mathbf{t}}\,d\mathbf{x} \tag{2.1}$$

and

$$\mathbf{P}_s(\mathbf{t};\alpha) = (2\pi)^{-n/2}\int_{R^n} \mathbf{p}_s(\mathbf{x};\alpha)e^{i\mathbf{x}\cdot\mathbf{t}}\,d\mathbf{x}. \tag{2.2}$$

The results obtained are the following:

(i) For $n\geqq 2$, $\alpha>0$ and $s\geqq 0$,

$$i\mathbf{t}\mathbf{P}_s(\mathbf{t};\alpha) = \left(2s+1+\alpha\frac{d}{d\alpha}\right)P_s(\mathbf{t};\alpha). \tag{2.3}$$

Thus, $\mathbf{P}_s$ can be found directly from P_s, eliminating thereby the intricate integration hitherto used.

(ii) For $n\geqq 2$ and $\alpha>0$ the Fourier integral P_s can be defined for $s=-1$ pointwise everywhere, except at $|\mathbf{t}|=0$. Moreover, at each point $\mathbf{t}$,

$$i\mathbf{t}P_{-1}(\mathbf{t};\alpha) = (n-1)\mathbf{P}_0(\mathbf{t};\alpha); \tag{2.4}$$

with the help of (2.3) we can write this result in the form

$$P_{-1}(\mathbf{t};\alpha) = (1-n)\,|\,\mathbf{t}\,|^{-2}\left(1+\alpha\frac{d}{d\alpha}\right)P_0(\mathbf{t};\alpha), \qquad (|\,\mathbf{t}\,|\neq 0). \tag{2.5}$$

REMARK. When n is sufficiently large, P_s can be defined for integers $s<-1$. This is demonstrated through repeated integrations by parts of boundary terms that do not vanish, and estimates similar to (3.5).

(iii) For $s\geqq 0$, $n\geqq 3$ and positive α, we have

$$P_s(\mathbf{t};\alpha) = \alpha^{-2s-1}C_s e^{-\alpha|\mathbf{t}|}\sum_{\nu=0}^{s}\binom{s}{\nu}\frac{(2s-\nu)!}{s!}(2\alpha\,|\,\mathbf{t}\,|\,)^{\nu}, \tag{2.6}$$

where

$$C_s = \frac{\sqrt{\pi}}{2^{n/2+2s}\Gamma\left(\dfrac{n+1}{2}+s\right)}\cdot \tag{2.7}$$

Formula (2.6) was obtained by Bochner [2], who used Bessel functions and the Fourier inversion formula; Horváth [5] evaluated (2.6) and the left hand side of (2.3) for the special case $s=0$ utilizing a formula of Leray, the Reimann-Liouville integral and the Fourier inversion formula. The present method is elementary: it is based on a transformation to spherical coordinates and change of variables.

3. Proof. We proceed now to establish the results listed above.

(i) An easy computation shows that

$$\sum_j \frac{\partial}{\partial x_j} \mathbf{p}_s(\mathbf{x};\alpha) = \sum_j [p_s(\mathbf{x};\alpha) - (2s+n+1)x_j^2 p_{s-1}(\mathbf{x};\alpha)] \tag{3.1}$$
$$= -\left(2s+1+\alpha\frac{d}{d\alpha}\right)p_s(\mathbf{x};\alpha).$$

Multiplying (2.2) by $i\mathbf{t}$ and performing an integration by parts with respect to x_j we have

$$\begin{aligned} i\mathbf{t}\mathbf{P}_s(\mathbf{t};\alpha) &= (2\pi)^{-n/2}\sum_j \int_{R^n} it_j x_j p_s(\mathbf{x};\alpha)e^{i\mathbf{x}\cdot\mathbf{t}}\,d\mathbf{x} \\ &= (2\pi)^{-n/2}\sum_j \lim_{r_j\to\infty} \int_{R^{n-1}} e^{i\mathbf{x}^j\cdot\mathbf{t}^j}\,d\mathbf{x}^j \int_{-r_j}^{r_j} x_j p_s(\mathbf{x};\alpha)\frac{\partial}{\partial x_j}e^{ix_jt_j}\,dx_j \\ &= (2\pi)^{-n/2}2i\sum_j \lim_{r_j\to\infty} \sin t_jr_j \int_{R^{n-1}} (|\mathbf{x}^j|^2 + r_j^2 + \alpha^2)^{-(n+1)/2-s} e^{i\mathbf{x}^j\cdot\mathbf{t}^j}\,d\mathbf{x}^j \\ &\quad - (2\pi)^{-n/2}\sum_j \int_{R^n} \frac{\partial}{\partial x_j}[x_jp_s(\mathbf{x};\alpha)]e^{i\mathbf{x}\cdot\mathbf{t}}\,d\mathbf{x} = I_1 + I_2. \end{aligned} \tag{3.2}$$

Since the limits in I_1, as $r_j\to\infty$, are independent of one another, we may replace each r_j by r. An easy estimate shows now that for each integral I_j of I_1,

$$\begin{aligned} |I_j| &\leqq A\int_0^\infty \rho^{n-2}(\rho^2+r^2+\alpha^2)^{-(n+1)/2-s}\,d\rho \\ &= \frac{1}{2}AB\left(\frac{n-1}{2}, s+1\right)(r^2+\alpha^2)^{-s-1} = o(1) \end{aligned} \tag{3.3}$$

as $r\to\infty$, A being a constant. Hence, $I_1\to 0$ as $r\to\infty$, and applying (3.1) to I_2 now yields the desired result.

(ii) We now consider (2.1) for the special case $s=-1$. Upon multiplication by it_j and an integration by parts with respect to x_j we find that

$$\begin{aligned} it_jP_{-1}(\mathbf{t};\alpha) &= (2\pi)^{-n/2}\lim_{r\to\infty}\int_{R^{n-1}} e^{i\mathbf{x}^j\cdot\mathbf{t}^j}\,d\mathbf{x}^j\int_{-r}^{r} p_{-1}(\mathbf{x};\alpha)\frac{\partial}{\partial x_j}e^{ix_j\cdot t_j}\,dx_j \\ &= 2i\lim_{r\to\infty}\sin rt_j\int_{R^{n-1}} (|\mathbf{x}^j|^2+r^2+\alpha^2)^{-(n-1)/2}e^{i\mathbf{x}^j\cdot\mathbf{t}^j}\,d\mathbf{x}^j \\ &\quad - \int_{R^n}\frac{\partial}{\partial x_j}p_{-1}(\mathbf{x};\alpha)e^{i\mathbf{x}\cdot\mathbf{t}}\,d\mathbf{x} = 2i\lim_{r\to\infty} I_1 + I_2. \end{aligned} \tag{3.4}$$

When we carry out the differentiation in I_2 we obtain the jth component of the right side of (2.4). Thus, it remains only to show that I_1 in (3.4) tends to zero

as $r\to\infty$. To demonstrate this it is necessary to perform an additional integration by parts. Doing this, we find that

$$
\tag{3.5}
\begin{aligned}
|I_1| \leqq 4 \lim_{q\to\infty} \int_{R^{n-2}} (|\mathbf{x}^{jk}|^2 + r^2 + q^2 + \alpha^2)^{-(n-1)/2}\, d\mathbf{x}^{jk} \\
- (n-1)\int_{R^{n-1}} |x_k|\,(|\mathbf{x}^j|^2 + r^2 + \alpha^2)^{-(n+1)/2}\, d\mathbf{x}^j = o(1)
\end{aligned}
$$

as $r\to\infty$. Thus, (2.4) is established.

(iii) We now come to the only integral we have actually to evaluate. Let θ denote the angle between the vectors $\mathbf{t}$ and $\mathbf{x}$. We construct an orthogonal coordinate system whose first coordinate coincides with the direction of $\mathbf{t}$. (For a derivation of n-dimensional spherical coordinates, divorced from geometric intuition, see e.g. [1]). With $|\mathbf{x}| = \rho$ we obtain for (2.1)

$$
P_s(\mathbf{t};\alpha) = (2\pi)^{-n/2}\omega_{n-1}\int_0^\infty p_s(\rho;\alpha)\rho^{n-1}\,d\rho \int_0^\pi e^{i|\mathbf{t}|\rho\cos\theta}\sin^{n-2}\theta\, d\theta,
$$

$\omega_{n-1} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$ denoting the area of the $n-1$ dimensional unit sphere. Setting $\xi=\rho\cos\theta$, and observing that the integrand of the inner integral is an even function of ξ, we get for $P_s(\mathbf{t};\alpha)$

$$
\begin{aligned}
&2(2\pi)^{-n/2}\omega_{n-1}\int_0^\infty p_s(\rho;\alpha)\rho\, d\rho\int_0^\rho (\rho^2-\xi^2)^{(n-3)/2}\cos|\mathbf{t}|\,\xi\, d\xi \\
&\quad = 2(2\pi)^{-n/2}\omega_{n-1}\int_0^\infty \cos|\mathbf{t}|\,\xi\, d\xi \int_\xi^\infty (\rho^2-\xi^2)^{(n-3)/2}(\rho^2+\alpha^2)^{-(n+1)/2-s}\rho\, d\rho,
\end{aligned}
$$

the change of the order of integration being permissible. With the transformation $\tau^2=(\rho^2-\xi^2)/(\rho^2+\xi^2)$ the inner integral is seen to be

$$
(\xi^2+\alpha^2)^{-s-1}\int_0^1 (1-\tau^2)^s\tau^{n-2}\, d\tau = \frac{1}{2}B\left(\frac{n-1}{2}, s+1\right)(\xi^2+\alpha^2)^{-s-1}
$$

and hence

$$
P_s(\mathbf{t};\alpha) = (2\pi)^{-n/2}\omega_{n-1}B\left(\frac{n-1}{2}, s+1\right)\int_0^\infty (\xi^2+\alpha^2)^{-s-1}\cos|\mathbf{t}|\,\xi\, d\xi.
$$

A variety of methods is available for evaluating the last integral, one using the Fourier inversion formula ([2], pp. 49–61). It may also be evaluated by elementary means (a method due to Laplace) and an iteration ([4], pp. 222–227). Still another method is furnished by the calculus of residues ([3], pp. 129–132). In particular, formula (2.6) follows directly from ([2], (28), p. 61).

The author wishes to use this opportunity to thank A. Douglis for the numerous useful suggestions that carried this project through.

This research was supported by the United States Air Force through the Air Force Office of Scientific Research and Development Command. under Contract No. AF49(638)-590.

References

1. L. E. Blumenson, A derivation of n-dimensional spherical coordinates, this MONTHLY, 67 (1960) 63–66.
2. S. Bochner, Lectures on Fourier integrals, Princeton Univ. Press, 1959, 49–61, 235–245.
3. E. T. Copson, An introduction to the theory of functions of a complex variable, Oxford, 1957, 129–132.
4. J. Edwards, A treatise on the integral calculus, Chelsea, New York, 1954, 222–227.
5. J. Horváth, Sur les fonctions conjuguées à plusieurs variables, Nederl. Akad. Wetensch. Indag. Math., 15 (1) (1953) 21, 24.

CONFIGURATIONS OF FINITE JUNCTION-CLOSED SETS

J. B. ROBISON, Wesleyan University

If four points in the real affine plane are the vertices of a nonsingular convex quadrilateral, let their junction be defined as the point formed by the intersection of the diagonals. A junction-closed set is one such that the junction of every four-point subset as defined above is an element of the set. This note considers what possible configurations of points generate, by the operation of junction, a finite junction-closed set.

THEOREM. *There are two general forms of sets generating junction-closed sets:* (1) *any finite number of points on a line, plus at most one point on each side of the line*; (2) *six points consisting of three vertices of a triangle and one point on each of three alternating extended sides (i.e., the sides must be extended in such a way that each vertex is interior to exactly one extended side, as in Fig.* 1).

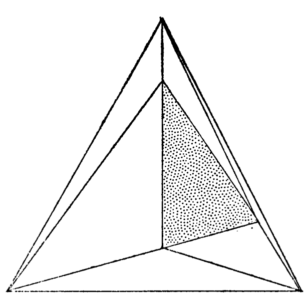


FIG. 1

Note that the junction-closed set generated by form (1) will have at most one additional point. In form (2) as well as in some cases of form (1) the condition for junction-closure is vacuously satisfied. Moreover, a set of either form is generated only by a set of the same form.

LEMMA 1. *The junction-closed set generated by any five-point set contains three collinear points.*

Proof. Consider the convex hull of such a set. If the hull forms a quadrilateral or a pentagon, the diagonals form a junction point collinear with the end-points of the diagonals. If the hull is a triangle, both other points are contained in the interior of the triangle or the collinearity of three points is immediate. When the fourth point (labeled D) is connected to the vertices, three interior triangles are formed (Fig. 2). Again, the fifth point, E, must be in the interior

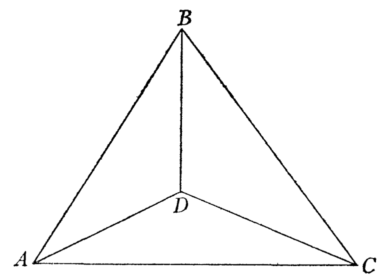


FIG. 2

of one of the three triangles, say ABD, or collinearity is again immediate. If it is inside ABD, line segment EC must intersect ABD somewhere, forming three collinear points. (Note that "line segments WY and XZ intersect" is equivalent to "$WXYZ$ is a convex quadrilateral.")

LEMMA 2. *If a finite junction closed set S contains three collinear points A, B, and C, then all points of S on a given side of line ABC, taken together, are collinear with the interior point B of line segment AC.*

Proof. If the junction-closed set is finite, on each side of line ABC there exists a point X closest to line ABC.

Assume there exists another point Y on the same side of ABC as X. Then consider line XY. If XY is not parallel to ABC, and X and Y are not collinear with B, then XY must intersect ABC on one side or another of B, say on the same side as A (Fig. 3). Call that intersection Q. Pasch's axiom applies to triangle

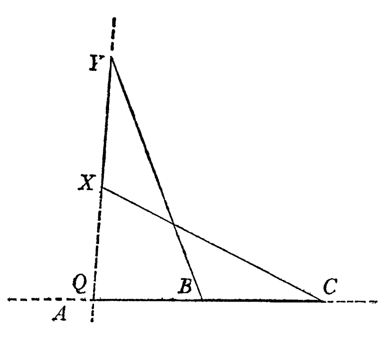


FIG. 3

YQC and line segment XC must intersect line segment YB to form a junction. This point must be closer to ABC than X, contradicting the hypothesis.

If XY is parallel to ABC, we get immediately a convex quadrilateral, in fact, a trapezoid, and the junction point generated is closer to ABC than X, again contradicting the hypothesis.

LEMMA 3. *If four points are on a line in a finite junction-closed set, there can be at most one point on either side of the line, and the set must be of form* (1).

Proof of the Theorem. Any finite junction-closed set not containing a five-point set as described in Lemma 2 is a case of form (1), as is the five-point set itself.

To that system of five points we can add any finite number of points on BXY, giving us form (1). Or we can add a point Z on XA on the other side of ABC, or we can add such a point on XC (Fig. 4). Either gives us form (2). Apply-

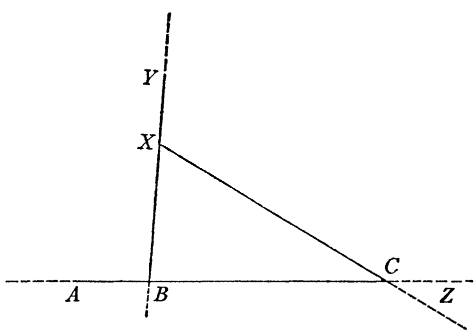


FIG. 4

ing Lemma 2 to line BXY as well as to line ABC, we see that these are the only possibilities. Applying Lemma 2 to line XZ and Lemma 3 to line BXY, we see that the three possibilities are mutually exclusive.

Thus, only the forms described above can generate a finite junction-closed set.

TESTS FOR SINGULAR POINTS

J. P. KING, Lehigh University

1. Introduction. Consider a function element $f(z)$ defined by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{1}$$

having positive radius of convergence. Since the circle of convergence of the series passes through the singular point of the function which is nearest to the origin, the modulus of that singular point can be determined from the sequence

$\{a_n\}$ in a simple manner. The problem of determining the exact position of the singular points on the circle of convergence is, however, more difficult. If a particular point on the circle of convergence is considered, tests can be devised to determine whether or not that point is a singular point of the function defined by the series. It may be supposed, without loss of generality, that the radius of convergence of the series is 1 and that the point to be considered is $z=1$.

In this note two theorems are given, each of which provides necessary and sufficient conditions that $z=1$ be a singular point of the function defined by the series (1). The sufficient conditions are particularly amenable to calculation since they can be phrased in terms of the Euler summability, $E(r)$ (see [1]), or the Taylor summability, $T(r)$ (see [2]), of the sequence of coefficients $\{a_n\}$. Special cases of Theorem 1 and of Theorem 2 for $r=\frac{1}{2}$ are contained in Titchmarsh [4, p. 216] and Hille [3, p. 7], respectively.

2. Results.

THEOREM 1. *A necessary and sufficient condition that $z=1$ be a singular point of the function defined by the series* (1) *is that*

$$\limsup \left| \sum_{m=0}^{n} \binom{n}{m} r^m (1-r)^{n-m} a_m \right|^{1/n} = 1$$

for some $0<r<1$.

Proof. Consider the function

$$F(t) = \frac{1}{1-(1-r)t} f\left(\frac{rt}{1-(1-r)t}\right).$$

$F(t)$ is regular in the region

$$D_r = \left\{ t: \left| \frac{rt}{1-(1-r)t} \right| < 1 \right\}.$$

Furthermore $z=1$ is a singular point of $f(z)$ if and only if $t=1$ is a singular point of $F(t)$. A simple calculation gives:

$$D_r = \{t: \operatorname{Re}(t) < 1\},$$

$$D_r = \left\{ t: \left| t - \frac{1-r}{1-2r} \right| > \frac{r}{1-2r} \right\},$$

$$D_r = \left\{ t: \left| t - \frac{1-r}{1-2r} \right| < \frac{r}{2r-1} \right\},$$

for $r=\frac{1}{2}$, $0<r<\frac{1}{2}$, and $\frac{1}{2}<r<1$, respectively.

In each case $t=1$ is on the boundary of D_r and D_r contains all points of the closed unit disk except $t=1$. If we write

$$F(t) = \sum_{n=0}^{\infty} b_n t^n$$

it follows that $t=1$ is a singular point of $F(t)$ if and only if the radius of convergence of the series is exactly 1. That is, if and only if $\limsup |b_n|^{1/n}=1$.

The function $F(t)$ is given by

$$\begin{aligned} F(t) &= \frac{1}{1-(1-rt)} \sum_{m=0}^{\infty} a_m \left[\frac{rt}{1-(1-r)t}\right]^m \\ &= \sum_{m=0}^{\infty} a_m r^m t^m \sum_{n=m}^{\infty} \binom{n}{m} (1-r)^{n-m} t^{n-m} \end{aligned}$$

provided that $(1-r)|t|<1$. It is easy to verify the interchange of summation in the last expression. Hence,

$$F(t) = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \binom{n}{m} r^m (1-r)^{n-m} a_m.$$

Therefore,

$$b_n = \sum_{m=0}^{n} \binom{n}{m} r^m (1-r)^{n-m} a_m. \tag{2}$$

This completes the proof.

The sequence $\{b_n\}$ defined by (2) is precisely the $E(r)$-transform of the sequence $\{a_m\}$ [1]. Furthermore, $\lim b_n = b \neq 0$ implies $\limsup |b_n|^{1/n}=1$. These remarks give

COROLLARY 1. *If the sequence $\{a_n\}$ is $E(r)$-summable, $0<r<1$, to a nonzero constant then $z=1$ is a singular point of the function defined by the series* (1).

Notice that, since $E(r)$ is regular for $0<r<1$ (see [1]), the condition $\lim a_n = a\neq 0$ is sufficient to insure that $z=1$ be a singular point of $f(z)$.

THEOREM 2. *A necessary and sufficient condition that $z=1$ be a singular point of the function defined by the series* (1) *is that*

$$\limsup \left| \sum_{n=m}^{\infty} \binom{n}{m} r^{n-m} (1-r)^{m+1} a_n \right|^{1/m} = 1$$

for some $0<r<1$.

Proof. Consider the function $G(t)=(1-r)f(r+(1-r)t)$. $G(t)$ is regular in the region $R_r=\{t: |r+(1-r)t|<1\}$. A simple calculation gives

$$R_r = \left\{t: \left|t - \frac{r}{r-1}\right| < \frac{1}{1-r}\right\}.$$

The point $t=1$ is on the boundary of R_r and R_r contains all points of the closed unit disk except $t=1$. If we write

$$G(t) = \sum_{n=0}^{\infty} c_n t^n$$

it follows that $z=1$ is a singular point of $f(z)$ if and only if $\limsup |c_n|^{1/n}=1$.

The function $G(t)$ is given by

$$\begin{aligned} G(t) &= (1-r)\sum_{n=0}^{\infty} a_n(r+(1-r)t)^n \\ &= (1-r)\sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} r^{n-m}(1-r)^m t^m \\ G(t) &= \sum_{m=0}^{\infty} t^m \sum_{n=m}^{\infty} \binom{n}{m} r^{n-m}(1-r)^{m+1}a_n. \end{aligned}$$

Hence,

$$c_m = \sum_{n=m}^{\infty} \binom{n}{m} r^{n-m}(1-r)^{m+1}a_n. \tag{3}$$

This completes the proof.

The sequence $\{c_m\}$ defined by (3) is the $T(r)$-transform of the sequence $\{a_n\}$ [2]. This gives

COROLLARY 2. *If the sequence $\{a_n\}$ is $T(r)$-summable, $0<r<1$, to a nonzero constant then $z=1$ is a singular point of the function defined by the series* (1).

The $T(r)$ method is also regular for $0<r<1$ [2] so that $\lim a_n \neq 0$ again insures that $z=1$ is a singular point of $f(z)$.

References

1. R. P. Agnew, Euler transformations, Amer. J. Math., 66 (1944) 318–338.
2. V. F. Cowling, Summability and analytic continuation, Proc. Amer. Math. Soc., 1 (1950) 536–542.
3. Einar Hille, Analytic Function Theory, vol. II, Ginn and Co., New York, 1962.
4. E. C. Titchmarsh, The Theory of Functions, Oxford, London, 1952.

ON THE DETERMINATION OF NUMBERS BY THEIR SUMS OF FIXED ORDER

NATHAN ELJOSEPH, University of Tel Aviv

1. Introduction. We wish to treat the following

Problem. Given a set of $\binom{n}{s}$ numbers $\{\sigma_{i_1\cdots i_s}\}$ with $0<i_1<i_2<\cdots<i_s\leqq n$. Under what conditions does there exist a unique set of numbers $\{x_1, \cdots, x_n\}$ so that $\sigma_{i_1\cdots i_s}=x_{i_1}+\cdots+x_{i_s}$?

A similar problem was treated by J. L. Selfridge and E. G. Straus (Pacific J.

Math., 8 (1958) 847–856) and by A. S. Fraenkel, B. Gordon and E. G. Straus (Pacific J. Math., 12 (1962) 186–196). In these papers, however, the set $\{\sigma\}$ was not assumed to be indexed.

2. Solution. If $s=0$ or $s=n>1$ then we have either no condition on the x_i or one condition, so that in these cases $\{\sigma\}$ does not determine $\{x\}$ uniquely. In all other cases we have the following.

THEOREM. *If $1\leqq s\leqq n-1$ then our problem has a solution if and only if*

$$\binom{n-2}{s-1}\sigma_{i_1\cdots i_s}=\sigma^{(i_1)}+\cdots+\sigma^{(i_s)}-\frac{s-1}{n-1}s\sigma \tag{1}$$

for all $0<i_1<\cdots<i_s\leqq n$, where σ is the sum of all $\sigma_{i_1\cdots i_s}$, while $\sigma^{(j)}$ is the sum of all $\sigma_{i_1\cdots i_s}$ in which one of the indices is j.

The solution, always unique, is given by

$$\binom{n-2}{s-1}x_i=\sigma^{(i)}-\frac{s-1}{n-1}\sigma,\qquad i=1,\cdots,n. \tag{2}$$

Proof. To prove necessity we note that, if $\sigma_{i_1\cdots i_s}=x_{i_1}+\cdots+x_{i_s}$ then

$$\sigma=\binom{n-1}{s-1}\sum x_i, \tag{3}$$

since each x_i appears in exactly

$$\binom{n-1}{s-1}$$

summands $\sigma_{i_1\cdots i_s}$ on the left of (3). Hence

$$\begin{aligned}\sigma^{(i)}&=\binom{n-1}{s-1}x_i+\binom{n-2}{s-2}\sum_{j\neq i}x_j\\&=\binom{n-2}{s-1}x_i+\binom{n-2}{s-2}\sum_{j=1}^{n}x_j\\&=\binom{n-2}{s-1}x_i+\binom{n-2}{s-2}\sigma\Big/\binom{n-1}{s-1}\\&=\binom{n-2}{s-1}x_i+\frac{s-1}{n-1}\sigma,\end{aligned} \tag{4}$$

which proves (2). Substituting the values of x_i from (2) we obtain

$$\binom{n-2}{s-1}\sigma_{1_1\cdots i_s}=\sigma^{(i_1)}+\cdots+\sigma^{(i_s)}-\frac{s-1}{n-1}s\sigma \tag{5}$$

which proves (1).

To prove sufficiency, assume (1) satisfied and define x_i by (2). Set

$$\overset{*}{\sigma}_{i_1 \cdots i_s} = x_{i_1} + \cdots + x_{i_s};$$

then

$$\binom{n-2}{s-1} \overset{*}{\sigma}_{i_1 \cdots i_s} = \sigma^{(i_1)} + \cdots + \sigma^{(i_s)} - \frac{s-1}{n-1} s\sigma = \binom{n-2}{s-1} \sigma_{i_1 \cdots i_s},$$

which completes the proof.

Note that the conditions (1) are not independent since summation over all s-tuples which contain a fixed index i yields identities.

ON THE CLASS OF RINGS WHICH DO NOT CONTAIN NONZERO SEMI-SINGULAR IDEALS

KWANGIL KOH, North Carolina State University

1. Introduction. An ideal Z of a ring R is called *semi-singular* if for each element x of Z, the right annihilator of x, denoted by x_r, is not zero. Let

$C_1 = \{R \mid R$ is an integral domain (not necessarily commutative)$\}$,
$C_2 = \{R \mid$ no nonzero ideal of ring R is semi-singular$\}$ and
$C_3 = \{R \mid R$ is a prime ring with zero right singular ideal$\}$.

The main results in this note are:

(i) If $R \in C_2$, each pair of nonzero right ideals of R has nonzero intersection (i.e., R is a right uniform ring) if and only if for each right R-module M the set

$$tM = \{m \text{ in } M \mid m_r \neq 0, \text{ where } m_r = \{r \text{ in } R \mid mr = 0\}\}$$

is a submodule of M. (E. R. Gentile proposed this problem in this MONTHLY, Advanced Problem Section, Dec., 69 (1962) for the Class C_1.)

(ii) $R \in C_2$ and has a maximal complement right ideal if and only if $R \in C_3$ with the ascending chain condition on annihilator right ideals and on complement right ideals.

For notations and definitions, we refer the reader to the paper of R. E. Johnson [2] and to the paper of A. W. Goldie [1].

2. Since any simple ring with unity belongs to C_2, it is clear that C_1 is properly contained in C_2. If $R \in C_2$ then $R \in C_3$, for if S_1, S_2 are nonzero ideals of R then $S_1 S_2 \neq (0)$ and by definition of C_2, the right singular ideal of R is zero. There exists a primitive ring with a minimal right ideal such that the sum of all minimal right ideals of R (the socle of R) is a semi-singular ideal (for example see [5, p. 77]). Hence $C_2 \not\supseteq C_3$.

LEMMA 2.1. *If R is a regular ring with unity and $R \in C_2$, then R is a simple ring.*

Proof. If S is a nonzero ideal of R then S contains an element x such that $x_r=0$. There exists r_0 in R such that $xr_0x-x=x(r_0x-1)=0$, hence $r_0x=1\in S$.

LEMMA 2.2. *If $R\in C_2$ then R' (its maximal right quotient ring) is simple with unity.*

Proof. Since $R\in C_2$, the right singular ideal of R is zero, hence by [2, Th. 3] R' is a regular ring. We will show that $R'\in C_2$. Suppose there exists a nonzero semi-singular ideal S' in R'. Then $S=S'\cap R\neq(0)$ since if $0\neq s\in S'$ then there exist a and b in R such that $sa=b\neq 0$. S is an ideal in R, and if $s\in S$ then $(s)_r\cap R\neq(0)$, thus $R\notin C_2$, which is absurd.

LEMMA 2.3. *If $R\in C_3$ and R is right uniform, then $R\in C_1$.*

Proof. Since $R\in C_3$, the right singular ideal of R is zero. Thus if $a\in R$, $a\neq 0$, then $a_r=0$; for if $a_r\neq 0$, then a_r would have nonzero intersection with each nonzero right ideal of R (since R is right uniform) and the right singular ideal of R would be nonzero.

This lemma has the obvious corollary:

COROLLARY 2.4. *If $R\in C_3$ and R is right uniform, then R is an Ore-domain.*

Let M be a nonzero (right) R-module of a given ring R. If N is a submodule of M such that N has nonzero intersection with each nonzero submodule of M, then N is called *large*. If M' is an R-module which is an extension of M such that M is a large submodule of M' then M' is called an *essential* extension of M.

THEOREM 2.5. *If $R\in C_2$ then R is right uniform if and only if for each right R-module M the set tM is a submodule of M.*

Proof. Suppose R is right uniform. Then for each right R-module M, tM is the singular submodule of M [2], which is clearly a submodule of M. Conversely, assume that for each right R-module M the set tM is a submodule of M. As an R-module, R' is an essential extension of R. Now $tR'=0$, for if not then $T=tR'\cap R\neq 0$, and T is a nonzero semi-singular ideal of R. Since R' is regular and $tR'=0$, evidently R' is a division ring. Thus R is an Ore-domain and hence right uniform.

3. Let R be a member of C_2. It is not necessarily true that R contains a maximal complement right ideal. For example, the set R_1 of all polynomials in noncommutative indeterminates x and y over a field is a member of C_2. But R_1 does not contain a maximal complement right ideal. For, if it did, it would have a uniform right ideal, say U, and if I_1 and I_2 are nonzero right ideals of R_1, and $u\in U$, $u\neq 0$, then $u(I_1\cap I_2)=uI_1\cap uI_2\neq 0$, since $u_r=0$. Thus $I_1\cap I_2\neq 0$, and R_1 would be a right uniform ring. This contradicts $xR_1\cap yR_1=(0)$.

THEOREM 3.1. *If $R\in C_2$ then R has a maximal complement right ideal if and only if R is a member of C_3 with the ascending chain condition on annihilator right ideals and on complement right ideals.*

Proof. If $R \in C_2$ has a maximal complement right ideal, say C, then a relative complement of C is a uniform right ideal. Let $L(R)$ be the set of all right ideals in R. It is known that if $R \in C_3$, then for each $A \in L(R)$ the mapping $s: A \to A^s$ of $L(R)$, where A^s is the unique maximal essential extension of A in $L(R)$, is a closure operation on $L(R)$ (for example see [4]), and if R contains a uniform right ideal, say U, then it is easy to see that U^s is minimal element (atom) of $L^s(R)$. Let $A \in L^s(R)$. If $A \neq (0)$ then $A \cdot U^s \neq 0$ and there is an element $a \in A$ such that $a U^s \neq 0$ and $(a \cdot U^s)^s \subseteq A^s = A$. Since $(a \cdot U^s)^s$ is an atom by [3, Th. 4.1], $L^s(R)$ is atomic. By [4, Corollary 2.6] and Lemma 2.2, R' is a simple ring with minimum condition on right ideals. By [4, Th. 4.2] R is a right order in R'. Thus, by A. W. Goldie's theorem [1, Th. 4.4], R is a ring with the ascending chain condition on annihilator right ideals and on complement right ideals.

Conversely if $R \in C_3$ and has ascending chain condition on annihilator right ideals and on complement right ideals, then each large right ideal in R contains a regular element, by [1, Th. 3.9]; and since any nonzero ideal in a prime ring is large right ideal, $R \in C_2$. By the maximum condition on complement right ideals of R, R contains a maximal complement right ideal.

References

1. A. W. Goldie, Semi-prime rings with maximum condition, Proc. London Math. Soc., 10 (1960) 201–220.
2. R. E. Johnson, The extended centralizer of a ring over a module, Proc. Amer. Math. Soc., 2 (1951) 891–895.
3. ———, Prime rings, Duke Math. J., 18 (1951) 799–809.
4. ———, Quotient rings of rings with zero singular ideal, Pacific J. Math., No. 4, 11 (1961) 1385–1392.
5. Nathan Jacobson, Structure of rings, Amer. Math. Soc. Colloquium Publ. Vol. 37.

ON EXTENSIONS OF TAYLOR'S FORMULA

T. V. Lakshminarasimhan, Madras Christian College, Tambaram (India)

In this note we give an extension of Taylor's theorem which contains results due to Hummel and Seebeck ([2], p. 243), Rajagopal ([3], p. 71) and Grimshaw ([1], p. 377). The interest in the theorem lies in the fact that its proof needs a lighter hypothesis than the one used in one of the special cases ([2], p. 243) mentioned above.

In what follows, f, g, and h denote functions which are defined and continuous on the finite closed interval $[a, x]$. Further, it is assumed that all the derivatives up to and including the $(m+n)$-th, $(p+q)$-th and $(r+s)$-th orders, respectively, exist and are continuous on the closed interval $[a, x]$. Also, let $f^{(m+n+1)}$, $g^{(p+q+1)}$, $h^{(r+s+1)}$ exist in the open interval (a, b).

THEOREM. *Under the conditions imposed on f, g, h, there exists a number ξ, $a < \xi < x$, such that*

$$\begin{vmatrix} \dfrac{(\xi-a)^n(x-\xi)^m}{(m+n)!}f^{(m+n+1)}(\xi) & (-1)^nS_{m,n}f(a,x) & f^{(n)}(x) \\ \dfrac{(\xi-a)^q(x-\xi)^p}{(p+q)!}g^{(p+q+1)}(\xi) & (-1)^qS_{p,q}g(a,x) & g^{(q)}(x) \\ \dfrac{(\xi-a)^s(x-\xi)^r}{(r+s)!}h^{(r+s+1)}(\xi) & (-1)^sS_{r,s}h(a,x) & h^{(s)}(x) \end{vmatrix} = 0,$$

where

$$S_{m,n}f(a,x) = \sum_{k\geq 0}\frac{(m+n-k)!}{(m+n)!}\left[(-1)^k\binom{n}{k}f^{(k)}(x) - \binom{m}{k}f^{(k)}(a)\right](x-a)^k$$

with similar definitions for $S_{p,q}\,g(a,x)$ and $S_{r,s}\,h(a,x)$. In all the summations the convention that the binomial coefficient $\binom{p}{q}=0$ for $p<q$ is adopted.

Proof. Let $\phi(t)$, $\psi(t)$ and $\chi(t)$ be defined by

(1) $$\phi(t) = t^n(1-t)^m, \qquad \psi(t) = t^q(1-t)^p, \qquad \chi(t) = t^s(1-t)^r$$

for $0 \leqq t \leqq 1$.

Let $F(t)$, $G(t)$, $H(t)$ be given by

$$F(t) = \sum_{k=0}^{m+n}(-1)^{m+n+k}(x-a)^k\phi^{(m+n-k)}(t)f^{(k)}(a+t(x-a))$$

$$G(t) = \sum_{k=0}^{p+q}(-1)^{p+q+k}(x-a)^k\psi^{(p+q-k)}(t)g^{(k)}(a+t(x-a))$$

$$H(t) = \sum_{k=0}^{r+s}(-1)^{r+s+k}(x-a)^k\chi^{(r+s-k)}(t)h^{(k)}(a+t(x-a)).$$

Consider the function

(2) $$\Delta(t) = \begin{vmatrix} \dfrac{F(t)-F(0)}{(m+n)!} & (-1)^nS_{m,n}f(a,x) & f^{(n)}(x) \\ \dfrac{G(t)-G(0)}{(p+q)!} & (-1)^qS_{p,q}g(a,x) & g^{(q)}(x) \\ \dfrac{H(t)-H(0)}{(r+s)!} & (-1)^sS_{r,s}h(a,x) & h^{(s)}(x) \end{vmatrix}.$$

It follows from our definitions of F, G, and H that ([4], p. 125)

$$F(1) - F(0) = (-1)^n(m+n)!\,S_{m,n}f(a,x)$$

$$G(1) - G(0) = (-1)^q(p+q)!\,S_{p,q}g(a,x)$$

$$H(1) - H(0) = (-1)^s(r+s)!\,S_{r,s}h(a,x)$$

so that

$$\Delta(1) = \Delta(0) = 0.$$

Further, $\Delta'(t)$ exists in the open interval (0, 1). Hence by Rolle's theorem there exists θ such that

$$\Delta'(\theta) = 0, \qquad 0 < \theta < 1. \tag{3}$$

But ([4], p. 125)

$$F'(t) = (x - a)^{m+n+1}\phi(t)f^{(m+n+1)}(a + t(x - a)) \tag{4}$$

$$G'(t) = (x - a)^{p+q+1}\psi(t)g^{(p+q+1)}(a + t(x - a)) \tag{5}$$

$$H'(t) = (x - a)^{r+s+1}\chi(t)h^{(r+s+1)}(a + t(x - a)). \tag{6}$$

Hence, by differentiating (2) with respect to t, and putting $t=\theta$ in the resulting expression we get from (3) in conjunction with (1), (4), (5) and (6) the required result by taking $a+\theta(x-a)=\xi$.

In the above theorem, if $n=q=s=0$, we get Rajagopal's theorem mentioned at the outset.

Further, in the theorem, take

$$g(y) = \frac{y^{p+q+1}}{(p + q + 1)!}, \qquad h(y) = \frac{y^s}{s!}, \qquad a \leqq y \leqq x.$$

Then, $g^{(p+q+1)}(y)=1$, $h^{(s)}(y)=1$, $h^{(r+s+1)}(y)=0$ and

$$\begin{aligned} S_{p,q}g(a, x) &= \frac{(-1)^q}{(p + q)!}[G(1) - G(0)] \\ &= \frac{(-1)^q}{(p + q)!}\int_0^1 G'(t)\,dt \\ &= \frac{(-1)^q(x - a)^{p+q+1}}{(p + q)!}\int_0^1 t^q(1 - t)^p\,dt \\ &= \frac{(-1)^q(x - a)^{p+q+1}p!q!}{(p + q)!(p + q + 1)!}. \end{aligned}$$

Now, taking $p=m$, $q=n$ and simplifying, the theorem reads

$$S_{m,n}f(a, x) = \frac{(-1)^n(x - a)^{m+n+1}m!n!}{(m + n)!(m + n + 1)!}$$

which is Hummel and Seebeck's generalization of Taylor's theorem.

It is to be observed here that only the existence of $f^{(m+n+1)}$ in the open interval (a, x) was used which is certainly more liberal than the one used by the above authors ([2], Theorem).

Finally, let $\alpha(y')$ be a finite function which is nonvanishing and L-integrable

on the interval $a \leqq y' \leqq y < x$ and further let

$$\alpha(y) = \frac{d}{dy}\int_a^y \alpha(y')\,dy'.$$

Now let $g(y)$ be defined by

$$g(y) = \frac{1}{p!}\int_a^y (y - y')^p \alpha(y')\,dy'$$

and let

$$g(x) = \frac{1}{p!}\int_a^x (x - y')^p \alpha(y')\,dy'$$

be convergent for some positive integer p. Since $\alpha(y')$ is integrable, the function $g(y)$ which is defined for $a \leqq y < x$ and convergent for $y = x$ is continuous for $a \leqq y \leqq x$. Further, since $\alpha(y)$ is the derivative of its indefinite integral, we have, for $a \leqq y < x$,

$$g^{(p)}(y) = \int_a^y \alpha(y')\,dy', \qquad g^{(p+1)}(y) = \alpha(y).$$

Thus, taking g as above and h as in previous cases, we get from our theorem, with $q = 0$, $p = m$,

$$\begin{vmatrix} \dfrac{(\xi - a)^n f^{(m+n+1)}(\xi)}{(m+n)!} & (-1)^n S_{m,n} f(a, x) \\ \dfrac{\alpha(\xi)}{m!} & S_{m,0} g(a, x) \end{vmatrix} = 0. \tag{7}$$

But

$$\begin{aligned} m! S_{m,0} g(a, x) &= G(1) - G(0) \\ &= (x - a)^{m+1} \int_0^1 (1 - t)^m \alpha(a + t(x - a))\,dt \\ &= \int_a^x (x - y)^m \alpha(y)\,dy \\ &= m! g(x). \end{aligned}$$

Hence, from (7) we get

$$S_{m,n} f(a, x) = \frac{(a - \xi)^n m!}{\alpha(\xi)(m + n)!}\, g(x) f^{(m+n+1)}(\xi).$$

which is Grimshaw's theorem when $n = 0$, $x = b$.

References

1. M. E. Grimshaw, On Taylor's theorem, Proc. Cambridge Philos. Soc., 52 (1956) 376–378.
2. P. M. Hummel and C. L. Seebeck, Jr., A generalization of Taylor's expansion, this MONTHLY, 56 (1949) 243–247.
3. C. T. Rajagopal, On the remainder in Taylor's theorem, Math. Student, 14 (1946) 71–73.
4. E. T. Whittaker and G. N. Watson, A Course in Modern Analysis, Cambridge University Press, New York, 1958.

REDUCED PARTIAL PRODUCTS

HOMER BECHTELL, Bucknell University

A group G is said to be the *partial product* of A by B, if for the normal subgroup A of G there exists a proper subgroup B of G such that $G=AB$. We consider in this article only finite groups. It is evident that B contains a subgroup H, $H \leqq B$, such that $G=AH$ and for no other proper subgroup K of H does $G=AK$. Clearly H may not be unique. If $H=B$ then the product is called a *reduced partial product* of G over A by B and the purpose of this note is to discuss several of its immediate properties. These properties relate to the Frattini subgroup, $\Phi(G)$, which for finite groups is the intersection of the maximal subgroups of the group G. The alternative definitions and elementary properties of the Frattini subgroup will be assumed known to the reader (see M. Hall [**1**]).

THEOREM. (i) *If $G=AB$ is a reduced partial product of A by B, then $A \cap B \leqq \Phi(B)$.* (ii) *If $G=AB$ is a reduced partial product of A by B, then $\Phi(G/A) \cong A\Phi(B)/A$.* (iii) *If A is the kernel of a homomorphism θ of a group G, then G contains a subgroup B such that $\Phi(B)\theta \cong \Phi(G\theta)$.*

Proof. For (i) suppose $C=A\cap B \nleqq \Phi(B)$. Then B contains a proper subgroup K such that $B=CK$. Thus $G=AK$ contradicts the role of B and so $C \leqq \Phi(B)$.

Since $G/A \cong B/A\cap B$, then $\Phi(G/A)\cong\Phi(B/A\cap B)$. From part (i), $A\cap B \leqq \Phi(B)$, which implies $\Phi(B/A\cap B)\cong\Phi(B)/A\cap B$. Thus $\Phi(G/A)\cong\Phi(B)/A\cap B$ and hence for $W\leqq G$, $W/A\cong\Phi(G/A)$, it is concluded that $W=A\cdot\Phi(B)$. This proves (ii).

Then for (iii) note that if $A\leqq\Phi(G)$, $\Phi(G/A)\cong\Phi(G)/A$, i.e., $B=G$. If $A\nleqq\Phi(G)$, G may be expressed as a reduced partial product $G=AB$ to which (ii) is applied.

Part (iii) answers the question as to whether or not there exists an inverse image S of $\Phi(G\theta)$ consisting of nongenerators in some subgroup of G. In fact more is shown in that there exists a subgroup B of G such that S is its Frattini subgroup and thus by (ii), $\Phi(G\theta)\cong\Phi(B)/A\cap\Phi(B)$ relative to the kernel A of the homomorphism θ. Note that there is no implication that S will consist of nongenerators of G. This condition occurs if and only if S is a subnormal subgroup of G.

Let $G=AB$ be a partial product over A by B and suppose W is a subgroup

of G for which $\Phi(G/A)\cong W/A$. Then in general $A\cdot\Phi(B)\leqq W$ i.e., the containment may be proper. Moreover, a minimal set of generators for B may not necessarily be coset representatives of a minimal set of generators for the quotient group of G over A. However, each such product is associated with a reduced partial product for which, by (ii), equality holds in the first statement and a minimal set of generators of B is a set of coset representatives for a minimal set of generators of G/A.

We find that (i) lends itself to the theory of group extensions. In the Schreier method it is known that the factor system F of the extension represented by a partial product $G=AB$ over A by B is contained in $A\cap B$. Thus in a reduced partial product the subgroup generated by F is nilpotent and contained in $\Phi(B)$.

Reference

1. M. Hall, The Theory of Groups, Macmillan, New York, 1959.

GENERATING FUNCTIONS FOR A CLASS OF ARITHMETICAL FUNCTIONS

R. L. DUNCAN, Pennsylvania State University and Lock Haven State College

Let $n=p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ be the representation of n as a product of powers of distinct primes and define $\Omega_k(n)=\alpha_1^k+\cdots+\alpha_r^k$ for every nonnegative integer k. This class of functions has been considered in previous notes [1, 2]. By using the identity

$$\sum_p p^{-s}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log\zeta(ns)\qquad(s>1)\tag{1}$$

it can be shown [3] that

$$\sum_{n=1}^{\infty}\omega(n)n^{-s}=\zeta(s)\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log\zeta(ns)\qquad(s>1)\tag{2}$$

and

$$\sum_{n=1}^{\infty}\Omega(n)n^{-s}=\zeta(s)\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\log\zeta(ns)\qquad(s>1)\tag{3}$$

where $\omega(n)=\Omega_0(n)$ and $\Omega(n)=\Omega_1(n)$.

The following theorem generalizes (2) and (3).

THEOREM 1. $\sum_{n=1}^{\infty}\Omega_k(n)n^{-s}=\zeta(s)\sum_{n=1}^{\infty}(\alpha_k(n)/n)\log\zeta(ns)$, $(s>1)$, *where* $\alpha_k(n)=\sum_{d/n}\{d^k-(d-1)^k\}d\mu(n/d)$.

Proof. Since $\sum_{n\leq x}\Omega_k(n)\sim x\log\log x$ for all $k\geqq 0$, [1], it follows that the series $\sum_{n=1}^{\infty}\Omega_k(n)n^{-s}$ converges absolutely for $s>1$. Hence, using (1), we get

$$\begin{aligned}\sum_{n=1}^{\infty} \Omega_k(n)n^{-s} &= \sum_{n=1}^{\infty} \sum_{p^m/n} \{m^k - (m-1)^k\} n^{-s} \\ &= \sum_{p^m} \sum_{v=1}^{\infty} \{m^k - (m-1)^k\} (vp^m)^{-s} \\ &= \zeta(s) \sum_{m=1}^{\infty} \sum_{p} \{m^k - (m-1)^k\} p^{-ms} \\ &= \zeta(s) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{m^k - (m-1)^k\} \frac{\mu(n)}{n} \log \zeta(mns).\end{aligned}$$

Since this last double series converges absolutely, it may be rearranged to give

$$\begin{aligned}\sum_{n=1}^{\infty} \Omega_k(n)n^{-s} &= \zeta(s) \sum_{v=1}^{\infty} \sum_{d/v} \frac{1}{v} \{d^k - (d-1)^k\} d\mu\left(\frac{v}{d}\right) \log \zeta(vs) \\ &= \zeta(s) \sum_{n=1}^{\infty} \frac{\alpha_k(n)}{n} \log \zeta(ns),\end{aligned}$$

where $\alpha_k(n) = \sum_{d/n} \{d^k - (d-1)^k\} d\mu(n/d)$.

It is obvious that $\alpha_0(n) = \mu(n)$ and $\alpha_1(n) = \phi(n)$.

THEOREM 2. *If* $k \geqq 1$, *then* $\phi(n) \mid \alpha_k(n)$.

Proof. For $k \geqq 1$ it is clear that $\alpha_k(n)$ is a linear combination with integral coefficients of functions of the form $n^r \sum_{d/n} \mu(d)d^{-r}$, where r is a positive integer. Since these functions are multiplicative, we obtain

$$n^r \sum_{d/n} \mu(d)d^{-r} = n^r \prod_{p/n} (1 - p^{-r})$$

which is an integer divisible by $\phi(n)$. Thus $\alpha_k(n)$ is divisible by $\phi(n)$.

Actually, as indicated above, we have

$$\alpha_k(n) = \sum_{v=0}^{k-1} (-1)^v \binom{k}{v+1} J_{k-v}(n),$$

where $J_r(n)$ is Jordan's generalization of Euler's function [4].

THEOREM 3. *If* $k \geqq 2$, *then* $\alpha_k(n) \sim kJ_k(n)$.

Proof. Using the above expression for $\alpha_k(n)$ we see that

$$\frac{\alpha_k(n)}{kJ_k(n)} = 1 + \sum_{v=1}^{k-1} (-1)^v \binom{k}{v+1} \frac{J_{k-v}(n)}{kJ_k(n)}.$$

Thus $\lim_{n\to\infty} \alpha_k(n)/kJ_k(n) = 1$ since $n^k/\zeta(k) < J_k(n) < n^k$ for $k \geqq 2$.

We conclude with the following result.

THEOREM 4. *If* $k \geqq 1$, *then* $\limsup_{n\to\infty} \Omega_k(n)/(\log n/\log 2)^k = 1$.

Proof. If $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $k \geqq 1$, then

$$\log n = \alpha_1 \log p_1 + \cdots + \alpha_r \log p_r \geqq (\alpha_1 + \cdots + \alpha_r) \log 2.$$

Hence $(\log n/\log 2)^k \geqq (\alpha_1 + \cdots + \alpha_r)^k \geqq \alpha_1^k + \cdots + \alpha_r^k = \Omega_k(n)$ and the result follows since equality holds for all numbers of the form $n = 2^m$.

References

1. R. L. Duncan, A class of additive arithmetical functions, this MONTHLY, 69 (1962) 34–36.
2. R. L. Duncan, Note on a class of arithmetical functions, this MONTHLY, 69 (1962) 991–992.
3. E. C. Titchmarsh, The Theory of the Riemann Zeta Function, Oxford, 1951, Chapter I.
4. L. E. Dickson, History of the Theory of Numbers, Vol. 1, Carnegie Institution of Washington (reprinted by Chelsea), 1919, Chapter 5.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

A UNIVERSAL SUBALGEBRA THEOREM

STEVEN FEIGELSTOCK, New York University

A type of theorem of interest to algebraists is the subalgebra theorem. "Subgroups of free groups are free," "subgroupoids of free groupoids are free," are examples of subalgebra theorems which have already been proved (see Kurosh, *Theory of Groups*).

Universal algebra is concerned with the notion of an "algebra" in general, and looks upon groups, rings, etc. as types of algebras. A subalgebra theorem of interest to the student of universal algebra should therefore be that, "subalgebras of free algebras are free." It is the object of this paper to prove this theorem.

The reader interested in pursuing the study of universal algebra is referred to Neumann's *Special Topics in Algebra* (*Universal Algebra*), N.Y.U. Lecture Notes. Because of the relative unavailability of this set of notes we present the following list of definitions and lemmas of universal algebra.

DEFINITION 1. *An n-ary operation* ω *in a set* X *is a mapping of* X^n *into* X.

If $(x_1, \cdots, x_n) \in X^n$, then $(x_1, \cdots, x_n)\omega \in X$. Throughout this paper, for

the sake of simplicity, all the operations will be considered to be defined for all $(x_1, \cdots, x_n) \in X^n$; thus the domain $d(\omega)$ of ω is X^n.

DEFINITION 2. *An algebra A is a pair $A = (X, \Omega)$, where X is a nonempty set called the carrier of A, and Ω is a set of operations in X.*

Note: The operations $\omega \in \Omega$ need not necessarily be n-ary with fixed n. There may exist an $\omega_m \in \Omega$, and $\omega_n \in \Omega$, where ω_m is m-ary, ω_n is n-ary, and $m \neq n$.

Notation: The carrier of A is denoted $|A|$.

DEFINITION 3. *An algebra $B = (Y, \Omega)$ is a subalgebra of an algebra $A = (X, \Omega)$ if $Y \subseteq X$ and, for all $(x_1, \cdots, x_n) \in |B|$, and all n-ary $\omega \in \Omega$, $(x_1, \cdots, x_n)\omega \in Y$.*

Notation: "B is a subalgebra of A" is denoted $B \leq A$.

DEFINITION 4. *An equivalence relation c in X is termed a congruence in $A = (X, \Omega)$, if whenever $(x_i, x_i') \in c$ (for $i = 1, \cdots, n$) then $((x_1, \cdots, x_n)\omega, (x_1', \cdots, x_n')\omega) \in c$. By $(x_i, x_i') \in c$ it is meant that x_i and x_i' belong to the same c-equivalence class.*

DEFINITION 5. *A/c the quotient algebra of $A = (X, \Omega)$ by a congruence c denotes the algebra $(X/c, \Omega)$. The elements of X/c, denoted xc, are the sets of all $y \in X$, such that $(x, y) \in c$, where x is also an element of X. In a quotient algebra the following equality must always be satisfied: $(x_1c, \cdots, x_nc)\omega = (x_1, \cdots, x_n)\omega c$.*

DEFINITION 6. *Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of algebras with Ω the set of operations for each A_λ, and $X_\lambda = |A_\lambda|$. The cartesian product $\prod_{\lambda \in \Lambda} (X_\lambda, \Omega)$ is such that $\lambda[(\phi_1, \phi_2, \cdots, \phi_n)\omega] = (\lambda\phi_1, \lambda\phi_2, \cdots, \lambda\phi_n)\omega$, where ϕ_i is an element of the cartesian product of the sets X_λ defined in the usual way, i.e., a mapping of*

$$\Lambda \to \bigcup_{\lambda \in \Lambda} X_\lambda \ni \lambda\phi_i \in X_\lambda, \qquad \forall \lambda \in \Lambda.$$

DEFINITION 7. *If $A = (X, \Omega)$ and $B = (Y, \Omega)$, then a mapping $\mu : |A| \to |B|$, i.e. $\mu : X \to Y$, is a homomorphism from A to B if $[(x_1, \cdots, x_n)\omega]\mu = (x_1\mu, \cdots, x_n\mu)\omega$ for all $(x_1, \cdots, x_n) \in |A|$, and n-ary $\omega \in \Omega$.*

Example: Let $\oplus \in \Omega$ be a 2-ary operation. Put $(x, y) \oplus = x \oplus y$. If μ is a homomorphism, then $[(x, y) \oplus]\mu = (x\mu, y\mu) \oplus$, or $[x \oplus y]\mu = x\mu \oplus y\mu$.

DEFINITION 8. *A homomorphism μ, defined as above, is an isomorphism if it maps $|A| \to |B|$ in a one-one, onto fashion.*

DEFINITION 9. *A set C, of algebras, is called an abstract class of algebras if whenever $A \in C$ and B is isomorphic to A then $B \in C$, i.e., C is closed under isomorphism.*

DEFINITION 10. *An abstract class of algebras C is a variety if it is closed under the formation of subalgebras, quotient algebras, and cartesian products.*

DEFINITION 11. *If* $\{A_\lambda = (X_\lambda, \Omega)\}$ *is a set of algebras with* $\bigcap_{\lambda\in\Lambda} X_\lambda \neq \varnothing$, *then the intersection*

$$\bigcap_{\lambda\in\Lambda} A_\lambda = \left(\bigcap_{\lambda\in\Lambda} X_\lambda, \Omega\right).$$

DEFINITION 12. $S \subseteq |A|$ *is a generator of the algebra* A *if* $\bigcap\{B : B \leqq A$ *and* $S \subseteq |B|\} = A$; *i.e. the smallest subalgebra of* A *whose carrier contains* S *is* A *itself. The algebra generated by* S *is denoted* $Alg(S)$.

DEFINITION 13. *Let* C *be a class of algebras.* $A \in C$ *is C-free if it satisfies the following two conditions:*

(i) $\exists X \subseteq |A| \ni X$ *generates* A.

(ii) $\forall B \in C$, *every mapping* $\theta: X \to |B|$ *can be extended to a homomorphism* $\phi: A \to B$.

The X *referred to in conditions* (i) *and* (ii) *is called a C-free generator of* A.

The reader may have an intuitive idea of what a free algebra is from his knowledge of free groups. His intuition will be realized in the following lemma.

LEMMA 1. *If* C *is the variety of all* Ω*-algebras,* (*i.e. the algebras having* Ω *as their set of operations*) *then given any set* $X \neq \varnothing$, X *generates C-freely an algebra* $A \in C$.

The algebra A which will be constructed in the proof of Lemma 1 is called the free anarchic algebra on X.

Proof. Let $X_0 = X$. Define $X_{i+1} = X_i \bigcup_{n=0}^{\infty} X_i^n \times \Omega_n$, where $\omega \in \Omega_n$ is n-ary. Put $Y = \bigcup_{i=0}^{\infty} X_i$. If $y \in Y$, then $\exists i \ni y \in X_i$. If $i > 0$, then $y = (y_1, \cdots, y_n)\omega$, where $y_j \in X_{i-1}$ $(j = 1, \cdots, n)$, or $y \in X_{i-1}$. If $(y_1, \cdots, y_n) \in Y$, and $\omega \in \Omega$ is n-ary, then $(y_1, \cdots, y_n)\omega$ is defined, because $\exists m \ni y_j \in X_m$, $(j = 1, \cdots, n)$; therefore $(y_1, \cdots, y_n)\omega \in X_m^n \cup \Omega_n \subset Y$ and $A = (Y, \Omega)$ is an algebra.

It must now be shown that X satisfies conditions (i) and (ii) of Definition 13.

(i) It will be shown inductively that X generates A. $|Alg(X)| \supseteq X_0$. Suppose that $|Alg(X)| \supseteq X_i$. If $(y_1, \cdots, y_n) \in X_i$ and $\omega \in \Omega_n$, then $(y_1, \cdots, y_n)\omega \in |Alg(X)| \Rightarrow |Alg(X)| \supseteq X_{i+1}$ $\therefore Alg(X) = A$.

(ii) It will be shown that X is a free generator of A. Let $B = (Z, \Omega) \in C$, and let $\theta: X \to Z$. Put $\theta = \theta_0$, and define θ_i as the mapping from X_i into Z. If $x \in X_{i+1} - X_i$, then $x = (y_1, \cdots, y_n)\omega$, and $x\theta_{i+1}$ is defined to be $(y_1\theta_i, \cdots, y_n\theta_i)\omega$. If $x \in X_i$ then $x\theta_{i+1} = x\theta_i$. Define a mapping $\phi = \bigcup_{i=0}^{\infty} \theta_i$, such that for $y \in Y$, $y\phi$ is defined to be $y\theta_i$ if $y \in X_i$. Let $(y_1, \cdots, y_n) \in X_i$, and let $\omega \in \Omega_n$, then

$$[(y_1, \cdots, y_n)\omega]\phi = (y_1\theta_i, \cdots, y_n\theta_i)\omega = (y_1\phi, \cdots, y_n\phi)\omega.$$

Hence ϕ is a homomorphism, and the lemma is proved.

COROLLARY. *The set of free Ω-algebras is isomorphic to the set of free anarchic Ω-algebras.*

LEMMA 2. *If $A=(X, \Omega)$ is a free anarchic algebra, then*

$$(x_1, \cdots, x_k)\omega = (x_1', \cdots, x_k')\omega \Rightarrow x_i = x_i' \qquad (i = 1, \cdots, k).$$

Proof. The validity of Lemma 2 follows immediately from the construction of free anarchic algebras given in Lemma 1.

With the background material now having been presented, the proof of the subalgebra theorem will be given.

THEOREM. *Subalgebras of free algebras are free.*

Proof. The corollary to Lemma 1 establishes the sufficiency of proving the theorem for free anarchic algebras.

Let $A=(E, \Omega)$ and $B=(F, \Omega)$ with $B\leqq A$. Let $\mu_1, \cdots, \mu_k, \cdots$ be the elements of Ω, with μ_i an n_i-ary operation. Let μ_k be the operation at the end of an element $u_k\in|B|$, then

$$\exists(x_1, \cdots, x_{n_k}) \in |A| \ni (x_1, \cdots, x_{n_k})\mu_k = u_k.$$

Define u_k as irreducible in B if $x_i\notin|B|$ for some $i=1, \cdots, n_k$.

Define the length (l) of an element of $|A|$ (or $|B|$) as the number of symbols in the element.

$$l(u_k) = 1 + \sum_{i=1}^{n_k} l(x_i) < \infty.$$

Define the set $T=\{Z|Z \text{ is irreducible in } B\}$.

To prove the theorem it suffices to show that T generates B freely. To do this it is necessary to show that T satisfies conditions (i) and (ii) of Definition 13.

(i) T generates B. Let $u_k\in|B|$. If u_k is irreducible then $u_k\in T$. If not, then $x_i\in|B|$ (for $i=1, \cdots, n_k$). $l(x_i)<l(u_k)<\infty$; therefore, inductively, there exists an $a\in|A|$ in $u_k\ni a$ is irreducible in $|B|$. T therefore generates B.

(ii) T generates B freely. Let T^* be a set in one-one correspondence with T, such that $t^*(\in T^*)\rightarrow t(\in T)$, and let B^* be a free anarchic algebra on T^*. Let θ be the mapping, $\theta: t^*\rightarrow t$. Extend θ to an epimorphism (onto map) ϕ of B^* onto B. It will be shown by induction that ϕ is a one-one mapping of B^* onto B.

Assume that ϕ is a one-one map of elements of B^* of length $\leqq n$, onto B. By the one-one correspondence between T^* and T, n may be assumed to be greater than 1. Suppose $b^*\in B^*$ is of length $n+1$. Let $b^{*i}=(x_1^{*i}, \cdots, x_{n_k}^{*i})\mu_k$ (where $i=1, 2$). $l(x)_j^{*i}<n$ (for $j=1, \cdots, n_k$). Suppose $b^{*1}\phi=b^{*2}\phi$, then

$$b^{*1}\phi = (\overset{1}{x_1}, \cdots, \overset{1}{x_{n_k}})\mu_k = (\overset{2}{x_1}, \cdots, \overset{2}{x_{n_k}}) = b^{*2}\phi \Rightarrow \overset{1}{x_j} = \overset{2}{x_j} \quad (\text{for } j = 1, \cdots, n_k)$$

by Lemma 2. This implies that $b^{*1}=b^{*2}$. Thus T is a free generator of B, and the proof of the theorem is completed.

It should be noted that this theorem does not imply the subalgebra theorem for special types of algebras, i.e., groups, rings, etc. The reason for this is that the property of being free is relative to the variety under consideration.

The author is indebted to Mr. Paul Zuckerman for many helpful suggestions.

Reference

1. P. M. Cohn, Universal Algebra, Harper and Row, New York, 1965

NONNORMALITY OF LINEAR COMBINATIONS OF NORMALLY DISTRIBUTED RANDOM VARIABLES

LLOYD ROSENBERG, Hudson Laboratories, Columbia University and School of Engineering and Science, New York University

In a first course in mathematical statistics or probability theory it is usually proved that linear combinations of independent normally distributed random variables are normally distributed [2]. If the summands have a multivariate normal distribution then linear combinations of these random variables are also normally distributed [1]. As a result many students loosely state that "linear combinations of normally distributed random variables are normally distributed." In this note we will give an example of two normally distributed dependent random variables whose sum is not normally distributed. The author has found that this example helps to clarify the conditions under which linear combinations of normally distributed random variables are normally distributed. It also exhibits a simple bivariate distribution which has marginal normal distributions but is not a bivariate normal distribution which emphasizes the importance of not merely specifying marginal distributions.

Gumbel has given in [3] the following expression for a family of joint two-dimensional density functions having the marginal densities of $f_1(x_1)$ and $f_2(x_2)$, and marginal distribution functions $F_1(x_1)$ and $F_2(x_2)$:

$$f_X(x_1, x_2) = f_1(x_1)f_2(x_2)(1 + \alpha[2F_1(x_1) - 1][2F_2(x_2) - 1]) \qquad |\alpha| \leqq 1. \tag{1}$$

Now let the vector $X = (X_1, X_2)$ have the joint density function given by (1) where

$$f_i(x_i) = \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x_i^2), \qquad F_i(x_i) = \int_{-\infty}^{x_i} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}t^2)\, dt, \qquad i = 1, 2. \tag{2}$$

Thus X_1 and X_2 are normally distributed random variables with mean zero and variance one. We will show that the random variable $W = X_1 + X_2$ is not normally distributed except in the case $\alpha = 0$.

Let us assume that W is normally distributed. Since a normally distributed random variable is determined by its mean and variance, we know that under the assumption of normality the moment generating function of W will be

$$\phi_W(s) = \exp\left(\mu_W s + \sigma_W^2 \frac{s^2}{2}\right), \tag{3}$$

where

$$\mu_W = E(X_1) + E(X_2) = 0, \tag{4}$$

$$\sigma_W^2 = \text{Var}\,(X_1) + \text{Var}\,(X_2) + 2\,\text{Cov}\,(X_1X_2) = 2\left(1 + \frac{\alpha}{\pi}\right) \tag{5}$$

since $\text{Cov}(X_1X_2) = \alpha/\pi$. This covariance may be computed easily since the computation reduces to finding the expected value of the minimum of two standardized normal random variables. From (3), (4) and (5) we obtain

$$\phi_W(s) = \exp\left[\left(1 + \frac{\alpha}{\pi}\right)s^2\right]. \tag{6}$$

Next we will compute the moment generating function of W directly from (1) without assuming normality. Thus

$$\begin{aligned} E(\exp\,(sX_1 + sX_2)) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{sx_1}e^{sx_2}f_X(x_1, x_2)\,dx_1dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{sx_1}e^{sx_2}f_1(x_1)f_2(x_2)\,dx_1dx_2 \\ &\quad + \alpha\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{sx_1}e^{sx_2}[2F_1(x_1) - 1][2F_2(x_2) - 1]f_1(x_1)f_2(x_2)\,dx_1dx_2 \\ &= \psi(s)^2 + \alpha\left[\int_{-\infty}^{\infty} e^{sx_1}[2F_1(x_1) - 1]f_1(x_1)\,dx_1\right]^2, \end{aligned} \tag{7}$$

where $\psi(s) = e^{s^2/2}$ is the moment generating function of a normal random variable with mean zero and variance one.

Now

$$\int_{-\infty}^{\infty} e^{sx_1}[2F_1(x_1) - 1]f_1(x_1)\,dx_1 = -2\int_{-\infty}^{\infty} e^{sx_1}[1 - F_1(x_1)]f_1(x_1)\,dx_1 + \psi(s) \tag{8}$$

and

$$\begin{aligned} 2\int_{-\infty}^{\infty} e^{sx_1}[1 - F_1(x_1)]f_1(x_1)\,dx_1 &= \int_{-\infty}^{\infty}\int_{x_1}^{\infty}\left(\frac{1}{\pi}\right)\exp\{-\tfrac{1}{2}[x_1^2 + t^2 - 2sx_1]\}\,dtdx_1 \\ &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{\exp\{-\frac{1}{2}[x_1^2 + (v + x_1)^2 - 2sx_1]\}}{\pi}\,dvdx_1 \\ &= \int_{0}^{\infty} \frac{\exp\{-v^2/2 + (v - s)^2/4\}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\exp\{-[x_1 + (v - s)/2]^2\}}{\sqrt{\pi}}\,dx_1dv \end{aligned} \tag{9}$$

$$= 2 \exp\left\{\frac{s^2}{2}\right\} \int_0^\infty \frac{\exp\{-\frac{1}{2}[(v+s)^2/2]\}}{2\sqrt{\pi}}\, dv = 2\psi(s)\Phi\left(\frac{s}{\sqrt{2}}\right),$$

where

$$\Phi(y) = \int_y^\infty \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}t^2)\, dt.$$

Upon substitution of (9) and (8) into (7) we obtain

$$\begin{aligned} E[\exp\{s(X_1 + X_2)\}] &= [\psi(s)]^2 + \alpha[-2\psi(s)\Phi[s/\sqrt{2}] + \psi(s)]^2 \\ &= (\psi(s))^2[1 + \alpha(1 - 2\Phi(s/\sqrt{2}))^2]. \end{aligned} \tag{10}$$

If W were normally distributed (10) should agree with (6) for all s and all α for which (1) is defined, so that

$$(\psi(s))^2 \exp\left\{\frac{\alpha}{\pi} s^2\right\} = (\psi(s))^2\left[1 + \alpha\left(1 - 2\Phi\left(\frac{s}{\sqrt{2}}\right)\right)^2\right]. \tag{11}$$

For $\alpha = 0$, which corresponds to the case where X_1 and X_2 are independent, the equality holds. The expression in brackets on the right of (11) is bounded from above by $1+\alpha$, whereas $\exp[(\alpha/\pi)s^2]$ is unbounded, so the equality cannot hold for all s and α. In fact, for any $\alpha > 0$ we can select an s sufficiently large so that $\exp[(\alpha/\pi)s^2]$ is greater than $1+\alpha$ and for any $\alpha < 0$ we can select an s sufficiently small so that $\exp[(\alpha/\pi)s^2]$ is greater than $1+\alpha$. Hence the only value of α for which the equality holds is $\alpha = 0$.

Hudson Laboratories of Columbia University Contribution No. 188.

This work was supported by the Office of Naval Research under Contract Nonr-266(84), and by Contract DA 18-108-CML-788(A). Reproduction in whole or in part is permitted for any purpose of the United States Government.

References

1. T. W. Anderson, An Introduction to Multivariate Statistical Analysis, Wiley, New York, 1958, p. 25.

2. H. Cramèr, Mathematical Methods of Statistics, Princeton University Press, Princeton, N. J., 1946, p. 212.

3. E. J. Gumbel, Distribution à plusieurs variables dont les marges sont données, C. R. Acad. Sci., Paris, 246 (1958) 2717–2720.

AN APPROACH TO DARBOUX-STIELTJES INTEGRATION

O. Shisha, Aerospace Research Laboratories, Wright-Patterson AFB, Ohio

1. Let $-\infty < a < b < \infty$, let f be a real function, defined and bounded on $[a, b]$, and let ϕ be a function, monotone nondecreasing (and finite) on $[a, b]$. There are two (nonequivalent) ways for defining an elementary Stieltjes integral $\int_a^b f(x)d\phi(x)$. The function f is said to be Riemann-Stieltjes (RS) integrable

$d\phi$ over $[a, b]$ iff there exists a real number I having the following property. For every $\epsilon > 0$ there exists a positive δ, such that if

$$a = x_0 < x_1 \cdots < x_n = b, \quad x_{j-1} \leqq \xi_j \leqq x_j \quad (j = 1, 2, \cdots, n), \quad \max_{1 \leqq j \leqq n} (x_j - x_{j-1}) < \delta,$$

then $\left| I - \sum_{j=1}^{n} f(\xi_j)[\phi(x_j) - \phi(x_{j-1})] \right| < \epsilon$. If f is RS integrable $d\phi$ over $[a, b]$, then the above I is unique and is denoted (RS) $\int_a^b f(x)d\phi(x)$.

Consider the set of all sums of the form $\sum_{j=1}^{n} \mathrm{M}_j[\phi(x_j) - \phi(x_{j-1})]$ where

$$a = x_0 < x_1 \cdots < x_n = b, \quad \mathrm{M}_j = \sup_{x_{j-1} \leqq x \leqq x_j} f(x) \qquad (j = 1, 2, \cdots, n),$$

n taking all values $1, 2, 3, \cdots$. The greatest lower bound of this set is denoted $\overline{\int_a^b} f(x)d\phi(x)$. Similarly, $\underline{\int_a^b} f(x)d\phi(x)$ is the least upper bound of the set of all sums of the form $\sum_{j=1}^{n} m_j[\phi(x_j) - \phi(x_{j-1})]$ where

$$a = x_0 < x_1 \cdots < x_n = b, \; m_j = \inf_{x_{j-1} \leqq x \leqq x_j} f(x) \qquad (j = 1, 2, \cdots, n),$$

n taking all values $1, 2, 3, \cdots$. The function f is said to be Darboux-Stieltjes (DS) integrable $d\phi$ over $[a, b]$ iff $\overline{\int_a^b} f(x)d\phi(x) = \underline{\int_a^b} f(x)d\phi(x)$. If the last equality holds, the common value of both members is denoted (DS) $\int_a^b f(x)d\phi(x)$.

If f is RS integrable $d\phi$ over $[a, b]$ then f is also DS integrable $d\phi$ over $[a, b]$. The converse, however, does not hold (cf. [1], p. 250). If f is RS integrable $d\phi$ over $[a, b]$ then (RS) $\int_a^b f(x)d\phi(x) = (\mathrm{DS})\int_a^b f(x)d\phi(x)$.

2. The purpose of the present note is to give an alternative approach for the introduction of DS integration. What we do here is a generalization of [2], Section 45.

THEOREM. *A necessary and sufficient condition for f to be* DS *integrable $d\phi$ over $[a, b]$ is the existence of a unique real function $\Delta(\alpha, \beta)$ whose domain is the set of all ordered pairs (α, β), $a \leqq \alpha < \beta \leqq b$, and which satisfies:*

A. $\Delta(\alpha, \beta) + \Delta(\beta, \gamma) = \Delta(\alpha, \gamma)$ *whenever* $a \leqq \alpha < \beta < \gamma \leqq b$.

B. $$\left[\inf_{\alpha \leqq x \leqq \beta} f(x)\right][\phi(\beta) - \phi(\alpha)] \leqq \Delta(\alpha, \beta) \leqq \left[\sup_{\alpha \leqq x \leqq \beta} f(x)\right][\phi(\beta) - \phi(\alpha)]$$

whenever $a \leqq \alpha < \beta \leqq b$.

If there exists such a unique real function $\Delta(\alpha, \beta)$, then (DS) $\int_a^b f(x)d\phi(x)$ *is* $\Delta(a, b)$.

REMARKS. 1. Thus the Theorem provides definitions of DS integrability and of DS integral.

2. One easily checks that $\Delta(\alpha, \beta) \equiv \overline{\int_\alpha^\beta} f(x)d\phi(x)$ and $\Delta(\alpha, \beta) \equiv \underline{\int_\alpha^\beta} f(x)d\phi(x)$ have properties A and B.

3. It is known that if f is DS integrable $d\phi$ over $[a, b]$, then it is also so over every $[\alpha, \beta]$ $(a \leqq \alpha < \beta \leqq b)$.

Proof of the Theorem. Sufficiency. The uniqueness of Δ implies, by Remark 2, that $\overline{\int_a^b} f(x)d\phi(x) = \underline{\int_a^b} f(x)d\phi(x)$.

Necessity. First observe, that $\Delta(\alpha, \beta) \equiv (\mathrm{DS})\int_\alpha^\beta f(x)d\phi(x)$ satisfies conditions A and B. Suppose now that $\Delta(\alpha, \beta)$ is an arbitrary function as in the Theorem. Let $a \leqq \alpha < \beta \leqq b$. Then by properties A and B, whenever $\alpha = x_0 < x_1 \cdots < x_n = \beta$,

$$\sum_{j=1}^{n}\left[\inf_{x_{j-1}\leqq x\leqq x_j} f(x)\right][\phi(x_j) - \phi(x_{j-1})]$$
$$\leqq \Delta(\alpha, \beta) = \sum_{j=1}^{n} \Delta(x_{j-1}, x_j) \leqq \sum_{j=1}^{n}\left[\sup_{x_{j-1}\leqq x\leqq x_j} f(x)\right][\phi(x_j) - \phi(x_{j-1})].$$

Taking least upper and greatest lower bounds, we get

$$(\mathrm{DS})\int_\alpha^\beta f(x)\,d\phi(x) \leqq \Delta(\alpha, \beta) \leqq (\mathrm{DS})\int_\alpha^\beta f(x)\,d\phi(x).$$

Thus $\Delta(\alpha, \beta) \equiv (\mathrm{DS})\int_\alpha^\beta f(x)d\phi(x)$.

References

1. H. Kestelman, Modern theories of integration, 2nd rev. ed., Dover, New York, 1960.
2. C. de la Vallée Poussin, Intégrales de Lebesgue; Fonctions d'ensemble; Classes de Baire, Gauthier-Villars, Paris, 1916.

ON THE SPECTRAL THEOREM FOR COMPACT SYMMETRIC OPERATORS

A. D. Martin, late of Carnegie Institute of Technology

Since invariant subspaces of a symmetric operator occur in orthogonal pairs, the essential part of a proof of the spectral theorem for compact operators is the determination of the existence of at least one eigenvalue for every compact symmetric operator. This note presents a discussion of this matter with a slight twist in the usual treatment.

Let H be any real Hilbert space, and $x \cdot y$ the inner product of $x, y \in H$. Let T be any linear operator on H. We term (α, λ) an ***eigenpair*** of T if $\alpha \neq 0$ and $T\alpha = \lambda\alpha$. As usual, α is an eigenvector and λ is an eigenvalue of T. For any nonzero vector α in the null space of T, $(\alpha, 0)$ is an eigenpair of T. Thus an operator lacks eigenpairs only when its null space consists of the zero vector alone.

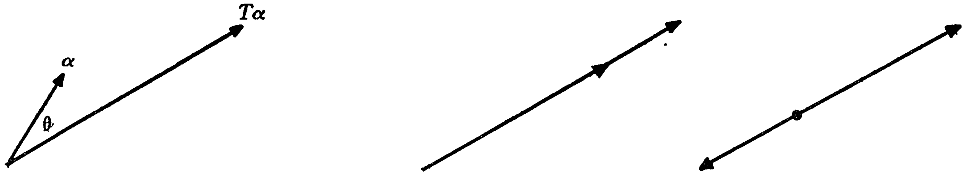


FIGURE 1

If the null space of T is zero then $\alpha\neq 0$ is an eigenvector of T when and only when the angle between α and $T\alpha$ is zero or π.

The angle θ between α and $T\alpha$ is given by the relation

$$\cos\theta = \frac{\alpha\cdot T\alpha}{|\alpha||T\alpha|}\,. \tag{1}$$

Thus α is an eigenvector of T if and only if the ratio $\alpha\cdot T\alpha/|\alpha|\ |T\alpha|$ is ± 1. This is the strong form of Schwarz's inequality:

$$-|\alpha||T\alpha| < \alpha\cdot T\alpha < |\alpha||T\alpha| \tag{2}$$

unless $T\alpha=\lambda\alpha$. This suggests the following well-known definitions:

$$\begin{aligned} \|T\| &= \sup_{|x|=1} |x\cdot Tx| \\ |T| &= \sup_{|x|=1} |Tx|, \end{aligned} \tag{3}$$

where $|x|$ is the length of x: $|x|=(x\cdot x)^{1/2}$. Either $\|T\|$ or $|T|$ may be infinite but Schwarz's inequality always gives

$$\|T\| \leqq |T|\,. \tag{4}$$

The following result is well known (see [1], Sec. 92, 93).

(5) Proposition. *If T is a compact operator then $\|T\|$ and $|T|$ are both finite and there are unit vectors α, β such that*

$$|\alpha\cdot T\alpha| = \|T\|, \qquad |T\beta| = |T|\,.$$

A consequence of (5) is the following important result.

(6) Theorem. *Let T be a compact operator such that $\|T\|=|T|$. Then T has at least one eigenpair. If α is a unit vector such that*

$$|\alpha\cdot T\alpha| = \|T\|$$

then (α, λ) is an eigenpair of T where $|\lambda| =\alpha\cdot T\alpha$.

Proof. The second statement of the theorem, together with Proposition (5), implies the first. The hypothesis gives

$$|\alpha\cdot T\alpha| = \|T\| = |T|\,. \tag{i}$$

Since α is unit we have by (3)

$$|T\alpha| \leqq |T| \tag{ii}$$

which combined with (i) gives $|\alpha\cdot T\alpha| \geqq |T\alpha|$. But by Schwarz's inequality $|\alpha\cdot T\alpha| \leqq |T\alpha|$ and therefore $|\alpha\cdot T\alpha| = |T\alpha| = |T\alpha|\ |\alpha|$. Thus the strong form of Schwarz's inequality gives $T\alpha=\lambda\alpha$ for some scalar λ. Thus (α, λ) is an eigenpair of T for which $\alpha\cdot T\alpha=\lambda$. The theorem is proved.

We now state the existence theorem for eigenpairs of symmetric compact operators. We denote by $[\alpha, \beta]$ the subspace generated by α and β.

(7) THEOREM. *Every compact symmetric operator has an eigenpair.*

Proof. Our demonstration has three stages: (i) If T is compact and symmetric then $\|T^2\| = |T^2|$ and hence T^2 has an eigenpair. (ii) If (α, λ) is an eigenpair of T^2 then $[\alpha, T\alpha]$ is an invariant subspace of T whose dimension is at most two. (iii) Every symmetric operator on a Hilbert space of dimension $\leqq 2$ has an eigenpair.

Statement (iii) is a trivial fact from linear algebra. Statement (ii) follows from the set inequality

$$T[\alpha, T\alpha] = [T\alpha, T^2\alpha] = [T\alpha, \lambda\alpha] \subset [\alpha, T\alpha].$$

Accordingly, (7) will follow from any proof of (i). The symmetry of T gives

$$\|T^2\| = \sup_{|x|=1} |x \cdot T^2x| = \sup_{|x|=1} Tx \cdot Tx = |T|^2.$$

Consequently, (i) will follow if we show that $|T^2| = |T|^2$. We have immediately $|T^2| \leqq |T|^2$, and

$$|T|^2 = \sup_{|x|=1} Tx \cdot Tx = \sup_{|x|=1} T^2x \cdot x \leqq \sup_{|x|=1} |T^2x| = |T^2|.$$

Thus $|T^2| = |T|^2$ and the theorem is proved.

Reference

1. F. Riesz and Béla Sz.-Nagy, Functional Analysis, Ungar, New York, 1955.

THE DENSITY OF CERTAIN CLASSES OF RATIONALS

PAUL SCHAEFER, State University of New York at Albany

In a recent issue of this MONTHLY, Lange and Thoro [2] showed that the set of all so-called Pythagorean rationals is dense in the set of all non-negative real numbers. Their first proof can be generalized in such a way as to demonstrate the density of other classes of rationals as well.

DEFINITION. *A positive rational number a/b is a P-rational if and only if the pair of integers a and b has property P.*

THEOREM. *Let $N(x, y)$ and $D(x, y)$ be polynomials in x and y which have integral coefficients and which are homogeneous of the same degree. Suppose that there exists a $t_0 \geqq 0$ such that $N(t_0, 1) = 0$ and for all $t \geqq t_0$, $N(t, 1) \geqq 0$ and $D(t, 1) > 0$. If for all positive integers x and y, with $x \geqq t_0 \cdot y$, $N(x, y)/D(x, y)$ is a P-rational, then the set of all such P-rationals is dense in the interval $[0, B)$, where*

$$B = \sup[N(t, 1)/D(t, 1) \mid t \geqq t_0].$$

Proof. Let $G(t)=N(t, 1)/D(t, 1)$. Then $G(t_0)=0$ and G is continuous for $t \geqq t_0$. If u and v are arbitrary real numbers such that $0 \leqq u < v < B$, then $G^{-1}[(u, v)]$ is a nonempty open set. Let p/q be any rational in $G^{-1}[(u, v)]$. Now, $G(p/q) = N(p, q)/D(p, q)$, since N and D are homogeneous of the same degree. Thus, the P-rational $N(p, q)/D(p, q)$ is between u and v. Since u and v are arbitrary in $[0, B)$, the set of all such rationals is dense in $[0, B)$.

In addition to the Pythagorean rationals a/b of [2], where P is the property that a^2+b^2 is the square of an integer, there are other classes of P-rationals to which the theorem applies.

EXAMPLE 1 [1, p. 229]. Consider the set of all P-rationals a/b where P is the property that a^2+b^2 is the cube of an integer. In this case, $N(x, y)=x^3-3xy^2$, $D(x, y)=3x^2y-y^3$, $t_0=\sqrt{3}$, and the set of all such rationals is dense in the set of all nonnegative numbers.

EXAMPLE 2 [1, p. 578]. If P is the property that a^3+b^3 is the square of an integer, then the set of all P-rationals a/b is dense in the set of all nonnegative numbers. Here $N(x, y)=x(x^3+y^3)$, $D(x, y)=y(x^3+y^3)$ and $t_0=0$.

In [1], one can find additional examples of Diophantine equations which can be used to define other classes of P-rationals to which the theorem applies.

References

1. L. E. Dickson, History of the Theory of Numbers, Vol. II (Diophantine Analysis), Stechert New York, 1934.
2. L. H. Lange, and D. E. Thoro, The density of Pythagorean rationals, this MONTHLY, 71 (1964) 664–665.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITORS: JOHN D. BAUM, Oberlin College, and JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

EFFECTING MATHEMATICS CURRICULUM CHANGE IN THE SECONDARY SCHOOL

JAMES HENKELMAN, University of Maryland

The extensive activity in mathematics education in the past decade has resulted in the publishing of lists of recommendations for secondary school mathematics and the writing of improved mathematics curriculum materials. This activity alone, however, will not result in widespread reforms in secondary school mathematics. Although it is generally recognized that there is a necessity to acquaint the teachers with the recommendations and to prepare them to

teach the new materials, there has been little careful analysis of the problem of effecting curriculum change in a school system through a program of intensive inservice training for the teachers.

The author, acting in the capacity of a consultant to a small city school system near Boston, Massachusetts, attempted to effect such a change. The status of mathematics education in this city was typical of that which the curriculum reform groups were attempting to correct. The textbooks in use did not reflect the spirit of the curriculum reforms, and the teachers were poorly prepared to teach the mathematics courses.

The activities of the program of curriculum improvement were kept intentionally flexible so that they could be designed to meet situations as they arose. The specific activities of the program were jointly planned with the administrative staff and the teachers of the schools involved. Considerable effort was concentrated on the establishment of a satisfactory relationship with each teacher and administrator.

An orientation meeting was held in the spring of 1962 to acquaint the fifteen secondary school mathematics teachers with the recommendations of the curriculum reform groups. During the 1962–63 school year, the teachers participated in an algebra inservice course. Demonstration classes were utilized in connection with the inservice course. During the second semester, the eighth grade teachers were organized into a team to introduce some new curriculum materials to accelerated students. The community was informed of the curriculum reforms through a demonstration class and an adult education workshop.

During the 1963–64 school year, new curriculum materials were introduced in selected classrooms. A program of algebra for accelerated students was initiated in the eighth grade with the prospect of a course of study leading to the advanced placement calculus course in the senior year in the high school. A geometry inservice course was offered to the high school teachers during the 1963–64 school year, and the junior high school teachers met separately in two teams to deal with the many facets of the new mathematics materials. The eighth grade team of teachers continued to make use of the large group instruction. In addition to this a variety of other supervisory techniques were utilized. The consultant worked in the classrooms with the teachers introducing new curriculum materials and taught a class of students daily. Formal contacts with the school system terminated at the close of the school year 1963–64.

Evaluative instruments seemed to indicate that the students using new curriculum materials improved their competence in the mathematics emphasized in the new materials; they also maintained their competence in the mathematics as measured on a test not reflecting curriculum reforms. Students using the new curriculum materials seemed to have about the same preferences and attitudes as other students, but they appeared to have a somewhat better perception of the nature of mathematics. The interest of the students in mathematics also increased during the program. The teachers generally showed subject mat-

ter improvement and increased interest in teaching mathematics. The curriculum reforms were generally accepted.

Two aspects of the program seemed to be important for gaining acceptance of the new curriculum programs. First, the consultant found it necessary to work directly with the poorly prepared teachers on the joint planning of lessons to show them the significance of the mathematics in the new materials. Team-teaching was an excellent organizational pattern within which to do this. Second, the teachers seemed to need the opportunity to be directly involved with the new curriculum materials in order to accept them [1].

Reference

1. J. H. Henkelman, A study of a program designed to effect mathematics curriculum change in the Saugus, Massachusetts, secondary schools (unpublished doctoral dissertation), Graduate School of Education, Harvard University, 1965.

THE NEW MATHEMATICAL LIBRARY

With the appearance of *Groups and their Graphs*, by Israel Grossman and Wilhelm Magnus, the total number of volumes available in Random House's New Mathematical Library reaches fourteen. This book is the latest in a series sponsored by the School Mathematics Study Group and written by professional mathematicians in order to make some important mathematical ideas interesting and understandable to a large audience of high school students and laymen. Each volume in the series contains problems, some of which may require considerable thought. Dr. Magnus is at New York University and Mr. Grossman is a teacher at Albert Leonard High School, New Rochelle, New York.

BRIEF COMMENT

In response to requests from readers of the MONTHLY, a new feature has been added to Mathematical Education Notes. Professor John D. Baum, of Oberlin College, has kindly agreed to review articles in other publications which are believed to be of interest to readers of the MONTHLY.

Mathematics for engineering students, W. G. BICKLEY, *The Mathematical Gazette*, 48 (1964) 379–383.

This deals with the problems of "How is mathematics to be taught to engineering students?" and "Who is going to teach mathematics to engineering students?" The author makes the point that if the mathematicians do not take the responsibility of teaching engineers, they will eventually lose the privilege of teaching engineers the sorts of mathematics that will be most useful to them. He points out that "there are only two really important things in teaching mathematics to engineers—to remember that they *are* engineers, and to teach them *mathematics.* If you can speak their language, and look at mathematics through their eyes, you will get a satisfying response, you will gain their con-

fidence, and you will then be able to lead them into some abstract, but interesting, fields of mathematics where the ultimate application may not be immediately apparent."

He makes the further point, which has occupied teachers of mathematics to engineers in this country that, "in the end, it is not the number of tricks which the student has learned, but rather the understanding of the concepts and the awareness of the relevance of the techniques, which is important—and, finally, his approach to the new learning which an encounter with a new problem may demand." The article should make valuable reading for anyone engaged in the mathematical training of engineers.

Which subjects in modern mathematics and which applications in modern mathematics can find a place in programs of secondary school instruction? J. G. KEMENY, *L'Enseignement Mathématique*, 10 (1964) 152–176.

This article is a summary of 21 national reports by subcommissions of the International Commission on Mathematical Instruction. The reports are beautifully summarized and the article makes fascinating reading, showing as it does the extent to which modern mathematics is being taught in schools throughout the world. "There seems to be fairly general agreement that some basic concepts from set theory and logic should be introduced, that geometry should be modernized, that some elements of modern algebra be introduced, and that probability and statistics are suitable for high school teaching. Even more important is the general agreement that much of traditional mathematics should be taught from a modern point of view."

The greatest hurdle to the accomplishment of the above objectives seems to be the lack of trained teachers and the absence of suitable text materials, although efforts are being made to overcome both of these problems. The author recommends that the following problems be investigated: (1) How can the teaching of applied mathematics be modernized in the schools? (2) To what extent should high school mathematics be axiomatized? and (3) How much probability theory should be introduced into the school curriculum and how should this be done?

Five views of the "New Math," E. E. MOISE, A. CALANDRA, R. B. DAVIS, M. KLINE, and H. M. BACON, Occasional Paper Number Eight, *Council for Basic Education*, Washington, D. C.; and *Newsweek*, Education Column, May 10, No. 19, 65 (1965) 112–117.

The five views of the "new math" consist of an article by Moise, which in abbreviated form appeared in November 1964 and four responses to this article. Although Calandra's response seems largely polemical, it suggests wisely that any curricular change made in school mathematics ought to be made with care and informed caution. Davis' response seems generally favorable toward the newer programs, while Kline's, though somewhat more carefully reasoned than Calandra's, does not seem to advance any reasons for his opposition to the new

programs beyond personal taste. Kline also promises programs more in line with his thinking, but is not explicit as to who is developing these. Bacon's response is the most carefully thought out and seems generally to be one of cautious optimism; it is worth quoting his concluding paragraph: "General agreement on exactly what should constitute the content and style of school mathematics programs will never be achieved. It is well that it should not be. Dissatisfaction with current practice is what leads to progress and improvement. For a great many years we have had an abundance of the former; it is heartening in our times to observe some, and to hope for more, of the latter."

The article in Newsweek is a review of the above mentioned "occasional paper." It seems to concentrate attention largely on the more controversial areas of disagreement.

Copies of the "occasional paper" are available from the Council For Basic Education, 725 Fifteenth St., Washington, D. C. 20005 at a cost of 25 cents each, or 20 cents each in lots of twenty or more.

Reflections on the Teaching of Mechanics, M. H. STONE, *The Mathematics Student* (Published by the Indian Math. Soc.) Nos. 3 and 4, July–December, 32 (1964).

Although the teaching of courses of mechanics in departments of pure mathematics has become rather rare in recent years in the United States, it seems apparent from the above article that such is not necessarily the case elsewhere. The article has interesting implications for the teaching of mechanics itself, and for the teaching of mathematics for students of the natural sciences. Not only is it made clear how strongly the development of certain branches of mathematics has been motivated by physical problems, but it is also stressed how a large variety of mathematical tools is necessary to attack physical problems. In conjunction with this, the following is worth quoting, "From the foregoing considerations we may conclude that the teaching of mechanics should involve both experimental and axiomatic treatments of a number of different models of axiom systems for the material world. These treatments necessarily involve knowledge of certain mathematical disciplines, including geometry, analysis, vector algebra, and probability theory. The teaching of mechanics needs to be adequately co-ordinated with these parts of mathematics."

Using P(rogramed) I(nstruction) in Mathematics Curriculum Development: The UICSM Project, O. ROBERT BROWN.

School Mathematics Study Group Programed Learning Project, LEANDER W. SMITH. Both appearing in *Programed Instruction*, December, No. 3, 4 (1964).

Both the above articles deal with the development of teaching machine programs for mathematical instruction at the school level, largely, if not entirely, in the secondary school. From both of the above it is clear that there is a clear secondary gain in writing a program. For example, Brown writes, "This

work . . . is a prime example of the new methodology of programed instruction assisting in the implementation of curriculum reform." Smith has this to say, "More significantly, the SMSG Programed Learning Project has brought about a thorough rethinking of the mathematical content and pedagogy in a first course in algebra. It has resulted in the production of test items and analysis of content with revision based on empirical data rather than on educated hunches."

Both of the above articles describe the materials which were developed, and the longer one (Brown) does draw some conclusions from the experiences had with the material. It seems evident that the development of programmed material is a much slower process than the development of regular text materials and that many more revisions and rethinkings are necessary to arrive at a satisfactory end product.

It is also interesting to see some of the problems mentioned by May in the CEM Study, "Programed Learning and Mathematical Education" recurring in these articles. Brown says, "a choice must be made between using the materials as the sole agents of instruction for research on particular variables or using them with greater effectiveness by letting the teacher instruct as he sees fit."

Programed learning and mathematical education, KENNETH O. MAY, *CEM Study* (Committee on Educational Media of the Mathematical Association of America) 1965.

The above article, contained in a pamphlet of 24 pages, was sent to each member of the Association; however, it is felt that it would be wise to call attention to it here as well. It is to be emphasized first of all, that the scope of the article is restricted strictly to college level mathematics, and second that the views expressed are strictly those of the author and do not carry the endorsement of the committee or the Association.

Upon first reading this study one comes away with the notion that the author finds little if anything to commend programmed materials for use in college mathematics instruction. This is, however, a false impression, and one is advised to read the summary with considerable care, for it is apparent that there are many areas in which programmed materials can be used effectively and to advantage. It appears that the author's main conclusion can be stated fairly thus: Programmed materials do not in and of themselves constitute an adequate or effective medium for mathematical instruction at the college level; they can be used to advantage, however, in a variety of ancillary roles.

After a definition of educational programming, the author investigates the claims for programmed materials, and proceeds to demolish the claims one by one. Despite this, in a later section (p. 20) the author says, "In sum we need a wider variety and larger quantity of auxiliary materials to assist students in their individual study outside of class. These auxiliaries need not be programmed according to any existing style, but they must offer more programming than the usual text. They should all have a double purpose: to help the student master

the material at hand, and to help him learn to master such material with less outside assistance. Programs should program themselves out of the student's life."

Copies of the above study are available from the Committee on Educational Media, PO Box 2310, San Francisco, California 94126.

Numerical analysis vs. mathematics, R. W. HAMMING, *Science*, April 23, No. 3669, 148 (1965) 473–475.

This is a most interesting article and points up clearly some deep-lying differences in aim between mathematics and numerical analysis and the differences which must consequently exist in the education of mathematicians as opposed to numerical analysts. The principal ideas of the article are best summarized in the author's own words, where for "noise" we read "small uncertainties in the initial data":

"I hope I have shown not that mathematicians are incompetent or wrong but why I believe that their interests, tastes and objectives are frequently different from those of practicing numerical analysts . . . I hope I have also shown that most of the 'art form' of mathematics consists of delicate 'noise-free' results, while many areas of applied mathematics, especially numerical analysis, are dominated by noise. Again, in computing the *process* is fundamental, and rigorous mathematical proofs are often meaningless in computing situations. Finally, in numerical analysis, as in engineering, choosing the right model is more important than choosing the model with the elegant mathematics."

Besides its inherent interest, the article has many implications for the education of people who will use the mathematics they learn in the sense of numerical analysis rather than in the sense of pure mathematics.

Progress report on the Mathematics Revolution, EVELYN SHARP, *Saturday Review*, March 20, No. 12, 48 (1965) 62 ff.

This article, written by a mathematics teacher at Holland Hall High School, Tulsa, Oklahoma, who is also the author of *A Parent's Guide to the New Mathematics*, is a brief report on the current status of the new mathematics in the school curriculum. It gives a good overview of individual and group projects presently being pursued. There is some discussion of the problems encountered in changing from more traditional programs; the following quotation gives a good notion of the flavor of the article:

"What works best is a changeover that is gradual, sound, and thorough. Crash programs run into difficulties in any line. A middle of the road approach, fusing the old math with the new, gives the best results. Of course, the middle keeps moving over—in some quarters SMSG is now regarded as conservative."

This would make an excellent article for students who are preparing to teach school mathematics to read. It gives them a fair and realistic picture of the present state of affairs.

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before February 28, 1966.

E 1815. *Proposed by D. M. Paine, Wells College, Aurora, N. Y.*

Can $2n$ people be paired differently on $2n-1$ occasions, so that each is paired with each other on exactly one occasion?

E 1816. *Proposed by J. Garfunkel, Brooklyn, N. Y.*

For $n>1$, prove

$$(n!)! > [(n-1)!]^{n!}.$$

E 1817. *Proposed by C. F. Blackmore, University of Arkansas*

Let $r=(2n+1)^k$, where n and k are nonnegative integers. Show that for any integer a, $a^r+(a+1)^r+\cdots+(a+2n)^r$ is always divisible by $(2n+1)^{k+1}$.

E 1818. *Proposed by I. J. Schoenberg, University of Wisconsin*

Let $A_0A_1\cdots A_7A_0$ be a convex octagon in the plane having all its angles equal. Show that, if all the sides A_jA_{j+1} $(j=0, 1, \cdots, 7; A_8=A_0)$ are commensurable, then opposite sides must be equal in pairs, e.g. $A_jA_{j+1}=A_{j+4}A_{j+5}$ $(j=0, 1, 2, 3)$.

E 1819. *Proposed by V. K. Rohatgi, University of Alberta*

Let p, q, k be positive integers such that $pk<q$. Consider a sequence consisting of p plus ones and q minus ones in a random arrangement. Find the probability that at some point in the sequence the number of minus ones up to that point is less than k times the number of plus ones up to that point.

E 1820. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Show that the relations $a+b+c=p$, $bc+ca+ab=q$, $a<b<c$, $D=p^2-3q\geqq 0$, where a, b, c, p, q are reals, imply that

$$\tfrac{1}{3}(p-2\sqrt{D})\leqq a<\tfrac{1}{3}(p-\sqrt{D})<b<\tfrac{1}{3}(p+\sqrt{D})<c\leqq\tfrac{1}{3}(p+2\sqrt{D}).$$

E 1821. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

By means of the Heaviside function, $H(x)=0(x<0)$, $H(0)=\frac{1}{2}$, $H(x)=1$ $(x>0)$, express the following function in a single formula:

$$\begin{array}{ll} f(x)=f_1(x)\ \ (x<a_1), & f(a_1)=b_1,\\ f(x)=f_2(x)\ \ (a_1<x<a_2), & f(a_2)=b_2,\\ \cdots\cdots\cdots\cdots & \cdots\\ f(x)=f_n(x)\ \ (a_{n-1}<x<a_n), & f(a_n)=b_n,\\ \multicolumn{2}{c}{f(x)=f_{n+1}(x)\qquad (x>a_n),} \end{array}$$

where a_k and b_k $(k=1,2,\cdots,n)$ are real numbers with $a_1<a_2<\cdots<a_n$.

E 1822. *Proposed by Necdet Ucoluk, Purdue University*

Let A, A_1 and B, B_1 be any two pairs of given points in the plane. Consider the locus of points N such that the angles ANA_1 and BNB_1 (with measures having absolute values α and β respectively) satisfy the equation $\alpha=k\beta$, where k is a given positive real number. (a) Determine the differentiability properties of this locus, and (b) when the tangent line exists give a geometric procedure (finite) for its construction.

E 1823. *Proposed by W. C. Brady, Washington and Jefferson College*

Show that

$$\begin{vmatrix} (a_0+b_0)^n & (a_0+b_1)^n & \cdots & (a_0+b_n)^n\\ (a_1+b_0)^n & (a_1+b_1)^n & \cdots & (a_1+b_n)^n\\ \cdots & \cdots & \cdots & \cdots\\ (a_n+b_0)^n & (a_n+b_1)^n & \cdots & (a_n+b_n)^n \end{vmatrix} = (-1)^{n(n-1)/2}\prod_{j=0}^{n}\binom{n}{j}\prod_{\substack{j<i\\ i=0}}^{n}(a_i-a_j)(b_i-b_j).$$

E 1824. *Proposed by Douglas Lind, Falls Church, Va.*

Let $p_1, p_2, \cdots, p_n$ be n primes, not necessarily distinct, let $p^{(n)}$ be the product of these primes, and let p' be the integer $p'=p^{(n)}(1-1/p_1-1/p_2-\cdots-1/p_n)$. Prove that

$$\frac{x^{p_n}}{p_n}+\frac{x^{p_{n-1}}}{p_{n-1}}+\cdots+\frac{x^{p_1}}{p_1}+\frac{p'x}{p^{(n)}}$$

is always an integer for integral x.

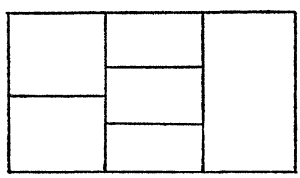
SOLUTIONS OF ELEMENTARY PROBLEMS

Planar Maps of Convex Countries

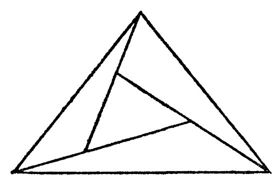
E 1726 [1964, 912]. *Proposed by Daniel I. A. Cohen, Princeton University*

Prove (or disprove): every planar map of convex countries can be colored with three colors.

I. *Solution by David Cohen, 9th Grade Student, Central High School, Philadelphia, Pennsylvania.* The following map (a) of convex countries needs four colors, so the theorem is false.



(a)



(b)

II. *Solution by Harry M. Gehman, SUNY at Buffalo.* The above map (b) disproves the statement even if "convex" is replaced by "triangular".

Comment by M. S. Klamkin, University of Minnesota. A map is *normal* if exactly three boundaries meet at each vertex, and a necessary and sufficient condition that a normal map be properly colorable with three colors is that each of its countries have an even number of boundaries. [L. I. Golovina and I. M. Yaglom, *Induction in Geometry*, Heath (1963), page 33, or E. B. Dynkin and V. A. Uspenskii, *Multicolor Problems*, Heath (1963), page 23].

Comment by R. F. Jolly, University of California, Riverside, California. The corresponding question for four colors, instead of three, is considerably more difficult. More tractable would be the following problem: prove that every planar map of triangular countries can be colored with four colors. Notice that this does not follow directly from the known result that every regular map containing at most one region of more than 6 edges can be colored in four colors, since the triangular boundaries could contain many edges.

Also solved by R. G. Albert, J. W. Baldwin, Robert Bart, D. A. Blaeuer, Andreas Blass, Carolyn Brauer, Emmeline G. de Pillis, W. G. Dotson, Jr., L. L. English, N. J. Fine, P. K. Garlick, Michael Goldberg, Mary L. Gomes, Faith M. Hammett, D. M. Hancasky, Robert Harvey, J. E. Homer, Jr., R. T. Hood, R. F. Jackson, R. A. Jacobson, Leroy Junker, Roman Kaluzniacki, M. S. Klamkin, Kenneth Kramer, E. S. Langford, F. Leuenberger (Switzerland), Robert McGuigan, Joseph Malkevitch, Norman Miller, J. B. Muskat and T. A. Van Wormer (jointly), J. C. Nichols, L. E. Pursell, G. P. S., Robin Sibson (England), J. L. Sieber, J. R. Swenson, Judith E. Tomlinson, B. R. Toskey, Alan Weinstein, B. J. Winter, and the proposer.

The proposer intended that rectangular (and not arbitrarily convex) countries be considered, and solution (a) above furnishes a disproof of this modified proposal.

An Algebraic Equation

E 1727 [1964, 912]. *Proposed by Paul Machuca, Fairleigh Dickinson University, Teaneck, New Jersey*

If the a_i are real and distinct show that the following equation is true for n real values of x:

$$\frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} = n.$$

Solution by Jim Campbell, Northeast Louisiana State College, Monroe, Louisiana. Consider the function y defined by

$$y(x) = \frac{a_1}{a_1 - x} + \frac{a_2}{a_2 - x} + \cdots + \frac{a_n}{a_n - x} - n.$$

Suppose without loss of generality that $0 < a_1 < a_2 < \cdots < a_n$. (Note: The equation as stated might have $a_i = 0$ for some i, in which case there would not be n roots.)

Now since $y(x) \to -\infty$ as $x \to a_i +$ and $y(x) \to +\infty$ as $x \to a_i -$, and y is continuous at each point except the a_i's, the equation $y(x) = 0$ has at least $n-1$ distinct real roots. But it has n roots since it is equivalent to a polynomial equation of degree n. Since the remaining root cannot be complex, $y(x) = 0$ for n real values of x, and they are distinct if 0 is not a repeated root.

Also solved by J. C. Abad, A. N. Aheart, R. G. Albert, W. A. Al-Salam and A. A. Gioia (jointly), E. B. Anders, J. W. Baldwin, Robert Bart, M. A. Bershad, J. L. Brown, Jr., J. A. Burslem, Victor Camillo, Leonard Carlitz, M. M. Chawla and K. M. Das (jointly) (India), Huseyin Demir (Turkey), M. S. Demos, Bob De Vore, Kenneth Dieter, W. G. Dotson, Jr., N. J. Fine, P. M. Flynn, John Fournier, Philip Fung, Michael Goldberg, S. H. Greene, D. M. Hancasky, E. R. Hansen, B. A. Hausmann, S. J., J. M. Horner, S. P. Hoyle, Jr., R. F. Jackson (two solutions), R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly), Roman Kaluzniacki, P. G. Kirmser, M. S. Klamkin, Kenneth Kramer, E. S. Langford, Lawrence Lessner, J. E. MacDonald, Jr., E. L. Magnuson, D. C. B. Marsh, M. R. Meck, C. B. A. Peck, George Purdy (England), Judith Richman, J. S. Rubie, G. P. Speck, J. J. Schneider, Robin Sibson (England), Robert Spira, Sidney Spital, R. A. Struble, R. L. Syverson, L. Y. L. Tong, B. R. Toskey, C. A. Tsonis, Van de Vyle, Simon Vatriquant (Belgium), Charles Wexler, Clement Winston, Theodore Zerger, and the proposer.

A Function and Its Derivative Implicitly Defined

E 1728 [1964, 912]. *Proposed by Jet Wimp, Midwest Research Institute*

For $t \geqq 0$, $0 < a \leqq 1$, let $f(t)$ be defined by

$$t = \frac{1}{\sqrt{2a}} \int_1^{f(t)} \frac{dx}{(\ln x - \ln a)^{1/2}} \cdot$$

Show that

$$t = \frac{1}{a} \int_0^{f'(t)} e^{x^2/2a}\, dx.$$

Solution by Miltiades S. Demos, Drexel Institute of Technology, Philadelphia, Pennsylvania. The problem is incorrect as stated, so we consider instead the function $f(t)$ defined for $t \geqq 0$ by

$$t = c \int_b^{f(t)} \frac{dx}{\sqrt{(\ln x - \ln a)}},$$

where $c \neq 0$ and $b \geqq a$. Since the integrand is nonnegative, $f(0) = b$. Differentiating with respect to t gives $1 = cf'(t)/\sqrt{(\ln f(t)/a)}$, whence $f'(0) = (1/c)\sqrt{\ln (b/a)}$ and $f(t) = a \exp\{c^2 f'^2(t)\}$. Differentiating again leads to $1 = 2ac^2 f''(t) \exp\{c^2 f'^2(t)\}$ since $f'(t) \neq 0$. But then

$$t = 2ac^2 \int_0^t f''(v) \exp\{c^2 f'^2(v)\} dv = 2ac^2 \int_{(1/c)\sqrt{\ln(b/a)}}^{f'(t)} e^{c^2 y^2} dy.$$

Also solved by A. Balfour (Scotland), M. M. Chawla and K. M. Das (jointly) (India), W. G. Dotson, Jr., A. C. Fang, A. D. Fine, B. A. Fusaro, Stuart Goff, S. H. Greene, Cornelius Groenewoud, Stephen Hoffman, J. M. Horner, R. F. Jackson, R. A. Jacobson, M. S. Klamkin, E. S. Langford, Lawrence Lessner, D. C. B. Marsh, S. M. Robinson, G. P. Speck, P. A. Scheinok, Richard Sinkhorn, Sidney Spital, R. A. Struble, R. E. Whitney, Dale Woods, and the proposer.

Arithmagic Squares

E 1729 [1964, 912]. *Proposed by Azriel Rosenfeld, The Budd Company and the University of Maryland*

A 3×3 *arithmagic* square is an arrangement of nine of the digits $0, 1, \cdots, 9$ into a 3×3 matrix such that if the rows and columns are regarded as three-digit numbers, then the sum of the first two rows is equal to the third row, and similarly for the columns. Prove there is only one such arithmagic square.

Solution by R. B. Eggleton, Avondale College, Cooranbong, N.S.W., Australia. Let

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

with elements all *distinct* digits. If M is arithmagic, the row and column requirements respectively lead to

$$100(a + d - g) + 10(b + e - h) + (c + f - i) = 0$$
$$100(a + b - c) + 10(d + e - f) + (g + h - i) = 0,$$

and the difference of these two equations is

$$(1) \qquad 90(d - b) + 101(c - g) + 11(f - h) = 0.$$

Thus $f - h \equiv g - c \pmod{90}$. But the sum of any two elements of M is positive

and not greater than $9+8=17$, so the congruence implies $c+f=g+h$. Then (1) implies $d-b=f-h=g-c$. Without loss of generality we can set $d-b>0$. Then the conditions to be satisfied are

(i) $c+f=g+h\equiv i \pmod{10}$, with $c<g$ and all five of c, f, g, h, i different, the first four being nonzero (so $5\leqq c+f\leqq 15$ is necessary);

(ii) $b+f=d+h$, with $b<d$, and both b and d different from c, f, g, h, i; and

(iii) $b<c$, $d<g$, and $a<\min(c, g)$ since $a+b\leqq c$, $a+d\leqq g$.

Systematic enumeration of the possibilities consistent with these conditions enables every arithmagic square to be determined. In fact, contrary to the assertion of the problem, there are four such squares (and, of course, their transposes):

$$M_1=\begin{bmatrix}1&4&6\\5&8&3\\7&2&9\end{bmatrix}\quad M_2=\begin{bmatrix}3&2&6\\5&8&4\\9&1&0\end{bmatrix}\quad M_3=\begin{bmatrix}3&4&8\\5&6&2\\9&1&0\end{bmatrix}\quad M_4=\begin{bmatrix}7&1&8\\2&3&6\\9&5&4\end{bmatrix}.$$

Moreover, if the digit 0 is allowed to appear as the first digit of a "three-digit number," then there are four other arithmagic squares:

$$M_5=\begin{bmatrix}0&2&3\\5&9&4\\6&1&7\end{bmatrix}\quad M_6=\begin{bmatrix}0&2&3\\6&9&5\\7&1&8\end{bmatrix}\quad M_7=\begin{bmatrix}0&4&5\\8&7&6\\9&2&1\end{bmatrix}\quad M_8=\begin{bmatrix}0&6&7\\8&5&4\\9&2&1\end{bmatrix}.$$

Also solved by A. N. Aheart, J. W. Baldwin, J. F. Dillon, R. A. Jacobson, J. D. E. Konhauser and E. L. Magnuson (jointly), E. S. Langford, D. C. B. Marsh, D. C. Mayne, Donald Quiring, J. J. Schneider, B. R. Toskey, and the proposer.

Arithmetic Progression of Integers Prime to n

E 1730 [1964, 912]. *Proposed by Robert Spira, Duke University*

Let $(a, b)=1$. If n is any integer, there is an x such that $(ax+b, n)=1$.

I. *Solution by Erwin Just and Norman Schaumberger, Bronx Community College, New York City.* If $(b, n)=1$, then $(an+b, n)=1$. If $(b, n)>1$, then $(a, n)=1$ since $(a, b)=1$, and so $(a(n+1)+b, n)=1$.

II. *Solution by Douglas M. Moore, Davidson College, Davidson, North Carolina.* Factor a, b, and n into prime factors. Let x be the product of all the prime factors of n which are not factors of a or of b. If there are no such factors let $x=1$.

Now every prime factor of n is a factor of either ax or of b but not of both and thus not of $ax+b$; therefore, $(ax+b, n)=1$.

We note that there are an infinite number of values for x such that $(ax+b, n)=1$, and they can be obtained by repeatedly multiplying the value defined above by any prime integer which is not a factor of both b and n.

III. *Solution by Charles Wells, Durham, North Carolina.* Let $d=(a, n)$. The problem is equivalent to showing that the equation $ax+b\equiv c \pmod{n}$ has a

solution x for at least one c in a reduced residue system (mod n). But such an equation has a solution if and only if $d|c-b$. Now $(d, b)=(d, c)=1$ for any c in a reduced residue system (mod n). Since $d|n$, a reduced residue system (mod n) contains a reduced residue system (mod d), which in turn contains an element $c\equiv b$ (mod d).

IV. *Solution by Vis Upatisringa, Oregon State University.* If p is a prime divisor of (a, n), then $p\nmid(ax+b)$ for each integer x. If P is the product of the prime divisors of n each of which fails to be a divisor of a, the Chinese Remainder Theorem gives a solution x of $ax+b\equiv 1$ (mod P).

V. *Solution by Allan Chuck and Peter Goldstein, San Francisco, California.* The result follows immediately from Dirichlet's Theorem: If $(a, b)=1$, then there are infinitely many primes of the form $ax+b$ with x a positive integer. Take any such prime greater than each prime divisor of n.

Also solved by J. C. Abad, A. N. Aheart, R. G. Albert, W. A. Al-Salam and A. A. Gioia (jointly), A. Ariff (Malaya), J. W. Baldwin, Robert Bart, M. A. Bershad, D. M. Bloom, Carolyn Brauer, Adelaide J. Brooks, J. L. Brown, Jr. (two solutions), J. D. Buckholtz, J. A. Burslem, Victor Camillo, Jim Campbell, John Christopher, D. I. A. Cohen, Jon Cole, Huseyin Demir (Turkey), Tom De Vore, G. W. Dinolt, D. M. Dribin, R. B. Eggleton (Australia), L. L. English, Philip Fung, P. K. Garlick, J. L. Gieser, Michael Goldberg, Jerry Goodman, R. N. Gupta (India), D. M. Hancasky, Ned Harrell, B. A. Hausmann, S. J., D. E. Helton, J. E. Homer, Jr., R. A. Jacobson, Erwin Just and Norman Schaumberger (jointly, second solution), Roman Kaluzniacki, M. S. Klamkin, J. D. E. Konhauser, E. S. Langford, Sim Lasher, Lawrence Lessner, C. C. Lindner, E. L. Magnuson, D. C. B. Marsh, Norman Miller, J. B. Muskat, R. N. Nelson, C. P. Peterson, R. W. Prielipp, T. L. Reynolds, F. W. Roush (two solutions), D. R. Shoemaker, Robin Sibson (England), Sidney Spital, L. Y. L. Tong, A. M. Vaidya, Van de Vyle, Simon Vatriquant (Belgium), Emanuel Vegh, Charles Wells (second solution), Charles Wexler, P. O. Wood, Jr., David Zeitlin, and the proposer.

Aheart and Brown found E1730 as Exercise 7 on page 54 of W. J. Leveque, *Elementary Theory of Numbers*, Addison-Wesley (1962); Garlick and Gieser located it as Exercise 8 on page 32 of Niven and Zuckerman, *An Introduction to the Theory of Numbers*, Wiley (1960); and Wells offers page 12 of L. E. Dickson, *Introduction to the Theory of Numbers*, Dover (1957). Finally, Muskat points out that it is Lemma 2 on page 32 of Harvey Cohn, *A Second Course in Number Theory*, Wiley (1962).

A Geometric Progression of Exponents

E 1731 [1964, 1041]. *Proposed by Gus Mavrigian, Youngstown University*

Let $x>0$, and let $x_1=[x(x)^b]^a$, $x_2=[x(x\cdot x_1)^b]^a$, $\cdots$, $x_n=[x(x\cdot x_{n-1})^b]^a$. Determine the function $f(x)=x_n$ and find $\lim_{n\to\infty} x_n$ if $ab<1$.

Solution by Vijay H. Arakeri, Student, Utah State University, Logan, Utah. It is easily seen that $x_1=x^{a(b+1)}$, $x_2=x^{a(b+1)(1+ab)}$, and $x_3=x^{a(b+1)(1+ab+a^2b^2)}$. It then follows by mathematical induction that

$$x_n = x^{a(b+1)(1+ab+a^2b^2+\cdots+a^{n-1}b^{n-1})}.$$

The exponent here converges (to $a(b+1)/(1-ab)$) if and only if $|ab|<1$. For such ab, then, $\lim_{n\to\infty} x_n=x^{a(b+1)/(1-ab)}$ because of the continuity of $g(t)=x^t$.

Also solved by J. C. Abad, A. N. Aheart, W. A. Al-Salam, M. A. Bershad, J. A. Burslem, F. A. Butter, Jr., Jim Campbell, P. L. Chessin, John Christopher, D. I. A. Cohen, Ragnar Dybvik (Norway), E. S. Eby, J. D. Elterich, A. C. Fang, N. J. Fine, Gary Ford, A. A. Gioia, Michael Goldberg, Ernest Gore, F. R. Gray and R. L. Price (jointly), S. H. Greene, Cornelius Groenewoud, Joel Gyllenskog, H. A. Heckart, Stephen Hoffman, J. M. Horner, R. F. Jackson, R. A. Jacobson, R. E. Jones, Jr., Erwin Just and Norman Schaumberger (jointly), Roman Kaluzniacki, Kenneth Kramer, E. S. Langford, J. D. Lockridge, D. C. B. Marsh, T. M. Morrisette, C. B. A. Peck, Walter Penney, J. M. Perry, R. N. Pesut, C. R. Rao, G. L. N. Rao (India), Simeon Reich (Israel), V. K. Rohatgi, Fred Rosenblum, H. D. Ruderman, P. A. Scheinok, R. N. Schneider, R. S. Seamons, R. Shantaram, Robin Sibson (England), Arnold Singer, Richard Sinkhorn, Mitchell Snyder, Al Somayajulu, R. P. Soni, Sidney Spital, C. A. Tsonis, Van de Vyle, Simon Vatriquant (Belgium), C. R. Wall, W. C. Waterhouse, J. E. Wynn, K. L. Yocom, David Zeitlin, and the proposer.

An Application of the Arithmetic-Geometric Mean Inequality

E 1732 [1964, 1041]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

If $x_i > 0$ $(i=1, 2, \cdots, n)$, show that

$$\sum_{j=1}^{n}\left\{\prod_{i=1}^{n} x_i\left[\left(\sum_{k=1}^{n} x_k\right) - x_j\right]\Big/ x_j\right\} \geqq n(n-1)\prod_{i=1}^{n} x_i.$$

Editor's comment. The term $\{\cdots\}$ in the left member above is of the form $\prod_{i=1}^{n} x_i a/b$, and the usual convention regarding the pi-notation requires it to be taken as $\prod_{i=1}^{n}(x_i a/b)$. On the other hand, rules of arithmetic (multiplication takes precedence over division takes precedence over addition takes precedence over subtraction) suggest that it be taken as $[\prod_{i=1}^{n}(x_i a)]/b$, in which case the best lower bound of the left member is $n(n-1)^n(\prod_{i=1}^{n} x_i)^2$. In any event, $\{\cdots\}$ above should never be construed as $(\prod_{i=1}^{n} x_i)a/b$, and yet this is precisely what the proposers had in mind. We therefore offer solutions using the first and third of these interpretations.

I. *Solution by Sidney Spital, California State Polytechnic College, Pomona, California.* Denoting the left member of the asserted inequality by S and interpreting $\{\cdots\}$ in S as

$$\prod_{i=1}^{n}\left[x_i\left(\sum_{k=1}^{n} x_k - x_j\right)\Big/ x_j\right],$$

we will prove that $S \geqq n(n-1)^n \prod_{i=1}^{n} x_i$.

Recalling that the geometric mean of positive numbers is no greater than their arithmetic mean, we see that

$$\left(\sum_{k=1}^{n} x_k - x_j\right)\Big/ x_j = \sum_{\substack{k=1\\k\neq j}}^{n} x_k/x_j \geqq \frac{n-1}{x_j}\left(\prod_{\substack{k=1\\k\neq j}}^{n} x_k\right)^{1/(n-1)} = \frac{n-1}{x_j^{n/(n-1)}} P^{1/(n-1)}$$

with $P = \prod_{k=1}^{n} x_k$. Applying the same inequality again gives

$$S \geqq \sum_{j=1}^{n} \frac{(n-1)^n P}{x_j^{n^2/(n-1)}} P^{n/(n-1)} \geqq n(n-1)^n P P^{n/(n-1)} \left[\prod_{j=1}^{n} 1/x_j^{n^2(n-1)} \right]^{1/n}$$

$$= n(n-1)^n P.$$

II. *Solution by Robert Maas, University of Santa Clara, Santa Clara, California.* We here interpret the left member of the asserted inequality as

$$S = \sum_{j=1}^{n} \left(\prod_{i=1}^{n} x_i \right) \left(\sum_{k=1}^{n} x_k - x_j \right) \Big/ x_j.$$

For $a, b > 0$, $a^2 + b^2 \geqq 2ab$ and hence $a/b + b/a \geqq 2$. By pairing fractions with their reciprocals, we therefore have

$$\sum_{j,k=1}^{n} x_k/x_j = n + \sum_{k<j} \left(\frac{x_k}{x_j} + \frac{x_j}{x_k} \right) \geqq n + \frac{n^2 - n}{2} \cdot 2 = n^2,$$

$$S \Big/ \prod_{i=1}^{n} x_i = \sum_{j=1}^{n} \sum_{k=1}^{n} x_k/x_j - \sum_{j=1}^{n} x_j/x_j \geqq n^2 - n,$$

and the result follows.

Also solved by A. N. Aheart, M. A. Bershad, J. L. Brown, Jr., J. A. Burslem, F. A. Butter, Jr., Jim Campbell, John Christopher, D. I. A. Cohen, Gary Ford, F. R. Gray and R. L. Price (jointly), Stephen Hoffman, R. F. Jackson, R. A. Jacobson, P. G. Kirmser, Kenneth Kramer, E. S. Langford, F. Leuenberger (Switzerland), E. L. Magnuson, D. C. B. Marsh, D. E. Myers, C. B. A. Peck, Harsh Pittie, V. K. Rohatgi, P. A. Scheinok, P. S. Schnare, R. Shantaram, Robin Sibson (England), R. P. Soni, C. A. Tsonis, Van de Vyle, K. L. Yocom, David Zeitlin, and the proposer.

Jacobson's and Tsonis' solutions were in the spirit of Solution I above, while all the rest were comparable to Solution II.

A Characterization of Finite Groups

E 1733 [1964, 1041]. *Proposed by E. F. Assmus, Jr., Wesleyan University*

Let G be a group and H a finite normal subgroup of G which is not trivial. Suppose H has the property that whenever an element of G commutes with some nontrivial element of H, it is, in fact, in H. Prove that G is finite.

I. *Solution by Azriel Rosenfeld, University of Maryland.* If $C(h) = \{g \in G \mid gh = hg\}$ for each $h \in G$, one of the hypotheses is

$$\bigcup_{\substack{h \in H \\ h \neq e}} C(h) \subset H \quad \text{or} \quad C(h) \subset H \quad \text{for some} \quad h \in H,\ h \neq e,$$

and it is not clear from the statement of the problem which is intended. In any event, we prove the result under the weaker assumption $C(H) \subset H$, where $C(H) = [g \in G \mid gh = hg$ for each $h \in H]$ is the centralizer of H in G.

Since H is normal, $f_x(h) = xhx^{-1}$ defines an automorphism of H for each $x \in G$.

Since H is finite, there are at most $n!$ such automorphisms, n being the order of H. If G were infinite, H would have infinitely many cosets, and there would exist x and y in different cosets such that $f_x=f_y$. But then $xhx^{-1}=yhy^{-1}$ or $h(x^{-1}y)=(x^{-1}y)h$ for each $h\in H$, so that $x^{-1}y\in C(H)\subset H$, contrary to the choice of x and y from different cosets.

II. *Solution by Robert J. Gregorac, Iowa State University.* In the notation of Solution I, we show that G has less than n^2 $(n=o(H))$ members under the assumption $C(h)\subset H$ for some $h\in H$, $h\neq e$. For suppose that $g_1H, g_2H, \cdots, g_nH$ are distinct cosets of H in G, and let $h\neq e$ be a member of H for which $C(h)\subset H$. Then $e\notin\{g_ihg_i^{-1}\mid i=1, 2, \cdots, n\}\subset H$, so that $g_ihg_i^{-1}=g_jhg_j^{-1}$ for some $i\neq j$. But then $g_j^{-1}g_i$ commutes with h and, hence, is an element of $C(h)\subset H$; i.e., $g_i\in g_jH$, a contradiction. Therefore, H has at most $n-1$ cosets and G accordingly at most $n(n-1)$ members.

Also solved by Shair Ahmad, D. R. Arterburn, R. A. Avelsgaard, R. W. Ball, C. W. Barnes, Homer Bechtell, J. L. Brown, Jr., S. H. Brown, R. J. Bumerot, P. L. Chabot, J. F. Dillon, E. W. Ewing, N. J. Fine, Hyman Gabai, R. W. Gilmer, Jr., M. G. Greening (Australia), Israel Grossman, R. R. Hathway, D. R. Hayes, Agatha Himmelfarb, Stephen Hoffman, J. E. Humphreys, Thomas Jefferson, Jr., Erwin Just and Norman Schaumberger (jointly), Victor Keiser, S. C. King, Kwangil Koh, E. S. Langford, C. C. Lindner, J. R. Porter, J. R. Purdy, Donald Quiring and Al Toubassi (jointly), J. D. Reid, Fred Rosenblum, Alan Saleski, P. S. Schnare, R. N. Schneider, Robin Sibson (England), Indranand Sinha, Richard Sinkhorn, D. A. Smith, R. T. Smythe (England), Al Somayajulu, E. R. Sparks, D. E. Tepper, A. M. Vaidya, W. C. Waterhouse, Paul Willig, S. V. Witt, Kenneth Yanosko, and the proposer.

This One Is Eeeeezy

E 1734 [1964, 1041]. *Proposed by Omar Khayyam, Jr., University of California, Berkeley*

The functions f and g are defined by the series:

$$f(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{m+n}m^nx^n}{m!n!},\qquad g(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{m^nx^n}{m!n!}\cdot$$

Prove that $(f(e))^{g(1)}(g(e))^{f(-1)}=1$.

Solution by D. I. A. Cohen, Student, Princeton University. Interchanging the order of summation in both series (they are absolutely convergent), we see that $f(x)=e^{-e^{-x}}$ and $g(x)=e^{e^x}\cdots$ from which the desired result follows easily.

Also solved by A. N. Aheart, W. A. Al-Salam, M. A. Bershad, W. H. Bonney, J. A. Burslem, C. G. Denlinger, E. S. Eby, R. V. Esperti, A. C. Fang, J. A. Faucher, Ernest Gore, F. R. Gray and R. L. Price (jointly), S. H. Greene, Cornelius Groenewoud, Stephen Hoffman, J. M. Horner, R. F. Jackson, Kenneth Kramer, E. S. Langford, W. R. McEwen, Robert Maas, D. C. B. Marsh, M. A. Mustafa (England), J. M. Perry, J. R. Purdy, C. R. Rao, V. K. Rohatgi and M. J. Rossin (jointly), P. A. Scheinok, Klaus Schmitt, J. J. Segedy, H. D. Shane, Robin Sibson (England), Arnold Singer, Richard Sinkhorn, F. C. Smith, Al Somayajulu, R. P. Soni, Sidney Spital, C. A. Tsonis, Vis Upatisringa, Mary C. Vioni, Van de Vyle, W. C. Waterhouse, Max Wyman, G. M. Yadao (India), K. L. Yocom, David Zeitlin, and the proposer.

Monotone Multiplicative Number-Theoretic Functions

E 1735 [1964, 1042]. *Proposed by William Emerson, University of California, Berkeley*

Determine all real valued functions f defined on the positive integers N which satisfy: (1) $(m, n)=1$ implies that $f(mn)=f(m)f(n)$; and (2) $m<n$ implies that $f(m)\leqq f(n)$ for all $m, n\in N$.

Solution by Louis Gordon and Sim Lasher, University of Illinois. Since $f(n)=f(n\cdot 1)=f(n)f(1)$, either $f(1)=0$ (and $f(n)\equiv 0$) or $f(1)=1$ and $f(n)\geqq 1$. We henceforth consider only this latter case, so that $f(n)\neq 0$ for $n\in N$.

We show below that

1. If p is a prime, there are sequences $\{s_n\}$ and $\{t_n\}$ for which $(s_n, p)=(t_n, p)=1$ and

$$f(p) = \lim_{\substack{n\to\infty \\ p\nmid n}} f(ps_n + 1)/f(s_n) = \lim_{\substack{n\to\infty \\ p\nmid n}} f(pt_n - 1)/f(t_n).$$

We then use (1) to deduce

2. If p is a prime and k is a positive integer, then

$$f(p^{k+1}) = f(p^k)f(p).$$

It follows from (2) that f is completely multiplicative. If k is then a fixed positive integer other than 1, and r and s are positive integers for which $n^s\leqq k^r$, then $\{f(n)\}^s\leqq\{f(k)\}^r$ and $f(n)\leqq\{f(k)\}^{r/s}$. Letting $r/s\to\log_k n$, we have $f(n)\leqq\{f(k)\}^{\log_k n}=n^\alpha$ with $\alpha=\log_k f(k)$. Similarly, there are positive integers ρ and σ for which $n^\sigma\geqq k^\rho$ and $\rho/\sigma\to\log_k n$, so that $f(n)\geqq n^\alpha$. In other words, $f(n)=n^\alpha$.

To prove (1), observe first that if $(n, p)=1$, there are

$$n - 1 - 2\left[\frac{n-1}{p}\right] + \left[\frac{[(n-1)/p]}{p}\right] = c(p, n)$$

multiples xp of p between n and $(n-1)p$ inclusive for which $(x, p)=1$. If n is sufficiently large, say $n>p^2$, we have

$$c(p, n) \geqq \left[\frac{[(n-1)/p]}{p}\right] \geqq n/\alpha,$$

where $\alpha=\alpha(p)$ is a positive number independent of n. In the product

$$\prod_{r=n}^{pn-1}\frac{f(r+1)}{f(r)} = \frac{f(pn)}{f(n)} = f(p),$$

there are then at least n/α factors $f(ps+1)/f(ps)$ with $(s, p)=1$ and hence at least one such factor for which $f(ps_n+1)/f(ps_n)\leqq\{f(p)\}^{\alpha/n}$, ($f$ is nondecreasing).

Letting $n \to \infty$ with $(n, p) = 1$ we have

$$1 \leqq f(ps_n + 1)/f(ps_n) \leqq \{f(p)\}^{\alpha/n} \to 1$$

and, hence, $f(ps_n+1)/f(s_n) \to 1$.

Similarly, for $n > p^2$ and $(n, p) = 1$, there is a positive number $\beta = \beta(p)$ and a t_n between $[n/p]$ and n for which $(t_n, p) = 1$ and $f(pt_n)/f(pt_n - 1) \leqq \{f(p)\}^{\beta/n}$. But then $f(pt_n - 1)/f(t_n) \to f(p)$.

With s_n and t_n as above and k a positive integer, observe that

$$(p^{k+1}, s_n) = (p^k, ps_n + 1) = (p^{k+1}, t_n) = (p^k, pt_n - 1) = 1,$$

$p^k(ps_n+1) > p^{k+1}s_n$, and $p^k(pt_n - 1) < p^{k+1}t_n$; we then have

$$\frac{f(pt_n - 1)}{f(t_n)} \leqq \frac{f(p^{k+1})}{f(p^k)} \leqq \frac{f(ps_n + 1)}{f(s_n)}$$

and (2) follows by letting $n \to \infty$, $(n, p) = 1$.

Also solved by J. C. Abad, Michael Goldberg, M. G. Greening (Australia), E. S. Langford, G. M. Leibowitz, Andrzej Makowski (Poland), Leo Moser, M. G. Murdeshwar, S. L. Segal and A. L. Somayajulu (jointly), Robin Sibson (England), and the proposer.

Two of these solutions simply stated that $f(n) = n^\alpha$. The proposer's solution was quite similar to that of Gordon and Lasher above. All others referred to P. Erdös, *On the distribution function of additive functions*, Ann. Math,. 47 (1946) 1–20, or to L. Moser and J. Lambek, *On monotone multiplicative functions*, Proc. Amer. Math. Soc., 4 (1953) 544–545. Abad calls attention to the December, 1963, William Lowell Putnam Examination, Part 1, Problem 2, and several solvers noted that Problem 5254 [1965, 84] is the same as E1735.

Other references offered by Langford or Makowski or Moser: I. J. Schoenberg, *On two theorems of P. Erdös and A. Renyi*, Ill. Jour. Math., 6 (1962) 53–58; M. A. S. Besicovitch, *On additive functions of a positive integer*, Studies in Mathematical Analysis and Related Topics, Stanford Univ. Press, (1962) 38–41; Ch. Pisot and I. J. Schoenberg, *Arithmetic problems concerning Cauchy's functional equation*, Ill. Jour. Math., 8 (1964) 40–56, and Ill. Jour. Math., 9 (1965) 129–136, and Compositio Mathematica, 16, 169–175.

Langford and Moser point out that Erdös also showed that $f(n) = n^\alpha$ under the weaker assumptions that f is multiplicative and bounded below by 1 and $f(n+1)/f(n) \to 1$ as $n \to \infty$.

It is perhaps worth noting that the Moser-Lambek solution to this problem arrives at $f(n) = n^\alpha$ without first deducing that f is completely multiplicative: Treating only the case where $f(1) = 1$, observe that for $m, n > 1$ and $m^k \leqq a \leqq n^l$,

$$\begin{aligned} f(a) &\geqq f(m^k) \geqq f(1 + m + m^2 + \cdots + m^{k-1}) \geqq f(m + m^2 + \cdots + m^{k-1}) \\ &= f(m)f(1 + m + m^2 + \cdots + m^{k-2}) \geqq \cdots \geqq \{f(m)\}^{k-1}, \end{aligned}$$

and, similarly, $f(a) \leqq f(n^l) \leqq f(-1 - n - n^2 - \cdots - n^l + n^{l+1}) \leqq \{f(n)\}^{l+1}$. Taking $k \leqq \log_m a < k+1$ and $l - 1 < \log_n a \leqq l$, and observing that $\log_r s = \log s/\log r$, it follows that

$$\{f(m)\}^{1/(\log m) - 2/(\log a)} \leqq \{f(a)\}^{1/(\log a)} \leqq \{f(n)\}^{1/(\log n) + 2/(\log a)}$$

and hence that $\{f(k)\}^{1/\log k}$ is a positive constant for $k \geqq 2$. If this constant is e^α, then $f(n) = n^\alpha$ (even for $n = 1$).

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before April 30, 1966.

5320. *Proposed by R. V. Moody, Toronto, Canada*

Let $PG(n, F)$ be the n-dimensional projective space over the field F. If F is finite (character $\neq 2$) it is well known that in any quadric in the space there are embedded subspaces of all dimensions $k \leqq \frac{1}{2}(n-2)$ if n is even, and $k \leqq \frac{1}{2}(n-1)$ or $\frac{1}{2}(n-3)$ if n is odd. (If n is odd there are two types of quadric allowing different maximum values of k.) How many subspaces of each dimension are there in each type of quadric? (The quadric is assumed to be nondegenerate.)

5321. *Proposed by Børge Jessen, University of Copenhagen, Denmark*

Let C denote the class of all pairs (f, g) of continuous functions on the interval $[0, 1]$ on $\mathcal{R}$, such that

$$\int_0^1 [f(t)]^n [g(t)]^m \, dt = \frac{1}{(n+1)(m+1)}$$

for all pairs (n, m) of nonnegative integers. Each pair $(f, g) \in C$ determines a continuous curve $(x, y) = (f(t), g(t))$, $t \in [0, 1]$, in $\mathcal{R}^2$. Prove that

(i) The class C is not empty.

(ii) For any pair $(f, g) \in C$, the set $\{(f(t), g(t)) \mid t \in [0, 1]\}$ is the unit square $[0, 1] \times [0, 1]$ in $\mathcal{R}^2$, i.e., the corresponding curve is contained in the square and contains all its points.

(iii) For any pair $(f, g) \in C$, the functions f and g are nowhere differentiable to the right in $[0, 1)$ and nowhere differentiable to the left in $(0, 1]$.

5322. *Proposed by John Brillhart, University of San Francisco*

Let $S(n, a) = \sum_{d \mid n} f(d)$, where $(n, a) = 1$, $f(1) = 1$, and $f(d) = \phi(d)/e(d)$, $d > 1$, ϕ the totient, and $e(d)$ the exponent to which a belongs mod d. Then show that $S(n, 2)$ is odd if $n \equiv \pm 1 \pmod 8$ and is even if $n \equiv \pm 3 \pmod 8$.

5323. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England*

There is a well-known problem in which two ladders rest between two walls. The lengths of the ladders and the height above the ground at which they cross are given, and it is required to find the distance between the two walls and the heights that the ladders reach up the walls. The complete solution when all these six quantities are integers is known (Mathematical Gazette, XLVII, 1963, 133–136). Is there a solution in which the distance between the tops of the ladders is also integral, other than the trivial case in which the ladders are of equal length?

5324. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England*

Following Hardy's visit to Ramanujan at Putney it is well known that 1729 is the smallest number that is the sum of two cubes in two different ways. (a) Show that 3368 is the difference of two cubes in three ways. (b) What is the smallest number that is the sum of two positive cubes in three different ways?

5325. *Proposed by D. J. Newman, Yeshiva University and W. E. Weissblum, AVCO Research*

Suppose $0 \leqq c_i \leqq 1$ and $\sum c_i = 2$. Prove that all the coefficients of $(\sum c_i x^i)^n$ are $\leqq \binom{n}{q}$, where $q = [n/2]$.

5326. *Proposed by C. S. Venkataraman and R. Sivaramakrishnan, Trichur India*

If $\sigma_k(n)$ denotes the sum of the kth powers of the positive divisors of a positive integer n and $\tau(n)$ is the number of positive divisors of n, prove that

$$\frac{\sigma_k(n)}{\tau(n)} \geqq \sqrt{n^k}.$$

5327. *Proposed by J. H. Conway, Gonville and Caius College, Cambridge, England.*

Show that the group generated by five generators a, b, c, d, e, subject only to the relations $ab = c$, $bc = d$, $cd = e$, $de = a$, $ea = b$, is cyclic of order 11.

5328. *Proposed by J. H. Conway, Gonville and Caius College, Cambridge, England*

Show that the plane may be filled with rational-sided triangles in such a way as to use just one of each type. (The triangles should be taken as closed, and "filled" means covered in such a way that the overlap has zero area.)

5329. *Proposed by Leopold Flatto, Yeshiva University*

Let $\omega(z)$ be an entire function such that there exists another nonconstant entire function $f(z)$ for which $f(z) = f(\omega(z))$. Show that this implies $\omega(z) = \zeta z + a$, where ζ is a root of unity.

SOLUTIONS OF ADVANCED PROBLEMS

Idempotent Element

3133 [1925, 261; 1962, 237; 1965, 324]. *Proposed by A. A. Bennett*

Prove that every finite semigroup contains an idempotent element.

III. *Notes by Albert Wilansky, Lehigh University.* The following theorems provide a strong generalization for this problem:

THEOREM I. *If a semigroup can be given a compact T_2 topology such that multiplication is continuous, then it has an identity.*

THEOREM II. *If the semigroup in Theorem I is not a group, then it has an idempotent element which is not an identity.*

These theorems appeared in a Yale dissertation by J. E. L. Peck. See K. Numukara, *On bicompact semigroups*, Math. J. of Okayama University, 1 (1952), and A. D. Wallace, *A note on mobs*, An. Acad. Brasil. Ci., 24 (1952). They were employed by J. G. Wendel, Proc. Amer. Math. Soc., (1954) 923–929.

Rings without Zero Divisors

5191 [1964, 440]. *Proposed by Kwangil Koh, University of North Carolina*

Given that R is a ring without divisors of zero but that R is not a right uniform ring. Prove that there is a right quotient ring Q of R which is a primitive ring without a minimal right ideal. (If, for arbitrary nonzero elements a, b of R, we have $aR\cap bR\neq\{0\}$, then R is called a right uniform ring. Q is a right quotient ring of R if Q has a subring isomorphic to R such that to every nonzero element q of Q there exist x, $y\in R$ such that $qx=y\neq 0$.) Refer to N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloquium Publications, vol. 37.

Solution compiled from the proposer's solution and solutions submitted by E. R. Gentile, University of Buenos Aires, Argentina, and W. H. Caldwell, Rutgers—The State University. Let E be the injective hull of R_R. Johnson and Wong [1] prove that (1) E_E is injective, (2) E is regular. It is also proved that if E is a sfield, then R is right uniform. We first obtain the following

LEMMA. *E is a simple ring (hence primitive, since $1\in E$) and every nonzero element of R has a left inverse in E.*

Proof. Let $r\neq 0$ be in R and consider rE. $rx=0$ implies that $r(xE\cap R)=0$, so that $x=0$; hence the map $f\colon rE\to E$, $f(rx)=x$, is a well-defined E-map. Since E_E is injective, there exists $m\in E$ such that $f(rx)=mrx$ for all $rx\in rE$. In particular $1=f(r\cdot 1)=mr$, so that m is a left inverse for r. Now let A be an ideal of E. $A\neq\{0\}$ implies $A\cap R\neq\{0\}$. Hence there is an element of A with a left inverse in E, whence $A=E$.

We show now that if E has a minimal right ideal, then E is a sfield, hence right uniform. Let eE be a minimal right ideal of E, $e=e^2\in E$ (E is regular). Let $r\in eE\cap R$, $r\neq 0$. Then $er=r$, and $re\neq 0$, as in the proof of the lemma. If r' is a left inverse of r, $(1-e)r're=(1-e)e=0$, so that $(1-e)r'(reE)=0$. But $0\neq reE\subseteq eE$, so that $reE=eE$ by minimality of eE. Therefore $(1-e)r'e=0$, so $r'e\in eE$. Thus $1=r'r=r'er\in eE$, and $eE=E$ is a field. The assertion of the problem now follows.

1. Johnson and Wong, *Self injective rings*, Canad. Math. Bull., 2 (1959) 167–173.

Mario Petrich points out that a similar result for semigroups is given in P. Dubreil, *Algèbre* vol. I, (1954) 269, Th. 1.

Multiple Sum-product Identities

5196 [1964, 440]. *Proposed by L. Carlitz, Duke University*

Prove the identities

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} \frac{(1-a)(1-qa)\cdots(1-q^{n-1}a)}{(1-q)(1-q^2)\cdots(1-q^n)} \tag{1}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2}a^n}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} \cdot \prod_{r=1}^{\infty}(1+q^r),$$

$$\sum_{n=0}^{\infty} (-1)^n q^{n^2} \frac{(1-a)(1-q^2a)\cdots(1-q^{2n-2}a)}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} \tag{2}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n^2}a^n}{(1-q)(1-q^2)\cdots(1-q^{2n})} \prod_{r=1}^{\infty}(1-q^{2r-1}).$$

Solution by the proposer. Consider the sum

$$\sum_{r=0}^{\infty} (-1)^r q^{r(r-1)/2} \frac{(a)_r}{(q)_r} x^r,$$

where $(a)_r = (1-a)(1-qa)\cdots(1-q^{r-1}a)$. Since

$$(a)_r = \sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix} q^{s(s-1)/2}a^s, \qquad \text{where} \qquad \begin{bmatrix} r \\ s \end{bmatrix} = \frac{(q)_r}{(q)_s(q)_{r-s}},$$

the above sum is equal to

$$\sum_{r=0}^{\infty} (-1)^r q^{r(r-1)/2} \frac{x^r}{(q)_r} \sum_{s=0}^{r} (-1)^s \begin{bmatrix} r \\ s \end{bmatrix} q^{s(s-1)/2}a^s$$
$$= \sum_{s=0}^{\infty} q^{s(s-1)/2} \frac{a^s x^s}{(q)_s} \sum_{r=s}^{\infty} (-1)^{r-s} q^{r(r-1)/2} \frac{x^{r-s}}{(q)_{r-s}}$$
$$= \sum_{s=0}^{\infty} q^{s(s-1)} \frac{a^s x^s}{(q)_s} \sum_{r=0}^{\infty} (-1)^r q^{r(r-1)/2} \frac{q^{rs}x^r}{(q)_r}$$
$$= \sum_{s=0}^{\infty} q^{s(s-1)} \frac{a^s x^s}{(q)_s} \prod_{r=0}^{\infty} (1-q^{r+s}x).$$

Thus we have the identity

$$\sum_{r=0}^{\infty} \frac{(-1)^r q^{r(r-1)/2}(1-aq)(1-aq^2)\cdots(1-aq^{r-1})\cdot x^r}{(1-q)(1-q^2)\cdots(1-q^r)}$$
$$= \sum_{r=0}^{\infty} \frac{q^{r(r-1)}a^r x^r E(x,q)}{(1-q)\cdots(1-q^r)(1-x)(1-xq)\cdots(1-xq^{r-1})},$$

where $E(x, q)=\prod_{n=0}^{\infty}(1-xq^n)$. Letting $x=-q$ we obtain (1). Replacing first q by q^2, and then letting $x=q$, we obtain (2).

Also solved by Sylvan Green and (partially) by Joseph Arkin.

Some Isometrically Isomorphic Banach Spaces

5226 [1964, 802]. *Proposed by E. O. Thorp, New Mexico State University*

Let S and T be arbitrary point sets. The Banach space $l_1(S)$ is the set of functions x on S such that $x(s)\neq 0$ for at most countably many s and such that the norm of x, defined by $|x| = \sum_{s\in S} |x(s)|$ is finite. The Banach space $l_\infty(S)$ is the set of bounded functions x on S with the norm of x defined by $|x| =\sup_{s\in S} |x(s)|$. When is $l_1(S)$ isometrically isomorphic to $l_\infty(T)$?

Solution by D. R. Arterburn, Montana State University. Let $|S|$ denote the cardinality of S. Since $\{e_s: s\in S\}$ is a total subset of $l_1^*(S)$ of least cardinality, as is $\{e_t: t\in T\}$ in $l_\infty^*(T)$, $|S| = |T|$ in order for them to be isometrically isomorphic.

CASE 1: Real scalars. The unit sphere of $l_1(S)$ has $2\cdot|S|$ extreme points, whereas that of $l_\infty(T)$ has $2^{|T|}$. Since an isometric isomorphism preserves extreme points, $|S| = |T| \leqq 2$. If $|S| = 0$ or 1, clear. If $|S| = 2$, define $f: l_1(S) \to l_\infty(T)$ by $f(1, 0)=(1, 1)$ and $f(0, 1)=(1, -1)$ and extend by linearity.

CASE 2: Complex scalars. Call $x\equiv y$ iff $x=\alpha y$, $|\alpha| = 1$. Then the unit sphere of $l_1(S)$ has $|S|$ equivalence classes of extreme points whereas that of $l_\infty(T)$ has $C^{|T|}$ if T is infinite, $C^{|T|-1}$ if T is finite. Since an isometric isomorphism preserves these equivalence classes,

$$|S| = |T| \leqq 1.$$

Also solved by N. T. Peck, and the proposer.

Primitive Duo Ring

5227 [1964, 922]. *Proposed by Jiang Luh, Indiana State College, Terre Haute*

Following E. J. Feller, a ring A is called a duo ring if every one-sided ideal of A is a two-sided ideal. Prove that a ring is duo and left (or right) primitive if and only if it is a division ring.

Solution by Harsh Pittie, Undergraduate, Swarthmore College. If A is a division ring, then the only ideals are the trivial ones, and they are two-sided. $\{0\}$ is therefore a maximal ideal which is modular, so that $(\{0\} : A)=\{0\}$; hence A is a primitive duo ring.

Conversely, if A is a left primitive duo ring, there exists a modular maximal ideal I such that $(I: A)=\{0\}$. If x is an element of I then $xA\subseteq I$; but $(I: A)$

$=\{0\}$ implies that x is in $\{0\}$, so that $x=0$ and, therefore, $I=\{0\}$. Since $\{0\}$ is a maximal ideal, the only other ideal in A is A itself. Since I is modular, there exists an element e in A such that for all y in A, $ey-y$ is in I, so that $(ey-y)A\subseteq I$; but $I=\{0\}$, so that $ey=y$ for all y. Hence A is a simple ring with unity and, thus, is a division ring.

Also solved by W. E. Coppage, A. G. Heinicke, Kwangil Koh, C. C. Lindner, L. M. Perry, Indranand Sinha, B. R. Toskey, Bertram Walsh, and the proposer.

The Partition Function $p(n)$ and Group Representations

5228 [1964, 923]. *Proposed by A. M. Vaidya, University of Colorado*

Prove: If $p(n)$ denotes the number of partitions of n, then there exist $p(n)$ integers $n_1, \cdots, n_{p(n)}$, not necessarily distinct, such that each n_i divides $n!$ and

$$\sum_{i=1}^{p(n)} n_i^2 = n!.$$

Solution M. G. Greening, University of New South Wales, Australia. It is known (see e.g., W. R. Scott, *Group Theory*, p. 331 ff.; also C. W. Curtis and Irving Reiner, *Theory of Finite Groups and Associative Algebras*, pp. 186–7, 236) that $k(m)$ integers $n_1, n_2, \cdots, n_{k(m)}$ exist such that $\sum_{i=1}^{k(m)} n_i^2=m$, for any group of order m. The n_i are the degrees of the irreducible representations of the group ring and $k(m)$ is the number of classes in the group. Taking the symmetric group on n symbols, we find $k(n)$ as the number of partitions of n, while $m=n!$, whence the result follows, because the degree of a representation divides the order of the group.

In fact, the additional condition that two, and only two, of the n_i are equal to 1 holds, by considering that there are exactly two Abelian representations, necessarily of degree one, viz. S_1 and S_2, isomorphic to the quotient groups S_n/S_n and S_n/A_n where A_n is the alternating group of degree n, which is the commutator subgroup of S_n.

Also solved by L. Carlitz, J. B. Kelly and the proposer.
It would be desirable to have a number-theoretic solution.

Rational Function Approximation

5229 [1964, 923]. *Proposed by H. R. van der Vaart, North Carolina State College, Raleigh*

Find a rational function of n with neither numerator nor denominator of degree greater than 3 which, for $n\geqq 6$, is less than $S_n=\sum_{r=0}^{n}(r/n)^n$, and differs from it by at most $7\cdot 10^{-5}$.

Solution by Ronald C. Read, University of the West Indies. Using the Euler-Maclaurin sum formula,

$$\sum_{r=0}^{n} p(r) = \int_0^n p(x)\,dx + \tfrac{1}{2}[p(0) + p(n)] + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}[p^{(2i-1)}(n) - p^{(2i-1)}(0)],$$

where $p(x) = (x/n)^n$, we have

$$\sum_{r=0}^{n}\left(\frac{r}{n}\right)^n = \int_0^n \left(\frac{x}{n}\right)^n dx + \frac{1}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\cdot\frac{n^{(2i-1)}n^{n-2i+1}}{n^n},$$

where $n^{(m)} = n(n-1)(n-2)\cdots(n-m+1)$. Thus

$$\begin{aligned} S_n &= \frac{n^{n+1}}{n^n(n+1)} + \frac{1}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}\cdot\frac{n^{(2i-1)}}{n^{2i-1}} \\ &= \frac{n}{n+1} + \frac{1}{2} + \frac{1}{12}\cdot\frac{n}{n} - \frac{1}{30\cdot 24}\cdot\frac{n^{(3)}}{n^3} + X \\ &= \frac{n}{n+1} + \frac{7}{12} - \frac{(n-1)(n-2)}{720n^2} + X, \end{aligned}$$

where

$$X = \sum_{i=3}^{\infty} \frac{B_{2i}}{(2i)!}\cdot\frac{n^{(2i-1)}}{n^{2i-1}}.$$

Now

$$\frac{B_{2i}}{(2i)!} = \frac{2(-1)^i}{(2\pi)^{2i}}\left(1 + \frac{1}{2^{2i}} + \frac{1}{3^{2i}} + \cdots\right).$$

Hence the terms in X alternate in sign and steadily decrease to zero. Hence X is numerically less than

$$\frac{B_6}{6!}\cdot\frac{n^{(5)}}{n^5} < \frac{1}{42\cdot 720} < 3.4(10^{-5}),$$

provided this term is not zero, i.e. for $n \geqq 6$.

Thus the rational function is

$$\frac{n}{n+1} + \frac{7}{12} - \frac{(n-1)(n-2)}{720n^2} = \frac{1139n^3 + 422n^2 + n - 2}{720(n^2)(n+1)}$$

which satisfies the required conditions.

Notice that this function is exactly S_n for $n = 2, 4, 5$; only for $n = 1$ and $n = 3$ does it differ from S_n by more than the specified tolerance.

Also solved by Mrs. A. C. Garstang, A. W. Johnson, Jr., J. H. van Lint (Netherlands), and the proposer.

Johnson, in his solution, noted that $(n-2)/720n^2(n+1) < 3.5(10^{-5})$ for $n \leqq 6$ and therefore $(1139n+422)/720\,(n+1)$ also satisfies the conditions of the problem.

Real Valued Lipschitzian Functions

5230 [1964, 923]. *Proposed by J. L. Denny, Indiana University*

Is there a real-valued Lipschitzian function f defined on $[0, 1]$ such that for each subinterval $[a, b]\subset[0, 1]$ the set of $y\in f[[a, b]]$, such that $f^{-1}\{y\}$ is uncountable, is of the second category in $(-\infty, \infty)$ and a residual set in $f[a, b]$?

Solution compiled from the solution by Solomon Marcus, University of Bucharest, Rumania, and the proposer's solution. A theorem due to Marcus (Sur les fonctions continues qui ne sont monotones en aucun intervalle. Rev. Math. Pures Appl., No. 1, 3 (1958) 101–105) states: *Let the real function f be continuous on the interval I. A necessary and sufficient condition that f be not monotonic in any subinterval of I is that for every interval J contained in I, the set* $\{y\in f(J) \mid J\cap f^{-1}(y)$ *is uncountable*$\}$ *is of the second category in* $(-\infty, \infty)$ *and a residual set in* $f(J)$.

An f satisfying the conditions of the theorem may be constructed as follows: Let A, B be two disjoint Lebesgue measurable sets, $A\cup B=[0, 1]$, such that for every $(a, b)\subset[0, 1]$, $A\cap(a, b)$ and $B\cap(a, b)$ have positive measure. Designating by ϕ_M the characteristic function of the set M, put

$$f(x) = \int_0^x [\phi_A(t) - \phi_B(t)]\, dt.$$

Then $f(x)$ is Lipschitzian, $f'(x)=1$ on A and $f'(x)=-1$ on B. This satisfies the conditions of the theorem and answers the problem in the affirmative.

The construction of such functions $f(x)$ may also be found in E. W. Hobson, *Theory of Functions*, vol. II.

Inversion of Convolutal Sequences

5231 [1964, 923]. *Proposed by H. W. Gould, West Virginia University*

Let x, z be real numbers. Prove that each of the following systems implies the other:

$$B_n = \sum_{k=0}^{n}\binom{z}{k}x^k A_{n-k}, \qquad A_n = \sum_{k=0}^{n}\binom{-z}{k}x^k B_{n-k}.$$

I. *Solution by Murray Klamkin, University of Minnesota.* If $F(x, z, k)$ and $G(x, z, k)$ satisfy

$$\sum_{k=0}^{\infty} F(x, z, k)t^k\cdot \sum_{k=0}^{\infty} G(x, z, k)t^k \equiv 1, \tag{1}$$

then either of the following systems implies the other:

$$B_n = \sum_{k=0}^{n} F(x, z, k)A_{n-k}, \qquad A_n = \sum_{k=0}^{n} B_{n-k}G(x, z, k). \tag{2}$$

These transform equations follow by direct substitution and noting that

$$\sum_{k=0}^{s} G(x, z, k)F(x, z, s-k) = \begin{cases} 1, & s = 0 \\ 0, & s > 0. \end{cases}$$

The special case for the present problem requires

$$F(x, z, k) = \binom{z}{k} x^k, \qquad G(x, z, k) = \binom{-z}{k} x^k,$$
$$\sum F \cdot t^k = (1 + xt)^z, \qquad \sum G \cdot t^k = (1 + xt)^{-z}.$$

II. *Solution by L. Carlitz, Duke University.* A more general result can be stated. Consider the linear transformation T_z defined by

$$b_n = \sum_{k=0}^{n} \binom{z}{k} x^k a_{n-k} \qquad (n = 0, 1, 2, \cdots), \tag{1}$$

that carries the sequence $\{a_n\}$ into the sequence $\{b_n\}$; the parameter x is held fixed. Let T_w carry $\{b_n\}$ into $\{c_n\}$:

$$c_n = \sum_{k=0}^{n} \binom{w}{k} x^k b_{n-k} \qquad (n = 0, 1, 2, \cdots). \tag{2}$$

Then

$$c_n = \sum_{k=0}^{n} \binom{w+z}{k} x^k a_{n-k},$$

so that T_{w+z} carries $\{a_n\}$ to $\{c_n\}$. We have therefore $T_{w+z} = T_w T_z$, i.e., the set of transformations $\{T_z\}$ is a group isomorphic to the additive group of the reals. Clearly T_{-z} is the inverse of T_z, which gives the original assertion.

III. *Solution by John Riordan, Bell Telephone Laboratories, Murray Hill, N. J.* With the substitutions $x^{-n}A_n = a_n$, $x^{-n}B_n = b_n$, the given inverse relations become

$$a_n = \sum_{k=0}^{n} \binom{-z}{n-k} b_k, \qquad b_n = \sum_{k=0}^{n} \binom{z}{n-k} a_k,$$

whose orthogonality relation is

$$\sum_{k=m}^{n} \binom{-z}{n-k}\binom{z}{k-m} = \sum_{j=0}^{n-m} \binom{-z}{n-m-j}\binom{z}{j} = \binom{0}{n-m} = \delta_{nm},$$

as required. These relations are the instance $b = 1$ of

$$a_n = \sum_{k=0}^{n} \binom{z+bk-k}{n-k} b_k, \qquad b_n = \sum_{k=0}^{n} (-1)^{n+k} \frac{z+bk-k}{z+bn-k} \binom{z+bn-k}{n-k} a_k.$$

Also solved by W. A. Al-Salam, P. T. Bateman, M. A. Bershad, A. P. Boblétt, J. L. Brown, Jr., R. G. Buschman, N. J. Fine, M. G. Greening, Eldon Hansen, Stephen Hoffman, A. S. B. Holland and A. Meir (jointly), J. M. Horner, R. R. Janić (Yugoslavia), E. S. Langford, M. G. Murdeshwar, A. G. P. M. Nijst (Netherlands), J. M. Perry, Stanton Philipp, V. K. Rohatgi, Y. P. Sabharwal (India), Arnold Singer, Sidney Spital, W. F. Trench, C. A. Tsonis, Benjamin Volk, G. M. Yadao (India), and the proposer.

Horner and Johnson noted that the formulas remain valid if x and z are any complex numbers. Brown noted that the result follows from exercise 74, p. 385 of *Orthogonal Polynomials* by G. Szegö (A. M. S. Colloquium No. 23).

Characterization of Hausdorff Spaces

5232 [1964, 923]. *Proposed by Alan Weinstein, Massachusetts Institute of Technology*

A topological space Y has the "unique extension property" if, for any topological space X, any two continuous functions from X into Y which agree on a dense subset of X are equal.

Every Hausdorff space has the unique extension property. Does this property characterize Hausdorff spaces; i.e., must a space with the unique extension property be Hausdorff?

Solution by Sim Lasher, University of Illinois. The answer is yes. Suppose, on the contrary, that Y is not Hausdorff. Choose distinct y_1, $y_2 \in Y$ such that $N(y_1) \cap N(y_2) \neq \phi$, where $N(y_1)$ and $N(y_2)$ are arbitrary neighborhoods of y_1 and y_2 respectively. Let $x_0 \notin Y$ and let $X = Y \cup \{x_0\}$, with a base of X consisting of the open sets of Y and all sets of the form $\{x_0\} \cup [N(y_1) \cap N(y_2)]$. Let $f(x) = g(x) = x$ for $x \in Y$, $f(x_0) = y_1$, $g(x_0) = y_2$, and Y does not have the unique extension property.

Also solved by Ethan Akin, W. L. Allen, D. R. Anderson, C. K. Denlinger, John Fournier, H. S. Fox, Fred Galvin, Hewitt Kenyon, Norman Levine, R. C. McKinney, M. D. Mavinkurve (India), Stewart Robinson, P. S. Schnare, Martin Tangora, L. E. Ward, W. C. Waterhouse, John Wenger, F. E. Willmore, and the proposer.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, University of Calif. and E. P. VANCE, Oberlin College

Elementary General Topology. By Theral O. Moore. Prentice-Hall, Englewood Cliffs, N. J., 1964. 174 pp. $5.95.

The author of this carefully written text states in his preface that he has kept in mind especially the undergraduate students who have not yet had advanced calculus. Thus, one of his aims is to develop mathematical maturity. He has been remarkably successful in his part of the task—the rest is up to the

kind of economizing seems excessive, even misleading, as on page 62 (the distribution of a random variable is defined for all Borel sets) and on page 191 (the equivalence of two forms of the strong law of large numbers).

In fact, these complaints are instances of a general lack of care in style and detail, which is hard to excuse in a book bent on economy of words. Too often the wording is vague, ambiguous, misleading, or wrong. Examples: page 73, "for all x in (a, b) either $h'(x)>0$ or $h'(x)<0$," by which they do not mean that $h'(x)$ never vanishes in (a, b); page 83, the Lebesgue-Stieltjes integral "has every property possessed by" the Riemann-Stieltjes integral (but not conversely!); page 89, omission of the hypothesis of independence from the statement of a theorem which obviously needs it. There are sentences which one could simply wish had been edited out: "One can wax metaphysical and call . . . , but of course one can also laugh at this point of view," (page 54).

When results are presented so extensively without proofs, the reader (and his teacher!) need to have rather considerable confidence in the reliability of the authors, or do an inordinate amount of work. The user of this book may well feel a bit nervous.

F. CUNNINGHAM, JR., Bryn Mawr College

Theory of Integration. By Ralph Henstock. Butterworth, London, 1963. 168 pp. $4.25.

In a series of articles appearing in the *Journal* and *Proceedings* of the London Mathematical Society in 1960 and 1961, the author of this little book defined and developed a concept which he called the Riemann-complete integral. This book presents the basic properties of this integral, and relates it to previous integrals.

After a brief, historical discussion of earlier integrals, the author gives a constructive definition (i.e., using approximating sums), and later a descriptive definition (i.e., in terms of its characteristic properties) of his integral. The integral so defined possesses all the standard properties. In particular, it is linear, nonnegative on nonnegative integrands, and, as a function of its upper limit, is differentiable almost everywhere; there is also a theorem on bounded convergence.

The text is generally well written, although in places, a little concise, and the typography is sometimes crowded. The misprints noted were too few and too obvious to create a problem. The exercises range from routine to frustrating; they are, however, especially welcome because much of the notation and terminology will be new to the reader who is not familiar with Henstock's work. Most of the book can be read after a sound course in advanced calculus.

The author has done a good job of presenting his material in textbook form. While one might hesitate to select this book for a course in integration theory until the Riemann-complete integral assumes a more prominent role in this field, the book might well be used for independent study, especially since it is

almost certain to get the student into the journals. The book would make a worthwhile addition to any mathematics library.

T. P. DENNEHY, Reed College

Foundations of Geometry. By C. R. Wylie, Jr. McGraw-Hill, New York, 1964. 338 pp. $8.50.

This is a very well written book on the foundations of conventional euclidean geometry, and is primarily intended for undergraduate teaching majors in mathematics, and for use in academic year and summer institutes for teachers of mathematics. It can certainly be used by mathematics majors.

A novel feature of the book is a chapter on four-dimensional euclidean geometry. This is a stimulating chapter, and should extend students by just the right amount in their quest for geometric experience. There is also a good chapter on plane hyperbolic geometry, and a final chapter on a euclidean model of the hyperbolic plane.

Felix Klein, in his lectures to school-teachers, would tell them that they needed to know something about non-euclidean geometry, because this is what the man in the street would always ask them about, and the teachers should be able to answer. This, of course, is not the reason for studying non-euclidean geometry nowadays. The man in the street may even be asking different questions in the nineteen-sixties. If he is asking about the axiomatic method, the first chapter in this book is a very good introduction to this topic.

I have noted with interest and approval that the author expects his students to be literate. There are many exercises which consist of extracts from books which mention mathematical topics, and the student is asked to comment, which he can hardly do without writing something on the subject. I wonder what kind of essays Prof. Wylie receives in response to such mathematico-literary assignments.

The format and production of the book are excellent, and it should admirably fulfil the purpose for which it was written.

D. PEDOE, University of Minnesota

Aufgabensammlung zur Infinitesimalrechnung. Band I: Funktionen einer Variablen. By A. Ostrowski. Birkhäuser, Basel and Stuttgart, 1964. 336 pp. with 9 fig. 38.50 Sw. fr.

This collection of problems forms a companion volume to the revised edition (1960) of the first volume of Professor Ostrowski's three-volume text on calculus. The first edition of the book appeared in 1945 and was reviewed by the present reviewer (Bull. Amer. Math. Soc., 52 (1946) 798–799). In the revised edition, which was reviewed in Bull. Amer. Math. Soc., 67 (1961) 336–337, the problems were all taken out, and they now appear separately in the volume under review. Some problems have been modified, and a number of wholly new problems have been added. The book is divided into three parts: Problems, Hints, and Solutions. The problems occupy 126 pages, and are accompanied, section by section,

errors introduced into it (through previous computation) will be greatly amplified in the solution of the subproblem (which may then become initial data for further computation).

Chapter 2 contains error analyses for the evaluation of truncated power series and polynomials, a discussion of the condition of zeros of polynomials and a series of error analyses on zero finding procedures including the methods of Bisection, Newton, Polynomial Deflation, Root Squaring, and Bairstow. All of these methods involve recursive procedures which, with perfect operations, are independent, in their regions of convergence, of initial values, and are, therefore, self correcting. Notwithstanding these properties, however, the accuracy of such a recursive method is not necessarily comparable to the accuracy of an iteration because of round off and ill conditioning problems.

Chapter 3 is especially noteworthy and divided into two main parts, one dealing with solving linear systems and inverting matrices, the other with eigenvalue computations. The first part includes a discussion of matrix norms; error analyses of matrix multiplication and vector orthogonalization. Then the condition of a linear system is discussed and the method of Gaussian elimination and variants are analyzed for solving linear systems and inverting matrices in a sequence of subanalyses. These subanalyses consider the reduction of linear systems to triangular form, solving triangular linear systems (with both single and double precision inner product operations), a discussion of left and right numerical inverses (they are surprisingly interchangeable), an analysis of Cholesky's method for inverting positive definite matrices and its generalization, and finally the use of an approximate inverse in an iterative procedure.

The remainder of Chapter 3 considers eigenvalue computations. The condition of eigenvalues is discussed, and then methods of estimating the accuracy of a proposed eigenvalue and eigenvector are developed. An error analysis for the calculation of eigenvectors of a tri-diagonal matrix with known eigenvalues is given (this handles the general case with known reductions). Finally, an analysis of the determination of eigenvalues of a lower Hessenberg matrix by the method of Hyman is given.

Dr. Wilkinson is a very clear writer and has supplied a profusion of numerical examples worked out in decimal arithmetic. The work is accessible, with some diligence, to almost any level of background expected in a practicing numerical analyst. And, as indicated above, the rewards will merit the diligence required. Much of the material of this book has been taken from notes and papers prepared separately and the organization and some repetition shows in this. This is not all deficit, however, because the various analyses can be studied almost independently once a few basic ideas are grasped from the first chapter.

HARLAN D. MILLS, International Business Machines Corp.

Interpolation and Approximation. By Philip J. Davis. Blaisdell, New York, 1963. x+393 pp. $12.50.

The author is well known for his work in interpolation and approximation

and this informative and well written book reflects his interests and pedagogical abilities. The book is a wonderful text book and can be read profitably by a beginning student with a modicum of knowledge of real and complex variables and matrix theory. There are 14 chapters in all. Chapter 1 is introductory in nature and contains basic material from matrix theory and linear algebra, real variables, complex variables and integration theory. Chapter 2 deals with polynomial interpolation. Chapter 3 is on remainder theory, Chapter 4 on convergence theorems for interpolatory processes, and Chapter 5 on infinite interpolation. Chapter 6 is on uniform approximation and treats such topics as the Weierstrass approximation theorem and the Bernstein polynomials. Chapter 7 is on best approximation, and Chapter 8 on least squares approximation. The elementary properties of Hilbert spaces are discussed in Chapter 9, and Chapter 10 is devoted to orthogonal polynomials. Chapter 11 deals with closure and completeness, Chapter 12 with orthogonal functions, Chapter 13 with degree of approximation and Chapter 14 with approximation of linear functionals. There is an appendix on orthogonal polynomials, giving tables of the Tschebyscheff and Legendre polynomials.

A vast amount of material is treated in this book but the author observes that it is only a small part of the information available. The author does not hesitate to give complete definitions for concepts which might not be familiar to a beginning student. In addition, complete proofs of important theorems are given. The book is consequently somewhat leisurely for the professional mathematician, but it is never tedious. The author's delightful literary style serves to maintain the reader's interest by such remarks as "*Zorn's Lemma.* Today a mathematics book without this lemma would be like an 18th century gentleman without his sword."

There are many excellent problems, chosen to develop the skill of the reader. The book is a valuable source book in the areas of interpolation and approximation and deserves a place in the library of the working mathematician, as well as in that of the numerical analyst.

MORRIS NEWMAN, National Bureau of Standards

A Sophisticate's Primer of Relativity. By P. W. Bridgman. Wesleyan University Press, Middletown, 1962. 191 pp. $4.50.

This is an interesting book, like the earlier books by Bridgman. It is addressed to the philosopher of science or to the interested layman. The modern physicist or applied mathematician, who is familiar with the theory of special relativity, will occasionally find a new and stimulating thought; on the whole, however, he will have known before what the author says. Adolf Grünbaum has written a short prologue and a long epilogue, which discusses and compares the points of view of Bridgman and Reichenbach.

ALFRED SCHILD, The University of Texas

Complex Numbers and Conformal Mappings. By A. I. Markushevich. Pergamon Press, New York, 1963. 53 pp. $1.50.

This booklet is based upon a lecture given by the author to high school students. It "aims at acquainting the reader with complex numbers and elementary functions of them . . . " using ideas no more sophisticated than geometric intuition and the angle-preserving properties of the conformal maps, (i) $z'=(z-a)/(z-b)$, (ii) $z'=z^2$ and (hence) the Joukovsky function (iii) $z'=\frac{1}{2}(z+1/z)$. No proofs are given.

The presentation is clear, the translation reads well and the text is free of misprints. Except for sections 27–34, which appear rather crowded and tedious, the author's exposition is well-motivated and expertly done.

J. L. GOLDBERG, University of Michigan

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Associate Secretary, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Francis P. Egan, State University College of New York at Oneonta, represented the Association at the inauguration of William L. Gragg as President of Fulton-Montgomery Community College on May 18, 1965.

University of Bridgeport: Professor W. G. Brady, Washington and Jefferson College, has been appointed Bernhard Professor of Mathematics and Chairman of the Department of Mathematics; Associate Professor Martin Lipschutz, Fairleigh Dickinson University, has been appointed Professor.

Professor H. W. Alexander, Earlham College, has been appointed Chairman of the Department of Mathematics at University College, Nairobi, Kenya, Africa.

Mr. D. R. Baker, Melpar, Inc., Falls Church, Virginia, has accepted a position on the technical staff of Wolf Research and Development Corporation, College Park, Maryland.

Dr. L. E. Blumenson, University of Chicago, has been appointed Senior Cancer Research Scientist at Roswell Park Memorial Institute.

Assistant Professor L. R. Bragg, Case Institute of Technology, has been promoted to Associate Professor.

Associate Professor B. F. Bryant, Vanderbilt University, has won the 1964–1965 Madison Sarratt Prize for excellence in undergraduate teaching.

Associate Professor John Dyer-Bennet, Carleton College, has been promoted to Professor.

Professor Melvin Henriksen, Purdue University, has been appointed Head of the Department of Mathematics at Case Institute of Technology.

Assistant Professor J. P. King, Lehigh University, has been promoted to Associate Professor.

Associate Professor G. J. Minty, University of Michigan, has been appointed Professor at Indiana University.

Mr. Herbert Nadler, Manhattan Life Insurance Company, New York, New York, has been promoted to Assistant Actuary.

Dr. Katharine E. O'Brien, Deering High School and University of Maine, Portland, has been awarded an honorary degree of Doctor of Humane Letters by Bowdoin College.

Assistant Professor Richard Rodgers, Earlham College, has been appointed Chairman of the Department of Mathematics.

Mr. Steven Rosencrans, Massachusetts Institute of Technology, has been appointed Assistant Professor at Tulane University.

Professor Alberto Saez has been appointed Head of the Physics Department of the Universidad Central de Venezuela, Caracas, Venezuela.

Dr. H. W. Vayo, Harvard University, has been apppointed Assistant Professor at the University of Toledo.

Mr. L. J. Adams, Santa Monica City College, died on April 24, 1965. He was a member of the Association for 31 years.

Mr. R. A. Bach, State University of New York at Binghamton, died on February 25, 1965. He was a member of the Association for 8 years.

Professor Emeritus W. E. Brooke, University of Minnesota, died on December 22, 1963. He was a charter member of the Association.

Professor José Gallego-Diaz, Universidad Central de Venezuela, Caracas, died on February 16, 1965. He was a member of the Association for 17 years.

Mr. R. L. Huckins, Beech Aircraft Corporation, Wichita, Kansas, died on October 3, 1964. He was a member of the Association for 13 years.

Mr. A. F. Payton, South Orange, New Jersey, died on April 21, 1965. He was a member of the Association for 15 years.

Mr. G. C. Wolf, Washington, D. C., died on December 24, 1964. He was a member of the Association for 6 years.

THE JOURNAL OF THE FRANKLIN INSTITUTE PREMIUM

The Journal of The Franklin Institute, the oldest continuing scientific publication in the United States, founded 1826, announces the founding of a special JOURNAL PREMIUM in the amount of $1000. It is to be awarded annually to the author of the outstanding paper published in the JOURNAL during the preceding year. The award will be presented, in general, to the recipient of the Louis E. Levy Medal.

Established in 1923, the Levy Gold Medal is awarded by The Franklin Institute "to the author of a paper of special merit published in the JOURNAL, preference being given to one describing the author's experimental and theoretical researches in a subject of fundamental importance."

The Journal of The Franklin Institute accepts contribution in all traditional branches of mathematics and the physical sciences, pure and applied, as well as in interdisciplinary fields or composite sciences that combine the philosophies of two or more disciplines.

Consideration for the JOURNAL PREMIUM will be given for the first time to papers published during the calendar year 1965.

Query: Mr. H. LONDON, Pennsylvania State University, McAllister Building, University Park, Pa. 16802, requests information from anyone on the solvability of the Diophantine equation $y^2=x^3+k$.

BELFER GRADUATE SCHOOL OF SCIENCE—YESHIVA UNIVERSITY

The fourth Annual Science Conference sponsored by the Belfer Graduate School of Science will be held on November 15 and 16, 1965 at the Hotel Astor in New York City. The sessions are open to all who wish to attend and admission is free. The schedule of the program will be:

Monday, November 15th—Physics: Professors Feza Gürsey, T. D. Lee, A. S. Wightman, P. W. Anderson, E. E. Salpeter, and G. C. Wick.

Tuesday, November 16th—Mathematics: Professors B. M. Dwork, C. E. Shannon, P. A. Smith, Kenkichi Iwasawa, G. D Mostow and Richard Brauer.

The annual award for distinguished service to science will be presented by the Belfer Graduate School of Science at that time.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE IOWA SECTION

The fifty-second regular meeting of the Iowa Section of the MAA was held at the University of Dubuque on April 23, 1965. Chairman Robert V. Hogg presided. Total attendance was 89, including 49 members of the Association. Routine business was considered during the afternoon meeting.

A report of the Iowa 1965 high school mathematics contest was given. The contest is sponsored by the Des Moines Actuaries Club.

The following officers were elected: Chairman, Donald E. Sanderson, Iowa State University, Ames; Vice-Chairman, Charles M. Lindsay, Coe College, Cedar Rapids; Secretary-Treasurer, Earle L. Canfield, Drake University, Des Moines. The following papers completed the Program:

A report of the MAA Cooperative Summer Seminar of 1964, by E. R. Mullins, Jr., Grinnell College.

A report of the first MAA Cooperative Summer Seminar for college teachers of mathematics held at Cornell University, Ithaca, N. Y., for eight weeks in the summer of 1964. The purpose, the structure and the subject matter of the seminar are the essentials of the report.

Central automorphisms of a finite p-group, by A. D. Otto, State University of Iowa.

Suppose that G is a finite p-group. The order of the group $A_c(G)$ of central automorphisms of G is studied to help determine when the order of G divides the order of $A_c(G)\cdot I(G)$, where $I(G)$ is the group of inner automorphisms. Attention is centered on those non-abelian p-groups which have no non-trivial abelian direct factors. For such a group G, $A_c(G)$ is a p-group and the order of $A_c(G)$ is between p^2 and p^{rs} where p^r is the order of the center and p^s is the order of the factor commutator group.

Function spaces of invertible spaces, by S. A. Naimpally, Iowa State University.

This will appear as a note in the MONTHLY.

Complete normality, the long line, and products, by D. E. Sanderson, Iowa State University.

The long line is completely normal (T_5) and its product with an interval, though not T_5, is surprisingly close to it. Any uncountable product of intervals contains a long line and a half-open

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 9

CONTENTS

NOVEMBER 1965

THE AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2}\times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $10.00

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

GOALS FOR MATHEMATICS INSTRUCTION

R. C. BUCK, University of Wisconsin

In recent years, the mathematical community has been well supplied with articles and brochures which set forth detailed recommendations for curricula and courses, both at the college level and for K-12. More often, the suggestions which follow a title such as the one above have been very specific, saying such things as: "The number line should be done before fractions," or "In linear algebra, one should treat matrices after one has discussed linear transformations," or "The central core of grades 1–5 should be Naive Set Theory and an introduction to formalized Intuitionistic Logic."

In the remarks to follow, I wish to present some goals which are less specific and I hope more universally applicable. They are intended to be more or less independent of courses or content; some may help to supply an answer to the person who asks: "Aside from its technological importance, what are the educational values of mathematics?"

GOAL 1: **To provide understanding of the interaction between mathematics and reality.**

The point here is one that has been said many times, but I feel it is important enough to repeat the plea, [1].

> "Regardless of his ultimate interests and career, a student of mathematics ought to understand something about the way in which mathematics is used in applications, and the complicated interaction between mathematics and the sciences. For many mathematicians, engaged in pure research, contact with other sciences may be infrequent; they are apt to see their subject as one that is largely self-sufficient, breeding its own sub-disciplines and creating its own research problems with only occasional stimulus from outside. Others who are more directly involved in neighboring disciplines may find it difficult to agree with this position, and indeed may lay great stress upon the role of the physical sciences as a source for mathematical ideas and techniques."

These apparently opposing views are, of course, merely two sides of the same coin, and it is certainly not fair to a student to present only one. Continuing the same quotation:

> "What is then the role of mathematics in the sciences? We think that mathematics offers the scientist a vast warehouse full of objects, each available as a model for various aspects of physical reality."

By 'model,' we do not mean the plaster form on the shelf but a mathematical structure which represents one of the possible abstractions from the concrete problem. The model can be a system of differential equations, or a group with its associated representations, or a differentiable manifold. A wide variety of illustrations may be found in the September 1964 issue of *Scientific American*, devoted to "Mathematics in the Modern World."

Finally, there is a philosophical question which must be posed. Why is mathematics useful? The physicist E. P. Wigner addressed himself to this in an article [12] with the title: "The unreasonable effectiveness of mathematics in the natural sciences." From our 'model oriented' point of view, a possible answer can be given [1]:

> "The richness and diversity of this supply [of mathematical structures] are central reasons for the importance of mathematics; another is that—along with these objects—mathematics offers a system for using the models to help raise or answer questions about physical reality, as well as techniques for exploring the behavior of the models themselves."

GOAL 2: **To convey the fact that mathematics, like everything else, is built upon intuitive understandings and agreed conventions, and that these are not eternally fixed.**

Ever since Euclid, students have been reminded that mathematics rests upon a collection of undefined terms and accepted axioms. It is perhaps less often pointed out that we must also agree upon a system of deduction. (Anyone in doubt about the last point should reread what the Tortoise said to Achilles! [2]). At the same time, however, I think we should point out the flexibility of these starting points, and that dogmatism in mathematics is not a virtue. An instance can be found in the notions of function and set.

During the last decades of the 19th century, mathematics went through a period of intensive self-examination and purification in the attempt (ultimately vain) to achieve a complete codification of its foundations. The notion of 'set' was selected as a basic undefined term. The rules of the game then require you to impose a collection of axioms, and then to define all later concepts in terms of the primitive ones. The purpose here is not primarily mathematical but philosophical. As mathematicians, we operate in a world of entities whose existence and properties we understand on the intuitive level. The approach of the formalist is to construct models for these entities within a selected system (say 'set theory') which mimic the platonic ideas in our mind, and then to propose these models as *definitions* for the concepts themselves, thus identifying the model with the object.

It is this process which led to the statement which has been so fashionable in the last several decades (mostly in very elementary books): "A function is a certain sort of class of ordered pairs." Indeed, if this is done in the proper 19th century spirit, one should also *define* the ordered pair $\langle a, b\rangle$ to be the set $\{\{a, b\}, \{a\}\}$, and it does not take much effort to see that a formalized presentation of analysis in this fashion is not very rewarding. A function of two variables becomes a class of pairs $\langle\langle a, b\rangle, c\rangle$, each of which has the set description:

$$\{\{\{\{a, b\}, \{a\}\}, c\}, \{\{\{a, b\}, \{a\}\}\}\}.$$

For example, it is tedious to check that another correct description of $\langle\langle a, b\rangle, c\rangle$ is:

$$\{\{\{\{a\}, \{a, b\}\}\}, \{\{\{a\}, \{a, b\}\}, c\}\}.$$

While there are times when it is necessary to work with formal models for 'ordered pair' and 'function,' most mathematicians find it far easier to leave these unformalized, and to treat both 'set' and 'function' as independent primitive concepts, defined only by their properties; the collection of ordered pairs associated with a function is then the *graph* of the function, and is regarded as something you construct from the function, not the function itself. In support of this position, Saunders MacLane recently wrote [7]:

> "For effectiveness in teaching mathematics, it would be more in order to follow a balanced approach, using the intuition of both 'set' and 'function.' Unfortunately, recent American reformers of elementary mathematics teaching have been over-zealous in propagandizing the notion of set. This has led to elementary books grossly overemphasizing sets (and logic). This overemphasis should be reversed: for example, one should no longer preach that a function *is* a certain sort of set of ordered pairs."

But, students should also know that this is not the only possible axiomatization of the foundations of mathematics. In his 1925 thesis [10], von Neumann proposed an alternate development in which "function" is the primitive concept, and "set" is derived. Moreover, it should also be pointed out that recent work in category theory has revived interest in these matters, and that it now appears that another fruitful approach [5] is to adopt "morphism" as the basic primitive. The moral is that there is no single approach, and that coming generations may adopt different views about the axiomatics of mathematics, but that the test is whether they agree with our intuitive picture of mathematics, not whether they are 'right' in some absolute sense.

GOAL 3: **To demonstrate that mathematics is a human activity and that its history is marked by inventions, discoveries, guesses, both good and bad, and that the frontier of its growth is covered by interesting unanswered questions.**

We speak here of the historicity of mathematics. Examples from the past can be given which make this clear, and which show individuals at work and the nature of their contributions. There is, for instance, Decartes' invention of the exponential notation for integral powers, followed by the emotional need to attach meaning to a^x for general choices of x which led to a considerable amount of research work culminating in the general treatment by Euler in 1750 which allowed complex x. Another illustration is to be found in the work of Hamilton who tried for ten years to solve the problem of making appropriate definitions of multiplication and addition so that 3-space became a division algebra, and who at last took the leap into 4-space and discovered quaternions [3]. The effect of Hilbert's 1900 speech on the directions of mathematical activity in the half century which followed is again such an illustration [4]. (In this, it is also interesting to speculate upon the effect of this speech, had Hilbert himself not been one of the major contributors to mathematics during this same period.)

We all recognise that it is difficult to present problems of current interest to an unsophisticated audience. In certain areas, however, this can and has been

done, and it is important for the future development of mathematics and its status in the world that more attention be given to exposition of this sort.

GOAL 4: **To contrast "argument by authority" and "argument by evidence and proof"; to explain the difference between "not proved" and "disproved," and between a constructive proof and a nonconstructive proof.**

In many fields of study, questions are settled by citation of authorities, and even by oratory and polemic. In the sciences, one more often finds argument which is based upon evidence, and is inductive or deductive in structure. Throughout all its history, mathematics has made "proof" itself a subject of study, and has a special contribution to make.

Let us examine a special situation. Suppose someone asks: "Is it possible to . . . ," where the blank is to be filled with some specific objective. In many areas of human concern, the answer may have to be: "No, that is impossible; assuming, of course, that our present understanding of the world is correct." (Unfortunately, too often the last qualification is omitted.)

As I have indicated, answers of this sort must be *time-bound*. Answer for yourself, each of the following:

1845 Can you see through a sheet of metal?
1950 Can argon form chemical compounds?
1965 Can you transmit information at a speed faster than light?

In mathematics, it seems possible for many questions to provide eternally negative answers. In 1965, we do not know how to square a circle; we believe, in fact, that we can prove that no one will ever be able to square the circle. We also believe that we can prove that no one will ever be able to find integers x, y, z such that $x^3+y^3=z^3$, $xyz\neq 0$. Fermat's conjecture, of course, remains "not proved."

For many people, a more interesting example is to be found in the so-called Arrow Paradox. (No relation to Zeno.) The problem posed is to obtain a completely fair method for arriving at a group consensus ranking of a set of alternatives when one has been given a collection of individual rankings of these alternatives. (Each judge has ranked the beauty queen contestants from 1 to 10; what should the final ranking be?) Being objective about this, we list a set of criteria which any admissible system should satisfy. (For example, if *each* judge ranks contestant C higher than contestant D, then the group ranking should also place C higher than D, although the interval need not be the same.) Having completed this, we seek those group ranking schemes which satisfy the criteria. The paradox is that, in spite of the apparent utter reasonableness of the criteria, there is no admissible solution! Here, again, we prove that no one will ever arrive at a completely 'fair' system for deciding such matters [6].

I also feel that the typical non-constructive proof of existence is peculiar to mathematical thought, and that it is therefore important for students to be made aware of this consciously at some time. En passant, it should be remarked

that the recent argument which was used in nuclear physics to deduce the existence of a particle by the *absence* of interactions sounds a lot like this mode of proof!

GOAL 5: **To demonstrate that the question "Why?" is important to ask, and that in mathematics, an answer is not always supplied by merely giving a detailed proof.**

Suppose we have just finished proving an interesting and perhaps somewhat surprising result. Is it meaningful to ask *why* it is true? In many cases, I think it is, and that a correct answer may not be to repeat the proof just given. What I would hope for is an insight which illuminates the entire matter, which goes to the heart and lets the student see why the whole result came out as it did. Such insights are seldom unique, for one may look at a situation from many viewpoints.

Perhaps several illustrations will explain what I mean. In number theory, Euler's generalization of Fermat's theorem asserts that if c is relatively prime to m, then

$$c^{\phi(m)} \equiv 1 \pmod{m}.$$

For me, the clarifying insight is that the integers between 1 and m which are prime to m form a multiplicative group. Since the order of this group is $\phi(m)$, the Euler relation becomes an instance of Lagrange's theorem on groups.

As another illustration, suppose we have just shown by the usual process that the real and imaginary parts of a holomorphic function obey the Cauchy-Riemann equations. Is there a viewpoint which makes this look sensible? For me, the following has been helpful. Suppose the function is given by $f: z \rightarrow w$, with $z = (x, y)$ and $w = (u, v)$. The differential of this mapping is represented by the matrix

$$df = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}.$$

The Cauchy-Riemann equations assert: $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. Looking at df, this means that the matrix has the form

$$df = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

The illuminating insight is that matrices of this form are precisely those which arise in the representation of the complex field as an algebra of real 2-by-2 matrices. (The actual connection is made by re-examining the nature of the differentiation process for complex valued functions.)

Finally, the change of variable formula for multiple integrals in elementary calculus is made more transparent for me by the following observations. Sup-

pose that T is an orientation preserving homeomorphism of class C', defined in a region R in n-space, and let D be a compact quadrable set in R, and D^* its image under T. Let f be a continuous function defined on D^*. We want to understand why it is true that

$$\int \cdots \int_{D^*} f = \int \cdots \int_D f(T(x))J(x)\, dx_1 dx_2 \cdots dx_n,$$

where $J(x)$ is the Jacobian of T at x.

First, introduce a set function F for suitable sets $S \subset R$ by

$$F(S) = \int \cdots \int_{T(S)} f.$$

(Note that we want to calculate the value of $F(D)$). As a general rule (the fundamental theorem of calculus), set functions are the integrals of their derivatives. If g is the point function which is the derivative of the set function F, i.e.

$$g(x_0) = \lim \frac{F(S)}{\text{vol}\,(S)},$$

where the limit is taken as the set S closes down on x_0, then we would have for any set S,

$$F(S) = \int \cdots \int_S g.$$

(Looking at the answer, we want to show $g(x) = f(T(x))J(x)$.) If $T(x_0) = y_0$ and S is closing down on x_0, then $T(S)$ is closing down on y_0. When $T(S)$ is small enough, f is essentially constantly $f(y_0)$ on $T(S)$, so that $F(S)$ is approximately $f(y_0)$ vol$(T(S))$. This gives us

$$g(x_0) = f(y_0) \lim \frac{\text{vol}\,(T(S))}{\text{vol}\,(S)}.$$

Now, the differential of T, dT, is a linear transformation which gives a local approximation to T. Linear transformations alter the volumes of sets by a multiplicative factor, namely their determinant. But, the determinant of dT is J, so we arrive at

$$\begin{aligned} g(x_0) &= f(y_0) \lim \frac{J(x_0)\,\text{vol}\,(S)}{\text{vol}\,(S)} \\ &= f(T(x_0))J(x_0). \end{aligned}$$

(In a simplified form, and with the details supplied, this approach is to be found in [1], 300.)

GOAL 6: **To show that complex things are sometimes simple, and simple things are sometimes complex; and that, in mathematics as well as in other fields, it pays to subject a familiar thing to detailed study, and to study something which seems hopelessly intricate.**

The physical sciences have made much of their progress by repeating older experiments with greater and greater precision. In a similar way, progress in mathematics has often been made by re-examining some apparently well-worked vein with totally new tools and viewpoint.

Surely, the nature of the real numbers system and the elements of calculus are sterile ground! The classical work of Weierstrass and others succeeded in erasing the embarrassing unrigor of Leibnitz and Cauchy, and we know where we stand. Nevertheless, those with an ear to the ground and an eye on Mathematical Reviews know that "infinitesimals" have once more crept out into the open, and with new and impressive credentials! The subject of nonstandard analysis owes its origin to recent work in mathematical logic, model theory, and the study of syntactical descriptions. The nonstandard reals not only have infinitesimals, but also their reciprocals, and we can construct tangents in nonstandard calculus by drawing secants through points which are infinitely close! Moreover, it even seems possible to work within nonstandard analysis and obtain solutions to previously unanswered problems. If significant new results in classical mathematics continue to emerge from beneath the cloak of this heresy, it may become necessary for us to re-examine the nature of the elementary calculus we teach in colleges. (See [8] and [9].)

In another direction, the amorphous collection of all social games must surely seem an unlikely arena in which to employ the tools of mathematics. The simple steps by which von Neumann analysed this, to emerge at the end [11] with the common abstraction of a game in normalized form, about which certain general statements could be made, is a perfect example of the last sentence in Goal 6. Another example on a much simpler level occurs when a student replaces

$$\left(\frac{2x^2+5}{x+1}\right)^2 + 4\,\frac{2x^2+5}{x+1} + 4$$

by

$$y^2 + 4y + 4$$

in order to obtain the simplification

$$\frac{(2x^2+2x+7)^2}{(x+1)^2}\,.$$

In another sense, the same pattern occurs whenever one divides out the radical of an algebra to get something one may hope to work with more easily, postponing the analysis of the radical and the original algebra until later. In applications, it often pays to isolate certain components of a complex structure

and study their interaction, before attempting to investigate the internal structure of each of the components.

I hope by these remarks that I have succeeded in showing that mathematics need be neither austere nor remote, that it has relevance for almost all human activities, and that it can be of value to any sincerely interested person.

References

1. R. C. Buck and E. F. Buck, Advanced Calculus, 2nd ed., McGraw-Hill, New York, 1965.

2. Lewis Carroll, What the Tortoise said to Achilles, The World of Mathematics, Simon & Shuster, New York, 1956, pp 2402–2404.

3. W. R. Hamilton, Lectures on Quaternions, Dublin, 1853.

4. David Hilbert, Mathematical problems, Bull. Amer. Math. Soc., (1902) 437–479.

5. F. W. Lawvere, An elementary theory of the category of sets, Proc. Nat. Acad. Sci., 52 (1964) 1506–1510.

6. R. Duncan Luce and Howard Raiffa, Games and Decisions, Wiley, New York, 1957 (Chap. 14).

7. S. MacLane, in Proc. Prelim. Meeting on College Level Math. Educ., US-Japan Prog. Sci. Coop., Katada, 1964, pp. 68, 69.

8. A. Robinson, Non-standard analysis, Nederl. Akad. Wetensch., 23 (1961) 432–440.

9. A. Robinson, Introd. to Model Theory and the Metamathematics of Algebra, North Holland, Amsterdam, 1963 (Chap. X).

10. John von Neumann, Die Axiomatisierung der Mengenlehre, Math. Zeit., 27 (1928) 669–752.

11. J. von Neumann and O. Morgenstern, The Theory of Games and Economic Behavior, Princeton University Press, Princeton, 1947.

12. E. P. Wigner, The unreasonable effectiveness of mathematics in the natural sciences, Comm. Pure Appl. Math., 13 (1960) 1–14.

A CLASS OF INTEGRAL EQUATIONS

A. G. MACKIE, Victoria University of Wellington, New Zealand

In a recent paper Buschman [1] solved the integral equation

$$\int_1^x P_n\left(\frac{x}{x_0}\right) g(x_0)\, dx_0 = f(x), \tag{1}$$

where $x > 1$ and P_n is a Legendre polynomial. In a later note Erdélyi [2] gave a more direct proof of the inversion formula with fewer restrictions on the function $f(x)$. These papers prompted the author to examine whether it would be possible to solve the integral equation (1) when the limits in the integral are from 0 to x instead of 1 to x. Because of the singularity at $x_0 = 0$ there will need to be certain restrictions on the f and g functions to ensure convergence of the integral but once these are met it is in fact possible to solve for $g(x_0)$ explicitly. Moreover, the solution is valid for any real value of n and not merely for positive integral values.

A very similar integral equation can arise in the following way. Let us consider the equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2N}{r}\frac{\partial\phi}{\partial r} - \frac{\partial^2\phi}{\partial t^2} = 0 \tag{2}$$

which may be thought of as the generalized radially symmetric wave equation in $2N+1$ variables. Formally, however, we do not need to take $2N$ integral. If Cauchy data

$$\phi = 0, \qquad \frac{\partial\phi}{\partial t} = g(r)$$

are given on $t=0$, then the solution at a point (r_0, t_0), where $0 \leqq t_0 \leqq r_0$, is given by Riemann's method as

$$\phi(r_0, t_0) = \frac{1}{2}\int_{r_0-t_0}^{r_0+t_0} R(r, 0; r_0, t_0) g(r)\, dr,$$

where $R(r, t; r_0, t_0)$ is the Riemann function for equation (2). In particular when $t_0 = r_0$ we have

$$\phi(r_0, r_0) = \frac{1}{2}\int_0^{2r_0}\left(\frac{r}{r_0}\right)^N P_{N-1}\left(\frac{r}{2r_0}\right) g(r)\, dr, \tag{3}$$

since this is the form (see, for example, Copson [3]) to which the Riemann function reduces when $t=0$ and $t_0=r_0$. If $\phi(r_0, r_0)$ is regarded as known, then (3) is essentially the integral equation (1) for the unknown function g but with the lower limit on the integral 0 instead of 1 and with the argument of the Legendre function inverted.

The manner in which this integral equation has arisen suggests the method by which it can be solved. We are given ϕ, say $\phi = f(r)$, on the characteristic $r=t$ and also we are given that $\phi=0$ on $t=0$. We seek the unknown function $g(r)$ which is the value of $\partial\phi/\partial t$ on $t=0$. We can find $g(r)$ by setting up the characteristic boundary value problem for ϕ in which $\phi = f(r)$ on $r=t$ and $\phi = -f(r)$ on $r=-t$, thus satisfying $\phi=0$ on $t=0$ automatically by symmetry. This boundary value problem also can be solved by using the Riemann function and the result of differentiating with respect to t and then setting $t=0$ is the solution of the integral equation.

We now proceed to the details. Equation (2) was used to supply the motivation but we can now establish a more general result. Moreover, the details are less cumbersome if we work throughout in characteristic co-ordinates. Accordingly, we shall consider the self-adjoint equation

$$\frac{\partial^2\phi}{\partial x\partial y} + c(x, y)\phi = 0, \tag{4}$$

where $c(x, y)$ is a function of x and y for which

$$c(y, x) = c(x, y). \tag{5}$$

The Riemann function $R(x, y; x_0, y_0)$ of equation (4) will satisfy equation (4) as a function either of x, y or of x_0, y_0. It is unchanged when x is interchanged with x_0 and y with y_0 simultaneously. Finally R has the value unity when $x=x_0$ for all y and when $y=y_0$ for all x.

If Cauchy data

$$\phi = 0, \qquad \frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} = 2g(x), \tag{6}$$

are given on $y=x$, $x>0$, where the factor 2 is included for later convenience, then the solution at a point $(x_0, 0)$ is given by a standard application of Riemann's method in the form

$$f(x_0) = \int_0^{x_0} R(x, x; x_0, 0)g(x)\,dx. \tag{7}$$

Let us now consider the boundary value problem in which $\phi=f(x)$ on $y=0$ $(x>0)$ and $\phi=-f(y)$ on $x=0$ $(y>0)$. Because of (5), symmetry ensures that $\phi=0$ on $y=x$. This characteristic boundary value problem also can be solved by Riemann's method and the solution is

$$\phi(x_0, y_0) = \int_0^{x_0} R(x, 0; x_0, y_0)f'(x)\,dx - \int_0^{y_0} R(0, y; x_0, y_0)f'(y)\,dy.$$

We must now form $\partial\phi/\partial x_0 - \partial\phi/\partial y_0$ and equate this to $2g(x_0)$ from (6). Contributions will come both from the limits and from differentiating the integrands. When use is made of the existing symmetry we obtain

$$g(x_0) = f'(x_0) + \int_0^{x_0} \left[\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0}\right) R(x, 0; x_0, y_0)\right]_{y_0=x_0} f'(x)\,dx, \tag{8}$$

since $R(x_0, 0; x_0, x_0)=1$ by definition. This, then, is the solution of the integral equation (7). The class of integral equations thus solved is formed by the set of Riemann functions of equations such as (4) provided (5) is satisfied. We illustrate by two examples, the first of which leads to the solution of our original equation involving Legendre functions.

If $c(x, y) = -n(n+1)/(x+y)^2$, where n is a constant, it can be shown [**3**] that the Riemann function is

$$R(x, y; x_0, y_0) = P_n\left\{1 + \frac{2(x - x_0)(y - y_0)}{(x + y)(x_0 + y_0)}\right\}.$$

Thus the associated integral equation (7) becomes

$$f(x_0) = \int_0^{x_0} P_n\left(\frac{x}{x_0}\right) g(x)\, dx \tag{9}$$

and the solution (8) gives

$$g(x_0) = f'(x_0) + \frac{1}{x_0}\int_0^{x_0} f'(x) P_n'\left(\frac{x_0}{x}\right) dx$$

or

$$g(x_0) = \frac{1}{x_0^2}\int_0^{x_0} P_n\left(\frac{x_0}{x}\right)\{x^2 f''(x) + 2xf'(x)\}\, dx \tag{10}$$

on integration by parts. Equation (10) therefore gives the solution of the integral equation (9). If we write

$$F(x) = x^2 f''(x) + 2xf'(x) = \frac{d}{dx}\{x^2 f'(x)\} \tag{11}$$

and $G(x_0) = x_0^2 g(x_0)$ we obtain from (10) a second integral equation

$$G(x_0) = \int_0^{x_0} P_n\left(\frac{x_0}{x}\right) F(x)\, dx$$

which differs from (9) in that the argument of the Legendre polynomial is inverted and is really equation (1) with the lower limit 1 replaced by 0. By (9) the solution of this is

$$f(x_0) = \int_0^{x_0} P_n\left(\frac{x}{x_0}\right)\frac{G(x)}{x^2}\, dx$$

or, from (11) after some simplification,

$$F(x_0) = \frac{d}{dx_0}\int_0^{x_0} \frac{x_0}{x^2} P_n\left(\frac{x}{x_0}\right)\{xG'(x) - G(x)\}\, dx.$$

As a second example we select $c(x, y) = m^2$, where m is a constant. The equation which results is the equation of telegraphy or, with suitable variable changes, the damped wave equation and the Riemann function [**3**] is

$$R(x, y; x_0, y_0) = J_0\{2m\surd[(x_0 - x)(y_0 - y)]\}.$$

When the necessary substitutions are made in (7) and (8) we recover the result that the solution of $f(x_0) = \int_0^{x_0} I_0\{2m\surd[x(x_0 - x)]\} g(x)\, dx$ is

$$g(x_0) = \frac{1}{x_0}\int_0^{x_0} J_0\{2m\surd[x_0(x_0 - x)]\}\{xf''(x) + f'(x)\}\, dx.$$

We have assumed in the previous work that the functions f and g are bounded and differentiable as required. In practice the restrictions on them are fairly

obvious since the requirements for the various integrals to exist are easily worked out from the asymptotic forms of the Legendre and Bessel functions. The formulae which involve $f''(x)$ can be written in forms which do not require $f(x)$ to be twice differentiable.

Finally we note that a further class of integral equations and their solutions is obtained by replacing the boundary conditions (6) by

$$\phi = g(x), \qquad \frac{\partial \phi}{\partial x} - \frac{\partial \phi}{\partial y} = 0.$$

The characteristic boundary value problem has now $\phi = f(x)$ on $y=0(x>0)$ and $\phi = f(y)$ on $x=0(y>0)$.

Since completing this note the author has become grateful to Professor Erdélyi for sending him the typescript of a paper which is shortly to appear in the SIAM Journal. In this paper the techniques of fractional integration are used to solve an integral equation involving Legendre functions and a special case of this is one of the results contained here. In a note appended to Professor Erdélyi's paper are to be found some interesting references to work involving integral equations of this general type. The method used in the present note does not appear to have been considered by any of the authors of these various papers.

References

1. R. G. Buschman, An inversion integral for a Legendre transformation, this MONTHLY, 69 (1962) 288–289.

2. A. Erdélyi, An integral equation involving Legendre's polynomial, this MONTHLY, 70 (1963) 651–652.

3. E. T. Copson, On the Riemann-Green function, Arch. Rational Mech. Anal., 1 (1958) 324–348.

FURTHER REMARKS ON CONCENTRIC POLYGONS

DAVID MERRIELL, University of California, Santa Barbara

1. Introduction. An article on concentric polygons, defined as polygons which have the same centroid, appeared recently in this MONTHLY [7]. There is a considerably larger literature on this subject than was indicated in that article. The purpose of this paper is to give some of the history of the subject and to state some additional theorems concerning particular polygons which should be of interest to teachers of elementary geometry.

2. Historical background. The earliest known theorem concerning concentric polygons was that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram. According to Tropfke, this theorem was first recorded by Varignon in 1731. It seems likely, however, that it was known to the Greeks. The next theorem to be discovered was that the centers of equilateral triangles, constructed either inwardly or outwardly on the sides of any triangle,

are vertices of an equilateral triangle. Although this theorem has been credited to Napoleon Bonaparte, there seems to be no definite historical evidence for this attribution. Two proofs of the theorem appeared in 1863 in [5] and another in 1870 in [6].

The key to the generalization of these theorems and to the discovery of many new properties of elementary geometric figures in the nineteenth century lay in the development of vector methods. Although forgotten today, the first of these was the method of equipollences invented by the self-taught Italian mathematician Giusto Bellavitis in 1832. His work was translated into French and popularized by C. A. Laisant and others. It particularly influenced the work of French and Belgian geometers of the time. Essentially, the method involves the use of complex coordinates to represent points in the plane.

In 1877, Laisant published an application of the method of equipollences to polygons [8]. The results for the triangle and quadrilateral were: (1) If P, Q, R are the centers of squares constructed on the sides AB, BC, CA of any triangle, then the segments AQ and PR are perpendicular and equal in length, and similarly for the segment pairs BR, PQ and CP, QR; (2) The centers of squares constructed on the sides of any quadrilateral are vertices of a quadrilateral whose diagonals are perpendicular and equal in length. (The first theorem may be considered a special case of the second where two of the vertices of the given quadrilateral coincide.) These theorems were proved about the same time by H. van Aubel of Antwerp [1] and by J. Neuberg of Liège [11]. They were also included in 1891 in a paper by Collignon [3] who apparently was unaware of their earlier appearance. Among other theorems in this paper was Theorem 4 of [7].

The idea of constructing similar triangles on the sides of a general polygon goes back at least to 1870 [6]. The general case was discussed by Laisant in [8, 9] who observed the concentric property of the polygon of vertices of the triangles. The general theorem which was attributed to Douglas in [7] was actually first proved by K. Petr of Prague in 1905 [14] using complex coordinates. Theorem 3 of [7] first appeared in [16].

The bibliography given in [7] and in this paper indicates the interest in these problems. There are doubtless other sources which should be included for completeness.

3. Pentagons. In the sections which follow, an n-gon with vertices P_1, $P_2, \cdots, P_n$ will be denoted by Π or by $\{P_j\}$. The notation of [7] will be used. In particular, P_j will designate both the vertex and its representation in the complex plane. Any subscripts exceeding n are to be reduced modulo n. The proofs which are given illustrate a general method of proof.

THEOREM 1 (DOUGLAS). *Let* $\{P_j\}$ *be any pentagon. Designate the median of the triangle* $P_{j-1}P_jP_{j+1}$ *from the vertex* P_j *by* P_jM_j. *Shorten each median to* Q_j *by* $1/\sqrt{5}$ *times its length so that* $Q_jM_j=P_jM_j/\sqrt{5}$. *On the sides of the new pentagon* $\{Q_j\}$ *construct isosceles triangles outwardly with vertex angle* 72°. *The vertices of*

the isosceles triangles form a regular convex pentagon. If the medians are lengthened by $1/\sqrt{5}$ *times their length and isosceles triangles with vertex angle* 144° *are constructed on the sides of the new pentagon, the vertices of the isosceles triangles form a regular star pentagon.*

THEOREM 2 (DOUGLAS). *Let* $\{P_j\}$ *be any pentagon and define the median of the pentagon from* P_j *to be the segment joining* P_j *to the midpoint* M_j *of* $P_{j+2}P_{j+3}$. *Extend each median to* Q_j *by* $1/\sqrt{5}$ *times its length, so that* $M_jQ_j = P_jM_j/\sqrt{5}$. *On the sides of the new pentagon* $\{Q_j\}$ *construct isosceles triangles with vertex angle* 72°. *The vertices of the triangles form a regular convex pentagon. If the medians are shortened by* $1/\sqrt{5}$ *times their length and isosceles triangles with vertex angle* 144° *are constructed on the sides of the new pentagon, the vertices of the triangles form a regular star pentagon.*

Proof. Since

$$M_j = (P_{j+2} + P_{j+3})/2,$$
$$Q_j = M_j + (M_j - P_j)/\sqrt{5} = (P_{j+2} + P_{j+3})(5 + \sqrt{5})/10 - P_j/\sqrt{5}.$$

The construction of isosceles triangles on the sides of $\{Q_j\}$ is the construction of the c-derived polygon $\{R_j\}$ for $c = (1 - i \cot 36°)/2$. If a similar construction with inward isosceles triangles of vertex angle 72° is carried out on the sides of $\{R_j\}$ and the resulting vertices coincide, then $\{R_j\}$ is a regular convex pentagon. The second construction is the derived polygon $\{S_j\}$ for $\bar{c} = (1 + i \cot 36°)/2$. Thus

$$S_j = (1 - c - \bar{c} + c\bar{c})Q_j + (c + \bar{c} - 2c\bar{c})Q_{j+1} + c\bar{c}Q_{j+2}.$$

Since $\cos 36° = (1 + \sqrt{5})/4$, one can verify that $c\bar{c} = (5 + \sqrt{5})/10$. Substituting for Q_j, Q_{j+1} and Q_{j+2} and simplifying, using $P_{j+5} = P_j$, one finds that $S_j = (\sum_{j=1}^{5} P_j)/5$. Thus the vertices of $\{S_j\}$ coincide in the centroid of $\{P_j\}$. A similar calculation establishes the second part of the theorem.

4. Hexagons.

THEOREM 3 (LAISANT). *On the sides of any hexagon, construct isosceles triangles outwardly with vertex angle* 120°. *The vertices of these triangles form a new hexagon. The midpoints of the diagonals joining opposite vertices of the new hexagon form an equilateral triangle.*

Proof. Let the given hexagon be $\{P_j\}$. The new hexagon is $\{Q_j\}$ where $Q_j = \bar{c}P_j + cP_{j+1}$ and $c = (1 - i/\sqrt{3})/2$. The midpoint of Q_jQ_{j+3} is

$$M_j = (Q_j + Q_{j+3})/2 = [\bar{c}(P_j + P_{j+3}) + c(P_{j+1} + P_{j+4})]/2.$$

On the sides of the triangle $\{M_j\}$, construct inwardly isosceles triangles of vertex angle 120°. The vertices are given by

$$N_j = cM_j + \bar{c}M_{j+1}$$
$$= [c\bar{c}(P_j + P_{j+2} + P_{j+3} + P_{j+5}) + (c^2 + \bar{c}^2)(P_{j+1} + P_{j+4})]/2.$$

Since $c\bar{c}=1/3$ and $c^2+\bar{c}^2=1/3$, $N_j=(\sum_{j=1}^{5} P_j)/5$, and $\{M_j\}$ is equilateral.

THEOREM 4 (ZACHARIAS). *On the sides of any hexagon, construct equilateral triangles outwardly. The vertices of these triangles form a new hexagon. The midpoints of the sides of this hexagon form a third hexagon whose main diagonals are equal and intersect at* 60°. (*The order of the two constructions can be reversed.*)

Proof. Let the given hexagon be $\{P_j\}=\Pi$ and the third hexagon be $\{R_j\}=\Pi_{c,d}$ where $c=(1-i\sqrt{3})/2$ and $d=1/2$. Then $R_j=(\bar{c}P_j+P_{j+1}+cP_{j+2})/2$. Using the relations $c\bar{c}=1$ and $\bar{c}^2=-c$, the main diagonals are $R_{j+3}-R_j=\bar{c}(R_{j+2}-R_{j-1})$. Since $|\bar{c}|=1$ and multiplication by $\bar{c}$ induces a rotation of 60°, the diagonals are equal and inclined at 60°.

THEOREM 5 (DOUGLAS). *Let* Π *be any hexagon and obtain the hexagon* Π' *by taking the midpoints of the sides of* Π. *Obtain a third hexagon* Π'' *by taking the centroids of the triangles of consecutive vertices of* Π'. *Then the vertices of equilateral triangles constructed on the sides of* Π'' *form a regular hexagon.* (*The order of the three constructions can be permuted.*)

5. Octagons.

THEOREM 6 (LAISANT). *Construct squares outwardly on the sides of any octagon. Their centers form a new octagon. Let* $\{M_j\}$ *be the midpoints of the four main diagonals of the new octagon. The diagonals of the quadrilateral* $\{M_j\}$ *are perpendicular and equal in length.*

Proof. Let the given octagon be $\{P_j\}$. The centers of the squares on its sides are $Q_j=\bar{c}P_j+cP_{j+1}$ where $c=(1-i)/2$. By definition

$$M_j = (Q_j + Q_{j+4})/2 = (\bar{c}P_j + cP_{j+1} + \bar{c}P_{j+4} + cP_{j+5})/2.$$

Then

$$M_{j+2} - M_j = [\bar{c}(P_{j+2} + P_{j+6} - P_j - P_{j+4}) + c(P_{j+3} + P_{j+7} - P_{j+1} - P_{j+5})]/2.$$

Since $i\bar{c}=-c$ and $\bar{c}=ic$, it follows that $M_4-M_2=i(M_3-M_1)$. Hence M_2M_4 and M_1M_3 are perpendicular and equal in length.

6. The centroid of an n**-gon.** The constructions in Theorems 1 and 2 give methods for obtaining concentric polygons which are different from c-derived polygons. A general construction which includes these as special cases is given in the next theorem.

THEOREM 7. *Let* $\{P_j\}$ *be any* n*-gon and let* M_j *be the centroid of the* k*-gon* $P_{j+m_1}P_{j+m_2}\cdots P_{j+m_k}$ *where* $k\leqq n$, $m_i\geqq 0$, $j+m_i$ *is reduced modulo* n *if greater than* n, *and the points* P_{j+m_i} *are distinct. Extend* P_jM_j *to* Q_j *by* r *times its length. Then the* n*-gon* $\{Q_j\}$ *is concentric with* $\{P_j\}$.

Proof. By the construction, $Q_j=M_j+r(M_j-P_j)=(1+r)(\sum_{i=1}^{k} P_{j+m_i})/k-rP_j$. Then

$$\sum_{j=1}^{n} Q_j = (1+r)\left(k \sum_{j=1}^{n} P_j\right) \Big/ k - r \sum_{j=1}^{n} P_j = \sum_{j=1}^{n} P_j.$$

If complex values are allowed for r, then the construction of the c-derived polygon is a special case of the above construction for $k=1$, $M_j=P_{j+1}$, and $r=c-1$.

An interesting application of this construction occurs in the next theorem.

THEOREM 8. *Let $\Pi=\{P_j\}$ be any n-gon and let M_j be the centroid of the $(n-1)$-gon obtained from Π by omitting the vertex P_j. Shorten the segment P_jM_j to Q_j by $1/n$ times its length, so that $Q_jM_j=P_jM_j/n$. Then all the points Q_j coincide in the centroid of Π.*

Proof. Here $r=-1/n$ and $k=n-1$, so

$$Q_j = (1 - 1/n)\left(\sum_{i\neq j} P_i\right) \Big/ (n-1) + P_j/n = \left(\sum_{j=1}^{n} P_j\right) \Big/ n.$$

This theorem is a generalization of the theorem that the centroid of a triangle is located on the median at a distance which is $1/3$ of the length of the median from the midpoint of the side of the triangle. The theorem gives a simple construction for locating the centroid of any n-gon $\{P_j\}$ in $n-1$ steps:

(1) Locate the midpoint C_1 of P_1P_2.

(2) Locate the point C_2 on C_1P_3 such that $C_1C_2=C_1P_3/3$.

(3) Locate the point C_3 on C_2P_4 such that $C_2C_3=C_2P_4/4$.

Continuing this construction, we see that the point C_{n-1} on $C_{n-2}P_n$ such that $C_{n-2}C_{n-1}=C_{n-2}P_n/n$ is the centroid of $\{P_j\}$.

References

1. H. van Aubel, Note concernant les centres des carrés construits sur les côtés d'un polygone quelconque, Nouv. Corresp. Math., 4 (1878) 40–44.
2. ———, Question 963, Mathesis, 6 (1896) 94–95.
3. É. Collignon, Sur certaines séries de triangles et de quadrilatères, Assoc. Franç. Avance. Sci., 20 (1891) 38–66.
4. Jesse Douglas, A theorem on skew pentagons, Scripta Math., 25 (1960) 5–9.
5. W. Fischer, Ein geometrischer Satz, Arch. Math. Phys., 40 (1863) 460–462.
6. É. Françoise, Application du calcul des équipollences à la résolution d'un problème de géométrie élémentaire, Nouv. Ann. Math., 9 (1870) 66–73.
7. P. J. Kelly and D. Merriell, Concentric polygons, this MONTHLY, 71 (1964) 37–41.
8. C. A. Laisant, Sur quelques propriétés des polygones, Assoc. Franç. Avance. Sci., 6 (1877) 142–154.
9. ———, Théorie et applications des équipollences, Gauthier-Villars, Paris, 1887.
10. C. Müsebeck, "Zu Über einen Lehrsatz vom Sechseck," Math. Naturw. Blätter, 2 (1905) 125.
11. J. Neuberg, Quelques propriétés du triangle, Nouv. Corresp. Math., 4 (1878) 142–146.
12. ———, Notes de géométrie, Assoc. Franç. Avance. Sci., 22 (1893) 26–45.
13. ———, Sur quelques quadrilatères spéciaux, Mathesis, 4 (1894) 268–271.
14. K. Petr, Ein Satz über Vielecke, Arch. Math. Phys., 13 (1908) 29–31.
15. V. Thébault, Quadrilatère bordé de carrés, Assoc. Franç. Avance. Sci., 56 (1932) 78–82.

16. ———, Quadrangle bordé de triangles isoscèles semblables, Ann. Soc. Sci. Bruxelles, 60 (1940) 64–70.
17. ———, Octogone bordé de carrés, Mathesis, 54 (1940) 114–117.
18. ———, Polygone de $2n$ côtés bordé de triangles isoscèles semblables, Mathesis, 54 (1940) 161–166.
19. M. Zacharias, Über einen Lehrsatz vom Sechseck, Math. Naturw. Blätter, 2 (1905) 86.
20. ———, Elementargeometrie und elementare nicht-Euklidische Geometrie in synthetische Behandlung, Encykl. der Math. Wiss. III Ab. 9, Teubner, Leipzig, 1913.
21. J. Garfunkel and S. Stahl, The triangle reinvestigated, this MONTHLY, 72 (1965) 12–20.

A NOTE ON DIMENSIONAL ANALYSIS

RUDOLF KURTH, Michigan State University

The method of dimensional analysis for establishing quantitative physical hypotheses has received attention again and again. Two recent outstanding contributions to the clarification of its foundations are the papers of Brand [2] and Drobot [3]. The present note tries to combine the latter's axiomatical approach (in Section I) with the former's simple algebra (in Section II) which, I feel, becomes more perspicuous by the logarithmic notation used in this note. Some of the axioms are similar to Birkhoff's "Assumptions" in his book [1].

I. Axioms.

1. "Physical quantity" is taken as an undefined primitive notion.

AXIOMS: I.1. *The positive real numbers are physical quantities, the nonpositive real numbers are not.*

I.2. *In the set Q of all physical quantities a binary operation is defined which is called "multiplication" (and denoted correspondingly) and with respect to which Q is a commutative group. For positive real numbers the operation is identical with their numerical multiplication.*

Examples: angle, area, velocity, etc.

The exclusion of the nonpositive numbers, though not necessary, appears both mathematically convenient and physically appropriate —"measurement," in a strict sense, can only yield a positive number as its result.

2. Thus, if q is a physical quantity and α is a positive real number, then αq is a physical quantity. Any two physical quantities q and r for which qr^{-1} is a positive real number are said to be "comparable." Comparability is an equivalence relation and defines, therefore, a partition of the set Q into classes $C_1, C_2, \cdots$ of comparable physical quantities. The set of all these classes is a homomorphic image of the group Q and, therefore, a commutative group itself.

3. Let u be any fixed physical quantity belonging to the class C, and q be an arbitrary physical quantity of this class. Then u is called a "unit" of (the

physical quantities belonging to) the class C, and the positive real number $\lambda = qu^{-1}$ is said to be the "measure" of q (with respect to the unit u). The units form a group which is isomorphic to that of the classes C.

4. Let u and $\bar{u}$ be any two units of the same class C, and λ and $\bar{\lambda}$ be the measures of a physical quantity q of the class C with respect to these units. Then

$$\bar{u} = \tau u,$$

where τ is a positive real number, and

$$\bar{\lambda} = \tau^{-1}\lambda.$$

5. Axioms: II.1. *In any class C of physical quantities a binary operation is defined which is called "addition" (and denoted correspondingly). C is closed under this operation which is commutative and associative.*

II.2. *If q and r are any two physical quantities belonging to the same class and s is any third physical quantity, then*

$$(q + r)s = qs + rs.$$

6. Axioms: III.1. *If any two powers of a physical quantity q with distinct exponents belong to the same class, then q is a real number.*

III.2. *There is a finite set of distinct classes $B_1, B_2, \cdots, B_s$ of physical quantities of the following kind. Let $b_1, b_2, \cdots, b_s$ be units of these classes, and u be a unit of any class C. Then the transformation of the units b:*

$$\bar{b}_\sigma = \tau_\sigma b_\sigma, \qquad \sigma = 1, 2, \cdots, s,$$

where the τ's are any positive real numbers, implies the transformation

$$\bar{u} = \tau_1^{\alpha_1} \tau_2^{\alpha_2} \cdots \tau_s^{\alpha_s} u$$

of the unit u where the α's are constant real numbers which are specific for the class C of the unit u.

The s-vector $(\alpha_1, \alpha_2, \cdots, \alpha_s)$ is called the "dimension" of the class C, or of the physical quantities of the class C, with respect to the basis $\{B_1, B_2, \cdots, B_s\}$. The set of all dimensions is an s-dimensional vector-space; it is a homomorphic image of the group of units.

7. The unit transformations

$$\bar{u} = \tau_1^{\alpha_1} \cdots \tau_s^{\alpha_s} u$$

form an s-parametric transformation group operating on the units of any given classes; i.e., if $\{C_1, C_2, \cdots\}$ is a given finite or infinite sequence of classes and $\{u_1, u_2, \cdots\}$ is a corresponding sequence of units, $u_n \in C_n$, then the group of unit transformations operates on the set of all $\{u_1, u_2, \cdots\}$ with $u_n \in C_n$.

There is, by Section 4, an isomorphic group, Γ, of "measure transformations," viz.

$$\bar{\lambda} = \tau_1^{-\alpha_1} \cdots \tau_s^{-\alpha_s} \lambda.$$

8. AXIOMS: IV.1. *If there is a functional relationship between the physical quantities* $q_1, q_2, \cdots, q_n$, *and these quantities only, then it can be given the form*

$$\phi(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0,$$

where ϕ *is a numerical function of the measures* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *of the physical quantities* $q_1, q_2, \cdots, q_n$ *(with respect to a given system of units* $u_1, u_2, \cdots, u_n$*).*

IV.2. *Such a relationship* $\phi(\lambda_1, \cdots, \lambda_n) = 0$ *is invariant under the group* Γ *of measure transformations, i.e., if* $\phi(\lambda_1, \cdots, \lambda_n) = 0$ *and*

$$\bar{\lambda}_\nu = \tau_1^{-\alpha_{\nu 1}} \cdots \tau_s^{-\alpha_{\nu s}} \lambda_\nu, \qquad \nu = 1, \cdots, n,$$

then

$$\phi(\bar{\lambda}_1, \cdots, \bar{\lambda}_n) = 0.$$

Axiom IV.1 may appear as self-evident: when the units have been fixed, any relationship between physical quantities can be only one between their measures. It seems that Axiom IV.2 also is sometimes taken as a matter of course: but I do not think that it is. It is conceivable that a relationship

$$\phi(\lambda_1, \cdots, \lambda_n) = 0,$$

referring to given units, is only a special case of the relationship

$$\Phi(\bar{\lambda}_1, \cdots, \bar{\lambda}_n; \bar{u}_1, \cdots, \bar{u}_n) = 0$$

holding for arbitrary units. Hence, first, IV.2 is independent of IV.1, and secondly, it needs a justification. It seems that this can be given only by physical experience.

Axioms I–III describe the algebra of physical quantities, and Axioms IV may be considered as an exposition of the notion of a physical law.

II. The Pi Theorem.

9. Dimensional Analysis is an application of the Axioms I, III and IV (without II). Its basic problem reads: to determine all the possible functional relationships between n physical quantities $q_1, q_2, \cdots, q_n$ of given classes $C_1, C_2, \cdots, C_n$: i.e., to determine all the equations

$$\phi(\lambda_1, \lambda_2, \cdots, \lambda_n) = 0$$

which are invariant under the group Γ.

It is convenient to introduce the following definitions:

$$A = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1s} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{ns} \end{bmatrix}, \qquad x = \begin{bmatrix} \log \lambda_1 \\ \vdots \\ \log \lambda_n \end{bmatrix}, \qquad y = \begin{bmatrix} \log \tau_1 \\ \vdots \\ \log \tau_s \end{bmatrix},$$

$$\psi(x) = \phi(\lambda_1, \cdots, \lambda_n).$$

Thus all the equations $\psi(x) = 0$ are to be determined which for arbitrary y and $\bar{x} = x - Ay$ imply that $\psi(\bar{x}) = 0$.

10. The solution of this problem is given by the following variant of the Pi Theorem as formulated by Brand [2]:

Let r be the rank of the $n \times s$ "dimension matrix" A, and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a partition of it, such that A_{11} is a nonsingular $r \times r$ matrix; let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where x_1 and y_1 are r-columns, and $\chi(z)$ be an arbitrary function of the variable $(n-r)$-column z. Then, any equation of the form

$$\chi(x_2 - A_{21}A_{11}^{-1}x_1) = 0$$

is invariant under the group Γ. Conversely, if an equation $\psi(x) = 0$ is invariant under Γ, then (by a suitable choice of y_1) it can be put in the form $\chi(z) = 0$, where $z = x_2 - A_{21}A_{11}^{-1}x_1$.

For the proof of either statement Brand's identity

$$A_{22} = A_{21}A_{11}^{-1}A_{12}$$

is used [2]. It is a necessary and sufficient condition that the matrix A have the rank r. The first part of the theorem is shown by direct verification. As to the second part, note that $\psi(\bar{x}) = 0$ or

$$\psi\begin{pmatrix} x_1 - A_{11}y_1 - A_{12}y_2 \\ x_2 - A_{21}y_1 - A_{22}y_2 \end{pmatrix} = 0$$

for any y implies that

$$\psi\begin{pmatrix} 0 \\ x_2 - A_{21}y_1^{(0)} - A_{22}y_2 \end{pmatrix} = 0,$$

where $y_1^{(0)}$ is defined by

$$x_1 - A_{11}y_1^{(0)} - A_{12}y_2 = 0.$$

Hence, by substituting for $y_1^{(0)}$ in $\psi(\cdots)=0$ and applying Brand's identity, the second assertion of the theorem is obtained.

11. One of the so-called "paradoxes of dimensional analysis" [**3**] occurs when $r=n$. Then $A_{21}=0$, $A_{22}=0$, $x_2=0$ and the invariance of the equation $\psi(x)=0$ implies that $\psi(x)$ vanishes identically; i.e., there is no (genuine) invariant relationship between the physical quantities.

Other "paradoxes" have successfully been cleared up by Drobot and Brand in their papers quoted.

References

1. G. Birkhoff, Hydrodynamics, Chapter III, Princeton University Press, 1950.
2. L. Brand, The Pi Theorem, Arch. Rational Mech. Anal., 1 (1957) 35–45.
3. S. Drobot, On the foundations of dimensional analysis, Studia Math., 14 (1953) 84–99.

TWO-DIMENSIONAL HONEYCOMBS

M. N. BLEICHER AND L. FEJES TÓTH, University of Wisconsin

The honeycomb of hive-bees consists of prismatic cells having regular hexagonal openings while the bottom is closed by three equal rhombuses. The cells are situated between two parallel planes filling the space bounded by them without interstices (Fig. 1).

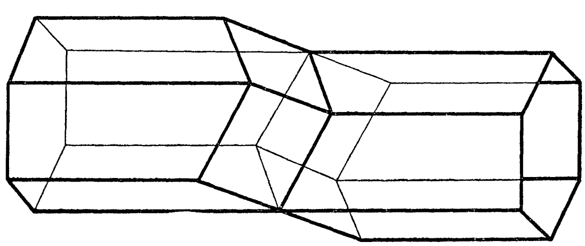


FIG. 1

Pappus tried to explain the hexagonal shape of the bee-cells by the fact that among the regular plane-fillers of equal area (i.e., among the regular triangle, quadrangle and hexagon) the hexagon has the least perimeter. Since generally the depth of a cell is considerably larger than the diameter of its opening, the bottom makes only a small contribution to the surface-area of a cell. Therefore, the observation of Pappus implies that the shape of the bee-cells are favorable

from the point of view of minimizing the surface-area and thus the wax used in constructing a cell of given volume.

In the 18th century, interest turned to the problem of choosing the rhombuses at the bottom of the cells so as to make the surface-area of the cells as small as possible. The result, which amazed the mathematicians and natural scientists, was that, in close accordance with the real shape of the bee-cells, the rhombuses must meet in an angle equal to 120° [1].

After these preliminaries it is reasonable to formulate the isoperimetric problem of the bee-cells as follows [2]: Divide the space bounded by two parallel planes of given distance d into congruent convex cells of given volume v with a distinguished face called its base. Each cell has its base on one of the two planes and has no face in common with the other plane. Consider the cells as open vessels, the surface-area of which is composed only of the area of the internal walls. Find the cell of least surface-area.

Obviously, this problem cannot be considered as being settled by the above results. For in the traditional "solution" first the question of the most economical cross-section (opening), then the question of the most economical bottom-figure are considered separately (both under further special conditions). This is of course by no means justified and it is interesting to note that for all values of d and v we know better cells than those of the bees [3]. The problem seems to be rather difficult, and it has not been solved as yet.

In this paper we concern ourselves with the analogous two-dimensional problem. We define a two-dimensional honeycomb, shortly a *comb*, as a set of congruent convex polygons, called *cells*, lying in a parallel strip without overlapping and without interstices in such a way that

1. Each cell has a side, called *base* (or opening) on one line bounding the strip, but does not have sides on both lines.
2. In the congruence of the cells, their bases correspond to each other.

The distance between the parallel lines is the *width* of the comb.

First we shall give an enumeration of all combs. This will enable us to solve the isoperimetric problem concerning combs, i.e., to select from all polygons of given area generating a comb of given width the one of least perimeter.

We call a comb whose cells have common points with both lines bounding the strip a *reduced comb*. The enumeration of the combs rests on the remark that the cells of a reduced comb are triangles.

In order to see this we note that in a reduced comb each cell has exactly one point on one of the lines bounding the strip. If this point is C, we denote the cell by c_C (Fig. 2). If AB is the base of c_C then the cells adjacent to c_C (and meeting at C) are c_A and c_B. Since the cells are convex, AC and CB are common sides of c_A and c_C, and c_C and c_B, respectively. Thus c_C is the triangle ABC.

Now we consider a nonreduced comb. Let l and m be the lines bounding the comb. Let l' be the line parallel to l and nearest to it which has common points with cells having their bases on m. In a similar way we define m' (Fig. 3). Since

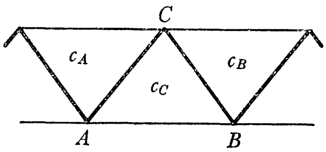


FIG. 2

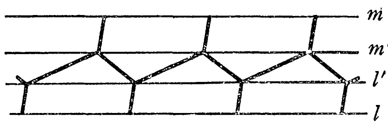


FIG. 3

the cells lying along l are congruent in a base preserving manner, they are either translates or mirror images of one another with respect to a line perpendicular to l. Obviously, the same holds for the parts of the cells lying between l and l'. Furthermore, since these partial cells are convex and they fill the strip bounded by l and l' without overlapping and without gaps, they must be parallelograms.

If l' and m' do not coincide, the part of the comb between l' and m' is a reduced comb and its cells are triangles. If, on the other hand, l' and m' coincide, the cells are parallelograms. Thus we can state the following

THEOREM 1. *A cell of a comb is either a triangle or a parallelogram or the union of a triangle and a parallelogram.*

In the last case the base is the side of the parallelogram opposite to the triangle.

FIG. 4

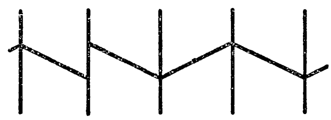


FIG. 5

Obviously, any convex polygon of one of the above types generates a comb. The comb is uniquely determined if the cell is a triangle, except (i) a right triangle having a leg as base (Fig. 4), or the union of a triangle and a parallelogram,

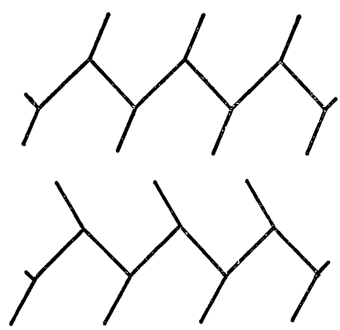


FIG. 6

except (ii) a right triangle and a rectangle adjoining to a leg (Fig. 5), or (iii) an isosceles triangle and a nonrectangular parallelogram adjoining to the base (Fig. 6). There are exactly two combs in case (iii), a denumerable set of combs in

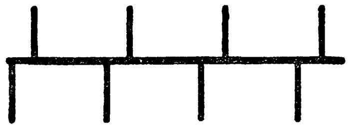


FIG. 7

each of the cases (i) and (ii), a continuous set of combs if the cell is a rectangle (Fig. 7), and two continuous sets of combs if the cell is a nonrectangular parallelogram (Fig. 8).

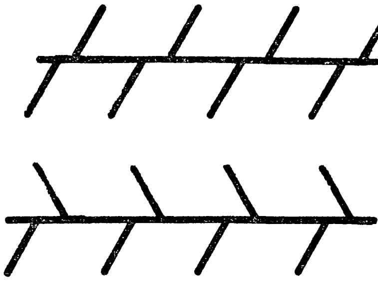


FIG. 8

Before considering the isoperimetric problem, we note that between the width w of a comb, the area a of a cell and its base-length b we have the relation $2a=bw$. This is obvious if the cell is a triangle or a parallelogram. If the cell consists of a triangle and a parallelogram, we transform it into a parallelogram of the same base and area and observe that it generates a comb of the same width as the original cell (Fig. 9). This relation enables us to consider cells of given base-length b generating a comb of given width w, instead of cells with given area.

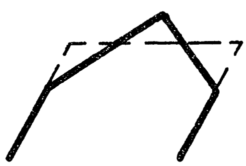


FIG. 9

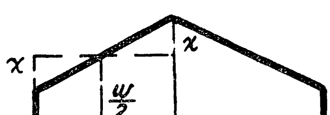


FIG. 10

We start with pentagonal cells. To minimize the perimeter, we can restrict ourself to a cell composed of a rectangle and an isosceles triangle. If the length

of a "vertical" side equals $\frac{1}{2}w-x$ (Fig. 10), then the total length t of the internal sides is

$$t = w - 2x + \sqrt{(b^2 + 16x)}.$$

The condition of a free minimum is

$$\frac{dt}{dx} = -2 + \frac{16x}{\sqrt{[b^2 + 16x]}} = 0, \quad \text{i.e.,} \quad 2x = \sqrt{\left[\left(\frac{b}{4}\right)^2 + x^2\right]}.$$

This means that the angles of the cell different from a right angle must be 120°.

In view of $\frac{1}{2}w-x>0$, this condition is fulfilled only if $b<\sqrt{12}w$. If $b\geqq\sqrt{12}w$ the best cell is a triangle. This completes the proof of

THEOREM 2. *Among the cells of given area generating a comb of given width the cell having the least perimeter is either an isosceles triangle having an angle* $\geqq 120°$ *at its apex or a pentagon composed of a rectangle and an isosceles triangle having an angle equal to* 120° *at its apex.*

References

1. D'Arcy W. Thompson, On Growth and Form I-II, 2nd ed., Cambridge, 1952.
2. L. Fejes Tóth, Regular Figures, Oxford Univ., New York, 1963.
3. ———, What the bees know and what they do not know, Bulletin Amer. Math. Soc., 70 (1964) 468–481.

HISTORICAL NOTE ON A RECURRENT COMBINATORIAL PROBLEM

WILLIAM G. BROWN, University of British Columbia

1. In two forthcoming papers [45, 46] the author enumerates dissections of the disc into triangles and quadrangles. These results generalize a much studied problem whose history will be described briefly in this paper.

A *dissection* of the disc will be a cell complex whose polyhedron is the closed disc B^2. (The reader who is unfamiliar with this concept can consult, for example, Chapter I of *Riemann Surfaces* by L. Ahlfors and L. Sario, Princeton Univ. Press, 1960.)

It will be required that

a) every edge be incident with two distinct vertices (called its *ends*);
b) no two edges have the same ends;
c) every vertex be incident with at least two edges; and
d) every edge not in the boundary of B^2 be incident with two distinct faces (2-cells).

Alternatively, a dissection of the disc can be thought of as a map on the sphere from which one face whose boundary is a simple polygon is removed; (cf. for example, Chapter 21 of *Introduction to Geometry* by H. S. M. Coxeter, Wiley, 1961; or Chapter 3 of *Generators and Relations for Discrete Groups* by

H. S. M. Coxeter and W. O. J. Moser, Springer, 1957). Conditions a) to d) will be retained.

A dissection is *rooted* if an edge in the boundary of B^2 is oriented and designated as the *root*.

Two rooted dissections will be said to be *isomorphic* if there exists a homeomorphism of B^2 with itself carrying vertices and edges of one dissection respectively onto vertices and edges of the other, and preserving the root (including its orientation).

A dissection will be said to be of type $[n, m]_k$ if

(i) each of its faces is incident with exactly k edges;
(ii) exactly n of its vertices lie in the interior of B^2;
(iii) exactly $m+k$ of its vertices lie in the boundary of B^2.

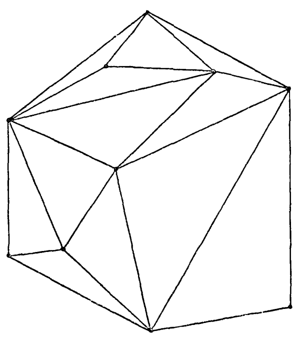


FIG. 1

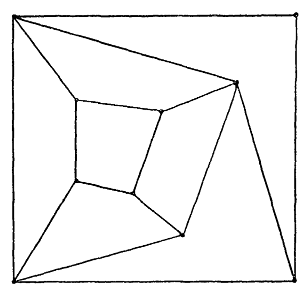


FIG. 2

Examples of dissections of types $[4, 2]_3$ and $[6, 0]_4$ are shown in Figures 1 and 2 respectively. The number, up to isomorphisms, of rooted dissections of type $[n, m]_k$ will be denoted by $D^{(k)}_{n,m}$. In [**45**, **46**] it is shown that

$$D^{(3)}_{n,m} = \frac{2(2m + 3)!(4n + 2m + 1)!}{(m + 2)!m!n!(3n + 2m + 3)!} \tag{1.1}$$

$$D^{(4)}_{n,m} = \begin{cases} \dfrac{3(3p + 4)!(3n + 3p + 2)!}{(2p + 3)!p!n!(2n + 3p + 4)!} & (m \text{ even}; \ p \underset{\text{def}}{=} m/2) \\ 0 & (m \text{ odd}) \end{cases} \tag{1.2}$$

for $n \geqq 0$, $m \geqq 0$.

2. The problem of computing the number of ways of dissecting a convex polygonal region into triangular regions by means of its diagonals appears to have been first posed by Euler to Segner, who developed a recurrence for the numbers which we have denoted by $D^{(3)}_{0,m}$. In his memoir on the subject [2] Segner included a table of values of $D^{(3)}_{0,m}$ for $m < 18$; because of an unfortunate computational error, however, the values for $m > 11$ were incorrect. This latter fact was pointed out by Euler, who published the correct values for $m < 23$; Euler also stated without proof the equivalent of our formula (1.1) for $n = 0$, viz.

$$D_{0,m}^{(3)} = \frac{(2m+2)!}{(m+1)!(m+2)!}$$

The more general problem of computing the number of dissections of a convex n-gonal region into m-gonal regions by means of its diagonals was posed by Pfaff to N. von Fuss, who generalized Segner's recurrence [**3**].

The Euler-Segner problem reappeared in the years 1838–1839 in a series of papers in Liouville's Journal [**4–10** inclusive] in which it was solved in various ways, notably by Binet, who used generating functions, and Rodrigues, who provided a very elegant direct solution. The Pfaff-Fuss problem was again considered in [**11, 12, 13**].

Numerous authors have observed [**5, 7, 21**, etc.] the equivalence of the Euler-Segner problem with the following algebraic problem: In how many ways can a product of n factors be interpreted in a non-commutative non-associative algebra? This latter problem and its generalizations are posed periodically in the Problems section of this MONTHLY [**23, 24, 27, 32, 35, 43**] and have also been considered in [**18, 22, 25, 28, 29, 30, 36, 38, 39, 40**].

The equivalence of the Euler-Segner problem to yet other algebraic and combinatorial problems has been demonstrated in [**21, 26, 30, 31, 34, 41**, etc.].

Generalizations of the Pfaff-Fuss problem (still involving dissections of a polygonal region by its diagonals) have been considered in [**19, 20, 31**]. Further generalizations (allowing internal vertices) were considered at great length by Kirkman [**14, 15, 16**, etc.] who developed numerous recurrences in an attempt to enumerate polyedra (sic).

The accompanying bibliography (undoubtedly incomplete) traces the appearances of these problems in print in chronological order of publication. The author will be indebted to any reader who can supply missing references.

References

1. L. Euler, Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae, **7** (1758–1759) 13–14.

2. J. A. v. Segner, Enumeratio modorum, quibus figurae planae rectilineae per diagonales dividuntur in triangula, *ibid.*, 203–209.

3. N. v Fuss, Solutio quaestionis, quot modis polygonum n laterum in polygona m laterum per diagonales resolvi queat, Nova Acta Acad. Sci. Imperialis Petropolitanae, 9 (1791).

4. G. Lamé, Extrait d'une lettre de M. Lamé à M. Liouville sur cette question: un polygone convexe étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales? J. Math. Pures Appl., 3 (1838) 505–507.

5. E. Catalan, Note sur une équation aux différences finies, *ibid.*, 508–516.

6. O. Rodrigues, Sur le nombre de manières de décomposer un polygone en triangles au moyen de diagonales, *ibid.* 547–548.

7. ———, Sur le nombre de manières d'effectuer un produit de n facteurs, *ibid.* 549.

8. J. Binet, Réflexions sur le problème de déterminer le nombre de manières dont une figure rectiligne peut être partagée en triangles au moyen de ses diagonales, J. Math. Pures Appl., 4 (1839) 79–91.

9. E. Catalan, Solution nouvelle de cette question: un polygone étant donné, de combien de manières peut-on le partager en triangles au moyen de diagonales? *ibid.* 91–94.

10. E. Catalan, Addition à la note sur une équation aux differences finies insérée dans le volume précédent, page 508, *ibid.*, 95–99.

11. J. A. Grunert, Über die Bestimmung der Anzahl der verschiedenen Arten, auf welche sich ein n-eck durch Diagonalen in lauter m-ecke zerlegen lässt. . . . Arch. Math. Phys. Grunert, 1 (1841) 192–203.

12. L. Liouville, Remarques sur un mémoire de N. Fuss, J. Math. Pures Appl., 8 (1843) 391–394.

13. J. Binet, Note, *ibid.* 394–396.

14. T. P. Kirkman, On the trihedral partitions of the x-ace, and the triangular partitions of the x-gon, Mem. Proc. Manchester Lit. Philos. Soc., 15 (1857) 43–74.

15. ———, On the partitions and reticulations of the r-gon, *ibid.* 220–237.

16. ———, On the K-partitions of the R-gon and R-ace, Philos. Trans. Roy. Soc. London, 147 (1857) 217–272.

17. A. Cayley, On the analytical forms called trees, Part II, Philos. Mag., (4) 18 (1859) 374–378.

18. E. Schröder, Vier combinatorische Probleme, Z. Math. Phys., 15 (1870) 361–376.

19. H. M. Taylor and R. C. Rowe, Note on a geometrical theorem, Proc. London Math. Soc., (First Series) 13 (1881–1882) 102–106.

20. A. Cayley, On the partitions of a polygon, *ibid.*, 22 (1890–1891) 237–262.

21. E. Lucas, Théorie des nombres, I Paris, (1891) 90–96, 489–490.

22. P. Quarra, Calcolo delle parentesi, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat., 53 (1918) 1044–1047.

23. P. Franklin, Problem 2681 (proposed), this MONTHLY, 25 (1918) 118.

24. C. F. Gummer, Problem 2681 (solution), *ibid.* 26 (1919) 127–128.

25. J. H. Wedderburn, The functional equation $g(x^2)=2x+[g(x)]^2$, Ann. of Math., (2) 24 (1922) 121–140.

26. E. Netto, Lehrbuch der Combinatorik, 2nd ed., Teubner, Leipzig and Berlin, (1927) 192.

27. G. Birkhoff, Problem 3674 (proposed), this MONTHLY, 41 (1934) 269.

28. I. M. H. Etherington, Non-associative powers and a functional equation, Math. Gaz., 21 (1937) 36–39.

29. ———, On non-associative combinations, Proc. Roy. Soc. Edinburgh, 59 (1939) 153–162.

30. ———, Some problems of non-associative combinations, I, Edinburgh Math. Notes, No. 32 (1940) 1–6.

31. ——— and A. Erdélyi, Some problems of non-associative combinations II, *ibid.*, 7–12.

32. O. Ore, Problem 3954 (proposed), this MONTHLY, 48 (1940) 245.

33. P. Erdös and I. Kaplansky, Sequences of plus and minus, Scripta Math., 12 (1946) 73–75.

34. T. Motzkin, Relations between hypersurface cross ratios and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, Bull. Amer. Math. Soc., 54 (1948) 352.

35. H. W. Becker, Problem 4277 (solution), this MONTHLY, 56 (1949) 697–699.

36. N. Jacobson, Lectures on abstract algebra I, Van Nostrand, Princeton, N. J., (1951) 18–19.

37. G. Pólya, Mathematics and plausible reasoning I, Princeton Univer. Press, 1954, 102.

38. H. Bateman, Higher transcendental functions, Bateman manuscript project, Volume 3, McGraw-Hill, New York, 1955, 230.

39. N. Bourbaki, Théorie des ensembles, Actualités Sci. Ind., 1243 (1956) 88.

40. G. N. Raney, Functional composition patterns and power series reversal, Trans. Amer. Math. Soc., 94 (1960) 441–451.

41. H. G. Forder, Some problems in combinatorics, Math. Gaz., 45 (1961) 199–201.

42. P. Lafer and C. T. Long, A combinatorial problem, this MONTHLY, 69 (1962) 876–883.

43. J. B. Kelly, Problem 4983 (solution), *ibid.* 931.

44. J. W. Moon and L. Moser, Triangular dissections of n-gons, Canad. Math. Bull., 6 (1963) 175–178.

45. W. G. Brown, Enumeration of triangulations of the disk, Proc. London Math. Soc., (3rd Series) 14 (1964) 746–768.

46. ———, Enumeration of quadrangular dissections of the disc, Canad. J. Math., 17 (1965) 302–317.

ON SOME PROPERTIES OF SHORTEST HAMILTONIAN CIRCUITS

L. V. QUINTAS, St. John's University, Jamaica, New York and
FRED SUPNICK, City College, City University of New York

1. Introduction. In this paper proofs are given of the following "folk theorems."

By the *covertex class* of a polygon is meant the class of all polygons having the same vertex set as the given polygon.

THEOREM 1. *A spherical or planar polygon which is shortest of its covertex class and whose vertices are noncogeodesic cannot intersect itself.*

THEOREM 2. *A spherical or planar polygon which is shortest of its covertex class and whose vertices are noncogeodesic must contain the vertices on the boundary of its convex hull in their cyclic order.*

When on a 2-sphere we shall assume that the vertex set of a polygon does not contain a pair of antipodal points and that all edges are minor geodesic arcs.

The symbols for polygons $[V_1 \cdots V_n]$ are to be considered cyclic and symmetric. The symbol

$$[V_1 \cdots V_{i-1}(V_i \cdots V_j)V_{j+1} \cdots V_n]$$

will denote the *arc-inversion* that operates on the polygon

$$[V_1 \cdots V_{i-1}V_i \cdots V_{j-1}V_jV_{j+1} \cdots V_n]$$

to produce

$$[V_1 \cdots V_{i-1}V_jV_{j-1} \cdots V_iV_{j+1} \cdots V_n].$$

2. Proof of Theorem 1. Let h denote a polygon which intersects itself. Then there is a closed edge of h which intersects an (open) edge of h. Let P_aP_b be the open edge and denote the closed edge by $\mathrm{Cl}(P_cP_d)$, where

$$h = [P_aP_b \cdots P_cP_d \cdots].$$

We have the following cases:

1. The intersection of P_aP_b and $\mathrm{Cl}(P_cP_d)$ is a point, i.e., the intersection point is P_c, P_d, or a point of P_cP_d.

2. The intersection of P_aP_b and P_cP_d is a geodesic arc and when P_aP_b and

P_cP_d are thought of as directed arcs, P_aP_b and P_cP_d have the *same direction.* By this we mean P_aP_b and P_cP_d lie on the same geodesic and induce the same orientation of this geodesic.

3. The intersection of P_aP_b and P_cP_d is a geodesic arc and P_aP_b and P_cP_d have opposite direction.

In Cases 1 and 2, the arc-inversion

$$[P_a(P_b \cdots P_c)P_d \cdots] \tag{2.1}$$

yields a polygon which is strictly shorter than h. This is clear for a plane. For a 2-sphere, this is seen by noting that, in Case 1, the edges

$$P_aP_b,\ P_cP_d,\ P_aP_c, \quad \text{and} \quad P_bP_d, \tag{2.2}$$

all lie in a hemisphere. Since the triangle inequality is valid for geodesic triangles on a 2-sphere, our assertion for Case 1 follows. Note that by a *geodesic triangle* on a 2-sphere we mean a triangle whose sides are minor geodesic arcs (by *minor* we mean strictly less than a semigreat circle in length) and whose area is less than the area of a hemisphere. In Case 2, all of the edges (2.2) lie on the same geodesic. In particular, we have

$$P_aP_c \cup P_bP_d \subsetneq P_aP_b \cup P_cP_d.$$

Thus, our assertion for Case 2 is verified.

In Case 3, we have a different situation. All of the edges (2.2) lie on the same geodesic, but there are distributions of P_a, P_b, P_c, and P_d for which the arc-inversion (2.1) yields a polygon which has the same length as h. This occurs, for example, when $P_cP_d \subset P_aP_b$.

We shall now show that for every possibility included in Case 3, there is an arc-inversion which yields a polygon which is strictly shorter than h.

Let each edge of h be thought of as a directed arc having the direction induced by the orientation of h compatible with the direction of P_aP_b. Let

$$P_sP_t \cdots P_aP_b \cdots P_uP_v \tag{2.3}$$

denote the maximal path of edges in h which contains P_aP_b and all of whose edges have the same direction as P_aP_b. Let

$$P_yP_z \cdots P_cP_d \cdots P_wP_x \tag{2.4}$$

denote the maximal path of edges in h which contains P_cP_d and all of whose edges have the same direction as P_cP_d. Note that both paths, (2.3) and (2.4), are contained in the same geodesic and that each edge of (2.3) has direction opposite to every edge of (2.4).

(i) If $P_s \neq P_x$, then either P_s is contained in an edge $P_{c'}P_{d'}$ having the same direction as P_cP_d, or P_x is contained in an edge $P_{a'}P_{b'}$ having the same direction as P_aP_b. We shall resolve the former situation and note that the latter is handled similarly.

Let P_{s_1} denote the vertex of h which immediately precedes P_s (in the ordering of the vertices induced by the orientation of h that we have chosen). Then, since (2.3) is maximal, $\mathrm{Cl}(P_{s_1}, P_s) \cap P_{c'}P_{d'} = \{P_s\}$ or $P_{s_1}P_s$ has the same direction as $P_{c'}P_{d'}$. Thus, the arc-inversion

$$[P_{s_1}(P_s \cdots P_{c'})P_{d'} \cdots]$$

yields a polygon which is strictly shorter than h.

(ii) If $P_s = P_x$, we consider the other two ends of the maximal paths (2.3) and (2.4). Since not all of the vertices of the polygon under consideration lie on the same geodesic, we must have $P_v \neq P_y$. Thus, this case reduces to the situation we have resolved in (i).

This completes the proof of Theorem 1.

3. Proof of Theorem 2. A set W of points on a 2-sphere is *convex* if every minor geodesic arc joining points of W is contained in W. The *convex hull* of a set W' of points on a 2-sphere or on a plane is the intersection of all the convex sets containing W'.

LEMMA. *Every proper convex subset W on a 2-sphere S^2 is contained in a closed hemisphere of S^2.*

Proof. If W is dense in S^2, then $W = S^2$. For, if P is a point in $S^2 - W$, then there exist three points of W defining a geodesic triangle which contains P in its interior. Since W is convex this triangle together with its interior would lie entirely in W. If W is not dense in S^2, then let D denote a maximal closed circular region on S^2 such that the interior of D does not intersect the closure $\mathrm{Cl}(W)$ of W. If D does not contain a great circle, then D is contained in the interior of a closed hemisphere. Since D is maximal, D has at least two points A, B of $\mathrm{Cl}(W)$ on its boundary which are not antipodal with respect to S^2. The minor geodesic arc AB intersects the interior of D. Since $\mathrm{Cl}(W)$ is convex, AB must be in $\mathrm{Cl}(W)$. This contradicts the fact that the interior of D and $\mathrm{Cl}(W)$ are disjoint. Thus, D must contain a great circle and W is contained in a closed hemisphere.

Proof of Theorem 2. Let h denote any spherical polygon. If the convex hull of h is the entire 2-sphere, then the boundary B of its convex hull is null. If the convex hull of h is a proper subset of S^2, then by the Lemma h must lie in a closed hemisphere. Assume that the vertices of h which are contained in B are not in cyclic order. Now consider all the closed paths in h which start and end at vertices in B and which do not contain any vertices of B other than their endpoints. Since the vertices of h which are contained in B are not in cyclic order, B must contain at least four vertices and at least one of the paths defined above does not join adjacent vertices of B. Denote this path by p. Then, the endpoints of p separate B into two components, B_1 and B_2, each of which contains vertices of B (exclusive of the endpoints of p). Since p lies in the convex hull of h, any path in h joining a vertex in B_1 to a vertex in B_2 must intersect p. Thus, h must

intersect itself and by Theorem 1 cannot be shortest of its covertex class.

For planar polygons, let h denote a planar polygon and delete the second and third sentences in the above proof for spherical polygons.

This completes the proof of Theorem 2.

COROLLARY 1. (*The convex case for shortest polygons*). *Let* $P_1, P_2, \cdots, P_n$ *denote* n *points on a* 2-*sphere or on a plane which fall on the boundary* B *of their convex hull in the stated order* (*cyclic, or linear if the points are collinear on a plane*). *Then* $B=[P_1P_2 \cdots P_n]$ *is shortest of its covertex class.*

COROLLARY 2. (*The almost-convex case for shortest polygons.*) *Let* $n-1$ *points of* n *points on a* 2-*sphere or on a plane fall on the boundary* B *of their convex hull. Let* Q *denote the point in the interior of the convex hull,* $B=[P_1P_2 \cdots P_{n-1}]$, *and* $d(S, T)$ *the length of the edge* ST. *Then, if*

$$d(Q, P_j) + d(Q, P_{j+1}) - d(P_j, P_{j+1})$$

is least for $j=j_0$ *for all* $j=1, 2, \cdots, n-1(P_n \equiv P_1)$, *then* $[P_1 \cdots P_{j_0}QP_{j_0+1} \cdots P_{n-1}]$ *is shortest of its covertex class.*

ACKNOWLEDGMENT OF PRIORITY

Most of the general results of our paper, *A Supplement to the Sturm Separation Theorem, with Applications*, this MONTHLY, 72 (1965) 359–366, are contained in previous work of Marston Morse and Walter Leighton, *Singular Quadratic Functionals*, Trans. Amer. Math. Soc., 40 (1936) 252–286, esp. Section 2, pp. 253–257. In particular, the concept of "zero-maximal" solution found in our paper was introduced by Morse and Leighton under the name "focal" solution. Later, Leighton (*Principal Quadratic Functionals*, Trans. Amer. Math. Soc., 67 (1949) 253–274, esp. p. 261) termed such a solution "principal."

We thank Professor Leighton for calling this earlier work to our attention.

LEE LORCH AND DONALD J. NEWMAN

Mathematical Swifties

"My, this is for real!" cried Tom irrationally, raising π to the t-th.
"An unconditionally convergent series converges," Tom said. "Absolutely!"
"I find your work such a bore, Harald," said Tom almost periodically.
"I can't find a trace of it," said Tom characteristically.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

A SINGLE AXIOM FOR GROUPS

JOSÉ MORGADO, Universidade do Recife, Brasil

Michael Slater [1] characterizes a group by a single axiom, in terms of two operations. He considers a nonempty set $G=\{a, b, c, d, e, \cdots\}$ on which are defined: a binary operation $\cdot$ (multiplication) and a unary operation $'$ (inversion). He shows that the system $\langle G, \cdot, '\rangle$ is a group, if and only if

$$ab\cdot c = ad\cdot e \quad \textit{implies} \quad b = d\cdot ec'.$$

In [2], p. 6, Marshall Hall gives a definition of a group in terms of one of the inverse operations of the multiplication by means of four axioms. These axioms are not independent, however, as shown in [3]. In [3] we have defined a group in terms of the same operation by using only two axioms.

The purpose of this note is to characterize a group by means of one axiom, written in terms of one binary operation. The operation to be considered is just one of the inverse operations of the multiplication.

THEOREM. *Let $\langle G, \cdot\rangle$ be a groupoid satisfying the following condition:*

C: $$a(bb\cdot b)\cdot(cc\cdot c) = a(dd\cdot d)\cdot(ee\cdot e) \quad \textit{implies} \quad b = d\cdot ce.$$

Then, if one defines $a\odot b=a(bb\cdot b)$, for all a, b, in G, the groupoid $\langle G, \odot\rangle$ is a group.

Proof. Since $ab\cdot c=ab\cdot c$, one has, by condition C, $b=b\cdot cc$, for every b, $c\in G$. In particular, one has

$$bb = bb\cdot cc = bb\cdot bb,$$

hence

$$a[(bb\cdot bb)\cdot bb]\cdot(cc\cdot c) = a(cc\cdot c)\cdot[(bb\cdot bb)\cdot bb].$$

From this it follows that $bb=c(c\cdot bb)=cc$, i.e., bb does not depend on b.

1. Let us set $i=bb$. Since $ii=i$, one has clearly

$$a \odot i = a(ii\cdot i) = ai = a$$

and, consequently, i is a right identity of the groupoid $\langle G, \odot\rangle$.

2. It is obvious that $a(aa\cdot a)\cdot(ii\cdot i)=a(ii\cdot i)\cdot(aa\cdot a)$ and from this it results, by the condition C, that

(1) $$a = i\cdot ia$$

and therefore, $a\odot(ia)=a[(ia\cdot ia)\cdot ia]=a(i\cdot ia)=aa=i$, that is to say, in the

groupoid $\langle G, \odot\rangle$, for each element a, there is a right inverse element ia.

3. One has $a\odot(b\odot c)=a\cdot i(b\cdot ic)$ and $(a\odot b)\odot c=(a\cdot ib)\cdot ic$.

In order to establish the associativity of the operation $\odot$, it will be shown that

$$ab = ac\cdot bc, \quad \text{for every } a, b, c \in G. \tag{2}$$

Utilizing (1), one may write

$$ab = i(aa\cdot a)\cdot[(ib\cdot ib)\cdot ib] = i[(ab\cdot ab)\cdot ab]\cdot(ii\cdot i),$$

and so one has, by condition C,

$$a = ab\cdot(ib\cdot i) = ab\cdot ib, \quad \text{for every } a, b \in G. \tag{3}$$

Consequently, from (1) and (3) one obtains

$$a = i[(ab\cdot ab)\cdot ab]\cdot(bb\cdot b) = i[(ac\cdot ac)\cdot ac]\cdot(cc\cdot c)$$

and hence, by condition C, it follows that $ab=ac\cdot bc$, as was claimed.

In particular, one has

$$i\cdot ab = bb\cdot ab = ba, \quad \text{for every } a, b \in G. \tag{4}$$

Then we can write $a\odot(b\odot c)=a(ic\cdot b)$.

On the other hand, one has from (1) and (2),

$$a = i\cdot ia = (i\cdot ib)\cdot(ia\cdot ib) = b(ia\cdot ib), \quad \text{for every } a, b \in G, \tag{5}$$

and therefore, employing (5), (4) and (2), we see that

$$\begin{aligned} a(ic\cdot b) &= (a\cdot ib)\cdot\{i[a(ic\cdot b)]\cdot i(a\cdot ib)\} \\ &= (a\cdot ib)\cdot[(ic\cdot b)a\cdot(ib\cdot a)] = (a\cdot ib)\cdot[(ic\cdot b)\cdot ib] \\ &= (a\cdot ib)\cdot ic, \end{aligned}$$

which proves the associativity of the operation $\odot$.

In short, the groupoid $\langle G, \odot\rangle$ is a group.

Furthermore, if $a\cdot b=a\odot b'$, where b' denotes the inverse of b in the group $\langle G, \odot\rangle$, then the groupoid $\langle G, \cdot\rangle$ is such that

$$a \odot b = a(bb\cdot b), \quad \text{for all } a, b \text{ in } G$$

and if $a(bb\cdot b)\cdot(cc\cdot c)=a(dd\cdot d)\cdot(ee\cdot e)$, that is to say, if $(a\odot b)\odot c=(a\odot d)\odot e$, then one has obviously

$$b = (d \odot e) \odot c' = d \odot (e \odot c') = d\cdot(e \odot c')' = d\cdot(c \odot e')$$

and hence $b=d\cdot ce$. This means that the group axioms imply the single axiom above. Consequently, the single axiom above is equivalent to the group axioms.

References

1. Michael Slater, A single postulate for groups, this MONTHLY, 68 (1961) 346–347.
2. Marshall Hall, Jr., The Theory of Groups, Macmillan, New York, 1959.
3. José Morgado, Nota sobre quasigrupos subtractivos, 92–93 (1963) 11–17.

THE DIFFERENTIABILITY OF LOCALLY RECURRENT FUNCTIONS

J. G. MAULDON, Corpus Christi College, Oxford

In a recent paper [1] S. Marcus has asked whether there exists a nonconstant continuous locally recurrent function possessing a dense set of points of differentiability. [A *locally recurrent* function $f(\cdot)$ is one such that, for all real x and all positive ϵ, there exists y with $f(y)=f(x)$ and $0<|y-x|<\epsilon$.] I show here that such a function may be differentiable almost everywhere, though ([2] p. 364, Section 267) it must certainly be nondifferentiable at a nonenumerable set of points. In fact I exhibit two such functions $f_1(\cdot)$ and $f_2(\cdot)$, each of which is differentiable except on a set of measure zero, of which $f_1(\cdot)$ has the very much stronger property of being almost everywhere *locally constant*—that is, except on a set of measure zero, every real number is an interior point of an interval throughout which $f_1(\cdot)$ is constant. By way of contrast $f_2(\cdot)$, though continuous everywhere and differentiable almost everywhere, is not of bounded variation on any interval. An open question is

Does there exist a nonconstant continuous locally recurrent function of bounded variation?

Let any real number x be expressed as an ordinary nonterminating decimal

$$x = m\cdot d_1d_2d_3 \cdots = m + \sum_{r=1}^{\infty} 10^{-r}d_r, \tag{1}$$

where m and the d_r are integers, $0 \leqq d_r \leqq 9$, and there is an infinity of nonzero d_r. We now derive from the sequence $d_1d_2d_3 \cdots$ a new sequence $c_1c_2c_3 \cdots$ by inserting, at certain points, the symbol θ. In fact, whenever $3 \leqq d_r \leqq 6$, we insert between d_r and d_{r+1} a set of $k=k_\alpha$ symbols θ $(\alpha=1, 2)$, where the value $k=k_1$ $[k=k_2]$ is to be used in the definition of the function $f_1(\cdot)$ $[f_2(\cdot)]$. The value chosen for k_1 is

$$k_1 = \infty, \tag*{$(2)_1$}$$

so that the effect is to delete all the d_s after the first $d_k=3$, 4, 5 or 6. This means that a separate definition of $f_1(\cdot)$ could be given more simply, without introducing θ, but there is overall economy in dealing with the two functions $f_1(\cdot)$ and $f_2(\cdot)$ simultaneously. For the function $f_2(\cdot)$ it is convenient to take

$$k_2 = 13. \tag*{$(2)_2$}$$

We now define inductively a sequence $\{b_1, b_2, b_3, \cdots\}$ of zeros and ones by writing $b_r=0$ if and only if either

$$\begin{aligned} &c_r = \theta \text{ or } 1 \text{ or } 8 && (r \geqq 1) \\ &\left.\begin{aligned} &\text{or } c_r = 0 \text{ or } 9 \text{ and } c_{r-1} = \theta \\ &\text{or } c_r = 0 \text{ or } 9,\ c_r = c_{r-1} \text{ and } b_{r-1} = 0 \\ &\text{or } c_r = 0 \text{ or } 9,\ c_r \neq c_{r-1} \text{ and } b_{r-1} \neq 0 \end{aligned}\right\} && (r \geqq 2). \end{aligned} \tag{3}$$

In all other cases $b_r = 1$. Finally we define $f_1(\cdot)$ and $f_2(\cdot)$ by

$$f_\alpha(x) = \sum_{r=1}^{\infty} 2^{-r} b_r \qquad (\alpha = 1, 2) \tag{4}$$

the equation $(2)_\alpha$ being used in the definition of $f_\alpha(\cdot)$. Since $f_\alpha(\frac{1}{2}) = \frac{1}{2}$ and $f_\alpha(1) = 1$ it follows that $f_\alpha(\cdot)$ is *nonconstant* ($\alpha = 1, 2$) and we proceed to show that each function $f_\alpha(\cdot)$ has the other properties claimed for it.

It is readily verified that if x admits a *terminating* decimal expansion, then the use of this instead of (1) would yield the same sequence $b_1 b_2 b_3 \cdots$ except possibly when the sequence $d_1 d_2 d_3 \cdots$ of (1) ends in $0\dot{9}$ or $1\dot{9}$ or $7\dot{9}$ or $8\dot{9}$. If the sequences $b_1 b_2 b_3 \cdots$ obtained from the alternative expressions for x differ, however, they differ only in that $\cdots 0\dot{1}$ is replaced by $\cdots 1\dot{0}$, so that the corresponding values of the function $f_\alpha(\cdot)$ are identical. It follows that the value of the function $f_\alpha(x)$ (independent of m) is unaltered if in (1) we replace d_r by $9 - d_r$ throughout, and hence,

$$f_\alpha(\cdot) \text{ is an even function,} \qquad (\alpha = 1, 2). \tag{5}$$

Since the binary "decimal" digits for $f_\alpha(x)$ are determined recursively by the decimal digits for x, and since we used the nonterminating expansion (1), it follows that $f_\alpha(\cdot)$ is continuous on the left and hence, from (5), that $f_\alpha(\cdot)$ is *continuous* ($\alpha = 1, 2$).

Since the pairs of values (1, 8), (2, 7), (3, 6) and (4, 5) of any particular d_r are interchangeable without altering the sequence $b_1 b_2 b_3 \cdots$, $f_\alpha(\cdot)$ is certainly locally recurrent unless all but a finite number of the d_r have the value 0 or 9. If, in the case $\alpha = 1$, there is a $d_k = 3$, 4, 5 or 6, local recurrence is trivial, and so finally we may suppose that, for all $r > N$, $d_r = 0$ or 9 and the value of d_r determines that of b_{r+K}. Now choose large $n > N$ and, whenever $r > n$, replace d_r by $b_{r+K} + 1$ ($= 1$ or 2). Then x is varied by a nonzero quantity whose absolute value does not exceed 10^{-n}, whereas the sequence $b_1 b_2 b_3 \cdots$ is unaltered. It follows that, for each $\alpha = 1, 2$, the function $f_\alpha(\cdot)$ is *everywhere locally recurrent.*

If there exists k with $d_k = 4$, then $f_1(\cdot)$ is constant throughout the interval $(x - 10^{-k}, x + 10^{-k})$. Hence $f_1(\cdot)$ is *locally constant* except perhaps on the set of real numbers containing no digit 4 in their decimal expansion, and this set [**3**, Theorem 145] is of measure zero as required. This completes our study of $f_1(\cdot)$ and we go on to consider $f_2(\cdot)$.

In fact it is true [**3**, Theorem 148], not merely that almost all real numbers contain a 4 in their decimal expansion, but that almost all real numbers are *normal*, in the sense that every digit occurs with exactly its expected frequency 1/10. Hence, for almost all real x, there exists $N = N(x)$ such that, whenever $n > N$, the number of zeros and nines in the sequence $d_1 d_2 \cdots d_{4n}$ is less than n and the number of digits 3, 4, 5 or 6 in the sequence $d_1 d_2 \cdots d_{3n}$ is greater than n. It follows that d_{4n} cannot be immediately preceded by as many as n zeros or nines and also that, in the case $(2)_2$, the sequence $c_1 c_2 \cdots c_m$ derived from $d_1 d_2 \cdots d_{3n}$ contains at least $13n$ symbols θ and consequently that $m \geqq 16n$.

For any sufficiently small nonzero increment δx of x we may suppose that $|\delta x| \in [10^{-4(n+1)}, 10^{-4n}]$ where $n > N$. Thus if we vary x by an amount δx we shall not alter the sequences $d_1 d_2 \cdots d_{3n}$ and $b_1 b_2 \cdots b_m$ and so $|\delta f_2| \leqq 2^{-m} \leqq 16^{-4n}$ and $|\delta f_2/\delta x| \leqq A\beta^n$, where $A = 10^4$ and $\beta = (10/16)^4 < 1$. Since $n \to \infty$ as $\delta x \to 0$, it follows that the function $f_2(\cdot)$ is *differentiable* almost everywhere.

We now define the sets

$$P = \left\{ x = \sum_{r=1}^{\infty} 10^{-r} d_r : d_r \in \{1, 2, 8\}, \quad r = 1, 2, 3, \cdots \right\},$$

$$P_M = \{ x \in P : d_r = 1 \text{ for } r > M \}, \qquad M = 1, 2, 3, \cdots.$$

Then, by making use of the finite partition P_M, we see that the variation of $f_2(\cdot)$ on the interval (0, 1) is at least $(3^M - 1)2^{-M}$. Since M is arbitrary, it follows that $f_2(\cdot)$ *is not of bounded variation on the interval* (0, 1). [The same argument proves the same result for $f_1(\cdot)$.]

Finally, since any nondegenerate interval contains some interval $(m/10^N, (m+1)/10^N)$, where m and $N > 0$ are integers, and since throughout the subinterval

$$\{ y = (10m + 1 + x)/10^{N+1} : 0 < x < 1 \},$$

we have $f_2(y) = A + Bf_2(x)$, where A and $B \neq 0$ are independent of x, it follows that $f_2(\cdot)$ *is not of bounded variation on any interval.*

By using the construction of [4] we may arrange that the exceptional set implied by the phrase "almost everywhere" is not merely of *measure* zero but actually of *dimension* zero. The ideas of the present note can also be used to simplify the exposition of [4] by taking $u_k = 0$ if and only if

$$\begin{array}{llll} & \text{either } x_k = 1 \text{ or } k - 1 & & \\ & \text{or} \quad x_k = 0,\ x_{k-1} = 0 & \text{and} & u_{k-1} = 0, \\ (6) & \text{or} \quad x_k = 0,\ x_{k-1} \neq 0 & \text{and} & u_{k-1} \neq 0, \\ & \text{or} \quad x_k = k,\ x_{k-1} = k - 1 & \text{and} & u_{k-1} = 0, \\ & \text{or} \quad x_k = k,\ x_{k-1} \neq k - 1 & \text{and} & u_{k-1} \neq 0, \end{array}$$

and $u_k = 1$ otherwise. For the purposes of the present note, the alternative

$$\text{or} \quad 2 < x_j < j - 2 \quad \text{for some } j < k$$

should be added to (6).

References

1. S. Marcus, On locally recurrent functions, this MONTHLY, 70 (1963) 822–826.
2. E. W. Hobson, The Theory of Functions of a Real Variable, vol. 1, Cambridge, 1927.
3. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1945.
4. J. G. Mauldon, Locally recurrent functions, this MONTHLY, 69 (1962) 896–899.

COVERING A RECTANGLE WITH T-TETROMINOES

D. W. WALKUP, Boeing Scientific Research Laboratories, Seattle, Washington

The principal objective of this note is to prove the following:

THEOREM 1. *A necessary and sufficient condition for an $a \times b$ rectangle to be dissectable into T-tetrominoes is that a and b be integral multiples of* 4.

A T-tetromino is a plane figure constructed from four unit squares arranged in the form of a T; it has a boundary of length ten with six outside corners and two inside corners. Since the 4×4 rectangle can be dissected into four T-tetrominoes, the sufficiency of the condition in Theorem 1 is clear. Certainly it is necessary that a and b be integers. A result of the above type is asked for in an editorial note to the solution of Elementary Problem E1543, this MONTHLY 70 (1963) 760–761.

Let the cartesian plane be marked off into unit squares by the lines with integral coordinates. Let R be a rectangle bounded by the axes and the lines $y=a$ and $x=b$. If R can be dissected into T-tetrominoes, then, by a simple repetition scheme, so can the entire quadrant between the positive axes. In Lemma 1 certain properties of dissections of the quadrant will be established, from which Theorem 1 will follow. For convenience we will use some special terminology:

DEFINITION 1. *A* **segment** *is a line segment of length* 1 *forming the edge of a unit square of the quadrant. A segment is a* **cut segment** (*or* **cut**) *if in every dissection of the quadrant it is one of the ten boundary segments of some T-tetromino. A point with nonnegative integer coordinates is a* **cornerless** *point if it does not lie at any one of the six outside corners of any T-tetromino in any dissection of the quadrant. A point is a type-A point if its coordinates are nonnegative and congruent modulo* 4 *to* (0, 0) *or* (2, 2); *it is a type-B point if its coordinates are nonnegative and congruent modulo* 4 *to* (0, 2) *or* (2, 0). *A* **translate** *of a point, segment, or T-tetromino is another point, segment, or T-tetromino in the quadrant obtained from the first by a displacement of $2k$ in y and $-2k$ in x, where k may be any positive or negative integer.*

Note that every segment on the axes is a cut segment and any translate of a type-A or type-B point is again a point of the same type.

LEMMA 1. *Every type-B point is cornerless and each of the* 2, 3, *or* 4 *segments incident on a type-A point is a cut.*

Proof. For each nonnegative integer λ let $P(\lambda)$ be the proposition that the lemma holds for all type-A and type-B points on or below the line $x+y=4\lambda$. $P(0)$ is true as the two segments incident on the origin are necessarily cuts. We shall demonstrate that $P(\lambda)$ implies $P(\lambda+1)$ using, for clarity, Figure 1, which shows the situation for $\lambda=2$. Certain cut segments and cornerless points required by $P(\lambda)$ are indicated in Figure 1 by heavy lines and dots.

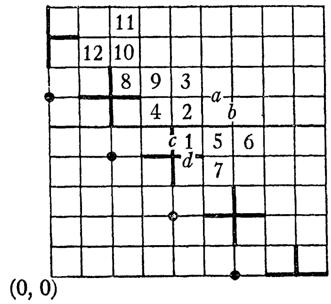


FIG. 1

If a dissection of the quadrant contained the T-tetromino 1-2-3-4 (Figure 1), it would also contain the T-tetromino 8-10-11-12 and hence, by induction, all upward translates of 1-2-3-4, since no other arrangement could cover squares 8 and 9. But, because of the relative position of the y-axis, not all upward translates of 1-2-3-4 can be in a dissection. Therefore, neither 1-2-3-4 nor, by symmetry, 1-5-6-7 can be in any dissection of the quadrant. The four remaining T-tetrominoes containing square 1 which do not overlap the cut segments c and d have segments a and b as edges. It follows that a, b, and their translates are all cut segments so that the diagrammed conditions of Figure 2 hold.

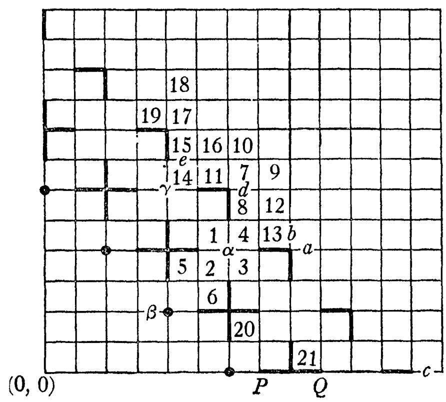


FIG. 2

We wish to prove the type-B point α is cornerless. (See Figure 2.) Suppose, to the contrary, that there is a dissection containing a T-tetromino having α at an outside corner. In this tetromino the unit square forming the outside corner at α cannot be 1, 2, or 3 because of the nearby cut segments. Consequently, either 1-2-3-5 or 1-2-3-6 is a tetromino of the supposed dissection. But no tetromino could then contain square 6 or 5 respectively, as β is a cornerless point. This proves that α is cornerless. By the same or similar arguments all translates of α, either inside the quadrant or on the axes, are cornerless.

We shall have shown $P(\lambda+1)$ true and consequently proven the lemma if we show that segments a, b, (Figure 2) and their translates are cuts. By symmetry, we need only show a and its translates are cuts. Since the border segment c must be a cut, it is sufficient to show that if some segment a is a cut so also is its adjacent translate d. Suppose, for the contradiction, that a is a cut and a dissection exists involving a tetromino containing 7 and 8. This tetromino cannot contain 4 since α is cornerless. It must contain two of the three squares 9, 10 and 11. If it contained 9 there would be no way of assigning 12 and 13. The tetromino must be 7-8-10-11. The squares 14 and 15 must be parts of different tetrominoes because γ is cornerless. But the only way to fill 15 and 16 then is to include 15-17-18-19 in the supposed dissection. It follows that all upward translates of 7-8-10-11 would be in the dissection. The absurdity of this is apparent at the y axis. We conclude d is a cut segment if a is. Thus the lemma is proven.

Proof of Theorem 1. Only the necessity remains to be proven. For which values of b does there exist a dissection of the quadrant which is simultaneously a dissection of the rectangle R? We cannot have b congruent to 2 modulo 4 as this puts a corner of R at a type-B point which, by Lemma 1, is cornerless. Moreover, b cannot be congruent to 3 or 1 modulo 4. To see this, note that there is no way to cover square 20 or 21 if the bounding line $x=b$ stands at P or Q, respectively, in Figure 2. Consequently b, and by symmetry a, must be congruent to 0 modulo 4.

Through use of some additional construction, Lemma 1 may be made to yield further information about dissections of a rectangle.

DEFINITION 2. *Let R be an $a\times b$ rectangle dissected into T-tetrominoes. A* **block** *of R is a 2×2 square of R whose vertices have even coordinates. A* **chain** *of R is any minimal subset of R which is both a union of blocks and a union of T-tetrominoes.*

THEOREM 2. *Let R be an $a\times b$ rectangle dissected into T-tetrominoes. Color every other block of R in checkerboard fashion. Every T-tetromino of the dissection will have three unit squares in one block and one unit square in an adjacent block. Every chain consists of an even number of blocks which may be cyclically ordered in such a way that the blocks are alternately colored and uncolored and the T-tetrominoes of the chain contain three unit squares of one block and one unit square of the succeeding block.*

Proof. The three-and-one coloration of T-tetrominoes follows from Lemma 1 and, say, inspection of Figure 2. Note that the type-A and type-B points are the corners of the blocks of R. The rest of the theorem is straightforward.

ON THE RELATIVE EXTREMA OF THE TURÁN EXPRESSION FOR THE GENERAL LAGUERRE FUNCTION

S. K. LAKSHMANA RAO, Regional Engineering College, Warangal (A.P.), India

The general Laguerre function $L_n^{(\alpha)}(x)$ which is regular in the neighbourhood of the origin is defined as a solution of the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0 \tag{1}$$

for any real index $n > -1$ and real $\alpha > -1$ and may be explicitly represented as

$$L_n^{(\alpha)}(x) = \binom{n+\alpha}{n} \Phi(-n, \alpha + 1; x), \tag{2}$$

where $\Phi(a, b; x)$ denotes the confluent hypergeometric function in Humbert's notation. It reduces to the orthogonal polynomial $L_n^{(\alpha)}(x)$ when $n = 0, 1, 2, \cdots$. It was shown by G. Szegö [1] that the polynomials $L_n^{(\alpha)}(x)$ satisfy Turán's inequality, viz.,

$$\Delta_n(x) \equiv [L_n^{(\alpha)}(x)]^2 - L_{n+1}^{(\alpha)}(x)L_{n-1}^{(\alpha)}(x) \geqq 0, \tag{3}$$

and this result was extended by the author [2] for the general Laguerre function $L_n^{(\alpha)}(x)$, when $n > 0$, $\alpha \geqq 0$ and $x > 0$.

In this note we study the sequences of the relative maxima and relative minima of the expression $T_n(x) \equiv e^{-x}x^{\alpha-1}\Delta_n(x)$, associated with the Turán expression $\Delta_n(x)$ for the general Laguerre function $L_n^{(\alpha)}(x)$, where $n > 0$, $\alpha > 1$ and $x > 0$.

From [2] we know that

$$\frac{d}{dx} T_n(x) = e^{-x} x^{\alpha-2} L_n^{\alpha}(x)[L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)]$$

for $n > 0$ and $\alpha > -1$. Since n and α are both real, the functions $L_n^{(\alpha)}(x)$ and $L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)$ both have only a finite number of zeros on the real axis (cf. [3], p. 288). Let $(\beta): 0 < \beta_1 < \beta_2 < \cdots$ and $(\delta): 0 < \delta_1 < \delta_2 < \cdots$ denote the zeros of $L_n^{(\alpha)}(x)$ and $L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)$ respectively, on the positive real axis. We find that when $n > 0$ and $\alpha > 1$

$$\left\{\frac{d^2}{dx^2} T_n(x)\right\}_{x=\beta} = \frac{(\alpha - 1)(n + \alpha)}{n + 1} e^{-\beta}\beta^{\alpha-3}[L_{n-1}^{(\alpha)}(\beta)]^2 > 0$$

and

$$\left\{\frac{d^2}{dx^2} T_n(x)\right\}_{x=\delta} = -(\alpha - 1)e^{-\delta}\delta^{\alpha-3}[L_{n-1}^{(\alpha)}(\delta) - L_n^{(\alpha)}(\delta)]^2 < 0.$$

Hence we have

THEOREM 1. *When $n>0$, $\alpha>1$, and $x>0$, the relative maxima $M_{r,n}$ of the function $T_n(x)$ occur at the zeros of $L_{n+1}^{(\alpha)}(x)-2L_n^{(\alpha)}(x)+L_{n-1}^{(\alpha)}(x)$ and the relative minima $m_{r,n}$ occur at the zeros of $L_n^{(\alpha)}(x)$.*

Let $M_{r,n}=T_n(\delta_r)$ and $m_{r,n}=T_n(\beta_r)$ denote the rth relative maximum and minimum respectively, of $T_n(x)$ and let

$$\xi=\frac{(\alpha-1)(\alpha-3)}{2n+\alpha+1} \quad \text{and} \quad \eta=\frac{\alpha^2-1}{2n+\alpha+1}\cdot$$

THEOREM 2. *If $\alpha>3$, the sequence $\{M_{r,n}\}$, $r=1, 2, 3, \cdots$, is increasing when $\delta_r<\xi$ and decreasing when $\delta_r>\xi$. If $1<\alpha\leqq 3$, the sequence $\{M_{r,n}\}$ is decreasing. If $\alpha>1$, the sequence $\{m_{r,n}\}$, $r=1, 2, 3, \cdots$, is increasing when $\beta_r<\eta$ and decreasing when $\beta_r>\eta$.*

Introduce the functions

$$f(x)=T_n(x)+\tfrac{1}{2}(\alpha-1)e^{-x}x^{\alpha-3}\left[L_{n+1}^{(\alpha)}(x)-2L_n^{(\alpha)}(x)+L_{n-1}^{(\alpha)}(x)\right]^2$$

and

$$g(x)=T_n(x)-\frac{\alpha-1}{2(n+1)(n+\alpha)}e^{-x}x^{\alpha-1}\left[L_n^{(\alpha)}(x)\right]^2.$$

It is obvious that when $n>0$, $\alpha>1$ and $x>0$,

$$f(x)\geqq T_n(x) \quad \text{and} \quad g(x)\leqq T_n(x).$$

We note further that $f(x)$ coincides with $T_n(x)$ at the points (δ_r), and $g(x)$ coincides with $T_n(x)$ at the points (β_r). Differentiating the functions $f(x)$ and $g(x)$, we find that

$$\begin{aligned} f'(x)=&-\tfrac{1}{2}e^{-x}x^{\alpha-4}[(2n+\alpha+1)x-(\alpha-1)(\alpha-3)] \\ &\times\left[L_{n+1}^{(\alpha)}(x)-2L_n^{(\alpha)}(x)+L_{n-1}^{(\alpha)}(x)\right]^2 \end{aligned}$$

and

$$g'(x)=-\tfrac{1}{2}[(n+1)(n+\alpha)]^{-1}e^{-x}x^{\alpha-2}[(2n+\alpha+1)x-(\alpha^2-1)]\left[L_n^{(\alpha)}(x)\right]^2.$$

If $\alpha>3$, $f(x)$ is increasing in $0<x<\xi$ and decreasing in $x>\xi$, while in the case $1<\alpha\leqq 3$, $f(x)$ is decreasing in $x>0$. Also $g(x)$ is increasing in $0<x<\eta$ and decreasing in $x>\eta$. Hence the theorem.

I thank the referee for suggesting modifications.

References

1. G. Szegö, On an inequality of P. Turán concerning Legendre polynomials, Bull. Amer. Math. Soc., 54 (1948) 401–405.

2. S. K. Lakshmana Rao, Turán's inequality for the general Laguerre and Hermite functions, Math. Student, 26 (1958) 1–6.

3. A. Erdelyi, Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, 1953.

ABSOLUTELY INDEPENDENT AXIOMS FOR ABELIAN GROUPS

R. A. JACOBSON, Houghton College

In a recent note [1], it was suggested that there might exist an absolutely independent set of axioms describing an abelian group. The search for such axioms might well begin by adding the axiom $x * y = y * x$ to the previous axioms, followed by an attempt to build 2^4 models that satisfy all combinations for which the axioms "hold" or "never hold." Upon deploying such an attack, one quickly becomes aware of the fact that even after suitable modification of the axioms the task of finding all sixteen examples is still somewhat staggering.

In this note, we approach the problem by describing an abelian group with only three axioms and thus require but eight models to show that the axioms are absolutely independent. In the following, let $*$ be a binary operation over a set S, where e is a specific element of S and x, y, z are arbitrary elements of S. The set S is an abelian group if:

(i) for all x, $x \neq e$; $x * e = x$

(ii) for all x, y, z, $x \neq y$, $x \neq e$; $(x * y) * z = y * (x * z)$;

(iii) for all x, y, $x \neq y$, $x \neq e$; $x * w = y$ has a unique solution.

Before considering the complete denial of these axioms we pause to show that they do define an abelian group.

Letting $z = e$ in (ii), we have

$$(xy)e = y(xe) = yx \quad \text{for } x \neq y, \ x \neq e. \tag{1}$$

For $xy \neq e$, we utilize (i) and simplify (1) to read $xy = yx$. Letting $y = z = e$ in (ii) gives $xe = ex$. Then since $xx = xx$ for all x, it follows that $xy = yx$ for all x, y provided $xy \neq e$. If $xy = e$, we assume $yx \neq e$. This implies that $x \neq y$ and $y \neq e$; thus, employing (i) and (ii), we can write $yx = (yx)e = xy = e$. Hence our assumption $yx \neq e$ is false and $*$ is commutative for all x and y.

In view of (i) and (iii), it is clear that the system has a right identity and right inverses if $ee = e$. To show this, we choose an $x \neq e$, such that $xx \neq e$. It follows from (iii) that there exists a y, $y \neq x$, such that $xy = e$. The commutative property and (i) yield $ee = (xy)e = yx = xy = e$. Suppose there exists no such x as mentioned above; that is, all elements $x \neq e$ are such that $xx = e$. Let $ee = y$ and assume $y \neq e$. We can then employ (i) and (ii) to arrive at the contradiction, $e = yy = (ye)y = e(yy) = ee$. Thus $ee = e$.

We have yet to establish that the system is associative; that is, $(yx)z = y(xz)$ for all x, y, z. Since the identity element commutes, we see that

$$(ye)z = yz = y(ze) = y(ez) \quad \text{for } x = e.$$

For $x \neq e$, we use (ii) and the commutative property in order to establish that:

$$\begin{aligned} (yx)z &= (xy)z = y(xz) && \text{for } x \neq y; \\ (xx)z &= z(xx) = x(zx) = x(xz), && \text{for } x = y,\ x \neq z; \\ (xx)x &= x(xx) && \text{for } x = y = z. \end{aligned}$$

Thus, operation $*$ is associative and the axioms do indeed describe an abelian group.

Axioms (i), (ii), and (iii) are all of the form (a) "for all $x, y, z, \cdots$ (A is B)." The simple contradiction ($\tilde{a}$) is "for some $x, y, z, \cdots$ (A is not B)." We shall refer to the complete denial as being ($\bar{a}$) "for all $x, y, z, \cdots$ (A is not B)."

The simple contradiction of each axiom is:

($\tilde{\text{i}}$) for some x, $x \neq e$; $x * e \neq x$;
($\tilde{\text{ii}}$) for some x, y, z, $x \neq y$, $x \neq e$; $(x * y) * z \neq y * (x * z)$;
($\widetilde{\text{iii}}$) for some x, y, $x \neq y$, $x \neq e$; $x * w = y$ does not have a unique solution.

We assert that the complete denial of each axiom is:

($\bar{\text{i}}$) for all x, $x \neq e$; $x * e \neq x$;
($\overline{\text{ii}}$) for all x, y, z, $x \neq y$, $x \neq e$; $(x * y) * z \neq y * (x * z)$;
($\overline{\text{iii}}$) for all x, y, $x \neq y$, $x \neq e$; $x * w = y$ does not have a unique solution.

The following examples, in which S is the set of all integers and $e = 0$, indicates that the axioms are absolutely independent.

BINARY OPERATIONS	AXIOMS
$x * y = x + y$	(i), (ii), (iii)
$x * y = y$	($\bar{\text{i}}$), (ii), (iii)
$x * y = \begin{cases} -y + 1, & y > 0 \\ -y, & y < 0 \\ -x, & y = 0,\ x \neq 0 \\ -1, & y = 0,\ x = 0 \end{cases}$	(i), ($\overline{\text{ii}}$), (iii)
$x * y = \begin{cases} x + y; & xy = 0 \\ 1; & xy \neq 0 \end{cases}$	(i), (ii), ($\overline{\text{iii}}$)
$x * y = x$	(i), ($\overline{\text{ii}}$), ($\overline{\text{iii}}$)
$x * y = \begin{cases} \lvert x \rvert + \lvert y \rvert, & xy \neq 0 \\ \lvert x \rvert + 1, & x \neq 0,\ y = 0 \\ 1 + \lvert y \rvert, & x = 0,\ y \neq 0 \\ 2 & x = 0,\ y = 0 \end{cases}$	($\bar{\text{i}}$), (ii), ($\overline{\text{iii}}$)

$$x * y = \begin{cases} y+2, & x-y>2 \\ y+3, & x-y \leqq 2 \end{cases} \qquad (\bar{\text{i}}), (\overline{\text{ii}}), (\text{iii})$$

$$x * y = \begin{cases} -y^2, & x \geqq 0, \ |y| \geqq 2 \\ y^2, & x < 0, \ |y| \geqq 2 \\ -4, & x \geqq 0, \ |y| < 2 \\ 4, & x < 0, \ |y| < 2 \end{cases} \qquad (\bar{\text{i}}), (\overline{\text{ii}}), (\overline{\text{iii}})$$

REMARK. The above examples also hold for the somewhat stricter cases in which $(\bar{\text{i}})$ is replaced by "for all x, y, $x \neq y$; $x * y \neq x$."

As in the previous paper, we note that the set S is infinite and the axioms might depend on the cardinality of S. Hence, the problem of finding absolutely independent axioms for a finite abelian group still remains.

I wish to thank the referee for his comments.

Reference

1. R. A. Jacobson and K. L. Yocom, Absolutely independent group axioms, this MONTHLY, 72 (1965) 756–758.

OPEN EVERYWHERE DISCONTINUOUS FUNCTIONS

SOLOMON MARCUS, University of Bucharest

A function f is said to be open if for each open set G the set $f(G)$ is also open. In [3], Robert Spira considers the following problem: *Can a function be open and not continuous?*

Spira gives an example of a real function f, defined and open on $[0, 1]$ and such that f is continuous at $x \in [0, 1]$ if and only if x does not belong to the Cantor middle third set.

If A and B are two topological spaces and each subset of B is open, then every function f (continuous or not) which maps A into B is open. But we will consider the case $A = B =$ the set of reals with the usual topology. Spira observes that, in this case, "the discontinuities of an open function must be particularly violent and shattering to the nerves".

In the following we shall prove the existence of several types of open everywhere discontinuous functions. All results are obtained as corollaries of our theorems given in [1] and [2].

PROPOSITION 1. *There exists a real function f defined and open on $(-\infty, \infty)$, everywhere discontinuous and such that f is not measurable and its graph is connected.*

Proof. Theorem 1 of [2] asserts the existence of a real function f defined on $(-\infty, \infty)$, which possesses a connected graph and takes, on each real perfect set, each real value. Since each open set G contains a perfect set, one has $f(G)$

$=(-\infty, \infty)$ for each open set G; thus f is open and everywhere discontinuous. By using Lusin's theorem concerning the continuous restrictions of a measurable function, it is easy to see that f is not measurable.

REMARK. A sharper result than Proposition 1 can be obtained with Theorem 2 of [2].

We recall that a function f has the Darboux property if, for each connected set A, the set $f(A)$ is connected.

PROPOSITION 2. *There exists a real function f defined and open on $(-\infty, \infty)$, everywhere discontinuous and such that f is not measurable, has the Darboux property and its graph is totally disconnected.*

Proof. Theorem 4 of [2] asserts the existence of a real function f defined on $(-\infty, \infty)$, which takes every real value on each real perfect set and such that its graph is totally disconnected. As in the proof of Proposition 1, it follows that f is open, everywhere discontinuous and not measurable. Since each real connected set is an interval and each interval contains a perfect set, it follows that $f(I)=(-\infty, \infty)$ for each real interval I; thus, f has the Darboux property.

PROPOSITION 3. *There exists a real function f, defined and open on $(-\infty, \infty)$, everywhere discontinuous and such that: 1° f is measurable (in the sense of Lebesgue); 2° f is not a Borel function; 3° f has the Darboux property; 4° the graph of f is totally disconnected.*

Proof. Theorem 5 of [2] asserts the existence of a real function f defined on $(-\infty, \infty)$ and such that: a) it is measurable in the sense of Lebesgue; b) it is not a Borel function; c) it takes each real value in each real interval; d) its graph is totally disconnected. Since each open set G contains an interval, it follows that $f(G)=(-\infty, \infty)$ for each open G; thus f is open and everywhere discontinuous and has the Darboux property.

PROPOSITION 4. *Each real function defined on $(-\infty, \infty)$ is the sum of two open functions.*

Proof. The theorem of [1] asserts that each real valued function defined on the real line can be expressed as a sum of two functions each of which carries every perfect set of real numbers onto the set of all real numbers. Since each real open set contains a perfect set, Proposition 4 follows.

Open problems on open functions. 1° Does there exist an open function which does not have the Darboux property?

2° If the answer to 1° is affirmative, does there exist an open function which has the Darboux property in no interval?

3° Does there exist a Borel function which is open and everywhere discontinuous?

4° If the answer to 3° is affirmative, does there exist an open function of the second class of Baire, which is everywhere discontinuous?

REMARK. All problems concern real functions defined on the real line.

References

1. S. Marcus, On a theorem formulated by A. Lindenbaum and proved by W. Sierpiński (in Rumanian), Com. Acad. R.P.R., 10 (1960) 547–550.
2. S. Marcus, Functions with the Darboux property and functions with connected graphs, Math. Annalen, 141 (1960) 311–317.
3. R. Spira, An open discontinuous function, this MONTHLY, 69 (1962) 128–129.

ON MEASURES OF NON-NORMALITY FOR MATRICES

P. J. EBERLEIN, University of Rochester

1. Introduction. Given an $n\times n$ matrix A with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, let $\Lambda=\operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Then $\|\Lambda\|^2\leqq\|A\|^2$ (Euclidean norm), and it is well known that equality occurs if and only if A is normal. Since $\|A\|^2-\|\Lambda\|^2$ and $\|AA^*-A^*A\|$ both measure departure from normality it is of interest to obtain relations between them. In a recent paper [1] Peter Henrici proves that

$$\|A\|^2 - \|\Lambda\|^2 \leqq \sqrt{\frac{n^3-n}{12}}\,\|AA^* - A^*A\| \tag{1}$$

and exhibits for every n a matrix for which equality holds.

In this note we obtain an inequality in the reverse direction:

$$\|A\|^2 - \|\Lambda\|^2 \geqq \frac{1}{6}\,\frac{\|AA^* - A^*A\|^2}{\|A\|^2} \qquad (A \neq 0). \tag{2}$$

2. Proof of the inequality. We may assume that A is in Schur triangular form, i.e. $A=\Lambda+M$, where $M=(m_{ij})$ is upper triangular. Then

$$\begin{aligned}\|AA^* - A^*A\| &= \|(\Lambda+M)(\Lambda^*+M^*) - (\Lambda^*+M^*)(\Lambda+M)\|\\ &\leqq \|M\Lambda^* + \Lambda M^* - \Lambda^*M - M^*\Lambda\| + \|MM^* - M^*M\|\\ &= \Big[2\sum_{i<j} |m_{ij}|^2|\lambda_i-\lambda_j|^2\Big]^{1/2} + \|MM^*-M^*M\|.\end{aligned}$$

Hence

$$\|AA^* - A^*A\| \leqq \sqrt{2}s\|M\| + \sqrt{2}\|M\|^2, \tag{3}$$

where $s=\max_{ij}|\lambda_i-\lambda_j|$ and we have employed the inequality $\|MM^*-M^*M\| \leqq\sqrt{2}\|M\|^2$, valid for any matrix:

$$\begin{aligned}\|AA^* - A^*A\|^2 &= \operatorname{trace}(AA^*-A^*A)^2 = 2\operatorname{tr}(AA^*AA^*) - 2\operatorname{tr}(AA^*A^*A)\\ &= 2\|AA^*\|^2 - 2\|A^2\|^2 \leqq 2\|A\|^4.\end{aligned}$$

When $\Lambda=\lambda I$,

$$\|AA^* - A^*A\| = \|MM^*-M^*M\| \leqq \sqrt{2}\|M\|^2 = \sqrt{2}(\|A\|^2 - \|\Lambda\|^2), \text{ or}$$

$$\|A\|^2 - \|\Lambda\|^2 \geqq \|AA^*-A^*A\|/\sqrt{2} \geqq \|AA^*-A^*A\|^2/2\|A\|^2.$$

Hence we may assume $\Lambda \neq \lambda I$; then $s \neq 0$ and $\|\Lambda\|^2 \neq 0$. Let

$$x = \frac{\|\Lambda\|}{\|A\|}, \qquad y = \frac{\|AA^* - A^*A\|}{\sqrt{2}\|A\|^2}, \quad \text{and} \quad \alpha = \frac{s}{\sqrt{2}\|\Lambda\|}. \tag{4}$$

Then $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, $0 \leqq \alpha \leqq 1$. In this notation we want to show that

$$1 - x^2 \geqq \frac{y^2}{3}. \tag{2'}$$

Now (3) becomes: $y \leqq 1 - x^2 + \sqrt{2}\alpha x\sqrt{1-x^2}$, or

$$y - 1 + x^2 \leqq \sqrt{2}\alpha x \sqrt{1 - x^2}. \tag{5}$$

If $y-1+x^2<0$, $y^2/3 \leqq y < 1-x^2$ and we are done. If $y-1+x^2 \geqq 0$, square (5) and rearrange terms to obtain

$$P(x^2) \equiv (1 + 2\alpha^2)x^4 - 2(1 - y + \alpha^2)x^2 + (y^2 - 2y + 1) \leqq 0. \tag{6}$$

Since P has two real zeros, x^2 must be less than or equal to the larger:

$$x^2 \leqq \frac{1}{(1 + 2\alpha^2)} (1 - y + \alpha^2 + \alpha(\alpha^2 + 2y - 2y^2)^{1/2}). \tag{7}$$

Note that

$$(\alpha^2 + 2y - 2y^2)^{1/2} = \alpha\left(1 - \frac{2y^2}{\alpha^2} + \frac{2y}{\alpha^2}\right)^{1/2} \leqq \alpha\left(1 - \frac{y^2}{\alpha^2} + \frac{y}{\alpha^2}\right).$$

Use this inequality with (7) to obtain $x^2 \leqq 1 - [y^2/(1+2\alpha^2)]$ or, since $\alpha^2 \leqq 1$,

$$x^2 \leqq 1 - \frac{y^2}{3}. \tag{2'}$$

Reference

1. Peter Henrici, Bounds for iterates, inverses, spectral variation, and fields of values of non-normal matrices, Num. Math., 4 (1962) 24–40.

SEMI-TOPOLOGICAL GROUPS

ELWOOD BOHN, Miami University, Oxford, Ohio, AND JONG LEE, Indiana University

1. Introduction. Let $T = (X, \mathfrak{J})$ denote a topological space. In accordance with [1], let $A \subset X$ be semi-open if and only if there exists $O \in \mathfrak{J}$ such that $O \subset A \subset \bar{O}$, where $\bar{O}$ denotes the closure of set O in T. As in [1], let S.O.(T) denote the class of all semi-open sets in T. Some of the basic properties concerning semi-open sets are summarized in the remarks which follow, and formal proofs of the theorems are given in [1].

THEOREM 1. *If $A_\alpha \in$ S.O.(T) for each $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} A_\alpha \in$ S.O.(T).*

DEFINITION 1. *A set $M_x \subset X$ is a semi-neighborhood of a point $x \in X$ if and only if there exists $A \in$ S.O.(T) such that $x \in A \subset M_x$.*

THEOREM 2. *If $A \subset X$, then a necessary and sufficient condition that $A \in$ S.O.(T) is that A be a semi-neighborhood of each $x \in A$.*

Proof. The necessity is trivial. For the sufficiency, there exists $A_x \in$ S.O.(T) such that $x \in A_x \subset A$ for each $x \in A$. Thus, $\bigcup_{x \in A} A_x = A \in$ S.O.(T) by Theorem 1.

DEFINITION 2. *A function $f: (X_1, \mathfrak{J}_1) \to (X_2, \mathfrak{J}_2)$ is semi-continuous at $x \in X_1$ if and only if for each neighborhood $N_{f(x)}$ of $f(x)$, $f^{-1}(N_{f(x)})$ is a semi-neighborhood of x. f is semi-continuous on X_1 if and only if f is semi-continuous at each point of X_1.*

THEOREM 3. *A function $f: (X_1, \mathfrak{J}_1) \to (X_2, \mathfrak{J}_2)$ is semi-continuous on X_1 if and only if for each $x \in X_1$ and each neighborhood $N_{f(x)}$ there exists a semi-neighborhood M_x such that $f(M_x) \subset N_{f(x)}$.*

THEOREM 4. *Let $T_1 = (X_1, \mathfrak{J}_1)$ and $T_2 = (X_2, \mathfrak{J}_2)$ be topological spaces and let $T = (X_1 \times X_2, \mathfrak{J}_1 \times \mathfrak{J}_2)$ be their product space. If $A_1 \in$ S.O.(T_1) and $A_2 \in$ S.O.(T_2), then $A_1 \times A_2 \in$ S.O.(T).*

2. Semi-topological groups. We introduce the notion of a semi-topological group as a generalization of a topological group (see definition in [2]).

DEFINITION 3. *A triple $(G, \circ, \mathfrak{J})$ is termed a semi-topological group if and only if $(G, \circ)$ is a group, $T = (G, \mathfrak{J})$ is a topological space, and for each $x, y \in G$ and each neighborhood $N_{x \circ y^{-1}}$ there exist semi-neighborhoods M_x and M_y such that $M_x \circ M_y^{-1} \subset N_{x \circ y^{-1}}$ where M_y^{-1} is defined below.*

DEFINITION 4. *For subsets $A, B \subset G$ $A^{-1} \equiv \{a \mid a^{-1} \in A\}$ and $A \circ B \equiv \{a \circ b \mid a \in A$ and $b \in B\}$.*

THEOREM 5. *If $A \in$ S.O.(T), then $A^{-1} \in$ S.O.(T).*

Proof. Since $A \in$ S.O.(T) there exists $O \in \mathfrak{J}$ such that $O \subset A \subset \overline{O}$. The conclusion follows since $O^{-1} \subset A^{-1} \subset (\overline{O})^{-1} = \overline{O^{-1}}$.

THEOREM 6. *If $(G, \circ, \mathfrak{J})$ is a semi-topological group then the function $f: G \times G \to G$, where $f((x, y)) = x \circ y^{-1}$ is semi-continuous relative to the product topology for $G \times G$.*

Proof. For any $(x, y) \in G \times G$ let $N_{x \circ y^{-1}}$ be given. There exist semi-neighborhoods M_x and M_y such that $M_x \circ M_y^{-1} \subset N_{x \circ y^{-1}}$. Theorem 4 implies $M_x \times M_y$ is a semi-neighborhood of (x, y). Thus, $f(M_x \times M_y) = M_x \circ M_y^{-1} \subset N_{x \circ y^{-1}}$ implies f is semi-continuous on $G \times G$.

THEOREM 7. *If $(G, \circ, \mathfrak{J})$ is a semi-topological group then the functions $i: G \to G$, where $i(x) = x^{-1}$ and $m: G \times G \to G$, where $m((x, y)) = x \circ y$ are semi-continuous functions.*

Proof. Let $x \in G$, e be the unity of G, and $N_{x^{-1}}$ be given. Since there exist semi-neighborhoods M_e and M_x such that

$$i(M_x) = M_x^{-1} = e \circ M_x^{-1} \subset M_e \circ M_x^{-1} \subset N_{e \circ x^{-1}} = N_{x^{-1}},$$

i is semi-continuous. Now let $(x, y) \in G \times G$ and $N_{x \circ y}$ be given. There exist semi-neighborhoods M_x and $M_{y^{-1}}$ such that $M_x \circ M_{y}^{-1} \subset N_{x \circ y}$. By Theorem 5, M_y^{-1} is a semi-neighborhood of y, say M_y. Since $M_x \times M_y$ is a semi-neighborhood of (x, y) and since $m(M_x \times M_y) = M_x \circ M_y \subset N_{x \circ y}$, m is semi-continuous.

THEOREM 8. *If $A \in$ S.O.(T) and $B \subset G$, then $A \circ B$, $B \circ A \in$ S.O.(T).*

Proof. Let $x \in B$ and $z \in A \circ x$, $z = y \circ x$ for some $y \in A = (A \circ x) \circ x^{-1}$. Since there exist semi-neighborhoods M_z and M_x such that $M_z \circ x^{-1} \subset M_z \circ M_x^{-1} \subset A$, we have $M_z \subset A \circ x$; thus, $A \circ x \in$ S.O.(T). By Theorem 1 $A \circ B \in$ S.O.(T).

REMARK. The converses of Theorems 6 and 7 are not true. This can be seen by considering the following example: Let $(G, +)$ be the group of integers modulo 2, with the usual operation of addition, and let $\mathfrak{J} = \{\varnothing, \{0\}, G\}$. i is continuous on G. Similarly, $m = f$ is continuous at (0, 0), (1, 0), and (0, 1) and semi-continuous at (1, 1). However, in view of Theorem 8, since $\{1\}$ is not semi-open, $(G, +, \mathfrak{J})$ is not a semi-topological group.

References

1. N. Levine, Semi-open sets and semi-continuity in topological spaces, this MONTHLY, 70 (1963) 36–41.

2. L. Pontrjagin, Topological Groups, Princeton Univ. Press, Princeton, N. J., 1946.

ON NORMAL METRICS

S. G. MROWKA, Pennsylvania State University

Throughout this paper, by a function we mean a real-valued function.

It is well known that every completely regular space X admits a uniformity $\mathfrak{U}$ such that every continuous function on X is uniformly continuous with respect to the uniformity $\mathfrak{U}$. Even when the space X is metrizable, however, we cannot assert the existence of a metrizable uniformity with this property. That is, even if there exists a metric for the space X (i.e., a metric compatible with the topology in X), there need not exist a metric ρ for X such that every continuous function on X is uniformly continuous with respect to the metric ρ. Necessary and sufficient conditions for the existence of such a metric are given in Theorem 4.

DEFINITION. *Let X be a space and let ρ be a metric for X. ρ is said to be a normal metric for X provided that for every two disjoint closed subsets A and B of X we have $\rho(A, B) > 0$.*

THEOREM 1. *A metric ρ for X is normal if and only if every continuous function f on X is uniformly continuous with respect to ρ.*

Proof. Let d denote the standard metric for the reals R (that is, $d(x,y) = |x-y|$ for $x, y \in R$). It is well known that a function f on X is uniformly continuous with respect to the metric ρ iff the condition $\rho(A, B)=0$ implies $d(f[A], f[B])=0$ for every pair A, B of subsets of X.

Assume that ρ is a normal metric for X and let f be a continuous function on X. If A, B is any pair of subsets of X with $\rho(A, B)=0$, then $\rho(\overline{A}, \overline{B})=0$; hence in view of the normality of ρ, $\overline{A} \cap \overline{B} \neq \varnothing$. Let $p_0 \in \overline{A} \cap \overline{B}$. By the continuity of f,

$$f(p_0) \in \overline{f[A]} \cap \overline{f[B]}; \quad \text{hence} \quad d(f[A], f[B]) = d(\overline{f[A]}, \overline{f[B]}) = 0;$$

thus f is uniformly continuous with respect to ρ.

Conversely, assume that the metric ρ is not normal. Then there exists a pair of disjoint closed subsets A and B of X with $\rho(A, B)=0$. But, by the Urysohn lemma, there exists a continuous function f on X such that $f(p)=0$ for $p \in A$ and $f(p)=1$ for $p \in B$. Clearly,

$$d(f[A], f[B]) = d(0, 1) = 1 \neq 0;$$

thus the function f is not uniformly continuous. The theorem is proved.

THEOREM 2. *If ρ is a normal metric for X, then X is complete with respect to the metric ρ.*

Proof. Assume that ρ is a normal metric for X and let $\{p_n\}$ be a Cauchy sequence of points of X. We shall show that $\{p_n\}$ is convergent. Of course, we can assume that the terms of the sequence $\{p_n\}$ are distinct. Suppose that $\{p_n\}$ is not convergent. Then $\{p_n\}$ has no convergent subsequence; consequently, the sets $A=\{p_{2n}: n=1, 2, \cdots\}$ and $B=\{p_{2n-1}: n=1, 2, \cdots\}$ are both closed. But $A \cap B=\varnothing$; hence $\rho(A, B)=\epsilon>0$. But $\rho(p_{2n-1}, p_{2n}) \geqq \epsilon$ for every n, contrary to the assumption that $\{p_n\}$ is a Cauchy sequence. Thus $\{p_n\}$ is convergent.

THEOREM 3. *Let X be a metrizable space. Every metric for X is normal if and only if X is compact.*

Proof. Obviously, if X is compact, then every metric for X is normal. Conversely, assume that X is not compact. Then, according to a result of Niemytzki and Tihonov [1], there is a metric ρ for X such that X is not complete with respect to ρ. By Theorem 2 this metric is not normal.

The main theorem of the present paper is the following:

THEOREM 4. *Let X be a metrizable space. X admits a normal metric if and only if the set X' of all nonisolated points of X is compact.*

Proof. Assume that the set X' is not compact and let ρ be any metric for X. There exists a sequence $\{p_n\}$ of points of X' which has no convergent subsequence (note that X' is closed in X). Of course, we can assume that all p_n are distinct: $p_n \neq p_m$ for $n \neq m$. It follows that $c_n=\rho(p_n, Z_n)>0$ for every n, where $Z_n=\{p_m: m \neq n\}$. Since p_n is a nonisolated point of X, we can find a point $q_n \in X$ such that

$$q_n \neq p_n \quad \text{and} \quad \rho(p_n, q_n) < \min\left\{\frac{1}{n}, c_n\right\}. \tag{1}$$

Observe that if $m \neq n$ then, by (1),

$$\rho(p_m, q_n) \geqq \rho(p_m, p_n) - \rho(p_n, q_n) \geqq \rho(p_n, Z_n) - \rho(p_n\ q_n) = c_n - \rho(p_n, q_n)$$
$$> c_n - c_n = 0;$$

thus

$$p_m \neq q_n \quad \text{for} \quad m \neq n. \tag{2}$$

It follows from (1) and (2) that the sets of terms of the sequences $A = \{p_n: n=1, 2, \cdots\}$ and $B = \{q_n: n=1, 2, \cdots\}$, are disjoint. Now $\{p_n\}$ has no convergent subsequence and, by (1), $\rho(p_n, q_n) \to 0$; hence $\{q_n\}$ has no convergent subsequence; therefore the sets A and B are closed. But $\rho(A, B) \leqq \rho(p_n, q_n) < 1/n$ for every n and hence $\rho(A, B) = 0$. Thus the metric ρ is not normal.

Conversely, assume that the set X' is compact. Let ρ be any metric for X; we are going to construct a normal metric for X. Let us set

$$A_1 = \{p \in X: \rho(p, X') \geqq 1\}$$

and

$$A_k = \left\{p \in X: \frac{1}{2^k} \leqq \rho(p, X') < \frac{1}{2^{k-1}}\right\} \quad \text{for} \quad k = 2, 3 \cdots.$$

Clearly

$$X = X' \cup A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots; \tag{3}$$

moreover the sets occurring in the left-hand side of (3) are mutually disjoint.

Define the function $\rho'(p, q)$; $p, q \in X$, in the following way:

If $p = q$, then $\rho'(p, q) = 0$ and if $p \neq q$, then

$$\begin{aligned}
&\rho'(p, q) = 0, \text{ if } p, q \in X';\\
&\rho'(p, q) = 1/2^n, \text{ if } p, q \in A_n;\\
(4)\quad &\rho'(p, q) = 1/2^n, \text{ if } p \in X' \text{ and } q \in A_n \text{ or } p \in A_n \text{ and } q \in X';\\
&\rho'(p, q) = \left|\frac{1}{2^n} - \frac{1}{2^m}\right|,\\
&\qquad \text{if } p \in A_n \text{ and } q \in A_m \text{ or } p \in A_m \text{ and } q \in A_n, \text{ where } n \neq m.
\end{aligned}$$

It is easy to check that ρ' is a pseudo-metric for X (that is, ρ' satisfies all the conditions of a metric except this condition: $\rho'(p, q) = 0$ implies $p = q$).

Observe that, if $\rho(p_n, p) \to 0$ and $p_n \neq p$, then p is a nonisolated point of X; i.e., $p \in X'$. But, if $p \in X'$, then $\rho'(p_n, p) \leqq \rho(p_n, p)$. Indeed, if p_n is also a non-

isolated point of X, then $\rho'(p_n, p)=0$ and if p_n is an isolated point of X, then $p_n \in A_k$ for some k and hence

$$\rho(p_n, p) \geqq \rho(p_n, X') \geqq \frac{1}{2^k},$$

while $\rho'(p_n, p)=1/2^k$. It follows from the preceding that

(5) the condition $\rho(p_n, p) \to 0$ implies $\rho'(p_n, p) \to 0$.

Since ρ is a metric and ρ' is a pseudo-metric, the function $\rho_1(p, g)=\rho(p, g)+\rho'(p, g)$ is a metric. Moreover, from (5) we obtain immediately

(6) the conditions $\rho(p_n, p) \to 0$ and $\rho_1(p_n, p) \to 0$ are equivalent,

and it follows that the metric ρ_1 is compatible with the topology of X. The proof that ρ_1 is a normal metric for X can be based on the following facts:

If p_n and q_n are two sequences of points of X with $\rho_1(p_n, g_n) \to 0$ and $p_n \neq q_n$ for every n, then $\rho_1(p_n, X') \to 0$. Consequently, we can find a sequence $\{p_n'\}$ of points of X' such that $\rho_1(p_n, p_n') \to 0$. But X' is compact; consequently, the sequence $\{p_n'\}$ has a limit point, say p_0. Clearly, this point p_0 is a common limit point of both sequences $\{p_n\}$ and $\{q_n\}$. This proves Theorem 4.

From Theorem 4 we obtain immediately:

COROLLARY. *A dense-in-itself metrizable space X admits a normal metric if and only if X is compact.*

The above corollary is due to N. Levine and W. G. Saunders [2].

The author was supported by a National Science Foundation Research Grant No. GP 1843.

References

1. V. Niemytzki and A. Tihonov, Beweis des Satzes, dass ein metrischer Raum dann und nur dann kompakt ist, wenn er in jeder Metrik vollständig ist, Fund. Math., 12 (1928) 118.

2. N. Levine and W. G. Saunders, Uniformly continuous sets in metric spaces, this MONTHLY, 67 (1960) 153–156.

BARELY FAITHFUL ALGEBRAS

HARRIET FELL AND JOHN MATHER, National Bureau of Standards

We deal with nontrivial associative linear algebras over a field F. An algebra A will be called *faithful* if its regular representations are both isomorphisms, which is true if and only if A has no annihilators. Some texts, however, mention only the stronger sufficient condition that A have no zero-divisors. This suggests studying those algebras, to be called *barely faithful*, in which each nonzero element is both a right and left zero-divisor, but neither a right nor a left annihilator. It will be shown here that such algebras exist in all dimensions $\geqq 4$, but not in lower dimensions.

If A is barely faithful, then a barely faithful algebra of dimension $\dim(A)+k$ is obtained from the direct sum $A+B$, where B is any k-dimensional faithful algebra, by setting $xy=yx=0$ if $x\in A$ and $y\in B$. A 4-dimensional barely faithful algebra (a modification of a 5-dimensional example due to M. Newman) is given by the matrices over F

$$M_1 = \begin{bmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & d_1 & a_1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} a_2 & 0 & 0 \\ b_2 & 0 & 0 \\ c_2 & d_2 & a_2 \end{bmatrix}$$

so that

$$M_1M_2 = \begin{bmatrix} a_1a_2 & 0 & 0 \\ b_1a_2 & 0 & 0 \\ c_1a_2 + d_1b_2 + a_1c_2 & a_1d_2 & a_1a_2 \end{bmatrix}.$$

Each element M_1 of A is a left zero-divisor, since we can take $a_2=d_2=0$ and then choose nonzero b_2 and c_2 for which $d_1b_2+a_1c_2=0$; similarly each element of A is a right zero-divisor. No nonzero element M_1 of A is a left annihilator, for if $M_1M_2=0$ for all M_2 then we can take $a_2\neq 0$ and $d_2\neq 0$ to prove $a_1=b_1=0$, then take $a_2\neq 0$ and $b_2=0$ to prove $c_1=0$, then take $b_2\neq 0$ to prove $d_1=0$; similarly no nonzero element of A is a right annihilator.

Thus there are barely faithful algebras of all dimensions $\geqq 4$, and it suffices to rule out dimensions 1, 2, and 3. The remainder of the paper is devoted to showing that there exist no barely faithful algebras of dimensions 1, 2, or 3. Throughout the rest of the paper, A denotes a barely faithful algebra, Greek letters denote scalars, and Latin letters denote members of A. For $x\in A$, let r_x (l_x) denote the image of x in the right (left) regular representation. Nonzero elements of the (left ideal) null space $N(r_x)$ will be called *left nulls* of x; right nulls are defined similarly. Note that nulls (right or left) of x are linearly independent of x if $x^2\neq 0$. Note also that

$$\dim(N(r_x)) + \dim(\mathrm{im}(r_x)) = \dim(N(l_x)) + \dim(\mathrm{im}(l_x)) = \dim(A),$$

with all summands positive if $x\neq 0$. It follows that $\dim(A)>1$.

Suppose $\dim(A)=2$. There are two cases. *First* assume $x^2=0$ for some nonzero x. Then x generates $N(r_x)$. Choose y' so that $\{x, y'\}$ is a basis. Then $0\neq y'x\in N(r_x)$, so $y'x=\alpha x$ for some nonzero α and we can replace y' by $y=\alpha^{-1}y'$; $\{x, y\}$ is a basis and $yx=x$. A right null $\beta y+\delta x$ of y ($\{\beta, \delta\}\neq\{0\}$) leads to $\beta y^2+\delta x=0$, implying $\beta\neq 0$ and thus $y^2=\Gamma x$ with $\Gamma=-\delta\beta^{-1}$. Then

$$x = yx = y(yx) = \Gamma x^2 = 0,$$

a contradiction.

Second, assume $x^2\neq 0$ for all nonzero x. For such an x, choose a generator y' of $N(r_x)$. Since $(y')^2\in N(r_x)$, we have $(y')^2=\alpha y'$ for some nonzero α, and can replace y' by $y=\alpha^{-1}y'$. Then $\{x, y\}$ is a basis, $y^2=y$ and $yx=0$. Similarly we can

adjust x by a scalar multiplication so that $x^2=x$. Since $yx=0$ implies $(xy)^2=0$, the case hypothesis yields $xy=0$. It readily follows that $x+y$ is a multiplicative two-sided identity in A and thus not a zero-divisor. This completes the proof that $\dim(A)>2$.

Now assume $\dim(A)=3$. For nonzero $t\in A$, if the left ideal $N(r_t)$ is not a line it must be 2-dimensional, and then the left ideal $\operatorname{im}(r_t)$ is a line. This shows that some $x\in A$ generates a line $[x]$ which is a left ideal. Then $yx=x$ for some y. Since $\dim(\operatorname{im}(r_x))=1$, we have $\dim(N(r_x))=2$ and $N(r_x)\cup\{y\}$ spans A in the vector space sense. A left null $\alpha y-v$ of y (with $v\in N(r_x)$, α and v not both zero) leads to $\alpha y^2=vy$, and since $vy\in N(r_x)$ we have either $y^2\in N(r_x)$ or $\alpha=0$. The first alternative implies $0=y^2x=yx=x$ and is impossible; hence $\alpha=0$, so that $vy=0$ with $0\neq v\in N(r_x)$. Considering a right null of y leads similarly to $yw=0$ with $0\neq w\in N(r_x)$.

CASE 1. Suppose $x^2=0$. Then $\{x, w\}$ is a basis for $N(r_x)$ and $\{x, w, y\}$ a basis for A. Application of l_y to $w^2=\beta x+\Gamma w$ yields $w^2=\Gamma w$. Also $xy\in[x]$, for otherwise $\{x, xy\}$ would span $N(r_x)$ and v would be a left annihilator of A. Say $xy=\delta x$. Since $\operatorname{im}(l_x)\subset N(l_x)$, $N(l_x)$ is 2-dimensional like $N(r_x)$, and $N(r_x)\cap N(l_x)\supset[x]$.

CASE 1A. Suppose $N(r_x)=N(l_x)$. Then $\delta\neq 0$; $xw=0$, and $\Gamma\neq 0$ since w is not a right annihilator; we may assume $w^2=w$. Applying r_w to $wy=\lambda x+\mu w$ leads to $wy=\lambda x$, and application of l_y gives $wy=0$. Thus we have the indicated multiplication table, with $\zeta\neq 0$ since $y^2\notin N(l_x)$; it is easily shown that $x+w+y$ has no right null.

CASE 1B. $N(r_x)\cap N(l_x)=[x]$ is the only alternative to Case 1A. Then $N(r_x)\cup N(l_x)$ spans A, and we can take $y\in N(l_x)$, $y\notin N(r_x)$. Applying l_x to $xw=\lambda x+\mu w$, and noting that $xw\neq 0$ since x is not a left annihilator, we get $xw=\lambda x$ with $\lambda\neq 0$ and can adjust w so $xw=x$. It follows that $wy\in N(r_x)\cap N(l_x)$, and applying r_w to $wy=\nu x$ shows $wy=0$. $\Gamma\neq 0$ since w is not a left annihilator. Since $y^2\in N(l_x)$, we can write $y^2=\rho y+\sigma x$ and apply r_w to obtain $y^2=\rho y$; $\rho\neq 0$ since y is not a right annihilator. We have the indicated multiplication table, and again $x+w+y$ has no right null.

CASE 1A

	x	w	y
x	0	0	δx
w	0	w	0
y	x	0	$\xi x+\eta w+\zeta y$

CASE 1B

	x	w	y
x	0	x	0
w	0	Γw	0
y	x	0	ρy

CASE 2. Suppose $x^2\neq 0$, so that $N(r_x)\cup\{x\}$ spans A. We may assume $N(r_x)\subset N(l_x)$, since for any $u\in N(r_x)$, $u\notin N(l_x)$, $x'=xu$ would fall under Case 1. Dimensionality shows that $N(r_x)=N(l_x)=N$, a 2-dimensional algebra. A right or left annihilator of N would annihilate A, so there are none. A left null $n'+\alpha x$ of $n+x$, $0\neq n\in N$ (with $n'\in N$) exhibits n' as a left null of n in N and similarly for right nulls; hence N is barely faithful, which is impossible.

A FIXED POINT THEOREM FOR MAPPINGS WHICH DO NOT INCREASE DISTANCES

W. A. Kirk, University of California, Riverside

In this note we use a characterization of reflexivity due to Šmulian and a concept of Brodskiĭ and Milman to prove a fixed point theorem for mappings which do not increase distances.

Šmulian proved [4] that a necessary and sufficient condition that a Banach space X be reflexive is that:

(C) *Every bounded descending sequence (transfinite) of nonempty closed convex subsets of X have a nonempty intersection.*

For $S \subset X$ we denote the diameter of S by $\delta(S)$. A point $x \in S$ is a *diametral point* of S provided $\sup \{\|x-y\|: y \in S\} = \delta(S)$. A convex set $K \subset X$ is said to have *normal structure* (cf. [1]) if for each bounded convex subset H of K which contains more than one point, there is some point $x \in H$ which is not a diametral point of H.

We shall prove the following:

THEOREM. *Let K be a nonempty, bounded, closed and convex subset of a reflexive Banach space X, and suppose that K has normal structure. If ϕ is a mapping of K into itself which does not increase distances, then ϕ has a fixed point.*

Professor Klee has observed that this theorem is true, but apparently no mention of it has been made in the literature. We feel, however, that the theorem is not generally known, and is of sufficient interest to warrant its mention. We should also remark that the construction of Brodskii and Milman [1] yields a unique point of K which is fixed under every isometry of K onto itself. While our proof is a modification of their procedure, it is the case that the point they construct is not necessarily fixed under mappings of the type we consider.

We also obtain the following corollary.

COROLLARY. *If in the above theorem the condition that K be bounded is replaced by the requirement that the sequence $\{\phi^n(p)\}$ be bounded for some $p \in K$, then ϕ has a fixed point.*

The following lemmas will be helpful in proving the theorem.

Let F denote a nonempty, bounded, closed and convex subset of the Banach space X. Let

$$\begin{aligned} r_x(F) &= \sup\{\|x-y\|: y \in F\}, \\ r(F) &= \inf\{r_x(F): x \in F\}, \\ F_c &= \{x \in F: r_x(F) = r(F)\}. \end{aligned}$$

LEMMA 1. *If X is reflexive then F_c is nonempty, closed and convex.*

Proof. Let $F(x, n) = \{y \in F: \|x-y\| \leq r(F) + 1/n\}$. It is easily seen that the

sets $C_n = \bigcap_{x \in F} F(x, n)$ form a decreasing sequence of nonempty closed convex sets, and hence $F_c = \bigcap_{n=1}^{\infty} C_n$ is closed, convex, and by (C), nonempty.

LEMMA 2. *Let F be a closed convex subset of X which contains more than one point. If F has normal structure, then $\delta(F_c) < \delta(F)$.*

Proof. By normal structure F contains at least one nondiametral point x. Hence $r_x(F) < \delta(F)$. If z and w are any two points of F_c then $\|z-w\| \leqq r_z(F) = r(F)$. Hence

$$\delta(F_c) = \sup\{\|z - w\| : w, z \in F_c\} \leqq r(F) \leqq r_x(F) < \delta(F).$$

Proof of the theorem. Let $\mathfrak{F}$ denote the collection of all nonempty closed and convex subsets of K, each of which is mapped into itself by ϕ. By (C) and Zorn's lemma $\mathfrak{F}$ has a minimal element which we denote by F. We complete the proof by showing that F consists of a single point.

Let $x \in F_c$. Then $\|\phi(x) - \phi(y)\| \leqq \|x - y\| \leqq r(F)$ for all $y \in F$, and hence $\phi(F)$ is contained in the spherical ball $\overline{U}$ centered at $\phi(x)$ with radius $r(F)$. Since $\phi(F \cap \overline{U}) \subset F \cap \overline{U}$, the minimality of F implies $F \subset \overline{U}$. Hence $\phi(x) \in F_c$ and F_c is mapped into itself by ϕ; by Lemma 1, $F_c \in \mathfrak{F}$. If $\delta(F) > 0$ then, by Lemma 2, F_c is properly contained in F. Since this contradicts the minimality of F, $\delta(F) = 0$ and F consists of a single point.

Proof of the corollary. Let $S = \{\phi^n(p)\}$. Choose $r > 0$ so that S is contained in the spherical ball $\overline{U}(p; r)$ centered at p with radius r. For each n let $\overline{U}(\phi^n(p); r)$ denote the ball centered at $\phi^n(p)$ with radius r, and let $K_n = \overline{U}(\phi^n(p); r) \cap K$. Since ϕ does not increase distances, we have $\phi^n(S) \subset K_n$, $n = 1, 2, \cdots$.

Thus for each n, K_n is closed, convex and nonempty.

Let

$$W = \left\{ x \in K : x \in \bigcap_{n=k}^{\infty} K_n \text{ for some } k \right\}.$$

The sequence $\{K_n\}$ has the finite intersection property, so it follows from (C) that W is not empty. It is easily seen that the closure $\overline{W}$ of W is bounded, convex, and mapped into itself by ϕ—thus establishing the corollary.

Examples. Let B denote the unit ball in the Hilbert space l_2. This space is reflexive and it is easily seen that B has normal structure. Let k be any number greater than 1.

For $x = (x_1, x_2, \cdots) \in B$, let $\phi(x) = (t(1 - \|x\|), x_1, x_2, \cdots)$, where t is a constant chosen so that $t < 1$ and $0 < t \leqq (k^2 - 1)^{1/2}$.

It is easily verified that ϕ maps B into itself and has the property that for each $x, y \in B$,

$$\|\phi(x) - \phi(y)\| \leqq k\|x - y\|.$$

Clearly ϕ has no fixed point.

This example shows that our theorem cannot be extended to include map-

pings of the above slightly more general type. (Fixed point theorems for mappings of this type, with weaker space hypotheses but with certain additional conditions on the mappings, have been obtained by Edelstein [2; 3].)

Another example might be of interest. The Banach space $C[0, 1]$ of continuous functions is not reflexive. Let $K = \{f \in C[0, 1] : f(0) = 0, f(1) = 1, 0 \leqq f(x) \leqq 1\}$. K is bounded, closed and convex. Define the mapping ϕ as follows:

$$\phi(f(x)) = xf(x).$$

It is easily seen that ϕ maps K into itself, does not increase distances, and has no fixed point.

Various questions regarding additional weakenings of our hypotheses remain unanswered. In particular, we do not know whether normal structure is essential for the result of this paper.

References

1. M. S. Brodskiĭ and D. P. Milman, On the center of a convex set, Dokl. Akad. Nauk SSSR (N.S.), 59(1948) 837–840.
2. Michael Edelstein, An extension of Banach's contraction principle, Proc. Amer. Math. Soc., 12(1961) 7–10.
3. ———, On predominantly contractive mappings, J. London Math. Soc., 38(1963) 81–86.
4. V. Šmulian, On the principle of inclusion in the space of type (B), Mat. Sb. (N.S.), 5(1939) 327–328.

A NOTE ON CONVERGENT MATRICES

C. G. CULLEN, University of Pittsburgh

Let A be a square matrix over the complex field and let

$$\rho(A) = \max_i \{ |\lambda_i| : \lambda_i \text{ is a characteristic value of } A\}$$

be the spectral radius of A. It is a well-known fact that A is convergent ($\lim_{n\to\infty} A^n = 0$) if and only if all characteristic values of A have modulus less than 1, i.e., $\rho(A) < 1$. P. Stein [1] gave an alternate necessary and sufficient condition for A to be convergent when he proved that $\lim_{n\to\infty} A^n = 0$ if and only if there exists a positive definite Hermitian matrix H for which $H - A^*HA$ is also positive definite. The sufficiency is clear if one considers a characteristic vector associated with a characteristic value of A of maximum modulus. Stein's proof of the necessity is quite involved and it is my purpose here to present a much simpler proof.

We first observe that if $\{B_N\}$ is any sequence of square matrices which converges to the zero matrix then the scalar sequence $\{\rho(B_N)\}$ must also approach zero. Since $\lim_{n\to\infty} A^n = 0$, then $\lim_{n\to\infty} (A^*)^n = 0$ and hence, $\lim_{n\to\infty} A^{*n}A^n = 0$. From our remark above we can find a positive integer M such that, for $n \geqq M$, $\rho(A^{*n}A^n) \leqq 1/2$. Now

$$H = I + A^*A + A^{*2}A^2 + \cdots + A^{*M-1}A^{M-1}$$

is clearly positive definite and a direct calculation shows that $H-A^*HA = I-A^{M*}A^M$ which, since $\rho(A^{M*}A^M)\leqq 1/2$, is certainly positive definite.

Our construction also shows, as observed by Stein, that if A is real then H may also be taken to be real.

Reference

1. P. Stein, Some general theorems on iterants, J. Res. Nat. Bur. Standards, 48 (1952) 82–83.

CLASSROOM NOTES

Edited by Gertrude Ehrlich, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Maryland 20740.

TWO EXTENDED BOLZANO-WEIERSTRASS THEOREMS

J. E. Kimber, Jr., Utah State University

In this note the Bolzano-Weierstrass Theorem is extended in a form which is exceedingly convenient for applications. It involves the replacement of finite and infinite sets by *light* and *heavy* sets, where lightness and heaviness are undefined concepts satisfying several natural axioms. A number of applications are given, followed by a modified version of the new theorem.

We first give a preliminary definition. Let R^n be Euclidean n-space, with the n-tuples $(x_1, \cdots, x_n)$ of real numbers as its points. A pair of subsets A and B of R^n will be called *cleft* if there exists a coordinate hyperplane $x_i=c$ neither side of which contains points of both A and B. For example, in R^2 let A be the set $x_1^2+x_2^2\leqq 1$, B the set $x_2\geqq 1$, C the set $x_2=1$, and D the set $x_1+x_2>2$. Then the pairs A and B, A and C, B and C are cleft, while the pairs A and D, B and D, C and D are not.

The properties to be imposed on the notions of lightness and heaviness will now be given. We shall be given a subset S of R^n, and lightness and heaviness of the subsets of S are to satisfy the following axioms:

(i) *Every subset of S is light or heavy, but not light and heavy.*
(ii) *The empty set is light.*
(iii) *Two cleft subsets of S are light if and only if their union is light.*

Clearly (iii) is equivalent to the condition that if A and B are cleft then $A\cup B$ is heavy if and only if A or B is heavy. It would seem more natural to use the following condition in place of (iii):

(iii′) *Two subsets of S are light if and only if their union is light.*

The use of (iii) instead of (iii′), however, allows the utility of the ordering of the real numbers to come into play. Notions of heaviness that satisfy (iii′) will be called *topological.*

Examples. 1. Take the heavy subsets of S to be the infinite subsets.

2. Take a subset of S to be heavy if it is uncountable.

3. Call a subset of R^n heavy if it contains an open interval. (An open interval in R^n is a set of the form $a_1 < x_1 < b_1, \cdots, a_n < x_n < b_n$, where $a_1 < b_1, \cdots, a_n < b_n$.) In this example (iii′) is not satisfied, but (iii) is.

4. By an *interval of rationals* in R^1 we shall mean the set of all rationals that lie in some given interval in R^1. Call a subset of R^1 heavy if its intersection with some interval in R^1 is an interval of rationals. Since R^1 is light but the set of all rationals is heavy, this example shows that heaviness is not in general a monotonic property. We can even find subsets A, B, C of R^1 with $A \subset B \subset C$ but with B light and A and C heavy. However *a topological heaviness is monotonic,* since (iii′) implies that a subset of a light set is light.

It will be convenient to call a subset S of R^n provided with a notion of heaviness of its subsets a *weighted* set. A point b will be called a *barypoint* of S if every neighborhood of b has a heavy intersection with S. Clearly b is a point in $\bar{S}$, the closure of S. Using (iii) it is easy to see that *b is a barypoint of S if and only if every open interval in R^n containing b has a heavy intersection with S.* The extension of the Bolzano-Weierstrass Theorem is

THEOREM 1. *Every bounded heavy subset of R^n has a barypoint.*

(The proof that follows is based on the Dedekind Completeness Principle, the least upper bound property of the real numbers being obtained later as a corollary of Theorem 1. It is just as easy to base a proof of Theorem 1 on the least upper bound property.)

Proof. Let T be a weighted bounded heavy subset of R^1. Let L be the set of all real numbers x for which $(-\infty, x) \cap T$ is light, and let H be the set of all real numbers y such that $(-\infty, y) \cap T$ is heavy. It follows from (iii) that if $x \in L$ and $y \in H$ then $x < y$. Since T is bounded, L and H are both nonempty. By the Dedekind Principle there exists a real number c such that $x \leqq c \leqq y$ for all x in L and y in H. The set $(-\infty, z) \cap T$ is light or heavy according as $z < c$ or $z > c$. Let $\epsilon > 0$. Since

$$(-\infty, c+\epsilon) = (-\infty, c-\epsilon) \cup [c-\epsilon, c+\epsilon)$$

and $(-\infty, c+\epsilon) \cap T$ is heavy while $(-\infty, c-\epsilon) \cap T$ is light, it follows from (iii) that $[c-\epsilon, c+\epsilon) \cap T$ is heavy. If N is any neighborhood of c then N contains an interval $[c-\epsilon, c+\epsilon)$ with $\epsilon > 0$, and it follows easily from (iii) that $N \cap T$ is heavy. Therefore c is a barypoint of T.

Now assume the theorem is true for $n = k-1$, where $k \geqq 2$, and let S be a

weighted, bounded, heavy subset of R^k. Let Q and T be the images of S in R^{k-1} and R^1 under the projections $(x_1, \cdots, x_k) \to (x_1, \cdots, x_{k-1})$ and $(x_1, \cdots, x_k) \to x_k$ respectively, so that $S \subset Q \times T$. Weight T by taking a subset B of T to be heavy if $S \cap (Q \times B)$ is heavy, and let c be a barypoint of T. Call a subset A of Q heavy if $S \cap (A \times N)$ is heavy whenever N is a neighborhood of c. It is easy to verify that this weights Q, and Q is obviously heavy. Let b be a barypoint of Q. Then (b, c) is a barypoint of S.

Next we shall illustrate the utility of Theorem 1 with some applications in proofs of various classical theorems. It will be seen that proofs based on Theorem 1 are often simpler and easier to think up than proofs based on other principles. *In what follows, b will always denote some barypoint.*

Taking heaviness to mean infinite, a barypoint b is a limit point. This shows that *every bounded infinite subset of R^n has a limit point.* Taking heaviness to mean uncountable, b is a condensation point, so every bounded uncountable subset of R^n has a condensation point. It follows by usual arguments that *every uncountable subset of R^n has a condensation point.* Similarly if $\{x_n\}$ is a bounded sequence whose points make up the set S, taking a set to be heavy if it contains x_n for infinitely many values of n, every neighborhood of b contains x_n for infinitely many values of n. Hence a subsequence can be obtained which converges to b, showing that *every bounded sequence in R^n has a convergent subsequence.*

Let $\{X_\alpha\}$ be a collection of bounded sets in R^n with the finite intersection property, and let S be one of the sets X_β. Writing $Y_\alpha = S \cap X_\alpha$, $\{Y_\alpha\}$ has the finite intersection property. Call a subset A of S heavy if $\{A \cap Y_\alpha\}$ has the finite intersection property. Using the relation $(A \cup B) \cap (P \cap Q) \subset (A \cap P) \cup (B \cap Q)$, it is easy to verify that this weights S. Since every neighborhood of b must meet each Y_α we have $b \in \overline{Y}_\alpha$. Hence *the intersection of a collection of closed bounded subsets of R^n with the finite intersection property is nonempty.*

Let S be closed and bounded and let $\{W_\alpha\}$ be an open covering of S. Take a subset of S to be heavy if it is not covered by a finite subcollection of $\{W_\alpha\}$. If S were heavy it would have a barypoint b contained in some W_α, contradicting the fact that every neighborhood of b must contain a heavy subset of S. Hence *every open covering of a closed bounded subset of R^n contains a finite subcovering.*

Let S be a bounded nonempty set of real numbers. Call a subset of S heavy if it is cofinal in S (A is cofinal in S if every element of S is less than or equal to some element of A). Since $b \in \overline{S}$ and S can contain no numbers greater than b, b is the least upper bound of S. Hence, *every bounded nonempty set of real numbers has a least upper bound.*

Similarly let S be a compact subset of R^n and let f be a continuous (or upper semi-continuous) real-valued function on S. Call a subset A of S heavy if $f(A)$ is cofinal in $f(S)$. If $f(x) > f(b)$ for some $x \in S$ then some neighborhood of b would contain no heavy set. It follows that *a continuous (or upper semi-continuous) real-valued function on a compact set has a maximum value.*

Similarly let f be a continuous function from a compact subset S of R^n into R^m, and let w be a point in $\overline{f(S)}$ (we shall follow the convention that w could be ∞ if $f(S)$ were unbounded). Call a subset A of S heavy if $w \in \overline{f(A)}$. Since every neighborhood of b contains a set A with $w \in \overline{f(A)}$ we must have $w = f(b)$. Hence $w \neq \infty$, so $f(S)$ is closed and bounded. Hence *a continuous image of a compact set is compact.*

With f and S the same as in the preceding application, let $\epsilon > 0$, and call a subset of S heavy if it meets pairs $\{x, y\}$ of points of S with $|x-y|$ arbitrarily small but with $|f(x)-f(y)| \geqq \epsilon$. If f were not uniformly continuous then for some $\epsilon > 0$, S would be heavy. But since $f(z) \to f(b)$ as $z \to b$, some neighborhood of b would contain no heavy set. It follows that *a continuous function on a compact set is uniformly continuous.*

Let f and S be as before except that we also take f one-to-one. Let $\epsilon > 0$, let $a \in S$, and call a subset of S heavy if it contains points x with $|f(x)-f(a)|$ arbitrarily small but with $|x-a| \geqq \epsilon$. If the inverse of f were discontinuous at $f(a)$ then for some $\epsilon > 0$, S would be heavy. We then would have $|b-a| \geqq \epsilon$ but $|f(b)-f(a)| = 0$, a contradiction. Hence *a continuous one-to-one function on a compact set is a homeomorphism.*

Up to now all of the kinds of heaviness involved in the applications have been topological. We now take up some applications where (iii) is satisfied but not (iii′). Let f be a continuous real-valued function on a closed interval $[a, c]$ in R^1, and suppose that $f(a) < k < f(c)$. Call a subset of $[a, c]$ heavy if it contains an open interval (α, γ) such that $f(\alpha) \leqq k \leqq f(\gamma)$. It is not difficult to verify (iii). Since $f(x) \to f(b)$ as $x \to b$, $f(b) = k$. Hence *a continuous real-valued function on an interval takes on all values between any two values.*

Again let f be a continuous real-valued function on $[a, c]$ and let $\epsilon > 0$. Call a subset of $[a, c]$ light if whenever an open interval (α, γ) is contained in it there exists a step-function g on $[\alpha, \gamma]$ such that $|f(x)-g(x)| < \epsilon$ for all x in $[\alpha, \gamma]$. Suppose that for some $\epsilon > 0$ $[a, c]$ were heavy. Then f would be discontinuous at b. Hence *a continuous real-valued function on a closed interval can be approximated arbitrarily closely by a step-function.* The same result can be obtained for a continuous function on a compact subset of R^n. This theorem seems to have important pedagogical advantages over uniform continuity in the elementary theory of integration, especially when upper and lower integrals are defined as the greatest lower and least upper bounds respectively of integrals of step-functions that are greater than and less than the integrand.

Let I be an open interval in R^2 and let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ be a polynomial having no zeros in the boundary $\dot{I}$ of I. The *winding number* of $f(z)$ on $\dot{I}$ is the net number of counterclockwise revolutions that $f(z)$ makes about the origin as z traverses $\dot{I}$ once in the counterclockwise direction. If I_1 and I_2 are disjoint open intervals in R^2 such that $\overline{I_1 \cup I_2}$ is a closed interval I containing no zeros of f then the winding number of $f(z)$ on $\dot{I}$ is the sum of the winding numbers of $f(z)$ on $\dot{I}_1$ and $\dot{I}_2$. Let S be an open interval in R^2 with no zeros of f in $\overline{S}$. Call a subset of S heavy if it contains an open interval I such

that the winding number of $f(z)$ on $\dot{I}$ is nonzero. From the preceding remarks about $\dot{I}_1$ and $\dot{I}_2$ it is seen that (iii) is satisfied. If S were heavy, from $f(b)\neq 0$ we would have arg $f(z)\rightarrow$arg $f(b)$ as $z\rightarrow b$. Hence some neighborhood of b would contain no heavy set. It follows that S must be light. Now suppose that $f(z)$ is of positive degree and has no zeros in R^2. Since

$$f(z) = z^n(1 + a_1z^{-1} + a_2z^{-2} + \cdots + a_nz^{-n}),$$

$$\arg f(z) = \arg z^n + \arg (1 + a_1z^{-1} + \cdots + a_nz^{-n}).$$

It follows that if S is a sufficiently large square open interval with center at the origin the winding number of $f(z)$ on $\dot{S}$ equals n. Thus S is heavy, which we saw was impossible. Hence *every polynomial of positive degree has a zero.*

Let S be an open interval in R^n, let f be a continuous real-valued function on $\overline{S}$, and let $\epsilon>0$. If E is a closed subinterval of $\overline{S}$ let $\Delta(E)$ denote the difference between the upper and lower integrals of f on E, and let $m(E)$ be the volume of E. Call a subset of S heavy if it contains an open interval I such that $\Delta(\overline{I}) \geqq \epsilon m(\overline{I})$. From the additivity of upper and lower integrals (iii) is easily verified. If f were not integrable on $\overline{S}$ then for some $\epsilon>0$, S would be heavy. Then every neighborhood of b would contain an interval I with $\Delta(\overline{I})\geqq\epsilon m(\overline{I})$, which would imply that f is discontinuous at b. It follows that *a continuous function on a closed interval in R^n is integrable.*

For a subset S of an arbitrary topological space X we shall be concerned with topological heaviness. Suppose that $S\subset C\subset X$, where C is compact, and that S has no barypoint. Then each point x of C has an open neighborhood W_x having a light intersection with S. Then for some finite subcollection $W_y, \cdots, W_z$ of $\{W_x\}$ we have

$$S = (W_y \cap S) \cup \cdots \cup (W_z \cap S),$$

from which it follows that S is light. Following the convention that a subset of a compact set is called *topologically bounded,* this proves

THEOREM 2. *Every topologically bounded heavy subset of a topological space has a barypoint.*

Theorem 2 makes it possible for many of the applications of Theorem 1 to carry over into more general settings than Euclidean spaces. Both of the extended Bolzano-Weierstrass theorems have other important applications. Together they have a unifying effect in analysis and general topology, tying results of various sorts together by the common property that they are easy corollaries.

The converse of Theorem 2 is interesting. It will be convenient to first put the theorem in a different form. A subset of a topological space will be called a *baryset* if whenever it is weighted and heavy it has a barypoint. Theorem 2 states that *every topologically bounded set is a baryset.* The converse is true in a large class of spaces.

THEOREM 3. *In a space that is locally compact or regular, every baryset is topologically bounded.*

Proof. Let X be a topological space and S a subspace of X that is a baryset. Consider first the case when X is locally compact. Weight S by taking $A \subset S$ to be light if A is bounded. Suppose S were unbounded, and therefore heavy. If b were a barypoint there would exist a bounded neighborhood N of b, since X is locally compact. This would contradict the fact that $N \cap S$ must be heavy.

Finally consider the case when X is regular. Suppose that S is unbounded. Then there exists a covering $\{W_\alpha\}$ of $\overline{S}$ by open sets W_α which contains no finite subcovering of $\overline{S}$. Weight S by taking $A \subset S$ to be light if $\overline{A}$ is covered by a finite subcovering of $\{W_\alpha\}$. Since S is heavy, it has a barypoint $b \in \overline{S}$, and b is in some one of the W_α, say $b \in W_\beta$. Since X is regular, there exists a neighborhood N of b such that $\overline{N} \subset W_\beta$. But then $\overline{N \cap S}$ is light, which contradicts the fact that b is a barypoint.

Example. Let S be a bounded open interval in R^2, let X be the set of points that are in $\overline{S}$, and topologize X by taking its open sets to be the sets of the form $(S \cap W) \cup E$, where $E \subset \overline{S} - S$ and W is an open subset of R^2 which contains E. The space X is Hausdorff, but is neither locally compact nor regular. From the fact that S considered as a subspace of R^2 is a baryset, it follows easily that S considered as a subspace of X is a baryset. Since S is an unbounded subspace of X, this example shows that the hypothesis of Theorem 3 is needed.

Acknowledgments. I wish to thank the referee for suggestions which have led to extensive improvements in terminology and clarity over my original exposition. I also wish to thank Professor E. Halfar for a valuable suggestion.

ON THE IRRATIONALITY OF CERTAIN TRIGONOMETRIC NUMBERS

E. A. MAIER, University of Oregon

In this note we present an elementary proof of the following theorem. (See Niven, *Irrational Numbers*, Carus Mathematical Monograph No. 11, MAA, 1956, p. 41, for references to other proofs.)

THEOREM. *If θ is rational in degrees, then* (i) $\cos\theta = 0, \pm\frac{1}{2}, \pm 1$ *or is irrational,* (ii) $\sin\theta = 0, \pm\frac{1}{2}, \pm 1$ *or is irrational,* (iii) $\tan\theta = 0, \pm 1$, *is undefined or is irrational.*

LEMMA. *If there exists a positive integer n such that* $\cos n\theta$ *is integral, then* $\cos\theta = 0, \pm\frac{1}{2}, \pm 1$ *or is irrational.*

Proof. From the identity $2\cos(n+1)\theta = (2\cos\theta)\, 2\cos n\theta - 2\cos(n-1)\theta$, it follows by induction that if n is a positive integer then there exists a monic polynomial of degree n with integral coefficients such that $2\cos n\theta = f(2\cos\theta)$. Thus if $\cos n\theta$ is an integer, $2\cos\theta$ is a root of the monic polynomial $f(x) - 2\cos n\theta$. Hence $2\cos\theta$ is an integer or is irrational. Since $|2\cos\theta| \leqq 2$, the lemma follows.

Proof of the theorem. If θ is rational in degrees then for appropriate integers a and $b>0$ we have $\theta=\pi(a/b)$ in radian measure. Then $\cos n\theta$, $\cos n(\pi/2-\theta)$ and $\cos n(2\theta)$ are integers for $n=b$. Thus (i) follows directly from the lemma and (ii) and (iii) follow from the lemma and the identities $\sin\theta=\cos(\pi/2-\theta)$, $\tan^2\theta=(1-\cos 2\theta)/(1+\cos 2\theta)$. Note that the latter identity implies that if $\tan\theta$ is rational, then $\cos 2\theta$ is rational.

POSTSCRIPT

NATHAN ALTSHILLER COURT, University of Oklahoma, Norman

A recent note of mine [1] includes the following proposition:

Given two unequal chords of a circle, the ratio of the shorter chord to the longer is greater than the ratio of the minor arcs subtended respectively by the given chords.

At the time the note was offered for publication I was aware of no reference for that proposition.

When the note was already in the hands of the printer I came across a statement of the proposition in a very recent Polish history of mathematics [2]. It is quoted there from a book entitled *De Lateribus et Angulis Triangulorum* (On sides and angles of a triangle) which was published in Württemberg, in 1542. The author of the book is no lesser a person than Mikolaj Kopernik (Copernicus) (1473–1543). This is the only other book of his, besides the famous *De Revolutionibus Orbium Celestium*, the latter having come off the press in 1543.

The proposition in question is, however, far more ancient. It is to be found in the celebrated Almagest of Ptolemy (2nd cent. A.D.) (Courtesy of Raymond Haas, The Key School, Annapolis, Md.) [3]. Thus it happens that by his minor book Copernicus adds prestige, whether knowingly or not, to the very work he was destined to dethrone by his own epoch-making major opus. Such is the subtle irony of history.

References

1. N. A. Court, A Generalized Inequality, this MONTHLY, 71 (1964) 539–540.

2. Jadwiga Dianni and Adam Wachulka, Thousand Years of Polish Mathematical Thought, (In Polish) Warsaw, 1963, p. 52.

3. Claudius Ptolemy, The Almagest, pp. 19, 20; in "The Great Books of the Western World," No. 16, Chicago, 1952.

A DENUMERABILITY FORMULA FOR THE RATIONALS

GERALD FREILICH, City College of New York

Though it is well known that the set of rational numbers is denumerable, the usual demonstrations of this fact rely upon embedding the rationals as a proper subset of some denumerable set. To construct a one-to-one correspondence with the integers then requires an inductive procedure. Thus, the question of actually exhibiting a one-to-one correspondence between the integers and the rational numbers by a simple, noninductive formula is interesting. The purpose

of this note is to construct such a formula, based upon the Fundamental Theorem of Arithmetic.

Let f be the one-to-one mapping from the set of nonnegative integers onto the set of all integers defined by:

$$f(n) = \begin{cases} n/2 & \text{for } n \text{ even, } n \geqq 0 \\ -(n+1)/2 & \text{for } n \text{ odd, } n > 0. \end{cases}$$

If $m > 1$ is an integer, then m can be represented uniquely as:

$$m = \prod_{i=1}^{k} p_i^{n_i}$$

where $p_i < p_{i+1}$, each p_i is a prime, and each n_i is a positive integer. For such an m, define

$$g(m) = \prod_{i=1}^{k} p_i^{f(n_i)}.$$

Complete the definition by letting $g(1) = 1$, $g(0) = 0$, and $g(m) = -g(-m)$ for negative integral m. It is obvious that g is a one-to-one mapping from the set of all integers onto the set of all rational numbers. Superposition of g on f gives a mapping from the nonnegative integers onto the rationals, if desired.

MATHEMATICAL EDUCATION NOTES

Edited by John R. Mayor, AAAS and University of Maryland
Collaborating Editors: John D. Baum, Oberlin College, and
John A. Brown, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington D. C. 20005

THE NEW MATHEMATICAL LIBRARY

A PROJECT TO PROMOTE GOOD ELEMENTARY MATHEMATICAL EXPOSITION

Anneli Lax, New York University

The New Mathematical Library is a series of expository paperbacks, not texts, on various topics in elementary mathematics. These books are published by Random House, Inc. and The L. W. Singer Company for the Monograph Project of the School Mathematics Study Group, a project set up seven years ago by the founders of the SMSG to counteract the scarcity of good mathematical writing accessible to young people between the ages of about fourteen and twenty. The monographs are written by individual mathematicians of varied pedagogical persuasions. The authors' common goal is to involve the

reader in some meaningful mathematical activity, and their approaches to this task are independent of any specific path outlined by any particular curriculum reform movement. Consequently, the fifteen books published to date (see list at end) differ a great deal from each other in subject, style and level.

The purpose of the present article is to acquaint the mathematical community with the Monograph Project, to describe briefly its history, its main objectives, the actual processes involved in producing the NML books, and the strengths and weaknesses of the series to date.

At the time the School Mathematics Study Group was formed, almost the only mathematics books available to high school and beginning college students were textbooks, advanced treatises, and popularizations. An interested student had difficulty finding what he was after in the thick tomes of special libraries and, once having found the relevant reference, would easily become discouraged in his effort to understand the sophisticated concise presentation.

At the same time, he was not satisfied with the frequently superficial treatment in popularizations. In view of these conditions, the Monograph Project was created to bring about the publication of readable books on mathematical topics for interested young people whose mathematical background consisted mainly of what they had studied in the first few years of their high school careers. Professor E. G. Begle, director of the SMSG, appointed an editorial panel of about twelve members, among them high school teachers, university mathematicians and some mathematicians who work in industry. The first chairman was Professor L. Bers, then at the Courant Institute of Mathematical Sciences at New York University, who suggested that I take over the actual editing of the manuscripts. Members of the editorial panel serve for approximately three years; Lipman Bers was succeeded as chairman by Mark Kac of the Rockefeller Institute (1961–1964), and then by Ivan Niven of the University of Oregon who is currently chairman.

The founders and first members of the Monograph Project felt that the project should remain alive until the great need for good elementary expository books on mathematics is at least partially met, either by the publication of approximately thirty monographs in the New Mathematical Library, or by the appearance of such books published by commercial American publishers.

The major concern discussed at the very first meeting of the editorial panel (and at every subsequent meeting) was to find the proper level at which the NML monographs should be written. Clearly, this problem has no precise solution, and the search for approximate solutions continues.

A second major task was to find authors. In contrast to many European countries, where experienced research mathematicians traditionally write texts for secondary school students, American mathematicians had, for the most part, never written anything besides papers for professional journals, i.e., for other mathematicians (preferably working in the same field). So the editorial panel undertook not only to interest American authors in expository writing, but to offer help, guidance, and criticism to those who were willing to try their hands

at it. In the latter task the panel is assisted by some high school teachers and by enterprising high school students who read and comment on a preliminary version of a manuscript.

A third problem was to select topics in such a way that the series as a whole would eventually represent a fairly balanced diet of good mathematics, rather than an exploration of selected nooks and crannies. The main difficulties here are two-fold: It is difficult to agree on the importance of various topics. Not only do individual tastes play a role, but good mathematics often consists in certain ways of attacking a problem, certain ways of thinking about it, certain methods of analyzing its ingredients, etc., and the field or "branch" of mathematics to which these insights are applied seems of secondary importance. Secondly, even if a list of priorities is made (and such lists have been made), the authors who agree to write on these priority subjects often do not deliver the promised manuscripts. They are busy people and cannot be held to deadlines. Consequently, we publish the manuscripts in the order in which their authors submit them and not according to their place in a priority list.

Now that fifteen NML books (including three problem collections) have appeared and four more are in preparation, the panel is taking a particularly critical look at its achievements to date and is screening with special care the dozen or so additional books that it wishes to publish before it disbands. In trying to evaluate the effect of NML books on students, the panel members have little to go on except sales figures, letters from students and teachers, oral reactions and reviews. Letters as well as reviews have been almost exclusively favorable. High school teachers in charge of superior students are especially delighted with the series. College teachers report that, in certain courses, their students greatly profited from reading one or another NML book. The sales figures are surprisingly encouraging in spite of the extremely small amount of publicity and advertising of the NML, and many NML books are being translated into Turkish, Swedish, Italian, Spanish and Russian. Perhaps the most frequently leveled criticism of the books is that they are too difficult for the average high school student, and that relatively few of them actually motivate a person not interested in mathematics to study it.

The difficulties inherent in the panel's task have been dealt with, to some extent, successfully. The price of this partial success is our rather complicated procedure for processing manuscripts; it has to be paid by authors and editors. For example, the panel attacks the problem of finding the right level of sophistication by asking an author for a representative sample chapter, examining it, and then deciding whether or not the author should be encouraged to continue. The first draft of a manuscript is scrutinized and criticized by a few members of the panel; the final version is circulated to the whole panel and needs a favorable majority vote to be published. The wishes of the author are, of course, respected and prevail in the final version, but he has to put up with a great deal of advice, often on many seemingly trivial matters. I am grateful indeed to all who good-naturedly submitted to our procedure.

We try to persuade each author to begin his book at a leisurely pace and,

after taxing the reader, to reward him with interesting applications and illustrations. We advise the reader (in a Note to the Reader) not to expect to breeze through a mathematics book as through a novel, not to get discouraged when he gets stuck, and to return to difficult parts later. We would guess that many a student might read the first half of an NML book while he is in high school and the balance a year or two later, when he is in college. While the Monograph Project is aimed mainly at high school students, it is hitting also a good part of the college population. This is not surprising, since some of its monographs reach considerable depths.

The problem of finding authors has been met very satisfactorily, though no systematic method was employed. We have approached many individuals who were known to have a knack for expository writing. Among these, many have responded, motivated mainly by the conviction that they were contributing to a worthwhile cause. Many wanted to contribute but simply could not sacrifice the time. We have also received manuscripts from people who had heard of the Monograph Project and were interested in contributing to it.

Now that about two-thirds of the projected output has been reached (at least quantitatively), we feel that we have not only added significantly to the supply of readable books, but that we have begun to make expository writing at an elementary level a more respectable activity for mathematicians. We are watching eagerly the appearance of similar books published commercially, that is, without government subsidy. Some good ones have indeed appeared recently. We hope that projects of high quality will emerge and be self-supporting.

There are three editions: a school edition for pre-college students and teachers only, available for 90 cents per volume from the Singer Co., 249 W. Erie Blvd., Syracuse, N. Y. 13201; a trade edition available for $1.95 per volume from Random House, 457 Madison Ave., New York; and a hard bound library edition costing $2.95 per volume, available from the Random House School and Library Services.

List of NML volumes published to date

1. Numbers: Rational and Irrational—by Ivan Niven.
2. What is Calculus About?—by W. W. Sawyer.
3. Introduction to Inequalities—by E. Beckenbach and R. Bellman.
4. Geometric Inequalities—by N. D. Kazarinoff.
5. The Contest Problem Book (MAA High School Contests, 1950–1960).
6. The Lore of Large Numbers—by P. J. Davis.
7. Uses of Infinity—by Leo Zippin.
8. Geometric Transformations—by I. M. Yaglom, translated from the Russian by Allen Shields.
9. Continued Fractions—by C. D. Olds.
10. Graphs and Their Uses—by Oystein Ore.
11. Hungarian Problem Book I, based on the Eötvös Competitions, 1894–1905.
12. Hungarian Problem Book II, based on the Eötvös Competitions, 1906–1928.
13. Episodes from the Early History of Mathematics—by Asger Aaboe.
14. Groups and Their Graphs—by I. Grossman and W. Magnus.
15. Mathematics of Choice—by Ivan Niven.

The work of the School Mathematics Study Group has been supported by grants from the National Science Foundation.

BRIEF COMMENT

In response to requests from readers of the MONTHLY a new feature has been added to Mathematical Education Notes. Professor John D. Baum, of Oberlin College, has kindly agreed to review articles in other publications which are believed to be of interest to readers of the MONTHLY.

New textbooks for the "New Mathematics," RICHARD P. FEYNMAN, *Engineering and Science*, March, No. 6, 28 (1965) 9–15.

Professor Feynman is Richard Chace Tolman Professor of Theoretical Physics at the California Institute of Technology and last year was a member of the California State Curriculum Commission, during his tenure on which he spent considerable time on the selection of textbooks for a modified arithmetic course for grades 1 to 8. The article is the result of this experience.

The article is generally quite critical of the text materials available for grades 1 to 8. For example, "Many of the books go into considerable detail on subjects that are only of interest to pure mathematicians." Or again, "It is possible to give an illusion of knowledge by teaching the technical words which someone uses in a field (which sound unusual to ordinary ears) without at the same time teaching any ideas or facts using these words." It is clear, however, that Professor Feynman's aim in the indicated grades in the "new" mathematics is quite close to those of an informed mathematician or mathematical educator, since "in the 'new' mathematics, then, first *there must be freedom of thought*; second, *we do not want to teach just words*; and third, *subjects should not be introduced without explaining the purpose or reason.*" Much of the criticism in the article probably stems from looking at the first eight grades in isolation from the full school program, for much of the terminology introduced in the earlier grades is put to use in later grades.

The article is provocative, and well worth reading, in spite of the fact that one may disagree with much of what is said. It is unfortunate that some of the statements are either the result of misinformation or over-simplification; for example, "one of the best ways to solve complex algebraic equations is by trial and error." Or again, "all the elaborate notation for sets that is given in these books, almost never appear in any writings in theoretical physics, in engineering, in business arithmetic, computer design, or other places where mathematics is being used."

The journal, *Engineering and Science*, is a publication of the California Institute of Technology; individual copies of the journal are available at 50 cents each,,by writing to the offices of the journal at 1201 East California Blvd., Pasadena, California 91104.

Mathematics in education and industry, Supplement to *The Mathematical Gazette*, 17 (1963) No. 362, December.

This is a report of the experience of two teachers of mathematics, one a secondary school teacher, the other a college teacher, who spent some time in

industry (with British Petroleum) and of the conclusions they drew from that experience. The report is cogent to those of us who may be spending much of our time in the teaching of people who will be using mathematics as a tool, but whose main interest is not mathematics *per se*.

The following are pointed out as being significant needs of people who must use mathematics in industry: knowledge of basic mathematics and the skill to apply these basic concepts in problems from a variety of sources; knowledge of what a computer can do, not how it does it; a knowledge of statistics, a knowledge of numerical methods; the ability to set up differential equations. In regard to the last point concerning differential equations, it is pointed out that "these are continually needed in order to set up mathematical models of operations which are being investigated. The chief difficulty lies in the setting up of these equations, for approximate solutions are usually quite sufficient."

An interesting point is made that the approach to abstract concepts is best made via the recognition of the concept as a common pattern in a variety of real situations.

The supply and training of teachers of mathematics, A report prepared for the Mathematical Association, London, 1963.

This report prepared by a sub-committee of the Mathematical Association (of Great Britain) in conjunction with representatives of the Association of Teachers in Training Colleges and Departments of Education (Mathematics Section) details some of the acute shortages of teachers of mathematics in the British Isles at all levels. Since we face or will face similar problems, this report may have considerable value for us. The shortage is acute, for example, "the supply of potential university teachers is only half that required to maintain university staffs" with similar shortages at the secondary school level. The shortage is due to a variety of factors, among others, "the number of places available in universities for potential mathematics graduates is inadequate"; the too high standards at big universities discourage the matriculation of students as mathematics graduates; and the status of mathematics graduates is poor.

Among other interesting comments in the report one notes the emphasis on better training in mathematics for teachers whose specialty is other than mathematics, the strong stress on the interpersonal factors in teaching, or as it is put in the report, "a teacher is concerned first to teach a *person*, then a subject." The importance of discovery by the student as a teaching technique is also emphasized.

Education: scholars organize a National Academy intended to advance educational scholarship, JOHN WALSH, *Science*, April 9, No. 3667, 148 (1965) 202 ff.

"Establishment has been announced of a National Academy of Education, which its founders hope will parallel in prestige the National Academy of Sciences, but which will not have the quasi-governmental status of NAS."

Headed by Ralph Tyler, director of the Center for Advanced Study in the Behavioral Sciences, and chartered by the New York State Board of Regents, the Academy is intended to create "a forum which will set the highest standards for educational inquiry and discussion."

Of considerable interest to mathematicians will be the fact that Patrick Suppes of Stanford University is a charter member of the Academy.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, Bloomfield College, Bloomfield, N. J. 07003. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

All solutions of Elementary Problems should be sent to A. E. Livingston, University of Alberta, Edmonton, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before March 31, 1966.

E 1825. *Proposed by A. K. Austin, The University, Sheffield, England*

Find the number of noncongruent triangles with integral sides and perimeter n.

E 1826. *Proposed by A. K. Austin, The University, Sheffield, England*

Does there exist a quadrilateral $ABCD$ such that the sides and diagonals have rational length, but AO, BO, CO, DO are all irrational? O is the intersection of the diagonals.

E 1827. *Proposed by Joseph Arkin, Spring Valley, New York*

Given an integer $n \geqq 2$ and the system of $n-1$ equations:

$$0 = a_n x_n + \binom{n}{1} x_{n-1} + \binom{n}{2} x_{n-2} + \binom{n}{3} x_{n-3} + \cdots + \binom{n}{n-2} x_2 + 1$$

$$0 = a_{n-1} x_n + \binom{n-1}{1} x_{n-1} + \binom{n-1}{2} x_{n-2} + \binom{n-1}{3} x_{n-3} + \cdots + \binom{n-1}{n-2} x_2 + 1$$

$$0 = a_{n-2}x_n + \binom{n-2}{1}x_{n-1} + \binom{n-2}{2}x_{n-2} + \binom{n-2}{3}x_{n-3} + \cdots + 0 + 1$$

. . . .

$$0 = a_3x_n + \binom{3}{1}x_{n-1} + \binom{3}{2}x_{n-2} + 0 + \cdots + 0 + 1,$$

$$0 = a_2x_n + \binom{2}{1}x_{n-1} + 0 + 0 + \cdots + 0 + 1$$

in $n-1$ quantities $(x_2, x_3, \cdots, x_n)$. Determine the value of the one quantity x_n in terms of the a_j.

E 1828. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Form the differential equation for which the general solution is

$$y = \sum_{k=1}^{\infty} A_k e^{a_k x},$$

where the constants A_k and a_k are arbitrary.

E 1829. *Proposed by M. J. Pascual and S. D. Beck, Watervliet Arsenal, N. Y.*

If m is a positive integer and x is arbitrary, establish the following identity:

$$\sum_{k=m}^{2m-1} \binom{2m-1}{k} x^k(1-x)^{2m-1-k} = \sum_{k=m}^{2m-1} \binom{k-1}{m-1} x^m(1-x)^{k-m}.$$

E 1830. *Proposed by Michael Gemignani, University of Notre Dame*

Let $\{S, \cdot\}$ be a semigroup such that, given any $a \in S$, x and y can be found in S such that $(xa)a = a$, and $a(ay) = a$. Prove (1) Every element $a \in S$ has at least one two-sided identity, i.e., given a, $\exists e_a$ in S with $e_a a = a e_a = a$. (2) Each $a \in S$ has at least one two-sided inverse, say b_a, such that $b_a a = a b_a = e_a$, e_a being a two-sided identity for a.

E 1831. *Proposed by William Becker, New York University*

Let F and G be fields with G (properly) embedded in F. Let $P(x)$ be a polynomial with coefficients in G such that $x \in G$ implies $P(x) \in G$, and $x \in (F-G)$ implies $P(x) \in (F-G)$. Must $P(x)$ be linear? (Compare E 1642 [1964, 914] and E 1696 [1965, 550].)

E 1832. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England*

There is a well-known problem in which a ladder rests against a wall and just touches a cube having one face in contact with the wall. The lengths of the ladder and of an edge of the cube are given, and it is required to find the distances of the ends of the ladder from the base of the wall. When are all four quantities integers?

E 1833. *Proposed by Alan Sutcliffe, Congleton, Cheshire, England*

If $m=\min(r, s)$ and $M=\max(r, s)$, prove that

$$\sum_{r=0}^{N}\sum_{s=0}^{N}\frac{(N+m)!}{r!s!(N-M)!}=\sum_{r=0}^{N}\frac{(N+r+1)!}{r!(r+1)!(N-r)!}.$$

E 1834. *Proposed by Douglas Lind, Falls Church, Virginia*

Prove that

$$\pi=\sqrt{\{r+r'+2\sqrt{[2r+r'+3\sqrt{(3r+r'+\cdots)}]}\}},$$

where $r=\pi-3$ and $r'=r^2+3r+1$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Lagrange's Theorem

E 1712 [1964, 793]. *Proposed by Michael Gemignani, University of Notre Dame*

Let G be a finite group of order n, and let H be a subset of G containing m elements. Suppose $gH\cap g'H\neq\varnothing$ implies $gH=g'H$ for any g, g' in G. Then prove (a) $m|n$; (b) if $e\in G$ is the unit element of G and $e\in H$, then H is a subgroup of G.

Solution by Paul S. Schnare, Louisiana State University in New Orleans. We prove first the

THEOREM. *Let G be a (not necessarily finite) group and H a nonempty subset of G. Then H is a left-translate of a subgroup of G if and only if $gH\cap g'H\neq\varnothing$ implies that $gH=g'H$ for any g, $g'\in G$.*

Proof. For the sufficiency, suppose that $h\in H$, so that $e\in K=h^{-1}H$. If k, $k'\in K$, then $k'=k'k^{-1}k\in K\cap k'k^{-1}K=h^{-1}H\cap k'k^{-1}h^{-1}H$. By hypothesis, $h^{-1}H=k'k^{-1}h^{-1}H$ or $K=k'k^{-1}K$. In particular, $k'k^{-1}\in K$, so that K is a subgroup and $H=hK$.

The converse is trivial.

For the proposed problem, (b) is clearly true since $e\in H$ implies that H is the identity coset of some subgroup of G. And (a) is true since H is a left-translate of some subgroup with m elements, so that Lagrange's Theorem tells us that $m|n$.

We note also that Problem 5247 [1964, 1138] is a corollary of our Theorem above.

Also solved by Raymond Balbes, W. E. Bodden, J. L. Brown, Jr., M. M. Chawla and K. M, Das (jointly) (India), D. I. A. Cohen, Eleanor G. Dawley, M. J. DeLeon, J. F. Dillon, L. L. English, E. W. Ewing, Philip Fung, P. K. Garlick, A. A. Gioia, M. G. Greening (Australia), Israel Gross-

man, C. P. Gupta, C. V. Heuer, J. E. Homer, Jr., W. D. Jackson, R. A. Jacobson, Thomas Jefferson, Jr., Sister Mary Justin, Roman Kaluzniacki, Donald Kanzaki, M. L. Klasi, E. S. Langford, C. C. Lindner, D. C. B. Marsh, M. D. Mavinkurve (India), T. M. Morrisette, W. O. J. Moser, M. G. Murdeshwar, J. C. Nichols, T. M. O'Leary, Neal Parker, Harsh Pittie, P. J. Ryan, Gary Sampson and David Zitarelli (jointly), M. S. R. K. Sastry, Gerald Schrag, Frank Servas, Jr., R. Shantaram, D. R. Shoemaker, Ronald Shubert, Robin Sibson (England), Indranand Sinha, E. R. Sparks, Raymond Stockton, Helen Strassberg, L. Y. L. Tong, B. R. Toskey, J. E. Turner, William Vale, Van de Vyle, Emanuel Vegh, D. A. Wick, S. W. Williams, L. D. Wojnowski, Kenneth Yanosko, and the proposer.

The Euler-Mascheroni Constant

E 1736 [1964, 1042]. *Proposed by E. R. Barnes, Student, Morgan State College*

Evaluate

$$S = \sum_{p=2}^{\infty}\left(\frac{1}{p}\sum_{q=2}^{\infty}\left(\frac{1}{q}\right)^{p}\right).$$

Solution by William E. Kaufman, Student, University of Texas. It is well known that $\log(1-x) = -\sum_{n=1}^{\infty} x^n/n$ for $|x| < 1$. Inverting the order of summation in S (all terms are positive), we therefore have

$$\begin{aligned} S &= \sum_{q=2}^{\infty}\sum_{p=2}^{\infty}(1/q)^p/p = \sum_{q=2}^{\infty}\left(\log\frac{q}{q-1} - \frac{1}{q}\right) \\ &= \lim_{n\to\infty}\left(\sum_{q=2}^{n}\log\frac{q}{q-1} - \sum_{q=2}^{n} 1/q\right) \\ &= \lim_{n\to\infty}\left[1 - \left(\sum_{q=1}^{n} 1/q - \log n\right)\right] = 1 - \gamma \approx 0.422784, \end{aligned}$$

where γ is the Euler-Mascheroni constant.

Also solved by A. N. Aheart, W. A. Al-Salam, M. T. L. Bizley (England), T. J. Burke, F. A. Butter, Jr., F. V. Cavoto, P. L. Chessin, G. L. Cole, M. S. Demos, P. G. Engstrom, R. V. Esperti, A. C. Fang, J. A. Faucher, N. J. Fine, Murray Geller, Michael Goldberg, F. R. Gray and R. L. Price (jointly), M. G. Greening (Australia), L. O. Hagglund, Eldon Hansen, D. R. Hayes, Stephen Hoffman, J. M. Horner, R. F. Jackson, Erwin Just and Norman Schaumberger (jointly), Frank Kocher, J. D. E. Konhauser, E. S. Langford, Sim Lasher, D. N. Lehman, D. H. Lehmer, D. C. B. Marsh, Gus Mavrigian, M. A. Mustafa (England), D. E. Myers, F. D. Parker, J. M. Perry, C. R. Rao, M. S. R. K. Sastry, P. A. Scheinok, Klaus Schmitt, P. S. Schnare, H. D. Shane, Robin Sibson (England), Arnold Singer, Richard Sinkhorn, Al Somayajulu, K. Soni and R. P. Soni (jointly), G. P. Speck, Sidney Spital, R. L. Tangeman, C. A. Tsonis, Van de Vyle, W. C. Waterhouse, K. L. Yocom, David Zeitlin, and the proposer.

Chessin, Goldberg, Hansen, and Zeitlin observe that $S=\Sigma_{p=2}^{\infty}[\zeta(p)-1]/p$ and appeal to the known result $\Sigma_{p=2}^{\infty}[\zeta(p)-1](-z)^p/p = \log\Gamma(2+z) + \log(1+z) - z(1-\gamma)$ for $-1 < z \leq 1$. [Formula 6.1.33 on page 256 of the *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D. C. (1964), or equation 6.342 (1) in Ryshik and Gradstein, *Tables of Series, Products, and Integrals*, Plenum, New York.]

"Telescoping" Sums and Products

E 1737 [1964, 1042]. *Proposed by F. L. Bookstein, University of Michigan*

Define a telescoping product for a natural number t as a sequence of n integers $a_i > 1$, $1 \leqq i \leqq n$, such that a_i divides a_{i-1} for $2 \leqq i \leqq n$ and such that $\prod_{i=1}^{n} a_i = t$. Define a telescoping sum for a natural number t as a sequence of n positive integers a_i, $1 \leqq i \leqq n$, such that $a_i \leqq a_{i+1}$, $1 \leqq i \leqq n-1$, and such that $\sum_{i=1}^{n} a_i = t$.

Let q have prime factorization $p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$. Let c_i be the number of telescoping sums for b_i. Prove that the number of telescoping products for q is $\prod_{i=1}^{k} c_i$.

Solution by D. C. B. Marsh, Colorado School of Mines. There is a one-to-one correspondence between the telescoping sums for a natural number b and the telescoping products for p^b(p prime), namely: $(a_1, a_2, \cdots, a_n) \leftrightarrow (p^{a_n}, \cdots, p^{a_2}, p^{a_1})$. For $q = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$, the selection of a telescoping sum for each b_i may be made independently of the others, whence the total number of telescoping products for q is given by the product of the total number of telescoping sums for each b_i, i.e., $\prod_{i=1}^{k} c_i$.

Also solved by J. C. Abad, Andreas Blass, J. A. Burslem, D. I. A. Cohen, R. B. Eggleton (Australia), M. G. Greening (Australia), R. F. Jackson, Kenneth Kramer, E. S. Langford, Robin Sibson (England), Simon Vatriquant (Belgium), P. C. Yang, and the proposer.

A Difference Equation

E 1738 [1964, 1042]. *Proposed by Michael Fried, Bell Aerosystems*

Given $T_1 = 2$, $T_2 = 3$, $T_{2n} = T_{2n-1} + 2T_{2n-2}$, $T_{2n+1} = T_{2n} + T_{2n-1}$. Find an explicit expression for T_i.

I. *Solution by Paul S. Schnare, Louisiana State University in New Orleans.* For convenience define $T_{-2} = -1/2$, $T_{-1} = 3/2$, and $T_0 = 1/2$. Observing that

$$T_{2n+1} = (T_{2n-1} + 2T_{2n-2}) + T_{2n-1} = 2T_{2n-2} + 2T_{2n-1},$$

we express the recurrence equations by the matrix equation $X_n = AX_{n-1}$ $(n = 0, 1, 2, \cdots)$ with

$$X = \begin{pmatrix} T_{2n} \\ T_{2n+1} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}.$$

Thus, $X_n = A^{n+1} X_{-1}$. Since A satisfies its characteristic equation, we have $A^{n+1} = \alpha_n I + \beta_n A$ for appropriate scalars α_n and β_n. But the characteristic values of A are $2 \pm \sqrt{2}$, so we must have $(2 \pm \sqrt{2})^{n+1} = \alpha_n + (2 \pm \sqrt{2})\beta_n$, whence $\alpha_n = [(2-\sqrt{2})^n - (2+\sqrt{2})^n]/\sqrt{2}$ and $\beta_n = [(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}]/(2\sqrt{2})$ and then

$$X_n = \begin{pmatrix} \alpha_n + 2\beta_n & \beta_n \\ 2\beta_n & \alpha_n + 2\beta_n \end{pmatrix} X_{-1};$$

i.e., $T_{2n} = (\beta_n - \alpha_n)/2$ and $T_{2n+1} = 2\beta_n + 3\alpha_n/2$.

II. *Solution by Orville Goering, Southern Illinois University, Edwardsville, Illinois.* Starting with the matrix equation $X_n = AX_{n-1}$ ($T_0 = 1/2$, $n = 1, 2, 3, \cdots$) as in Solution I, if A were a diagonal matrix, the solution would be evident. Since it is not, we seek a linear transformation defined by $X_n = PY_n$ so that (of course) $Y_n = P^{-1}APY_{n-1}$ and $P^{-1}AP$ is diagonal. Since the characteristic roots of A are $2 \pm \sqrt{2}$, a suitable P is

$$\begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix},$$

and solution then follows routinely.

III. *Solution by Cornelius Groenewoud, 255 Roycroft Blvd., Buffalo, New York.* Let a, b, c, and d be fixed numbers and consider the problem

$$T_{2n} = aT_{2n-1} + bT_{2n-2}, \qquad T_{2n+1} = cT_{2n} + dT_{2n-1}$$

with initial terms T_1 and T_2 specified. Then $U_{n+1} - (b+ac+d)U_n + bdU_{n-1} = 0$ where $U_n \equiv T_{2n}$ or $U_n \equiv T_{2n-1}$. If r_1 and r_2 are the roots of $x^2 - (b+ac+d)x + bd = 0$, it follows routinely (consider $f(x) = \sum_{n=1}^{\infty} U_n x^n$) that

$$U_n = \begin{cases} \alpha r_1^n + \beta r_2^n & \text{if } r_1 \neq r_2, \\ \alpha(n+1)r^n + \beta n r^{n-1} & \text{if } r_1 = r_2 = r, \end{cases}$$

where α and β are determined by U_1 and U_2; so that $\alpha = (r_2U_1 - U_2)/r_1(r_2 - r_1)$ and $\beta = (U_2 - r_1U_1)/r_2(r_2 - r_1)$ if $r_1 \neq r_2$, while $\alpha = (2rU_1 - U_2)/r^2$ and $\beta = (U_2 - rU_1)/r$ if $r_1 = r_2 = r$.

Also solved by W. A. Al-Salam, Joseph Arkin, Mordecai Avriel, M. T. L. Bizley (England), A. P. Boblett, Paul Chessin, C. A. Church, Jr., and D. P. Roselle (jointly), R. B. Eggleton (Australia), Gary Ford, Michael Goldberg, S. H. Greene, M. G. Greening (Australia), Carl Harris, Stephen Hoffman, J. M. Horner, J. A. H. Hunter, R. F. Jackson, R. A. Jacobson, A. W. Johnson, Jr., Erwin Just and Norman Schaumberger (jointly), P. G. Kirmser, E. S. Langford, D. C. B. Marsh, T. M. Morrisette, Ralph Oravec, F. D. Parker, Richard Pavelle, J. J. Price, Joel Recht, H. D. Ruderman, P. A. Scheinok, P. S. Schnare, R. S. Seamons, Robin Sibson (England), Arnold Singer, F. C. Smith, G. P. Speck, Sidney Spital, C. A. Tsonis, R. E. Whitney, Jet Wimp, K. L. Yocom, David Zeitlin, and the proposer.

A Rectangle Associated with Four Concyclic Points

E 1739 [1964, 1132]. *Proposed by D. P. Ambrose, University of Colorado*

Let A, B, C, D be four points on a circle, and let I_A, I_B, I_C, I_D be the incenters of triangles BCD, CDA, DAB, ABC. Prove that I_A, I_B, I_C, I_D is a rectangle.

Solution by Hui-hsiang Kuo, National Taiwan University, Taipei, Taiwan, China. $\angle BI_AC = \pi - \frac{1}{2}(\angle DBC + \angle DCB) = \pi - \frac{1}{2}(\pi - \angle BDC) = \pi/2 + \frac{1}{2}\angle BDC.$

Similarly, $\angle BI_DC=\pi/2+\frac{1}{2}\angle BAC$. Since $\angle BDC=\angle BAC$, we have $\angle BI_AC=\angle BI_DC$, and, hence, B, C, I_A, I_D are concyclic and $\angle I_DBC+\angle I_DI_AC=\pi$. Similarly, $\angle I_BDC+\angle I_BI_AC=\pi$, whence $\angle I_DBC+\angle I_DI_AC+\angle I_BDC+\angle I_BI_AC=2\pi$. But $\angle I_DBC+\angle I_BDC=\frac{1}{2}(\angle ABC+\angle ADC)=\pi/2$ and $\angle I_DI_AI_B=2\pi-(\angle I_DI_AC+\angle I_BI_AC)$, so that $\angle I_DI_AI_B=\pi/2$. In more or less the same way, we find that the other angles in $I_AI_BI_CI_D$ are right angles.

Also solved by Leon Bankoff, M. G. Beumer (Netherlands), Marjorie Bicknell, Jim Campbell, K. W. Crain, Huseyin Demir (Turkey), L. R. De Ritter, Alwin Gran, D. M. Hancasky, Arun Lanyal (India), Ruth S. Lefkowitz, W. R. McEwen, D. C. B. Marsh, J. K. Moulton, J. M. Quoniam (France), Simeon Reich (Israel), Sister M. Stephanie, Simon Vatriquant (Belgium), and the proposer.

The problem is not new: Bankoff and Crain found it as a theorem on page 255 of R. A. Johnson, *Modern Geometry*, Dover (reprint); Beumer finds it attributed to J. Neuberg in 1875 [Max Simon, *Über die Entwicklung der Elementar-Geometrie im 19 Jahrhundert*, Leipzig (1906), p. 159]; Bicknell, De Ritter, Lefkowitz, and McEwen located it as a theorem on page 133 in R. A. Court, *College Geometry*, Barnes and Noble (1952); Demir calls attention to Antoine Dalle, *Problèmes et Exercices de Gêometrie* (1931); and Vatriquant cites page 194 of F. and F. V. Morley, *Inversive Geometry*.

Four Concyclic Points

E 1740 [1964, 1132]. *Proposed by D. P. Ambrose, University of Colorado*

Let A, B, C, D be four points on a circle, and let G_A, G_B, G_C, G_D be the centroids of the triangles BCD, CDA, DAB, ABC. Prove that G_A, G_B, G_C, G_D are also concyclic.

I. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.* This known property [see Antoine Dalle, *Problèmes et Exercices de Gêometrie*, (1931), p. 136] may be proved as follows. The figure $G_AG_BG_CG_D$ is homothetic to $H_AH_BH_CH_D$ where H's are respective orthocenters. So it will suffice to prove the property for the latter quadrangle. If (O) is the circumcenter and X is the midpoint of BC, the relation $\overrightarrow{DH_A}=2\overrightarrow{OX}=\overrightarrow{AH_D}$ shows that the figure AH_DH_AD is a parallelogram and hence $H_AH_BH_CH_D$ is congruent to $ABCD$ which is concylic.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* Establish coordinate axes so that the given circle has radius r and center at the origin. Relabel the points A, B, C, D as P_i ($i=1, 2, 3, 4$) with coordinates ($r\cos\theta_i$, $r\sin\theta_i$). Then the centroids

$$G_i=\left(\frac{r}{3}\sum_{\substack{j=1\\ j\neq i}}^{4}\cos\theta_j,\ \frac{r}{3}\sum_{\substack{j=1\\ j\neq i}}^{4}\sin\theta_j\right)$$

all obviously lie on a circle of radius $r/3$ with center at

$$\left(\frac{r}{3}\sum_{j=1}^{4}\cos\theta_j,\ \frac{r}{3}\sum_{j=1}^{4}\sin\theta_j\right).$$

III. *Solution by Michael Goldberg, Washington, D. C.* Let equal weights be placed at four points A, B, C, D in the plane (not necessarily concyclic). Let their center of gravity be E. Then E must lie on the line from G_A to A and divide it in the ratio of 3 to 1. Similarly, E divides G_BB, G_CC and G_DD in the same ratio. Hence, $G_AG_BG_CG_D$ is homothetic to $ABCD$ and, therefore, similar to $ABCD$. If $ABCD$ is a cyclic quadrilateral, then $G_AG_BG_CG_D$ is also a cyclic quadrilateral inscribed in a circle of one-third the diameter.

For any n weights, the centers of gravity of the n sets of $(n-1)$ weights will produce a polygon similar to the polygon of the original n weights. If the original set lies on a circle, then the derived set of points lies on a circle which is $1/(n-1)$ times the diameter of the original circle.

A similar condition holds for weights in three-space. The derived configuration is similar and homothetic to the original configuration.

Comment by M. G. Beumer, Information Systems, Inc., The Hague, Netherlands. The same result applies if "centroids" is replaced by "orthocenters," in which case the multiplication factor of $1/3$ in the solutions above is replaced by 1. Quite generally, this kind of problem is due to Steiner. References: Max Simon, *Entwicklung der Elementar-Geometrie im 19 Jahrhundert*, Leipzig (1906), p. 159, and J. Steiner, *Gesammelte Werke I*, p. 128.

Also solved by A. N. Aheart, Leon Bankoff, M. A. Bershad, Jim Campbell, Hui-hsiung Kuo (Taiwan), E. S. Langford, W. R. McEwen, S. R. Mandan (India), J. K. Moulton, J. M. Quoniam (France), Simeon Reich (Israel), Sidney Spital, Sister M. Stephanie, Simon Vatriquant (Belgium), and the proposer.

Bankoff found E 1740 as Problem 4 on page 234 of H. G. Forder, *Higher Course Geometry*, Cambridge University Press (1931), while Mandan calls attention to his paper *Harmonic chains of equal circles*, Scripta Mathematica, 26 (1961) 47–65.

$\int_0^a f(x)dx$ and $\int_b^\infty [g(t)]^n dt$ Converge and Diverge Together

E 1741 [1964, 1132]. *Proposed by Harold Peacock, Hyattsville, Maryland*

Suppose that $\lim_{x\to 0} f(x) = \infty$, where $f(x)$ is a nonincreasing continuous function defined on some interval $(0, a]$. Suppose that $\lim_{t\to\infty} g(t) = 0$, where $g(t)$ is a strictly decreasing continuous function defined on some interval $[b, \infty)$. Define $h(x) = g(f(x))$ and $h(0) = 0$. Suppose that, for some natural number n, the integral

$$\int_b^\infty [g(t)]^n \, dt$$

converges (diverges) and that the derivative d^nx/dh^n is finite (nonzero) when $h=0$. In the event that $n>1$, suppose also that the lower order derivatives are zero at $h=x=0$.

Under these conditions, prove that the integral $\int_0^a f(x)dx$ converges (diverges.)

Solution by E. S. Langford, U. S. Naval Postgraduate School, Monterey, California. With regard to differentiability, we assume only that $D^n h^{-1}(0)$ exists (finite) and, hence, that $D^k h^{-1} \in C[0, c)$ for some $c > 0$ $(k=0, 1, 2, \cdots, n-1)$. It follows that f is strictly decreasing near 0 and, hence, that h and h^{-1} are strictly increasing near 0 and ∞, respectively.

Applying Taylor's Theorem in its "local" form and recalling that $D^k h^{-1}(0) = 0$ for $k=0, 1, \cdots, n-1$, we have

$$0 \leqq h^{-1}(y) = D^n h^{-1}(0) y^n/n! + y^n \phi(y),$$

where $\phi(y) \to 0$ as $y \to 0$. [See Ch. J. de la Valleé Poussin, *Cours d'Analyse Infinitésimale*, Tome 1, Dover (reprint), (1946) pp. 79–80, or this MONTHLY, 70 (1963) p. 866.] With $y = h(x)$ and $x = f^{-1}(t)$, this becomes

$$0 < f^{-1}(t) = D^n h^{-1}(0)[g(t)]^n/n! + [g(t)]^n \psi(t),$$

where $\psi(t) \to 0$ as $t \to \infty$.

For $\int^\infty [g(t)]^n dt$ convergent, our differentiability assumption gives $0 < f^{-1}(t) < \{D^n h^{-1}(0)/n! + 1\} [g(t)]^n$ for all large t. But then $\int^\infty f^{-1}(t) dt$ is convergent and, hence, $\int_0^a f(u) du$ is convergent for some $a > 0$.

If $\int^\infty [g(t)]^n dt$ is divergent, our differentiability assumption requires that $D^n h^{-1}(0) \neq 0$. Since $\psi(t) = o(1)$, we have $f^{-1}(t) > A[g(t)]^n$, $A > 0$, for all large t; i.e., $\int^\infty f^{-1}(t) dt$ (and, hence, $\int_0^a f(u) du$ for some $a > 0$) is divergent.

Also solved by the proposer.

A Counting Problem

E 1742 [1964, 1132]. *Proposed by M. T. Salhab, Illinois Institute of Technology*

Arrange the forty playing cards, one through 10 of each unit, face up in four rows as follows:

1st row:	1,	2, $\cdots$,			10 of clubs
2nd row:	10,	1,	2, $\cdots$,		9 of diamonds
3rd row:	9,	10,	1, $\cdots$,		8 of hearts
4th row:	8,	9,	10,	1, $\cdots$,	7 of spades

Pick them up, face down, from right to left, a column at a time, always starting with the first row. The bottom card will be the 10 of clubs, and the top, the eight of spades. Now distribute them face down, in four equal rows from left to right. Find a formula which identifies the card in each of the forty positions.

Solution by A. M. Vaidya and V. A. Vaidya, Texas Technological College. We observe that the $(4q+1)$-st card $(q=0, 1, \cdots, 9)$ in the new arrangement is the $(q+1)$-st card from the left in the last row of the original arrangement and, hence, has for value the least positive residue of $8+q$ (modulo 10). Since the

next three cards in the new arrangement have consecutive values (modulo 10), we see that the card numbered $n=4q+r$ ($r=0, 1, 2, 3$) has for face value the least positive residue of $7+q+r$ (modulo 10) and is a club, spade, heart, or diamond according as $r=0$, 1, 2, or 3. (Of course, the least positive residue of $7+q+r$ (modulo 10) is $7+q+r-10[(6+q+r)/10]$.)

Also solved by D. A. Blaeuer, Walter Bluger, J. W. Burns, J. A. Burslem, Jim Campbell, C. C. Clever, D. I. A. Cohen, Gil De Bartolo, R. B. Eggleton (Australia), E. V. Gadecki, Cornelius Groenewoud, David Herlacher, R. A. Jacobson, Linda Kalver, F. A. Kros, E. S. Langford, Margaret M. La Salle, D. C. B. Marsh, Mrs. Merkel, M. J. Sheridan, Arnold Singer, Mitchell Snyder, G. P. Speck, C. R. Wall, and the proposer.

Many of these solutions were based on the observation that the card a_{ij} originally in the i-th row and j-th column ($i=0, 1, 2, 3; j=0, 1, \cdots, 9$) winds up in the $(4-r)$-th row and q-th column, where $10i+j=4q+r$ ($r=0, 1, 2, 3$). By observing next that a_{ij} is the least positive residue of $11-i+j$ (modulo 10) and that $i=0$, 1, 2, and 3 correspond to clubs, diamonds, hearts, and spades, respectively, it is then easy to identify card $(4-r, q)$ in the new array.

An Increasing Function

E 1743 [1964, 1133]. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove that $(\sinh x-x)\ \{\text{arc sec}\ (2+\operatorname{sech} x)-\pi/3\}$ is a steadily increasing function of x.

Solution by E. S. Langford, U. S. Naval Postgraduate School, Monterey, California. Let $f(x)=\sinh x-x$ and $g(x)=\sec^{-1}(2+\operatorname{sech} x)-\pi/3$. Since $\sec y$ and $\operatorname{sech} x$ are even, g is even, while f is odd because $\sinh x$ and x are odd. Thus, fg is odd, and it suffices to show that fg is increasing for $x\geqq 0$. We have

$$g(x) = \sec^{-1}(2+\operatorname{sech} x) - \sec^{-1} 2 = \int_2^{2+\operatorname{sech} x} h(t)\,dt,$$

where $h(t)=1/t(t^2-1)^{1/2}$. Since h is decreasing, $g(x)\geqq(\operatorname{sech} x)h(2+\operatorname{sech} x)>0$, and, of course, $\cosh x-1\geqq 0$. Thus,

$$\begin{aligned}(fg)' &= f'g + fg' \\ &\geqq (\cosh x - 1)(\operatorname{sech} x)h(2+\operatorname{sech} x) \\ &\quad - (\sinh x - x)(\operatorname{sech} x \tanh x)h(2+\operatorname{sech} x) \\ &= (\operatorname{sech}^2 x)h(2+\operatorname{sech} x)[\cosh^2 x - \cosh x - \sinh^2 x + x\sinh x] \\ &= (\operatorname{sech}^2 x)h(2+\operatorname{sech} x)[1 - \cosh x + x\sinh x].\end{aligned}$$

Since $(\operatorname{sech}^2 x)h(2+\operatorname{sech} x)>0$, it suffices to show that $k(x)=1-\cosh x+x\sinh x \geqq 0$ for $x\geqq 0$. But $k(0)=0$, and $k'(x)=-\sinh x+\sinh x+x\cosh x=x\cosh x\geqq 0$, with equality only if $x=0$. Therefore fg is strictly increasing on $(-\infty, \infty)$.

Also solved by August Florian (Austria), and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before May 31, 1966, to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903.

5330. *Proposed by M. Edelstein, Dalhousie University*

Let S be a subset of E^n and suppose that the distance $d(s_1, s_2)$ is rational for all pairs $s_1, s_2 \in S$. Prove, or disprove, that S is countable.

5331. *Proposed by M. Edelstein, Dalhousie University*

Let f be a mapping of l_2 into itself defined as follows. If $x=(x_1, x_2, \cdots, x_n, \cdots) \in l_2$, then $y=(y_1, y_2, \cdots, y_n, \cdots)=f(x)$ if, and only if, $y_n=e^{2\pi i/n!}(x_n-1)+1$. Is the sequence of iterates $f^n(0)$ of the origin bounded?

5332. *Proposed by R. W. Forcade, Southern Illinois University*

Let T be an operator on (all) the subsets of a set X, such that the image of an arbitrary union of subsets is the intersection of their images. Prove that if every nonempty subset has a nonempty intersection with its image, then there is exactly one subset of X which is fixed by T.

5333. *Proposed by Masakazu Aoyagi, Chiba University, Japan*

If $0<x_1<x_2< \cdots <x_n \leqq 1$, then prove that

$$\left(n \sum_{k=1}^{n} x_k\right) \Big/ \left(\sum_{k=1}^{n} x_k + n x_1 x_2 \cdots x_n\right) \geqq \sum_{k=1}^{n} \frac{1}{1+x_k}.$$

5334. *Proposed by L. A. Shepp, Bell Telephone Laboratories, Murray Hill, N. J.*

If $b=[b_{ij}]$ is a symmetric square matrix with $b_{ii}=1, \sum_{j\neq i} |b_{ij}| \leqq 1$ for each i, then $\det(b) \leqq 1$.

5335. *Proposed by James Chew, Virginia Polytechnic Institute*

In this MONTHLY (Dec. 1963) Doyle and Warne, *Some properties of groupoids*, the following theorem is given without proof: *The set of idempotents in a topological groupoid is* (*topologically*) *closed.* Supply a proof.

5336. *Proposed by D. J. Newman, Yeshiva University*

Let k be an integer greater than 1. Prove that

$$\sum \frac{1}{n_1 n_2 \cdots n_k(n_1+n_2+\cdots+n_k)} = k!,$$

the sum extending over all k-tuples of positive integers which have no nontrivial common factor.

5337. *Proposed by C. S. Venkataraman, Sri Kerala Varma College, Trichur, India*

Evaluate

$$\sum \phi(n/e)/\sum \phi(n/d),$$

where e runs through the even divisors and d runs through the odd divisors of n, and ϕ is Euler's function.

5338. *Proposed by J. W. Wyman, Pasadena College*

Let $f: D \to D$ be a continuous function on the unit disk D in the complex plane such that $f(z) \neq z$ for each $z \in D$. Let ∂D denote the boundary of D and let $\psi: D \to \partial D$ be defined as follows: If $z \in D$, the line determined by $f(z)$ and z inter sects ∂D in exactly two places. Let $\psi(x)$ be the point on ∂D such that $f(z)$, z and $\psi(z)$ are in that order. Then ψ is continuous. Supply a proof. (This result is stated without proof in Tucker, *Some topological properties of disk and sphere*, Proceedings of the First Canadian Mathematical Congress, 1946, pp. 285–309.)

5339. *Proposed by Necdet Ücoluk, Purdue University*

Let ABC be a given triangle in the plane, with a nonempty interior. Determine the position of a point N of the plane for which the product $NA \times NB \times NC$ attains an extreme value.

SOLUTIONS OF ADVANCED PROBLEMS

Self-adjoint Operators on Hilbert Space

5233 [1964, 923]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

Let p and q be linear operators on a Hilbert space, with $\|p\| < 1$ and q variable. Evaluate

$$f(p) = \sup_{\|q\| \leq 1} (1 - q^*p^*)^{-1}(1 - q^*q)(1 - pq)^{-1}.$$

Solution by Mary R. Embry, Charlotte College, Charlotte, N. C. If $\|q\| \leq 1$, let $h(q) = (1-q^*p^*)^{-1}(1-q^*q)(1-pq)^{-1}$. Observe that $h(p^*) = (1-pp^*)^{-1}$. Suppose there exists q such that $h(q) \gg h(p^*)$. Let $r = q - p^*$. Then

$$(1 - (r^* + p)(r + p^*) \gg (1 - r^*p^* - pp^*)(1 - pp^*)^{-1}(1 - pr - pp^*).$$

Simplifying and cancelling, one has

$$-r^*r \gg r^*p^*(1 - pp^*)^{-1}pr.$$

Therefore, $r = 0$ and $q = p^*$. Consequently, $f(p) = (1 - pp^*)^{-1}$.

Also solved by the proposer.

Rings Isomorphic to the Direct Product of Copies of R

5235 [1964, 924]. *Proposed by Hans Schneider, University of Wisconsin*

Let R be an associative ring with unity, and let I be a transfinite cardinal. If F is a left R-module which is the (weak) direct sum of I copies of R, then it is known that every other representation of F as a direct sum of copies of R contains I summands.

Now let I be an arbitrary cardinal. Show that there exists a ring R (depending on I) with unity, which is isomorphic as a left R-module to the (complete) direct product of I copies of R.

I. *Solution by the proposer.* Let J be a transfinite cardinal, with $J \geqq I$. Let $(E_i)_I$ be a family of vector spaces over some division ring D, and suppose $\dim E_i = J$, for $i \in I$. Set $F = \sum_I \oplus E_i$, and note that $\dim F = J$, since $I \cdot J = J$. Thus there exists a D-isomorphism σ_i of F onto E_i. Let α_i be the inverse isomorphism and extend α_i to F by defining $\alpha_i E_j = 0$, $j \neq i$. Thus

$$\alpha_i\sigma_i = id \text{ on } F, \qquad \sigma_i\alpha_i \mid E_j = 0 \text{ on } E_j, \quad j \neq i,$$

$$\sigma_i\alpha_i \mid E_i = id \text{ on } E_i, \qquad \alpha_j\sigma_i = 0 \qquad j \neq i.$$

Let R be the ring of D-linear transformations on F into itself. Define $\Gamma: R \to \Pi(R\sigma_i)_I$ by $\Gamma(\lambda) = (\lambda\sigma_i)_I$. Then

(1) $\quad R\sigma_i = R, \quad \text{since} \quad \lambda\alpha_i\sigma_i = \lambda,$

(2) $\quad \Gamma$ is a left R-module homomorphism, as may be computed,

(3) $\quad \Gamma$ is one to one,

for $\Gamma(\lambda) = 0 \Rightarrow \forall i \in I, \lambda\sigma_i = 0 \Rightarrow \forall i \in I, \lambda\sigma_i\alpha_i = 0 \Rightarrow \forall i \in I, \lambda \mid E_i = 0 \Rightarrow \lambda = 0.$

Finally

(4) $\quad \Gamma$ is onto,

for, let $(\lambda_i\sigma_i) \in \prod (R\sigma_i)_I$, and define λ by $\lambda \mid E_i = \lambda_i\sigma_i\alpha_i \mid E_i = \lambda_i \mid E_i$. Then $\lambda\sigma_i = \lambda_i\sigma_i\alpha_i\sigma_i = \lambda_i\sigma_i$, since $\alpha_j\sigma_i = 0$, $j \neq i$.

The requirements for R now follow from (1), (2), (3), (4).

II. *Solution by W. C. Waterhouse, Harvard University.* Let J be any infinite cardinal $\geqq I$, so that $J \cdot I = J$, and let R be the direct product of J copies of the integers. Then we can break up the product into a product of I copies of rings isomorphic to R, and the decomposition respects the module structure.

Integral Equations with Bessel Function Kernels

5236 [1964, 1046]. *Proposed by James F. Heyda, General Electric Company*

Solve in closed form the integral equations

I. $$\phi(x) + \int_0^x t\phi(t)F'(z)\,dt = g(x),$$

II. $$\int_0^x \phi(t)F(z)\,dt + f(0)F(x^2) = f(x),$$

where $z = x(x-t)$ and the prime indicates differentiation with respect to z. $F(z)$ is the modified Bessel function $I_0(2z^{1/2})$. The free functions $f(x)$, $g(x)$ are assumed to be continuously differentiable.

Solution by A. G. Mackie, Victoria University, Wellington, New Zealand. If $R(x, y; x_0, y_0)$ is the Riemann function associated with the equation $\partial^2\phi/\partial x\partial y + c(x, y)\phi = 0$, where $c(x, y) = c(y, x)$, and if

(1) $$f(x_0) = \int_0^{x_0} R(x, x; x_0, 0)g(x)\,dx,$$

then

(2) $$g(x_0) = f'(x_0) + \int_0^{x_0}\left[\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0}\right)R(x, 0; x_0, y_0)\right]_{y_0=x_0} f'(x)\,dx.$$

(This result may be found in A. G. Mackie, *A class of integral equations*, in this issue of the MONTHLY, pp. 956–960.)

If we take $c(x, y) = -1$, then

$$R(x, y; x_0, y_0) = I_0\{2\sqrt{(x_0 - x)(y_0 - y)}\}$$

and

$$\left[\left(\frac{\partial}{\partial x_0} - \frac{\partial}{\partial y_0}\right)R(x, 0; x_0, y_0)\right]_{y_0=x_0} = \frac{xI_0'\{2\sqrt{x_0(x_0 - x)}\}}{\sqrt{x_0(x_0 - x)}}.$$

Thus I is precisely (2) above with f' corresponding to ϕ. Hence from (1)

$$f'(x_0) = \int_0^{x_0} \frac{\partial R(x, x; x_0, 0)}{\partial x_0} g(x)\,dx + g(x_0),$$

(since $R(x_0, x_0; x_0, 0) = 1$), and the solution to I is found to be

Ι. $$\phi(x) = g(x) - \int_0^x \left(\frac{t}{x - t}\right)^{\frac{1}{2}} g(t)J_1\{2\sqrt{t(x - t)}\}\,dt.$$

In the final paragraph of the paper referred to above, it is noted that a further class of integral equations and their solutions, analogous to (1) and (2), can be obtained by considering a boundary value problem similar to but slightly different from that which motivated the derivation of equations (1) and (2). The equivalent pair of equations is

(3) $$f(x_0) = f(0)R(0, 0; x_0, x_0) + \int_0^{x_0} g'(x)R(x, 0; x_0, x_0)\, dx,$$

and

(4) $$g(x_0) = f(x_0) + \int_0^{x_0} \left[\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) R(x, y; x_0, 0)\right]_{y=x} f(x)\, dx,$$

$g(x)$ being the solution of (3) for which $g(0) = f(0)$.

If again we choose $c(x, y) = -1$, then II is precisely (3) with $g'(x)$ taking the role of $\phi(x)$. Hence by (4), in the notation of the problem, we have, again after some manipulation,

II. $$\phi(x) = f'(x) - \frac{d}{dx}\int_0^x \frac{xf(t)}{\sqrt{t(x-t)}} J_1\{2\sqrt{t(x-t)}\}\, dt.$$

Also solved by the proposer, (who used equations derived in his paper, *A Note concerning Hyperbolic Equations*, Quarterly J. of Appl. Math., Oct. 1960).

Sets Dense in $L^2(0, 1)$

5237 [1964, 1046]. *Proposed by I. I. Hirschman, Jr. and Guido Weiss, Washington University, St. Louis*

Let $F(x)$ be a complex measurable function on $0 < x < 1$ such that $F(x) \notin L^2(0, 1)$ and let P be the subset of $L^2(0, 1)$ consisting of those functions $\phi(x)$ for which $F(x)\overline{\phi(x)} \in L^1(0, 1)$ and $\int_0^1 F(x)\overline{\phi(x)}dx = 0$. Show that P is dense in $L^2(0, 1)$.

Solution by Sim Lasher, University of Illinois, Chicago. Let $\psi \in L^2$, $F(x) \notin L^2$, $\epsilon > 0$ be given. Then we construct a function $\phi(x) \in L^2$ such that $\int_0^1 F(x)\overline{\phi(x)}dx = 0$ and $||\phi - \psi|| < \epsilon$.

First, select a function $G(x)$, $G(x) \in L^2$, $FG \geqq 0$ and $\int_0^1 F(x)G(x)dx = \infty$ (see e.g., A. C. Zaanen, *Linear Analysis*, p. 137, I; P. Natanson, *Theory of Functions of a Real Variable*, p. 202, ex. 5). Let $E_n = \{x \mid FG \leqq n\}$, $\psi_n = \chi_{E_n} \cdot \psi$, $n = 1, 2, \cdots$, (χ is the characteristic function) and note that nE_n increases to 1 as $n \to \infty$. Choose m sufficiently large in order that $||\psi_m - \psi|| < \epsilon/2$ and $||\chi_{CE_m} \cdot G|| < \epsilon/2$. Let

$$\lambda = \int_0^1 F\psi_m\, dx = \int_{E_m} F\psi\, dx.$$

Choose K sufficiently large in order that $\int_{E_k - E_m} FGdx > |\lambda|$ and $H \subset (E_k - E_m)$ so that $\int_H FGdx = |\lambda|$. Denoting $\lambda = |\lambda|$ sgn λ, let $\phi(x) = \psi_m(x) - \text{sgn}\, \bar{\lambda} \cdot \chi_H \cdot G(x)$. It now follows that $\int_0^1 F\bar{\phi}dx = 0$ and $||\phi(x) - \psi(x)|| < \epsilon$.

Also solved by P. M. Chernoff, R. O. Davies (England), W. C. Waterhouse, the proposer, and J. P. Williams who noted that a corresponding result is possible if $F \notin L^2(-\infty, \infty)$ but $F \in L^2(-N, N)$ for all N. Then the subset of $L^2(-\infty, \infty)$ "orthogonal" to F is dense in $L^2(-\infty, \infty)$.

Identity for the Γ-function

5238 [1964, 1046]. *Proposed by Z. Govindarajulu, Case Institute of Technology*

Show that for $\alpha > -1$,

$$1 - \sum_{s=1}^{\infty} \frac{\alpha(1-\alpha)\cdots(s-1-\alpha)}{s!(2s+1)} = 2^{2\alpha}\frac{\Gamma^2(1+\alpha)}{\Gamma(2+2\alpha)}.$$

Solution by Max A. Bershad, Bureau of the Census, U. S. Department of Commerce. The left side of the identity is easily seen to be $\frac{1}{2}\int_{-1}^{1}(1-y^2)^\alpha dy$, and this becomes $2^{2\alpha}\int_0^1 x^\alpha(1-x)^\alpha dx$ upon using the transformation $1-y=2x$. The result now follows from the identity

$$\int_0^1 x^\alpha(1-x)^\alpha\, dx = \mathrm{B}(1+\alpha, 1+\alpha) = \frac{\Gamma^2(1+\alpha)}{\Gamma(2+2\alpha)}.$$

Also solved by W. A. Al-Salam, P. T. Bateman, M. G. Beumer (Netherlands), A. P. Boblétt, L. R. Bragg, R. G. Buschman, Mrs. A. C. Garstang, S. H. Greene, M. G. Greening (Australia), Eldon Hansen, E. S. Langford, A. E. Livingston, J. L. Pabrinkis, J. M. Perry, Stanton Philipp, J. M. Quoniam (France), S. K. Rangarajan (England), B. E. Rhoades, P. A. Scheinok and W. F. Trench, V. Seshadri, A. Sharma, R. L. Shively, Arnold Singer, F. C. Smith, Al Somayajulu, K. Soni and R. P. Soni, F. W. Steutel (Netherlands), J. van Yzeren (Netherlands), L. D. Wojnowski, G. M. Yadao (India), David Zeitlin, and the proposer.

Somayajulu solves the problem by setting $p=1$, $q=\frac{1}{2}$, $\beta=-\alpha$ in the known identity (Whittaker and Watson, *A course in Modern Analysis*)

$$\frac{\mathrm{B}(p-\beta, q)}{\mathrm{B}(p, q)} = 1 + \frac{p\cdot q}{p+q} + \frac{\beta(\beta+1)q(q+1)}{2!(p+q)(p+q+1)} + \cdots.$$

Livingston noted that the identity is valid for complex α, $R\alpha > -1$.

The Iterated ϕ-function

5239 [1964, 1047]. *Proposed by Douglas Lind, Falls Church, Va.*

Let $\phi_n(m)$ be the iterated ϕ function (Euler's) such that $\phi_1(m)=\phi(m)$, $\phi_n(m)=\phi(\phi_{n-1}(m))$, and let $\omega(m)=n$ denote the smallest n such that $\phi_n(m)=1$. Then prove that if $m<2^\alpha$, then $\omega(m)\leqq\alpha$.

Solution by Rosalind Ruggiero, Harpur College. Suppose $\phi_j(m)=1$ and j is the smallest such integer; then $\phi_{j-1}(m)=2$. Also $\phi_1(m)<m<2^\alpha$, and since $\phi_1(m)$ is even $\phi_2(m)\leqq\frac{1}{2}\phi_1(m)<2^{\alpha-1}$. Continuing this process we find that $\phi_{j-1}(m)<2^{\alpha-(j-2)}$ or $2<2^{\alpha-(j-2)}$. Thus $j<\alpha+1$, or $j\leqq\alpha$.

Also solved by M. G. Beumer (Netherlands), J. E. H. Elliott, N. J. Fine, M. G. Greening (Australia), S. K. Gupta (India), D. L. Silverman, A. M. Vaidya, and the proposer.

Fine and Vaidya proved that $\omega(m)\leqq\alpha-1$ if m is even. P. T. Bateman, Underwood Dudley, and the proposer (post facto) noted that the result is in the literature: H. N. Shapiro, this MONTHLY, 50 (1943), 18–30; S. S. Pillai, Bull. Amer. Math. Soc., 35 (1929) 837–841.

Convoluted Sums of Γ-functions

5240 [1964, 1047; 1965, 192]. *Proposed by D. G. Bourgin, University of Illinois*

Prove the following identity:

$$\frac{4\Gamma\left(\frac{3}{4}\right)\Gamma\left(n+\frac{5}{4}\right)}{\Gamma(2(n+1))} = \sum_{j=0}^{2n-1} (-1)^{[j/2]} \frac{\Gamma\left(\frac{j}{2}+\frac{3}{4}\right)\Gamma\left(n-\frac{j}{2}+\frac{1}{4}\right)}{(j+2)\Gamma(j+1)\Gamma(2n-j)},$$

Solution by Eldon Hansen, Lockheed Corporation, Los Altos Hills, Calif. Consider the sum

$$S = \sum_{k=1}^{n} \frac{(-1)^k}{k!(n-k)!}\binom{ak+b}{m} p(k), \tag{1}$$

where $p(k)$ is a polynomial of degree d in k and m is an integer such that $m+d < n$. Since $\binom{ak+b}{m}$ is a polynomial of degree m in k, we have

$$\binom{ak+b}{m} p(k) = \sum_{r=0}^{m+d} A_r k^r, \tag{2}$$

where the coefficients A_r are independent of k. Substituting (2) into (1) and interchanging the order of summation, we obtain

$$S = \sum_{r=0}^{m+d} A_r \sum_{k=1}^{n} \frac{(-1)^k}{k!(n-k)!} k^r.$$

Now $r < m+d < n$, by hypothesis, and hence

$$\sum_{k=1}^{n} \frac{(-1)^k}{k!(n-k)!} k^r = -\frac{\delta_{r0}}{n!},$$

where δ_{r0} is the Kronecker delta. (This is easily verified by induction on r and n, or see p. 100 of I. J. Schwatt, *An Introduction to the Operations with Series*, 1924.) Therefore

$$S = -\frac{A_0}{n!} = -\frac{p(0)}{n!}\binom{b}{m}. \tag{3}$$

The desired sum is a special case of S. To show this, let $p(k) = k-1$, $k = r+2$, $n = 2N+1$, $a = \frac{1}{2}$, $b = -5/4$, and $m = N$. Then, from equations (1) and (3), we easily obtain

$$\sum_{r=0}^{2N-1} \frac{(-1)^r\Gamma(r/2+3/4)}{(r+2)r!(2N-r-1)!\Gamma(r/2-N+3/4)} = \frac{\Gamma(-1/4)}{(2N+1)!\Gamma(-N-1/4)}.$$

Using the relations $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$ and

$$(-1)^r \sin \pi(r/2 - N + 3/4) = (-1)^{N+[r/2]}2^{-1/2},$$

we obtain

$$\sum_{r=0}^{2N-1} \frac{(-1)^{[r/2]}\Gamma(r/2+3/4)\Gamma(N-r/2+1/4)}{(r+2)r!(2N-r-1)!} = \frac{4\Gamma(3/4)\Gamma(N+5/4)}{(2N+1)!},$$

which is the desired result.

Also solved by the proposer who uses ideas in his paper, *Associated transforms and convolutions*, Revista de Ciencias, 1949, pp. 5–46.

Representation of Prime Numbers

5241 [1964, 1047]. *Proposed by Necdet Ucoluk, Purdue University.*

Let p_j be the jth prime (so that $p_1=2$, $p_2=3, \cdots$) and let m be the greatest even number less than n. Prove, or disprove, that it is possible to choose $\epsilon_j = \pm 1$ such that $p_n = \sum_{j=1}^{m} \epsilon_j p_j$.

Solution by J. Moser and J. Riddell, University of Alberta. From Bertrand's postulate that for every $n \geqq 1$ there is a prime p such that $n < p \leqq 2n$, it follows that

$$p_{n+1} < 2p_n \quad \text{for all } n. \tag{1}$$

We shall also use the result

$$p_{n+2} < 2p_n \quad \text{for } n \geqq 4. \tag{2}$$

This follows from, for example, the result of Nagura that for every $n \geqq 25$ there is a prime p such that $n < p \leqq 6n/5$. (See J. Nagura, *On the interval containing at least one prime number*, Proc. Japan Acad., 28 (1952) 177–181.)

We prove that if $m \geqq 6$, m even, then any odd number in $(-2p_m, 2p_m)$ can be expressed in the form

$$\pm p_m \pm p_{m-1} \pm \cdots \pm p_1. \tag{3}$$

By (2) it then follows that if m is the greatest even number less than $n (n \geqq 7)$, p_n is in the interval $(-2p_m, 2p_m)$, and hence p_n is representable in the desired form. For smaller primes, 5, 11, and 13 have the desired representation; 2, 3, and 7 do not.

First, it is easily checked that any odd number in $(-26, 26)$ can be expressed by $\pm 13 \pm 11 \pm 7 \pm 5 \pm 3 \pm 2$ for some choice of signs. Now suppose any odd number in $(-2p_{m-2}, 2p_{m-2})$ can be expressed

$$\pm p_{m-2} \pm p_{m-3} \pm \cdots \pm p_1 \qquad (m \geqq 8, \text{ even}). \tag{4}$$

To get the numbers (3), we add to (4) one of $\pm(p_m - p_{m-1})$, $\pm(p_m + p_{m-1})$. By (1), $p_{m-1} < 2p_{m-2}$ and $p_m - p_{m-1} < p_{m-1}$; hence by adding $\pm(p_m - p_{m-1})$ to the numbers (4) we can get any odd number in

$$(-(2p_{m-2} + p_m - p_{m-1}), 2p_{m-2} + p_m - p_{m-1}).$$

Then by adding $\pm(p_m+p_{m-1})$ or $\pm(p_m-p_{m-1})$ to the numbers (4) we will get any odd number in

$$(5) \qquad (-(2p_{m-2}+p_m+p_{m-1}),\ 2p_{m-2}+p_m+p_{m-1})$$

provided that $-2p_{m-2}+p_m+p_{m-1}\leqq 2p_{m-2}+p_m-p_{m-1}$ (so that there shall not be any gap). This inequality is equivalent to $2p_{m-1}\leqq 4p_{m-2}$, which follows from (1). Since $2p_{m-2}+p_m+p_{m-1}>p_{m-1}+p_m+p_{m-1}>2p_m$, the interval (5) contains $(-2p_m,\ 2p_m)$.

Connected Unions of Disconnected Sets

5242 [1964, 1047]. *Proposed by M. K. Fort, Jr., University of Georgia, and B. Peterson, Middleburg College*

If a topological space is the union of two closed totally disconnected subsets, is the space necessarily totally disconnected?

I. *Solution by Martin Tangora, Northwestern University.* No; we construct an example in which E is connected.

Let the set E be the set of all real numbers, to which we will assign a topology strictly finer than the usual one. Let X, Y, Z be three disjoint subsets whose union is E, each of which is everywhere dense (with respect to the usual topology); for example, let X be the dyadic rationals, Y the remaining rationals, Z the irrationals. We make X and Y open subsets of E, along with all subsets of X which are relatively open in X under the usual topology, and likewise the corresponding subsets of Y. A basis for the neighborhoods of a point z in Z will be provided by sets of the form $\{z\}\cup\{w\in X\cup Y: |w-z|<r\}$ for positive r; i.e., the usual neighborhoods in E but with all points of Z deleted apart from z itself.

In this way E becomes a topological space partitioned into two totally disconnected open subsets X, Y and a discrete subset Z. We put A and B equal to the complements of X and Y respectively, and thus obtain the desired example.

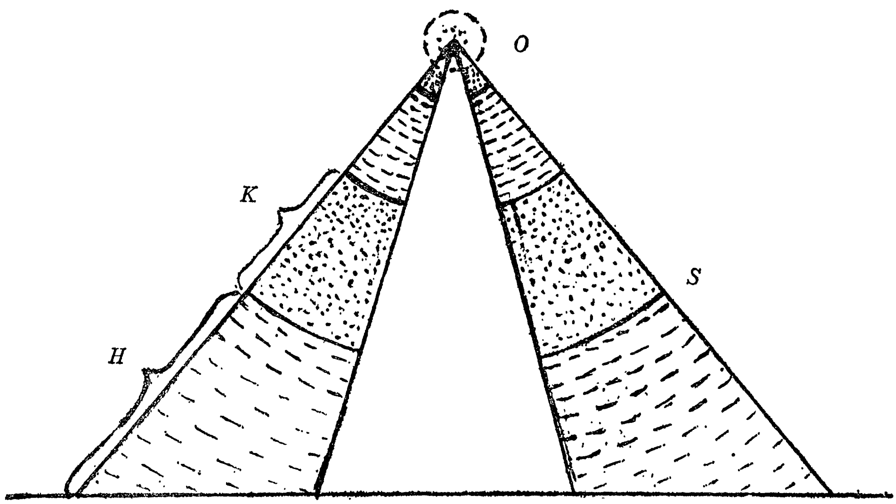
A and B are closed subsets: since X and Y are open.

A is totally disconnected: since X is everywhere dense in the usual topology, any two points of A are separated in E by a point of X. More precisely, every point of X divides A into two disjoint open and closed sets, using the usual ordering of the real numbers. Thus no two points of A can belong to the same component. Similarly for B.

E is connected: The usual argument that the real line is connected can be carried over to E with no essential modification: that is, if D were a nonempty open and closed subset of E whose complement were also nonempty, choose a point d in D and let p be the l.u.b. $\{e\in E-D: e<d\}$ (the usual ordering is intended). A study of the possible cases (with respect to X, Y, Z) shows that p cannot lie either in D or in its complement.

II. *Solution by F. Burton Jones, University of California, Riverside.* Let f be a discontinuous solution of the functional equation $f(x)+f(y)=f(x+y)$ whose graph F in the number plane is connected (see F. B. Jones, *Connected and disconnected plane sets*, Bull. Amer. Math. Soc., 48 (1942) 115–120). Let L be any line parallel to the x-axis. Since F is dense in the plane, between any two points of F there is a point of F belonging to L. Now consider F to be the space (giving it the relative topology of the plane) and let F_1 be the set of all points on F on or above the x-axis and let F_2 be the set of all points of F on or below the x-axis. Then $F=F_1\cup F_2$ and each of F_1 and F_2 is closed and totally disconnected. It is perhaps notable that, since f is an odd function, F_1 and F_2 are congruent.

III. *Solution by R. F. Jolly, University of California, Riverside.* Examples in a metric space of a connected point set S with a *dispersion* point are well known. (A dispersion point of a point set S is a point 0 such that $S-\{0\}$ is totally disconnected.) See R. Duda, *On biconnected sets with dispersion points*, Rozprawy Mat., 37 (1964) 1–60. With the relative topology S is a topological space. Let $\{R_n\}$ denote the sequence of open spheres in S with center O such that $R_O=S$ and R_n has radius $1/n$ for $n=1, 2, \cdots$. The two sets of the form $\bigcup(\overline{R}_n-R_{n+1})\cup\{0\}$, where n is taken to be even in one case and odd in the other, are clearly two closed and totally disconnected sets whose union is S. (Here $\overline{R}_n$ denotes the closure of R_n.) The figure may help to clarify the situation.



$S=H\cup K$

Also solved by John Cobb, William Huebsch, E. S. Langford, John Webb, and the proposers.

Jones notes the interesting and much more difficult situation if the topological space happens to be a complete metric space. Using Jolly's procedure and some results in a paper by Knaster and Kuratowski, (Rendiconti del Circolo Matematico di Palermo, 59 (1925), 382–386), an example may be given.

The End of a Knight

5243 [1964, 1047]. *Proposed by Erwin Just, Bronx Community College, and Herbert Sullivan, Lear Sigler, Inc.*

Let a knight's tour of a chess board be the number of random moves made by the knight before it moves off the board. If a knight's tour begins on one of the four center squares of the chess board, what is the probability that its tour will be n moves?

Solution by A. J. Bosch and J. van Yzeren, Technical University, Eindhoven, Netherlands. We first consider the same problem on a 6×6 board. If the 36 squares are classified as below, the matrix of mutual transition probabilities (apart from a factor $1/8$) is A. The probabilities of leaving the board (in *one* move) are complementary to the row totals. In the same way A^2, A^3, A^4 provide the probabilities of leaving the board in 2, 3, 4 moves. Starting from the squares 6 one successively finds probabilities 0, 15/32, 11/64, 147/1024; but the general rule is not immediately clear.

$$\begin{array}{cccccc}
4 & 1 & 2 & 2 & 1 & 4 \\
1 & 5 & 3 & 3 & 5 & 1 \\
2 & 3 & 6 & 6 & 3 & 2 \\
2 & 3 & 6 & 6 & 3 & 2 \\
1 & 5 & 3 & 3 & 5 & 1 \\
4 & 1 & 2 & 2 & 1 & 4
\end{array}
\qquad
\begin{array}{r|c}
\text{To} & \;\;1\;\;2\;\;3\;\;4\;\;5\;\;6 \\
\text{From} & \\
\hline
\begin{array}{c}1\\2\\3\\4\\5\\6\end{array} &
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 2 & 1 & 0 & 1 \\
0 & 0 & 2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 2 \\
2 & 2 & 2 & 0 & 2 & 0
\end{pmatrix}
\end{array}
= A, \quad
F(A) = \begin{pmatrix}
18 & 0 & -2 & -6 & -8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 2 & -2 & 0 & 0 \\
-12 & 0 & -4 & 12 & 8 & 0 \\
-16 & 0 & 0 & 8 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

The characteristic polynomial of A is

$$x(x+1)(x^4 - 3x^3 - 14x^2 + 14x + 32) = x(x+1)F(x).$$

We now calculate $F(A)$. The result appears to have a vanishing sixth row. Hence, the probabilities p_n of moving from 6 off the board in n moves are related by

$$p_n = \frac{3}{8}p_{n-1} + \frac{14}{8^2}p_{n-2} - \frac{14}{8^3}p_{n-3} - \frac{32}{8^4}p_{n-4}.$$

This leads to an explicit but very complicated expression for p_n as a combination of the nth powers of the zeros of $F(x)$ (which are not rational).

We restrict ourselves to finding the generating function

$$g(x) = \sum_{i=1}^{\infty} p_i x^i \text{ from } \left(1 - \frac{3}{8}x - \frac{14}{8^2}x^2 + \frac{14}{8^3}x^3 + \frac{32}{8^4}x^4\right)g(x)$$

$$= p_1 x + \left(p_2 - \frac{3}{8}p_1\right)x^2 + \left(p_3 - \frac{3}{8}p_2 - \frac{14}{8^2}p_1\right)x^3$$

$$+\left(p_4 - \frac{3}{8}p_3 - \frac{14}{8^2}p_2 + \frac{14}{8^3}p_1\right)x^4.$$

The result is

$$g(x) = \frac{120x^2 - x^3 - 6x^4}{256 - 96x - 56x^2 + 7x^3 + 2x^4}\cdot$$

The expansion

$$g(e^t) = 1 + 3\frac{53}{113}t + \left(\frac{600}{113} + \frac{179\cdot 784}{113^2}\right)\frac{t^2}{2} + \cdots$$

gives

$$E(n) = 3\frac{53}{113}, \quad \text{and} \quad \mathrm{Var}(n) = 4\frac{3396}{12769}\cdot$$

On an 8×8 board an analogous pattern of squares gives a 10×10 matrix A:

$$\begin{array}{cccccccc}
7 & 1 & 2 & 3 & 3 & 2 & 1 & 7\\
1 & 8 & 4 & 5 & 5 & 4 & 8 & 1\\
2 & 4 & 9 & 6 & 6 & 9 & 4 & 2\\
3 & 5 & 6 & 10 & 10 & 6 & 5 & 3\\
3 & 5 & 6 & 10 & 10 & 6 & 5 & 3\\
2 & 4 & 9 & 6 & 6 & 9 & 4 & 2\\
1 & 8 & 4 & 5 & 5 & 4 & 8 & 1\\
7 & 1 & 2 & 3 & 3 & 2 & 1 & 7
\end{array}
\qquad
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0\\
0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1\\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1\\
0 & 1 & 1 & 1 & 1 & 2 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0\\
2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 2\\
0 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 0
\end{pmatrix}.$$

Here we restrict ourselves to presenting the numerical values of $p_1, p_2, \cdots, p_{11}$:

0.0000, 0.1250, 0.2539, 0.1382, 0.1273,

0.0854, 0.0686, 0.0496, 0.0381, 0.0282, 0.0214.

In the latter p's the influence of the maximum latent root of the characteristic polynomial is predominant, giving practically

$$p_n = 0.0214\cdot\left(\frac{3}{4}\right)^{n-11} \qquad n \geqq 12$$

Using this approximation we find $E(n) \cong 5.41$.

Also solved by D. C. B. Marsh.

Although many basic homological concepts do not appear, this in no way lessens the usefulness of the book. Indeed, this monograph has already motivated many students to undertake a serious study of homological algebra. It is an excellent preparation for the later reading of, say, Cartan-Eilenberg or MacLane.

Each chapter, other than the first, contains a large number of good exercises, designed to illustrate and develop topics introduced in the text, and to test the student's comprehension of the material. There is also a large, pertinent bibliography.

The main fault of the text lies in the large number of typographical errors. For example, there are many incorrect subscripts and misdirected arrows. Correcting these, however, provides another excellent exercise for the reader.

HIRAM PALEY, University of Illinois, Urbana

Measure and Integration. By C. Carathéodory, translated by F. E. J. Linton. Chelsea, New York, 1963. 378 pp. $7.50.

The theory of measure and integration is usually constructed on a σ-ring S of subsets of a set. Later one is led to consider the measure on the associated quotient ring obtained by dividing S by the ideal of nullsets. It is then natural to ask how the theory would be altered if it were formulated from the outset for an abstract Boolean σ-ring A. This is the program which Carathéodory carried out, very elegantly, in his last book *Mass und Integral und ihrer Algebraisierung*, published posthumously in 1956, of which the book under review is a translation.

It turns out that while most of measure theory generalizes easily, integration is more complicated. The trouble, of course, is that point functions no longer exist. They are replaced by "place functions"—monotone mappings $S(y)$ of the real numbers into A. Algebraic operations on place functions have to be defined indirectly with the aid of the bounds $\alpha(X)=\sup \{y: XS(y)=0\}$ and $\beta(X) =\inf \{y: XS(y)=X\}$. The integral $\psi(X)=\int_X f d\phi$ of a nonnegative, ϕ-measurable place function f is defined to be a measure function ψ such that $\alpha(X)\phi(X)\leqq\psi(X) \leqq\beta(X)\phi(X)$ whenever $0<\phi(X)<\infty$.

In this connection the representation theorems of Stone and Loomis might have been used to simplify matters, at least in principle. These theorems are mentioned in a footnote in the Appendix, but they are not exploited, nor is their significance more than hinted at. They imply that the theory for a Boolean σ-ring is actually no more general than for a quotient σ-ring of sets. Nevertheless, since the representation theorems are not constructive, it is interesting and worth while to see how the theory can be developed without them, and new insight is gained thereby.

A somewhat unusual feature, independent of the Boolean generalization, is that the integral $\psi(X)$ is defined even when X is nonmeasurable. This is because ϕ is assumed to be only an outer measure. For example, when f is the character-

istic function of a ϕ-measurable set E, $\int_X f d\phi = \phi(EX)$ for all sets X, measurable or not.

The translation is literal and generally quite accurate. However, there are a few instances of logical inversion,* for example at the places indicated by the following page and line numbers: 71, 11; 79, 24; 83, 20–22; 87, 29; 300, 10.

JOHN C. OXTOBY, Bryn Mawr College

Introduction to Vector Analysis. By Frank Tiller. Ronald Press, New York, 1963. 215 pp. $6.00.

This is a textbook in which only the most basic topics of vector analysis and its applications are treated. Definitions are stated in an elementary manner (e.g. the definition of vector), and explanations are given in great detail throughout the book. Many drawings illustrate simple facts and relationships, and the author recommends memorizing certain formulas. The material treated is classical in every detail.

The book consists of six chapters. Chapter 1, which is concerned with addition and subtraction of vectors, contains some applications involving the principles of statics, electrical and gravitational fields, and complex numbers. Chapter 2 is about the applications of vector addition and subtraction to elementary geometry. Chapters 3 and 4 treat the subject of vector multiplication. The usual vector identities and applications of multiple vector products are presented. The subject of Chapter 5 is three-dimensional Euclidean geometry. Chapter 6 is an elementary treatment of vector calculus.

In the preface the author states clearly that he intends that the book be an introductory treatment and elementary in scope. In this he has succeeded, and consequently this book might well serve as a text for students to use either early in their mathematical careers or for self study. Very few errors were noted, and the exposition is good. There are numerous worked examples which serve as illustrations of important ideas, and there is an ample supply of exercises for the students to work.

RALPH L. SHIVELY, Oak Ridge National Laboratory

Principles of Modern Algebra. By J. Eldon Whitesitt. Addison-Wesley, Reading, Mass., 1964. 262 pp. $7.50.

Among the variety of texts in modern algebra published recently, this one is directed to a specific audience: prospective teachers of school algebra. The topics included draw their motivation from school algebra and are well chosen to throw conceptual light on all school mathematics. As a text for mathematics majors at a sophomore or higher level it might tell too much of the elementary part of the story and not suggest enough of the meatier portions.

The first chapter is about as clear a description of the terminology of sets, relations, functions, operations and partitions as one could find (although the section on logical terms is somewhat sparse). Then some specific systems are

* The logical inversions are due to interchanges of formulas on the part of the printer. The erratum sheet is available from the publisher.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Arthur Danese, State University of New York at Buffalo, represented the Association at the inauguration of Harold C. Martin as President of Union College on October 2, 1965.

Dean R. D. Larsson, Mohawk Valley Community College, represented the Association at the installation ceremonies of the Hamilton College Chapter of the Society of the Sigma Xi on September 24, 1965.

Dr. H. O. Pollak, Bell Telephone Laboratories, Murray Hill, New Jersey, represented the Association at the convocation on needed research in mathematical education sponsored by Teachers College, Columbia University, on October 30–31, 1965.

University of California, San Diego: Professor E. A. Bishop, University of California, Berkeley, has been appointed Professor; Dr. S. G. Williamson, University of California, Santa Barbara, has been appointed Assistant Professor; Dr. George Senge, University of California, Los Angeles, has been appointed Acting Assistant Professor; Dr. Gilbert Walter, University of Wisconsin, Milwaukee, has been appointed Visiting Assistant Professor.

Cornell University: Dr. Juris Hartmanis, General Electric Research Laboratory, Schenectady, New York, has been appointed Professor of Computer Science and Chairman of the Department of Computer Science; Assistant Professor P. C. Fischer, Harvard University, has been appointed Associate Professor of Computer Science.

Grand Valley State College: Professor Holmes Boynton, Northern Michigan University, has been appointed Professor; Professor Henry Hanson, Upsala College, has been appointed Lecturer.

Hampden-Sydney College: Assistant Professor D. B. Selden has been promoted to Associate Professor; Mr. M. A. Espigh has been promoted to Assistant Professor.

Shippensburg State College: Assistant Professor J. L. Sieber has been promoted to Associate Professor; Mr. Michael Varano has been promoted to Assistant Professor.

Professor A. G. Anderson, Western Kentucky State College, has been appointed Professor and Chairman of the Department of Mathematics, at Upsala College.

Assistant Professor Elaine Cook, Jamestown College, has been appointed Acting Chairman of the Department of Mathematics.

Associate Professor M. J. Hellman, Newark College of Rutgers, The State University, has been promoted to Professor.

Assistant Professor E. M. Hughes, Chadron State College, has been appointed Director of Institutional Research.

Mr. P. M. Kannan, Louisiana State University, Baton Rouge, has accepted a position on the staff of the Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Visiting Professor M. S. Klamkin, University of Minnesota, has joined the Ford Scientific Laboratory at Dearborn, Michigan, as a Principal Research Scientist.

Assistant Professor Visvanatha Krishnamurthy, University of Illinois, has been appointed Associate Professor at the Birla Institute of Technology and Science, Pilani, India.

Dr. W. F. Lucas, Princeton University, has been awarded a Fulbright grant to lecture at the Middle East Technical University, Ankara, Turkey, during the 1965–66 academic year.

Associate Professor A. F. Strehler, Carnegie Institute of Technology, has been appointed Associate Dean of Graduate Studies.

Professor Emeritus D. J. Struik, Massachusetts Institute of Technology, was appointed Visiting Professor at the University of Costa Rica from March 1, 1965 to June 30, 1965.

Professor T. Y. Thomas, Indiana University, has been granted a leave of absence to accept a Visiting Professorship in Engineering at the University of California, Los Angeles, during the academic year 1965–66.

Dr. J. A. Thorpe, Massachusetts Institute of Technology, has been appointed Assistant Professor at Haverford College.

Mr. R. F. Baker, Chevron Oil Company, New Orleans, Louisiana, died on April 18, 1965. He was a member of the Association for 6 years.

Professor Emeritus Claribel Kendall, University of Colorado, died on April 17, 1965. She was a charter member of the Association.

THE UNIVERSITY OF TEXAS—FOURTH ANNUAL SYMPOSIUM ON BIOMATHEMATICS AND COMPUTER SCIENCE IN THE LIFE SCIENCES

The Division of Continuing Education of The University of Texas Graduate School of Biomedical Sciences at Houston is pleased to announce the Fourth Annual Symposium on Biomathematics and Computer Science in the Life Sciences, which will be held at the Shamrock Hilton Hotel in Houston, Texas, on Thursday, Friday and Saturday, March 24, 25 and 26, 1966. The symposium will center its sessions around the following general topics: "Simulation and modeling of biological processes," "Training for bio-engineering and biomathematics," "Analysis of biological processes with time sharing computation," "Records and library information systems for a medical center," "Biostatistical techniques and biologically oriented computer languages."

For further information address the Office of the Dean, Division of Continuing Education, The University of Texas Graduate School of Biomedical Sciences at Houston, 102 Jesse Jones Library Building, Texas Medical Center, Houston, Texas 77025.

FELLOWSHIP AND RESEARCH OPPORTUNITIES IN MATHEMATICS

The Division of Mathematics, National Academy of Sciences—National Research Council, calls attention to a variety of fellowship and other support for basic research in mathematics at both the predoctoral and postdoctoral levels to be awarded during the year 1964–65. Copies of the complete announcement are available from the Division of Mathematics, National Academy of Sciences—National Research Council, 2101 Constitution Avenue, N. W., Washington, D. C. 20418.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 72 NUMBER 10

CONTENTS

DECEMBER 1965

THE AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

FREDERICK A. FICKEN, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, F. A. FICKEN, Department of Mathematics, New York University, New York, N. Y. 10453.

ADVERTISING CORRESPONDENCE should be addressed to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214.

NOTICE TO AUTHORS

The MONTHLY welcomes papers presenting valid mathematics, of rather general interest, at a level intelligible to persons with two years of full-time graduate study. Some novelty of content, viewpoint, or arrangement is essential. Expository articles are particularly desired. State the context and the principal aim of the paper early. Address yourself quite explicitly to the reader described above, communicating your ideas to him clearly and attractively.

The title should be brief and meaningful. Since the title will be quoted and reproduced by laymen, it should contain no symbols unfamiliar to laymen.

Articles should be typewritten, double-spaced, on $8\frac{1}{2} \times 11''$ paper of very good quality. Submit the original (and a duplicate if convenient) keeping a complete copy for yourself. To avoid loss and delay notify us of any change of address.

The typescript should be prepared with extreme care. Misprints are highly obnoxious; so are dangling participles. Put name and address between title and text. Put references at the end with bracketed citations in the text. Avoid footnotes; instead, use clearly designated remarks in the text. Put acknowledgments at the very end, just before the bibliography. Be generous with spacing and displays. *Keep notation simple.* For a matrix the notation $[a_{jk}]$ is recommended, with $\det[a_{jk}]$ or $|a_{jk}|$ for the determinant. On doubtful questions regarding format or notation, observe practices in current issues of the MONTHLY, or consult the applicable sections of the "Author's Manual" of the American Mathematical Society.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $6.00. For nonmembers the subscription price is $10.00

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Buffalo, N. Y. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

THE CIRCULAR TRACTRIX

W. G. CADY, 127 Power St., Providence, Rhode Island

Abstract. After a historical sketch the equations are derived for tractrix curves whose directrix is a circle. Two types are distinguished. In one, the curve has the form of a spiral. In the other, there is a succession of cusps forming a flower-like pattern; the conditions are considered under which the curve can close upon itself. A mechanical device for drawing the curves is described.

At a meeting in Paris in 1693 Claude Perrault laid his watch on the table, with the long chain drawn out in a straight line ([1] vol. 3). He showed that when he moved the end of the chain along a straight line, keeping the chain taut, the watch was dragged along a certain curve. This was one of the early demonstrations of the tractrix. The line along which the chain is pulled is called the directrix. Perrault did not know that the curve had already been recognized, and its equation derived, by Sir Isaac Newton [2] in 1676. This is said to have been the first example of the determination of a curve by the process of integration.

The theory of the tractrix was treated by Huygens and Leibniz, both of whom considered the case in which the directrix was any arbitrary curve. It was not until almost a century later that Euler [3] treated the problem so completely that little or nothing on the subject has appeared since. Among other things Euler derived equations for the case in which the directrix was a circle instead of a straight line.

Hitherto no one seems to have pointed out the fact that a tractrix can be drawn not only by pulling (whence the name "tractrix"), but also by pushing. That is, the string or chain can be replaced by a rigid bar; a mirror image of the same curve is then obtained. Such a reverse tractrix might appropriately be called a "trudrix," from the Latin *trudere*, to push. In the circular tractrix discussed here both pulling and pushing occur.

The ordinary tractrix, having a straight line as directrix, is the one treated in textbooks. It may be called a *linear* tractrix. The circular tractrix, though really more interesting, does not seem to have been considered at all in our day. It is interesting mechanically and aesthetically as well as mathematically and historically.

We now derive the equations for the circular tractrix in a form better adapted for comparison with the actual curves than that given by Euler.

Fig. 1 shows a linear tractrix, and for comparison part of a circular tractrix. As in the linear case, the curve is characterized by the fact that the length of the tangent from any point on the curve to the directrix is constant.

The differential equation of the circular tractrix is easily derived with the aid of Fig. 2, in which OA and AB are the driving and tracing arms. Initially the arms are in line, with $a=OA$ and $b=AB$. If γ is the angle between b and a, measured counter-clockwise from b, we have at the start $\gamma=\pi$. Now let a be rotated in the positive direction about the origin O to the position OA', through

the angle θ. The tracing point, initially at B, is dragged along the curve to B', where $A'B' = b$. B' thus represents any point on the curve, and b is tangent to the curve at B'. If a were rotated in the negative direction, the branch of the curve shown below OB would be traced. The angular amplitude of the radius vector at any point on the curve is ψ.

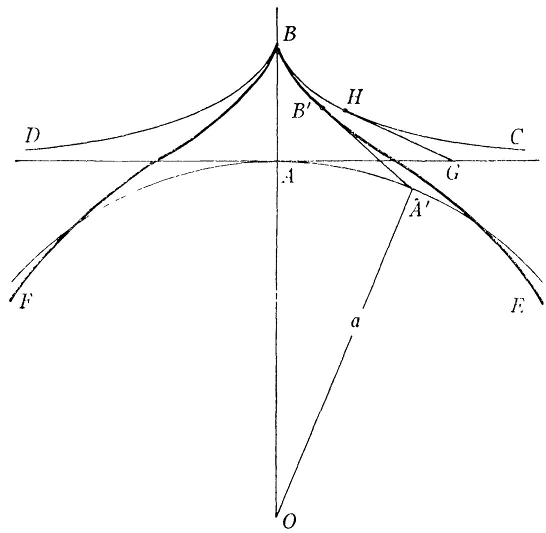


FIG. 1. A linear tractrix DBC with a straight line as directrix, and a circular tractrix FBE with a circle of radius a as directrix. The tangent lines AB, $A'B'$, and GH are of equal length.

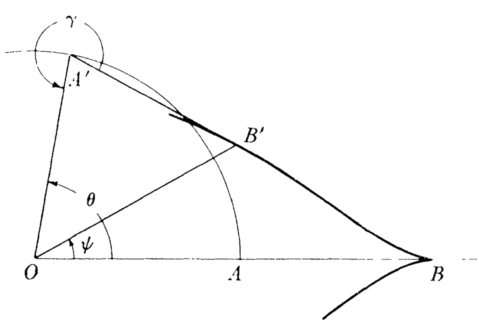


FIG. 2. Part of a circular tractrix generated by rotation of the driving arm a about the origin O. The circle with radius a is the directrix. When a has rotated through the angle θ, the point on the curve is at B', with $A'B' = b$. The angle between b and a is γ. ψ is the angular amplitude of any point on the curve. $OA = a$, $AB = b$.

Now suppose a to be rotated further positively through an infinitesimal angle $d\theta$. A' moves through the small distance $ad\theta$ along the a-circle, while B' moves a short distance along the curve. The arm b becomes rotated through the angle

$$d\beta = \frac{a d\theta}{b} \cos(\gamma - \pi) = -\frac{a}{b} d\theta \cos \gamma. \tag{1}$$

These changes in θ and β cause γ to change by the amount $d\gamma = d\theta + d\beta$, or from (1),

$$d\gamma = d\theta - \frac{a}{b} d\theta \cos \gamma. \tag{2}$$

Evidently the relation between γ and θ depends only on the ratio of the two arms. We now write $a/b = p$, and find from (2):

$$d\theta = \frac{d\gamma}{1 - p \cos \gamma}. \tag{3}$$

The integral of (3) assumes different forms according to whether $p > 1$, $p < 1$, or $p = 1$.

When $p > 1$, a is greater than b, and

$$\theta = \cot \alpha \ln \left| \frac{\sin \frac{1}{2}(\alpha - \gamma)}{\sin \frac{1}{2}(\alpha + \gamma)} \right| + C, \tag{4}$$

where $\alpha = \cos^{-1}(b/a) = \cos^{-1}(1/p)$, whence

$$\cot \alpha = \frac{1}{\sqrt{(p^2 - 1)}}. \tag{5}$$

To find C, set $\theta = 0$ and $\gamma = \pi$ in (4) and find that $C = 0$. Therefore for all values of θ, when $a > b$, we have

$$\theta = \cot \alpha \ln \left| \frac{\sin \frac{1}{2}(\alpha - \gamma)}{\sin \frac{1}{2}(\alpha + \gamma)} \right|. \tag{6}$$

When $p < 1$, a is smaller than b, and the integral is

$$\theta = \frac{2}{\sqrt{(1 - p^2)}} \tan^{-1} \left(\sqrt{\left(\frac{1 + p}{1 - p} \right)} \tan \frac{\gamma}{2} \right) + C. \tag{7}$$

To find C, set $\theta = 0$ and $\gamma = \pi$ as before. Then

$$C = \frac{\pi}{\sqrt{(1 - p^2)}} = \frac{\pi b}{\sqrt{(b^2 - a^2)}}.$$

The final form of (7) is therefore

$$\theta = \frac{1}{\sqrt{(1 - p^2)}} \left\{ 2 \tan^{-1} \left(\sqrt{\left(\frac{1 + p}{1 - p} \right)} \tan \frac{\gamma}{2} \right) + \pi \right\}. \tag{8}$$

Finally when $p = 1$, $a = b$, and the integral is simply

$$\theta = -\cot\frac{\gamma}{2}\cdot \tag{9}$$

Here, as is easily verified, the constant of integration vanishes.

Case 1. We consider first the case in which $a>b$, and therefore $p>1$. If the driving arm a continues to rotate counter-clockwise, the curve shown in Fig. 2 and expressed by (6) passes inside the a-circle and forms a spiral, approaching asymptotically a circle of radius $R=a^2-b^2$. This is illustrated in Fig. 3, in which the arms $a=OA$ and $b=AB$ are initially in line. When a rotates clockwise, one

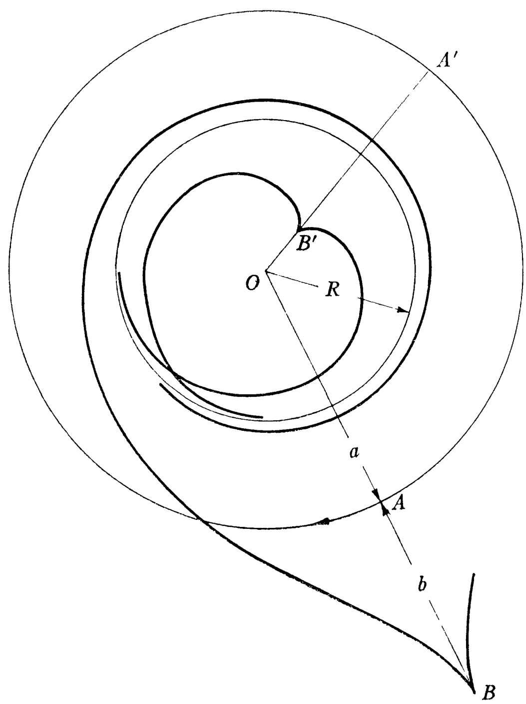


FIG. 3. Circular tractrix with the driving arm longer than the tracing arm.

branch of the curve is traced as shown, spiraling around the R-circle. A second branch would be traced if a rotated counter-clockwise. Moreover, if at the start the point B is located at B' instead of B, so that $B'A'=b$, two more spiral branches become traced, starting from an inwardly-facing cusp at B'. These branches approach the R-circle asymptotically from the inside. In all these cases the angle γ between b and a approaches the final value $\gamma=2\pi-\cos^{-1}(b/a)$. The starting point for the driving arm a can of course be anywhere on the a-circle, which is the directrix. If A' had been located in coincidence with A, B' would have fallen between O and A.

Case II. There are two extreme cases of (6). The first is when $a=b$ and $p=1$. This is the case expressed by (9), illustrated by Fig. 4. The radius R in Fig. 3 has become equal to zero, so that the curve converges asymptotically upon the pole. This curve was called by R. Cotes the Tractrix Complicata (see [4] p. 202).

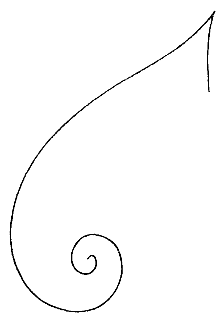


FIG. 4. Circular tractrix with driving and tracing arms equal.

Case III. The other extreme case is when, with b finite, a becomes infinite, so that $p=\infty$. The equation becomes that for the familiar linear tractrix, and the four branches of the circular tractrix mentioned above become the four branches of the linear tractrix.

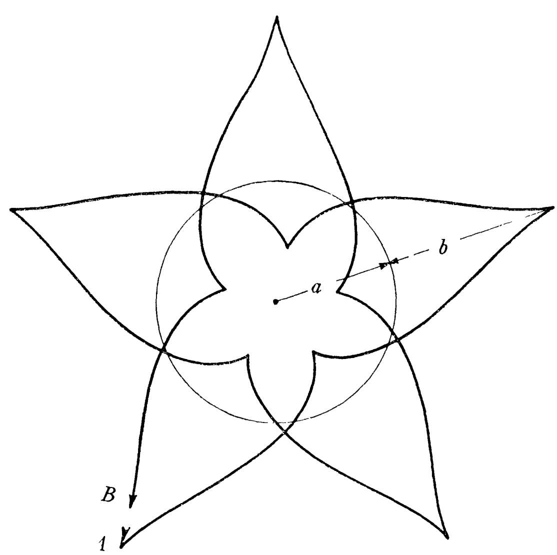


FIG. 5. Circular tractrix with the driving arm shorter than the tracing arm.

Case IV. The driving arm a is shorter than the tracing arm b, so that $p<1$. This is the case expressed by (8), yielding curves of the form shown in Fig. 5,

with a succession of positive and negative cusps, like a flower with petals spread out into a circle. The radial difference between positive and negative cusps is $2a$.

Figures 3, 4, and 5 are reproduced from actual curves traced with the apparatus described below.

With the aid of Eq. (8) the following relations can be derived. Since, at each positive cusp, b is an extension of a, it follows that between two successive positive cusps the angular rotation of γ is 2π; the angular rotation θ of the arm a is $2\pi/\sqrt{(1-p^2)}$; and the angular change in the radius vector is

$$d\psi = \frac{2\pi}{\sqrt{(1-p^2)}} - 2\pi = 2\pi\frac{1-\sqrt{(1-p^2)}}{\sqrt{(1-p^2)}} = \frac{2\pi}{m} \text{ radians,} \tag{10}$$

where

$$m = \frac{\sqrt{(1-p^2)}}{1-\sqrt{(1-p^2)}} = \frac{\sqrt{(b^2-a^2)}}{b-\sqrt{(b^2-a^2)}}. \tag{11}$$

In general the quantity m, which may have any value from zero to infinity, is an irrational number. This means that in general the circular tractrix does not become a closed curve for any finite rotation of the arm a. It is only when m is a rational fraction that the curve closes upon itself.

When $m=0$, $b=a$, and the curve has one positive (or one negative) cusp and a spiral converging on a point, as we have already seen. When m is an integer we have a closed curve with m positive cusps, completed in one revolution of the amplitude ψ. The curve then has m-fold symmetry.

When m is a rational fraction it may be written in lowest terms as $m=m_1/m_2$, where m_1 and m_2 are integers. Then, as the driving arm is rotated, m_1 positive cusps become described, while the covering operation is completed in m_2 revolutions of the curve.

If m is very small, $m_1 \ll m_2$, a is nearly as large as b, and there are few cusps in many revolutions of the curve. Fig. 5 illustrates this, for in this figure $m \approx 5/2$. The negative cusps come inside the a-circle.

If m is large, $m_1 \gg m_2$, $a \ll b$, and there are many small "petals" in few revolutions. If $m_2=1$, the curve closes on itself in one revolution.

Returning to Figure 5, we see that the curve apparently closes upon itself with $m_1=5$, $m_2=2$, so that $m=2.5$. Actually the curve was drawn with $a=31.8$ mm, $b=46.2$ mm, from which, by (11), $m=2.64$. The discrepancy is to be attributed to faulty mechanical tracking, together with possible errors in the measurement of a and b.

A mechanical device for drawing circular tractrix curves with any ratio of b to a is pictured in Fig. 6. A flat wooden base about 28 cm square, carefully leveled, carries a bridge below which there is a clearance of about 9 cm. Through the center of the bridge passes the vertical driving shaft, turned by a handle that plays over a graduated dial, so as to be set at any desired angle. The driving

arm a passes through a hole in this shaft, and by means of a set screw can be clamped at any desired length.

Fastened to the free end of the driving arm is a yoke bent out of a strip of sheet metal. The yoke carries a thin tracing shaft, at the lower end of which is a block for holding the tracing arm b. This tracing arm should be as close to the paper as practicable; it can be set at any length and clamped by a set screw. At the end of this arm is a block with vertical bore to hold the lead for drawing the curve. The softest type of lead should be used; it is pressed onto the paper by a weight of 3 or 4 ounces.

The device must be as free as possible from loose play, and from any friction except that between tracing point and paper.

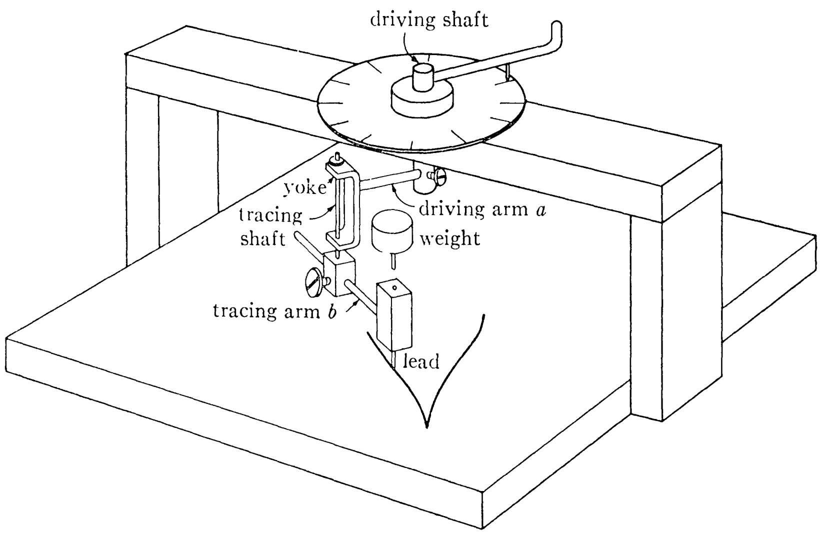


FIG. 6. Machine for drawing circular tractrix curves.

References

1. M. Cantor, Vorlesungen über Geschichte der Mathematik, Leipzig, 1908, Vol. 3, pp. 214, 215, 786; Vol. 4, p. 508.

2. The Correspondence of Isaac Newton, ed. by H. W. Turnbull, Cambridge Univ. Press, New York, 1961, Vol. 3, pp. 148, 160, 329.

3. L. Euler, Opera Omnia, Ser. II, Vol. 7, pp. 120–147, Lausanne, 1958; also Nova Acta Academiae Scientiarum Imperialis Petropolitanae II, pp. 3–27, 1788. Euler considers also the general case of the tractrix, in which the directrix may be of any form, but finds it insoluble.

4. G. Loria, Spezielle algebraische und transzendente ebene Kurven, vol. II, Teubner, Leipzig, 1911.

ON THE SOLUTIONS OF SOME "ISOMOMENT" FUNCTIONAL EQUATIONS

SAMUEL KOTZ, University of Toronto (Industrial Engineering)

1. Let $R_n^+ = \{ \mathfrak{X} \in R_n | x_i \geqq 0,\ i=1, \cdots, n \}$, where R_n is the n-dimensional Euclidean space and let $\mathfrak{X}$ be an element of R_n: $\mathfrak{X} = (x_1, x_2, \cdots, x_n)$, where x_i $(i=1, \cdots, n)$ are real numbers. Let $f(u)$ be a real valued function of a single real variable u, defined for all $u \in R_1$. By means of this function we define a transformation of R_n into itself:

$$f\colon (x_1, x_2, \cdots, x_n) \in R_n \to (f(x_1), f(x_2), \cdots, f(x_n)) \in R_n. \tag{1}$$

The r.h.s. of (1) can be denoted by $f\mathfrak{X}$; we also denote $fu = f(u)$ $(u \in R_1)$.

Consider the "isomoment" functional equations:

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^m\right) = \frac{\sum_{i=1}^{n}[f(x_i)]^m}{n}, \qquad m = 1, 2, \cdots, \quad n \geqq 2. \tag{2}$$

Introducing the operator μ_m over R_n, which corresponds to each $\mathfrak{X} \in R_n$ a real number

$$\mu_m \mathfrak{X} = \frac{1}{n}\sum_{i=1}^{n} x_i^m,$$

we can rewrite (2) in the form

$$f\mu_m\mathfrak{X} = \mu_m f\mathfrak{X}, \qquad m = 1, 2, \cdots \tag{2a}$$

In this note we obtain the set of all continuous solutions of the functional equations (2a) in the case of $\mathfrak{X} \in R_n^+$. In addition, some symmetry properties of regions $\{(x, y)\} \in R_2$ for which the equation $f(x+y) = f(x) + f(y)$ is satisfied are examined. The concluding remarks contain a discussion of the general formulation of problems considered in this note.

2. Let $\mathfrak{F}(R_n, m) = \{ f | f\mu_m\mathfrak{X} = \mu_m f\mathfrak{X},\ (\mathfrak{X} \in R_n) \}$. Choose $\mathfrak{X} = (c, c, \cdots, c)$, where c is an arbitrary real number, and consider some $f \in \mathfrak{F}(R_n, m)$. We get

$$f(c^m) = f^m(c). \tag{3}$$

Hence for every real $x \geqq 0$, $f(x^{1/m}) = f^{1/m}(x)$, and, in particular, for $c=1$, $f(1) = f^m(1)$. Hence $f(1)$ is either 0 or 1 (and possibly -1 in the case of odd m).

In a similar manner, we obtain (using $c=0$) that $f(0)$ is either 0 or 1 (and possibly -1 in case m is odd).

Consider now $\mathfrak{X} \in R_n^+$ and let $Y = (x_1^{1/m}, \cdots, x_n^{1/m})$, where $x_i^{1/m}$ is the non-negative root of $x_i \geqq 0$.

The correspondence $\mathfrak{X} \leftrightarrow Y$ is one-to-one of R_n^+ onto itself. Moreover,

$$\mathfrak{F}(R_n^+, m) \subseteq \mathfrak{F}(R_n^+, 1)$$

since if $f\in\mathfrak{F}(R_n^+, m)$, then for every $\mathfrak{X}\in R_n^+$, using (3) we obtain,

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)=f\left(\frac{1}{n}\sum_{i=1}^{n} y_i^m\right)=\frac{1}{n}\sum_{i=1}^{n} f^m(y_i)=\frac{1}{n}\sum_{i=1}^{n} f(y_i^m)=\frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

Hence, $f\in\mathfrak{F}(R_n^+, 1)$. Suppose now that $x_1\geqq 0$, $x_2\geqq 0$. Since $f\in\mathfrak{F}(R_n^+, 1)$, choosing $\mathfrak{X}=(x_1, x_2, 0, \cdots, 0)\in R_n^+$, we then get

$$f\left(\frac{x_1+x_2}{n}\right)=\frac{1}{n}f(x_1)+\frac{1}{n}f(x_2)+\frac{n-2}{n}f(0).$$

If we now choose $\mathfrak{X}=(x_1+x_2, 0, \cdots, 0)\in R_n^+$ then we have

$$f\left(\frac{x_1+x_2}{n}\right)=\frac{1}{n}f(x_1+x_2)+\frac{n-1}{n}f(0).$$

Hence, $f(x_1+x_2)=f(x_1)+f(x_2)-f(0)$, and defining $g(x)=f(x)-f(0)$, we get $g(x_1+x_2)=g(x_1)+g(x_2)$ for any $x_1\geqq 0$, $x_2\geqq 0$. The only solution of this equation, continuous at least at one point, is $g(x)=ax (x\geqq 0)$, where a is an arbitrary real number. Hence

$$f(x)=ax+b, \quad \text{where} \quad a=f(1)-f(0), \quad b=f(0).$$

Using the values of $f(0)$, $f(1)$ found above, we see that a is one of the numbers 0, ± 1, ± 2, and b is one of the numbers 0, ± 1. Substitution in (2) eliminates (for $m>1$) most of these functions, and we obtain the following theorem:

THEOREM 1. *Let $\mathfrak{F}(R_n^+, m)$ denote the set of all functions f, continuous at least at one point, which satisfy the functional equation*:

$$f\left(\frac{1}{n}\sum_{i=1}^{n} x_i^m\right)=\frac{1}{n}\sum_{i=1}^{n}[f(x_i)]^m, \qquad \text{with } x_i\geqq 0 \ (i=1, \cdots, n).$$

Then: $\mathfrak{F}(R_n^+, 2k)=\{0, 1, x\}$, *and* $\mathfrak{F}(R_n^+, 2k+1)=\{0, 1, -1, x, -x\}$ *for any integers* $k\geqq 1$ *and* $n\geqq 2$.

3. Consider now μ_1, operating on R_2, and let $\Omega_2(f, 1)$ be the set of all $\mathfrak{X}\in R_2$ for which the equation $f\mu_1\mathfrak{X}=\mu_1 f\mathfrak{X}$ is satisfied. Clearly $\Omega_2(f, 1)$ is symmetric with respect to the line l: $y=x$. Moreover, if the origin of R_2, $O\in\Omega_2(f, 1)$ then $f(0)=0$ and both x and y axes belong to $\Omega_2(f, 1)$. Also, f is an odd function at any point x_0, such that $(-x_0, x_0)\in\Omega_2(f, 1)$, provided $O\in\Omega_2(f, 1)$.

Put $\Omega(f)=\{(x, y)\in R_2 \mid f(x+y)=f(x)+f(y)\}$. Let x, y be two real numbers, both different from 0 and $x\neq y$. Consider the following set $\mathfrak{A}$ of 11 points in R_2.

$$\mathfrak{A}=\begin{Bmatrix}(x, y), (x, -x), (y, -y), (-x, -y), (x+y, -x-y), (0, 0), \\ (y, x), (-x, x), (-y, y), (-y, -x), (-x-y, x+y)\end{Bmatrix}.$$

Clearly, if the restricted set of 6 points

$$\mathfrak{A}^*=\{(x, y), (x, -x), (y, -y), (-x, -y), (x+y, -x-y), (0, 0)\}$$

belongs to $\Omega(f)$ then $\mathfrak{A}$ belongs to $\Omega(f)$. The following stronger result is also valid:

THEOREM 2. *If any* 5 *points of the set* $\mathfrak{A}^*$ *belong to* $\Omega(f)$ *then the whole set* $\mathfrak{A}$ *belongs to* $\Omega(f)$.

Proof. Set $f(0)=z_0$, $f(x)=z_1$, $f(y)=z_2$, $f(x+y)=z_3$, $f(-x)=z_4$, $f(-y)=z_5$, $f(-x-y)=z_6$. That $\mathfrak{A}^*$ belongs to $\Omega(f)$ can be expressed by the equations:

$$\begin{aligned}
E_1&: z_0 = 0\\
E_2&: z_0 - z_1 - z_4 = 0\\
E_3&: z_0 - z_2 - z_5 = 0\\
E_4&: z_0 - z_3 - z_6 = 0\\
E_5&: z_1 + z_2 - z_3 = 0\\
E_6&: z_4 + z_5 - z_6 = 0.
\end{aligned}$$

The theorem is equivalent to the statement that any 5 of these equations imply the remaining one. This is proved by showing that each equation E_i is a linear combination of the other ones. Indeed,

$$\begin{aligned}
E_1 &= E_2 + E_3 - E_4 + E_5 + E_6\\
E_2 &= E_1 - E_3 + E_4 - E_5 - E_6\\
E_3 &= E_1 - E_2 + E_4 - E_5 - E_6\\
E_4 &= -E_1 + E_2 + E_3 + E_5 + E_6\\
E_5 &= E_1 - E_2 - E_3 + E_4 - E_6\\
E_6 &= E_1 - E_2 - E_3 + E_4 - E_5
\end{aligned}$$

and this completes the proof.

4. Concluding remarks. The problem dealt with in this paper can be stated in more general form.

The operator μ_m acting on $Y\in R_n$ may be viewed as the mth sampling moment of a given sequence Y. A function f defined over R_1 may be called an *isomoment of order m* over some $\Omega_n\subseteq R_n$ if it satisfies equation (2a) for all $\mathfrak{X}\in\Omega_n$. Moreover, the set $\Omega_n(f, m)=\{\mathfrak{X}\in R_n | f\mu m\mathfrak{X}=\mu mf\mathfrak{X}\}$ may be referred to as the "*m-isomoment*" *region of f in* R_n. Using these definitions we can consider the class $\mathfrak{F}(\Omega_n, m)$, defined at the beginning of the note, as the class of all isomoment functions of order m over Ω_n. Clearly $f(x)=x$ and $f(x)=0$ both belong to $\mathfrak{F}(\Omega_n, m)$ for any $m\geqq 1$, $n\geqq 2$ and $\Omega_n\subseteq R_n$. In this note we characterized the class $\mathfrak{F}(\Omega_n, m)$ for the particular case of $\Omega_n=R_n^+$.

It would be interesting to investigate the structure of this class for various other subsets of R_n. Moreover, the inverse problem, namely: to find, for a given function f over R_1, its m-isomoment region $\Omega_n(f, m)$ in R_n, still remains open. We hope that our remarks will stimulate further investigation in this area.

Acknowledgements: The author wishes to express his sincere appreciation to the referee for his valuable comments. He also wishes to acknowledge a grant form the National Research Council of Canada which facilitated the completion of this article.

N-POINT COMPACTIFICATIONS

K. D. MAGILL, JR., State University of New York at Buffalo

1. Introduction. It is assumed that all topological spaces discussed in this paper are Hausdorff spaces. By a compactification $\zeta(X)$ of a space X, we mean a compact space containing X as a dense subspace. As is customary, we define two compactifications $\zeta(X)$ and $\eta(X)$ to be equivalent if there is a homeomorphism h from $\zeta(X)$ onto $\eta(X)$ such that $h(x)=x$ for each $x\in X$. We shall regard $\zeta(X)$ and $\eta(X)$ as being the same compactification if they are equivalent and different compactifications otherwise.

Let N be a positive integer. A compactification $\zeta(X)$ is called an N-point compactification if $\zeta(X)-X$ consists of N points, in which case we shall use the notation $\zeta_N(X)$. A characterization of those spaces which have an N-point compactification is given in the first theorem of the next section. In Theorem (2.3) it is shown that there is a one-to-one correspondence between the number of N-point compactifications of a locally compact space X and the number of equivalence classes of certain families of open subsets of X. Theorem (2.4) gives a necessary and sufficient condition that a space have precisely one N-point compactification. These results, along with some others obtained in Section 2, are applied to some well-known spaces in Section 3 to show, among other things, that the two familiar 1- and 2-point compactifications of the space of real numbers are the only N-point compactifications; the only N-point compactification of the complex plane is the 1-point compactification; and an infinite discrete space has infinitely many N-point compactifications for each positive integer $N>1$.

2. Theorems and Proofs. For $A\subset X$, we let $\mathcal{C}_X A$ denote the complement of A with respect to X. If no confusion can result, we use the simpler notation, $\mathcal{C}A$.

THEOREM (2.1). *The following statements concerning a space X are equivalent:*

(2.1.1). *X has an N-point compactification.*

(2.1.2). *X is locally compact and contains a compact subset K whose complement is the union of N mutually disjoint, open subsets $\{G_i\}_{i=1}^N$ such that $K\cup G_i$ is not compact for each i.*

(2.1.3). *X is locally compact and contains a compact subset K whose complement is the union of N mutually disjoint, open subsets $\{G_i\}_{i=1}^N$ such that $K\cup G_i$ is contained in no compact subset for each i.*

(2.1.4). *X is locally compact and contains N mutually disjoint, open subsets $\{G_i\}_{i=1}^N$ such that $\mathcal{C}[G_1 \cup G_2 \cup \cdots \cup G_N]$ is compact while for each i,*

$$\mathcal{C}[G_1 \cup \cdots \cup G_{i-1} \cup G_{i+1} \cup \cdots \cup G_N]$$

(which equals X if $N=1$) is not compact.

Proof. (2.1.1)$\Rightarrow$(2.1.2). Denote the points in $\zeta_N(X)-X$ by $\infty_1, \infty_2, \cdots, \infty_N$. Since $\zeta_N(X)$ is Hausdorff, there exists a mutually disjoint family $\{G_i'\}_{i=1}^N$ of open subsets of $\zeta_N(X)$ where $\infty_i \in G_i'$ for each i. Let $G_i = G_i' - \{\infty_i\}$. Since X is dense in $\zeta_N(X)$, each G_i is a nonempty open subset of X. Then

$$K = \mathcal{C}_X[G_1 \cup G_2 \cup \cdots \cup G_N] = \mathcal{C}_{\zeta_N(X)}[G_1' \cup G_2' \cup \cdots \cup G_N']$$

is a closed subset of $\zeta_N(X)$ and is therefore compact. Now

$$\begin{aligned} K \cup G_i &= \mathcal{C}_X[G_1 \cup \cdots \cup G_{i-1} \cup G_{i+1} \cup \cdots \cup G_N] \\ &= [\mathcal{C}_{\zeta_N(X)}[G_1' \cup \cdots \cup G_{i-1}' \cup G_{i+1}' \cup \cdots \cup G_N']] \cap [\mathcal{C}_{\zeta_N(X)}\{\infty_i\}]. \end{aligned}$$

So the assumption that $K \cup G_i$ is compact implies it is a closed subset of $\zeta_N(X)$ and hence that

$$\{\infty_i\} \cup [G_1' \cup \cdots \cup G_{i-1}' \cup G_{i+1}' \cup \cdots \cup G_N']$$

is an open subset of $\zeta_N(X)$. The intersection of this latter set with G_i' is $\{\infty_i\}$ which results in the contradiction that ∞_i is an isolated point of $\zeta_N(X)$. Therefore, $K \cup G_i$ is not compact. Finally, $X = \zeta_N(X) - \{\infty_i\}_{i=1}^N$ and is therefore an open subset of $\zeta_N(X)$. This implies X is locally compact.

(2.1.2)$\Rightarrow$(2.1.3). We need only show that if $K \cup G_i$ is not compact, then it is contained in no compact subset. But this is a consequence of the fact that

$$K \cup G_i = \mathcal{C}_X[G_1 \cup \cdots \cup G_{i-1} \cup G_{i+1} \cup \cdots \cup G_N]$$

and is therefore a closed subset of X.

(2.1.3)$\Rightarrow$(2.1.4). This also follows from the fact that

$$K \cup G_i = \mathcal{C}_X[G_1 \cup \cdots \cup G_{i-1} \cup G_{i+1} \cup \cdots \cup G_N].$$

(2.1.4)$\Rightarrow$(2.1.1). Denote $\mathcal{C}_X[G_1 \cup G_2 \cup \cdots \cup G_N]$ by K. Let $\{\infty_i\}_{i=1}^N$ denote N distinct elements not in the space X. For any subset H of X, we let H^i denote the set $H \cup \{\infty_i\}$. If H is an open subset of X and $[K \cup G_i] \cap \mathcal{C}_X H$ is compact, we say H has property P_i. Now, if H_1 and H_2 have property P_i, then

$$[K \cup G_i] \cap \mathcal{C}_X[H_1 \cap H_2] = [[K \cup G_i] \cap \mathcal{C}_X H_1] \cup [[K \cup G_i] \cap \mathcal{C}_X H_2]$$

is compact and thus $H_1 \cap H_2$ has property P_i. Therefore, the collection of all sets of the form H^i, where H has property P_i, is a neighborhood basis for ∞_i in $\zeta_N(X) = X \cup \{\infty_i\}_{i=1}^N$. We want to show that the topology on $\zeta_N(X)$ for which these sets, along with the open subsets of X, form a basis is Hausdorff and compact. Since

$$[K \cup G_i] \cap \mathcal{C}_X G_i = K \quad \text{for each } i,$$

G_i^j and G_j^i are disjoint open subsets of $\zeta_N(X)$ containing ∞_i and ∞_j respectively. Now let $x \in X$ and some ∞_i be given. Since X is locally compact, there is an open subset G^* and a compact subset K^* such that $x \in G^* \subset K^*$. Then $[K \cup G_i] \cap K^*$, being a closed subset of K^*, is compact. Therefore $[\mathcal{C}_X K^*]^i$ is an open set containing ∞_i which is disjoint from G^*. The remaining case follows from the fact that X is Hausdorff.

Now let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $\zeta_N(X)$. Then N of these sets $(U_1, U_2, \cdots, U_N)$ contain, respectively, sets of the form $H_1^1, H_2^2, \cdots, H_N^N$ where for each i, H_i is an open subset of X and $[K \cup G_i] \cap \mathcal{C}_X H_i$ is compact. Since $G_i \cap G_j = \emptyset$ for $i \neq j$,

$$\bigcap_{i=1}^{N} [G_1 \cup \cdots \cup G_{i-1} \cup H_i \cup G_{i+1} \cup \cdots \cup G_N] \subset [H_1 \cup H_2 \cup \cdots \cup H_N].$$

This implies

$$\mathcal{C}_X[H_1 \cup H_2 \cup \cdots \cup H_N] \subset \bigcup_{i=1}^{N} \mathcal{C}_X[G_1 \cup \cdots \cup G_{i-1} \cup H_i \cup G_{i+1} \cup \cdots \cup G_N]$$

and we get

$$\mathcal{C}_{\zeta_N(X)}[U_1 \cup U_2 \cup \cdots \cup U_N] \subset \mathcal{C}_X[H_1 \cup H_2 \cup \cdots \cup H_N]$$

$$\subset \bigcup_{i=1}^{N} \mathcal{C}_X[G_1 \cup \cdots \cup G_{i-1} \cup H_i \cup G_{i+1} \cup \cdots \cup G_N] = \bigcup_{i=1}^{N} [[K \cup G_i] \cap \mathcal{C}_X H_i]$$

which is compact. Thus $\mathcal{C}_{\zeta_N(X)}[U_1 \cup U_2 \cup \cdots \cup U_N]$ is a compact subset of $\zeta_N(X)$ and is covered by a finite subfamily of $\mathcal{U}$. Therefore $\zeta_N(X)$ is compact.

It remains to show that X is a dense subset of $\zeta_N(X)$. Let H^i be a basic open set containing ∞_i. Since $[K \cup G_i] \cap \mathcal{C}_X H$ is compact and $K \cup G_i$ is not, we must conclude $\mathcal{C}_X H \neq X$. Thus $H^i \cap X = H \neq \emptyset$.

DEFINITION (2.2). *We shall refer to a mutually disjoint, open family* $\mathcal{G} = \{G_i\}_{i=1}^N$ *of N subsets of X with the properties*

(2.2.1). $\mathcal{C}[G_1 \cup G_2 \cup \cdots \cup G_N]$ *is compact,*

(2.2.2). $\mathcal{C}[G_1 \cup \cdots \cup G_{i-1} \cup G_{i+1} \cup \cdots \cup G_N]$ *is not compact for each i,*

as an N-star of X. (The symbol $K_\mathcal{G}$ will be used to denote $\mathcal{C}[G_1 \cup \cdots \cup G_N]$.)

In proving that (2.1.4)⇒(2.1.1), we showed that each N-star of a locally compact space X gives rise to an N-point compactification of X. We shall refer to this as the compactification induced by the N-star. Now, let $\mathcal{S}_N(X)$ be the collection of N-stars of X. Define a relation $\mathcal{R}$ on $\mathcal{S}_N(X)$ by putting $\mathcal{G}\mathcal{R}\mathcal{H}$ if the elements $\{G_i\}_{i=1}^N$ and $\{H_i\}_{i=1}^N$ of $\mathcal{G}$ and $\mathcal{H}$ respectively can be ordered in such a

manner that $[K_{\mathcal{H}} \cup H_i] \cap \mathcal{C}G_i$ is compact for each i. Concerning this relation, we have

THEOREM (2.3). *If X is a locally compact space, then $\mathcal{R}$ is an equivalence relation on $\mathcal{S}_N(X)$ and there is a one-to-one correspondence between the equivalence classes of $\mathcal{S}_N(X)$ and the different N-point compactifications of X.*

Proof. Let $\mathcal{G}$ and $\mathcal{H}$ be elements of $\mathcal{S}_N(X)$, where $\mathcal{G} = \{G_i\}_{i=1}^N$ and $\mathcal{H} = \{H_i\}_{i=1}^N$. Let $\zeta_N(X)$ and $\eta_N(X)$ be the compactifications induced by $\mathcal{G}$ and $\mathcal{H}$ respectively (see the proof that (2.1.4)$\Rightarrow$(2.1.1)). Furthermore, denote the points of $\zeta_N(X) - X$ by $\{\infty_i\}_{i=1}^N$ and those of $\eta_N(X) - X$ by $\{\Delta_i\}$. Now, if $\mathcal{G}\mathcal{R}\mathcal{H}$, then for some ordering of the elements of $\mathcal{G}$ and $\mathcal{H}$, $[K_{\mathcal{H}} \cup H_i] \cap \mathcal{C}G_i$ is compact for each i. Hence, if U is any open subset of X such that $[K_{\mathcal{G}} \cup G_i] \cap \mathcal{C}U$ is compact, then $[K_{\mathcal{H}} \cup H_i] \cap \mathcal{C}U$ is compact since it is a closed subset of

$$[[K_{\mathcal{H}} \cup H_i] \cap \mathcal{C}G_i] \cup [[K_{\mathcal{G}} \cup G_i] \cap \mathcal{C}U].$$

This implies that if V is any open subset of $\zeta_N(X)$ which contains ∞_i, then $[V - \{\infty_i\}] \cup \{\Delta_i\}$ is an open subset of $\eta_N(X)$ which contains Δ_i. Therefore, the mapping h from $\eta_N(X)$ onto $\zeta_N(X)$ defined by $h(\Delta_i) = \infty_i$ and $h(x) = x$ for $x \in X$ is continuous. In fact, since h is a one-to-one continuous mapping of a compact space onto a Hausdorff space, h is a homeomorphism. Thus, we see that if $\mathcal{G}\mathcal{R}\mathcal{H}$, then the compactifications induced by $\mathcal{G}$ and $\mathcal{H}$ are equivalent.

We note that the converse also holds. That is, if the compactifications induced by $\mathcal{G}$ and $\mathcal{H}$ are equivalent, then there is an ordering (determined by the homeomorphism) for the elements of $\mathcal{G}$ and $\mathcal{H}$ such that $[K_{\mathcal{H}} \cup H_i] \cap \mathcal{C}G_i$ is compact for each i (and, of course, $[K_{\mathcal{G}} \cup G_i] \cap \mathcal{C}H_i$ is compact for each i). Summarizing, we see that to each N-star there corresponds an N-point compactification; to each compactification, there corresponds an N-star; and for two N-stars $\mathcal{G}$ and $\mathcal{H}$, $\mathcal{G}\mathcal{R}\mathcal{H}$ if and only if the corresponding compactifications are equivalent. It follows that $\mathcal{R}$ is an equivalence relation on $\mathcal{S}_N(X)$ and the resulting equivalence classes are in one-to-one correspondence with the different N-point compactifications of X.

From the previous two theorems, we get

THEOREM (2.4). *X has exactly one N-point compactification if and only if X is locally compact, has an N-star, and all other N-stars are equivalent to it.*

Using the facts that a space has a 1-star if and only if it is not compact, and any two 1-stars are equivalent, we obtain the well-known results that a space has a 1-point compactification if and only if it is locally compact and not compact and no space has more than one 1-point compactification.

From Theorem (2.1), we get

COROLLARY (2.5). *If X has an N-point compactification, then it has an M-point compactification for every positive integer $M < N$.*

THEOREM (2.6). *If X has the property that every compact subset of X is contained in a compact subset whose complement has at most N components (maximal connected sets), then X has no M-point compactification for $M > N$.*

Proof. Suppose X has an M-star $\{G_i\}_{i=1}^{M}(M > N)$. Let K^* be a compact subset containing $\mathcal{C}[G_1 \cup G_2 \cup \cdots \cup G_M] = K$ such that $\mathcal{C}K^*$ has r components H_1, $H_2, \cdots, H_r$ where $r \leqq N$. Then

$$[H_1 \cup H_2 \cup \cdots \cup H_r] \subset [G_1 \cup G_2 \cup \cdots \cup G_M].$$

Since $r < M$ and H_i is connected for each i, $G_j \cap [H_1 \cup H_2 \cup \cdots \cup H_r] = \emptyset$ for some j. Then $G_j \subset K^*$ and it follows that $[K \cup G_j] \subset K^*$ which contradicts (2.2.2). Hence, X has no M-point compactification for $M > N$.

Let us recall that a mapping is said to be compact if pre-images of compact sets are compact.

THEOREM (2.7). *If Y has an N-point compactification and is the image of X under a compact, continuous mapping, then X also has an N-point compactification.*

Proof. If Y has an N-point compactification, it is locally compact and has an N-star, $\{G_i\}_{i=1}^{N}$. It follows that if f is a compact, continuous mapping from X onto Y, then X is also locally compact and $\{f^{-1}[G_i]\}_{i=1}^{N}$ is an N-star of X.

LEMMA (2.8). *Let (X, d_1) and (Y, d_2) be two unbounded, connected metric spaces and denote their product by $X \times Y$. Let (x_0, y_0) be a point in $X \times Y$, let k be any positive number and put $K = \{(x, y) \in X \times Y : d_1(x, x_0) \leqq k$ and $d_2(y, y_0) \leqq k\}$. Then $\mathcal{C}K$ is a connected set.*

Proof. Suppose (x_1, y_1) and (x_2, y_2) are points of $\mathcal{C}K$. Then either $d_1(x_1, x_0) > k$ or $d_2(y_1, y_0) > k$ and either $d_1(x_2, x_0) > k$ or $d_2(y_2, y_0) > k$. We shall suppose $d_1(x_1, x_0) > k$ and $d_1(x_2, x_0) > k$. The arguments for the remaining cases are similar. Let $A_1 = \{(x_1, y) : y \in Y\}$. Since Y is an unbounded metric space, there exists a point $y_3 \in Y$ such that $d_2(y_3, y_0) > k$. Let $A_2 = \{(x, y_3) : x \in X\}$ and let $A_3 = \{(x_2, y) : y \in Y\}$. Let us note that A_1 and A_3 are homeomorphic to Y, and A_2 is homeomorphic to X. Hence, A_1, A_2, and A_3 are connected. Furthermore, since $(x_1, y_3) \in A_1 \cap A_2$ and $(x_2, y_3) \in A_2 \cap A_3$, $A_1 \cup A_2 \cup A_3$ is connected. Finally, $K \cap [A_1 \cup A_2 \cup A_3] = \emptyset$, and we see that each pair of points of $\mathcal{C}K$ is contained in a connected subset of $\mathcal{C}K$. Thus $\mathcal{C}K$ is connected.

THEOREM (2.9). *Let (X, d_1) and (Y, d_2) be two unbounded, connected metric spaces and suppose for every real number r and points $x_0 \in X$ and $y_0 \in Y$, the sets $\{x \in X : d_1(x, x_0) \leqq r\}$ and $\{y \in Y : d_2(y, y_0) \leqq r\}$ are compact. Then $X \times Y$ has no N-point compactification for $N > 1$.*

Proof. Let $K \neq \emptyset$ be a compact subset of $X \times Y$. Let P_X and P_Y denote the projection mappings into X and Y respectively. Then $P_X[K]$ and $P_Y[K]$ are compact subsets of X and Y respectively. Choose $x_0 \in P_X[K]$ and $y_0 \in P_Y[K]$.

Since compact subsets of metric spaces are bounded, there exists a positive number r such that $P_X[K] \subset \{x \in X: d_1(x, x_0) \leqq r\} = H_X$ and $P_Y[K] \subset \{y \in Y: d_2(y, y_0) \leqq r\} = H_Y$. By hypothesis, H_X and H_Y are compact, so $H_X \times H_Y$ is compact. Moreover, $K \subset H_X \times H_Y$ and by Lemma (2.8), $\mathcal{C}[H_X \times H_Y]$ has precisely one component. It now follows from Theorem (2.6) that $X \times Y$ has no N-point compactification for $N > 1$.

3. Discussions of some special spaces.

(3.1) *The Complex Plane.* It follows immediately from Theorem (2.9) that the Complex Plane has no N-point compactification other than the well-known 1-point compactification. More generally, Theorem (2.9) implies that the only M-point compactification of E^N ($N > 1$), the Euclidean N-space, is the 1-point compactification.

(3.2) *The Space R of Real Numbers.* Since every compact subset of R is contained in a bounded, closed interval, it follows from Theorem (2.6) that R has no N-point compactifications for $N > 2$. As we noted previously, two 1-point compactifications of any space are equivalent. For R, the same is true for any two 2-point compactifications. To see this, let $\mathcal{G} = \{G_1, G_2\}$ and $\mathcal{H} = \{H_1, H_2\}$ be two 2-stars. Then some bounded, closed interval $[a, b]$ contains $\mathcal{C}[G_1 \cup G_2]$ and $\mathcal{C}[H_1 \cup H_2]$. Therefore, $(-\infty, a)$ is contained in one of G_1, G_2 and one of H_1, H_2. Suppose, then, $(-\infty, a) \subset G_1$ and $(-\infty, a) \subset H_1$. Since $\mathcal{C}G_1$ and $\mathcal{C}H_1$ are not compact, we must have $(b, +\infty) \subset G_2$ and $(b, +\infty) \subset H_2$. Then $[K_{\mathcal{G}} \cup G_1] \cap \mathcal{C}H_1 = \mathcal{C}G_2 \cap \mathcal{C}H_1 = \mathcal{C}[H_1 \cup G_2] \subset [a, b]$. Similarly, $[K_{\mathcal{G}} \cup G_2] \cap \mathcal{C}H_2 = \mathcal{C}[G_1 \cup H_2] \subset [a, b]$. Therefore $\mathcal{G}$ and $\mathcal{H}$ are equivalent and the assertion now follows from Theorem (2.4).

We summarize our observations of R by stating that R has precisely one 1-point compactification, precisely one 2-point compactification and no N-point compactifications for $N > 2$.

(3.3) *The "S" Spaces.* This class of spaces is considered chiefly because it provides an answer to the question, "Given a positive integer N, does there exist a space which has an N-point compactification but no M-point compactification for $M > N$?" Let n be a positive integer and define

$$L_n = \left\{(x, y) \in R \times R: y = \frac{1}{n} x \text{ and } x \geqq 0\right\},$$

where R again denotes the space of real numbers. Thus L_n is a ray of the Euclidean plane which emanates from the origin. Define $S_N = \bigcup_{n=1}^{N} L_n$. If we let $K = \{(0, 0)\}$, we note that

$$\mathcal{C}_{S_N} K = \bigcup_{n=1}^{N} L_n', \quad \text{where} \quad L_n' = L_n - \{(0, 0)\}.$$

Furthermore, the family $\{L_n'\}_{n=1}^{N}$ is mutually disjoint and for each n, $K \cup L_n'$ is not compact. Therefore, by Theorem (2.1), S_N has an N-point compactification.

Now if K is any compact subset of S_N, there exists a positive number r such that

$$K \subset \{(x, y) \in S_N: \sqrt{(x^2 + y^2)} \leqq r\} = K^*.$$

K^* is compact and $\mathcal{C}_{S_N}K^*$ has precisely N components. Therefore, by Theorem (2.6), S_N has no M-point compactification for $M > N$.

It is well known that the product of two spaces having a 1-point compactification also has a 1-point compactification. This does not, in general, hold true for M-point compactifications, for it follows from Theorem (2.9) that $S_N \times S_N$ has no M-point compactification for $M > 1$.

(3.4) *Infinite Discrete Spaces.* Let X be an infinite discrete space and N a positive integer larger than 1. Let $\{G_{1j}\}_{j=1}^{N}$ be a mutually disjoint collection of infinite subsets whose union is all of X. Now, for each j where $1 \leqq j < N$, let $\{G_{ij}\}_{i=1}^{\infty}$ be a sequence of sets such that

(3.4.1). $G_{i+1,j} \subset G_{ij}$ for $i = 1, 2, \cdots$, and

(3.4.2). $G_{ij} - G_{i+1,j}$ contains infinitely many elements for $i = 1, 2, \cdots$.

Finally, for each $i > 1$, let $G_{iN} = \mathcal{C}[G_{i1} \cup G_{i2} \cup \cdots \cup G_{i,N-1}]$, and denote the family $\{G_{ij}\}_{j=1}^{N}$ by $\mathcal{G}_i$. Each $\mathcal{G}_i$ consists of N mutually disjoint subsets each containing an infinite number of elements. Moreover, $\bigcup_{j=1}^{N} G_{ij} = X$ for each i. Thus each $\mathcal{G}_i$ is an N-star. We want to show that if $i_1 \neq i_2$, then $\mathcal{G}_{i_1}$ is not equivalent to $\mathcal{G}_{i_2}$. It will be sufficient to show that for $i_1 < i_2$, and $j_1 \neq N$, then

$$\mathcal{C}[G_{i_1,1} \cup G_{i_1,2} \cup \cdots \cup G_{i_1,j_1-1} \cup G_{i_1,j_1+1} \cup \cdots \cup G_{iN}] \cap \mathcal{C}G_{i_2j_2}$$

is not compact, i.e., contains an infinite number of elements. But this set is equal to $G_{i_1j_1} \cap \mathcal{C}G_{i_2j_2}$ which, in the event $j_1 = j_2$, is infinite by (3.4.2). If $j_1 \neq j_2$,

$$G_{i_1j_1} \cap \mathcal{C}G_{i_2j_2} \supset G_{i_2j_1} \cap \mathcal{C}G_{i_2j_2} = G_{i_2j_1}$$

which is infinite. It now follows from Theorem (2.3) that X has infinitely many N-point compactifications for each $N > 1$.

We make one final remark. The latter procedure can be used to yield a more general result. If X is the union of an infinite number of mutually disjoint, open sets, a suitable modification (let each G_{ij} be the union of an infinite number of these sets) of the previous argument results in

(3.4.3). *If X is locally compact and the union of an infinite number of mutually disjoint, open sets, then for each $N > 1$, X has infinitely many N-point compactifications.*

Reference

1. J L. Kelley, General Topology, Van Nostrand, New York, 1955.

THE HOUSE PROBLEM

RAPHAEL FINKELSTEIN, University of Arizona

1. Introduction. A very well-known problem, found in many books of puzzles (see, e.g., [1]) is the following:

"When a professor went to make a social call on one of his former students, he got as far as the street on which the student's house was located before he realized he had forgotten the street-number of the house.

The professor knew his friend's house was in the numerical center of his side of the road. That is, the numbers from his house to one end of the street totalled the numbers from his house to the other end. The professor could see that the houses were numbered consecutively (1, 2, 3, etc.), starting on the right hand side, then turning back along the left hand. It was obvious that there were more than 200 but less than 500 houses on the right, and the professor knew the student's house was on that side. From this, he worked out the right house. Can you?"

In this article we shall solve this problem, and shall conduct an investigation of a more generalized version of it. We begin with the following:

DEFINITION. *Given the positive integer n, we shall call the positive integer N, ($N \leqq n$), a k-th-power numerical center for n in case the sum of the k-th powers of the integers from* 1 *to N equals the sum of the k-th powers of the integers from N to n.*

It is obvious that for any k, 1 is trivially a kth-power numerical center for $n=1$. Therefore, we ask the following question: For any given k, which positive integers $n>1$ possess kth-power numerical centers N? In this article we shall show that infinitely many integers n possess first-power numerical centers, and that the only positive integer n possessing a second-power numerical center is 1.

We now state without proof the following theorems of number theory, which we shall use in parts 2 and 3.

THEOREM A. *Pell's equation, $x^2-Dy^2=1$, where $D>0$ and D is not a square, has infinitely many solutions in positive integers $[x, y]$. If $[x_1, y_1]$ is the smallest such solution, then any positive integer solution $[x_q, y_q]$ is given by*

$$x_q + y_q\sqrt{D} = (x_1 + y_1\sqrt{D})^q,$$

where q is a positive integer.

THEOREM B (HEMER). *Let $K(\rho)$ be a cubic field over the field of rational numbers, and let $\alpha=A\rho^2+B\rho+C$ be an integer in the ring* $(1, \rho, \rho^2)$. *Suppose* $A\equiv B \equiv 0 \pmod{p^k}$, *where p is an odd rational prime, and* $(\alpha, p)=1$. *Further, suppose that* $tA+sB\not\equiv 0 \pmod{p^{2k}}$, *where t, s, and k are rational integers, and $k>0$. Then if $\alpha^n=A_n\rho^2+B_n\rho+C_n$, tA_n+sB_n is never zero for any $n\neq 0$.*

The proof of Theorem B may be found on page 97 of [3].

THEOREM C (NAGELL [4]). *The Diophantine equation* $Ax^3+By^3=C$ ($C=1$ *or* 3; $3\nmid AB$ *if* $C=3$; $A>B$; A, B, *positive integers*) *has at most one solution in nonzero integers* $[x, y]$. *There is the unique exception for the equation* $2x^3+y^3=3$, *which has exactly the two integral solutions* $[x, y]=[1, 1]$ *and* $[4, -5]$.

2. Solution for $k=1$. In this section we shall prove the following

THEOREM 1. *There are infinitely many positive integers* n *possessing first-power numerical centers* N. *If we set* $T=2n+1$, *then any such* T *and* N *may be obtained from the solution of the equation* $T^2-8N^2=1$.

Proof. Let N be any first-power numerical center for the positive integer n. Since

$$\begin{cases} \displaystyle\sum_{i=1}^{N} i = \frac{(N)(N+1)}{2}, \\ \displaystyle\sum_{i=N}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{N-1} i = \frac{(n+N)(n-N+1)}{2}, \end{cases}$$

the condition that N be a first-power numerical center for n requires

$$\frac{N(N+1)}{2} = \frac{(n+N)(n-N+1)}{2}. \tag{1}$$

On letting $T=2n+1$, we reduce (1) to

$$T^2 - 8N^2 = 1. \tag{2}$$

Now (2) is a Pell equation, and therefore it has infinitely many positive integral solutions. Clearly the smallest solution of (2) in positive integers is

$$[T_1, N_1] = [3, 1].$$

Hence, any positive integral solution $[T_q, N_q]$ is given by $T_q + N_q\sqrt{8} = (3+\sqrt{8})^q$, where q is a positive integer.

By actual computation, we deduce

$$[T_2, N_2] = [17, 6],\ n_2 = 8; \quad [T_3, N_3] = [99, 35],\ n_3 = 49;$$

$$[T_4, N_4] = [577, 204],\ n_4 = 288; \quad [T_5, N_5] = [3363, 1189],\ n_5 = 1681, \text{ etc.}$$

Thus we see that infinitely many positive integers n possess first power numerical centers, and that the required solution for $200<n<500$ is given by $N=204$, $n=288$, e.g.

$$\sum_{i=1}^{204} i = \sum_{i=204}^{288} i = 20{,}910.$$

It is also clear that the above argument is reversible, i.e., any N found in the above manner is a first power numerical center for the corresponding n.

3. Solution for $k=2$. In this part of this paper we shall prove the following

THEOREM 2. *The only positive integer n possessing a second-power numerical center is* 1.

Proof. Let N be any second-power numerical center for the positive integer n. Since the sum of the squares of the first N positive integers is given by

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6},$$

the condition that N be a second-power numerical center for n requires

$$\sum_{i=1}^{N} i^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{N-1} i^2,$$

i.e.,

$$\frac{N(N+1)(2N+1)}{6} = \frac{n(n+1)(2n+1) - N(N-1)(2N-1)}{6}. \tag{3}$$

On simplifying (3), and setting $T=2n+1$, $S=-2N$, we obtain

$$-2S^3 - 4S = T^3 - T. \tag{4}$$

We shall now prove several lemmas which together imply that the only integral solution of (4) subject to the conditions

$$T \text{ positive and odd, } S \text{ negative and even,} \tag{5}$$

is $[T, S]=[3, -2]$. From this, Theorem 2 will follow immediately.

LEMMA 1. *All the integral solutions of* (4), *subject to conditions* (5), *correspond to the integral solutions of the equations*

$$a^3 + 2b^3 = 11 \text{ and } 33. \tag{6}$$

Proof. Let $[T, S]$ be any integral solution of (4), subject to the conditions (5). We put $(T, S)=d$, where d is the greatest common divisor of T and S. Then, on substituting $T=ad$, $S=bd$ in (4) and simplifying, we have

$$d^2(a^3 + 2b^3) = -4b + a, \tag{7}$$

where

$$(a, b) = 1, \ a \text{ is positive and odd, } b \text{ is negative and even, and } d \text{ is odd.} \tag{8}$$

We set

$$a^3 + 2b^3 = c. \tag{9}$$

Then $(a, c)=1$, and $-4b+a=cd^2$, i.e.,

$$a = cd^2 + 4b. \tag{10}$$

Now we substitute (10) for a in (9) and expand, obtaining

$$c = c^3d^6 + 12c^2d^4b + 48cd^2b^2 + 66b^3. \tag{11}$$

But from (9),

$$66b^3 = 33c - 33a^3, \tag{12}$$

so we can substitute (12) in (11) to get $33a^3=c^3d^6+12c^2d^4+48cd^2b^2+32c$. Hence, $c|33a^3$, and thus $c|33$, because $(a^3, c)=1$. Also $c>0$, by (8) and (10). Thus $c=1$, 3, 11, or 33, and solving (4) in integers, subject to conditions (5), is equivalent to solving the set of equations

$$a^3 + 2b^3 = 1,\ 3,\ 11, \text{ or } 33,$$

subject to conditions (8). By Theorem C, however, the only nonzero integral solutions of the equations $a^3+2b^3=1$ or 3 are given by $[a, b]=[-1, 1]$, $[1, 1]$ and $[-5, 4]$, and none of these solutions satisfies the conditions (8), whence Lemma 1 follows.

LEMMA 2. *The only integral solution of the equation* $a^3+2b^3=11$ *is* $[a, b]=[3, -2]$.

Proof. The method to be used requires a discussion of the cubic field $K(\theta)$, where $\theta^3=2$. The necessary information for our problem is provided in articles by Nagell [4] and Dedekind [2], and is as follows:

(i) The integers of $K(\theta)$ take the form $\alpha=A+B\theta+C\theta^2$, where A, B and C are rational integers; i.e., $(1, \theta, \theta^2)$ is an integral basis for $K(\theta)$.

(ii) The ring of integers of $K(\theta)$ is a unique factorization domain.

(iii) By Dirichlet's theorem on units, there is only one fundamental unit of the field, which we designate by ϵ_0. From the table on page 120 of [2], the fundamental unit, ϵ_0 of $K(\theta)$, with $0<\epsilon_0<1$, is given by

$$\epsilon_0 = -1 + \theta.$$

All the units of the field are given by $\pm\epsilon_0^m$, where m is any rational integer. Any such power of ϵ_0 is of the form $u+v\theta+w\theta^2$, where u, v and w are rational integers. The product uvw is zero only when $m=0$ and $m=1$. All units of norm 1 of $K(\theta)$ are given by ϵ_0^m.

(iv) Since 11 is a rational prime of the form $3r+2$, the integer 11 splits into two primes in $K(\theta)$, one of the first degree and one of the second degree, because 11 does not divide the discriminant of the field, -108. We have, in fact,

$$11 = (3 - 2\theta)(9 + 6\theta + 4\theta^2), \tag{13}$$

and it is readily verified that each integer on the right-hand side of (13) is a prime of $K(\theta)$. Since the norm of $\alpha=A+B\theta+C\theta^2$ is given by

$$N(\alpha) = A^3 + 2B^3 + 4C^3 - 6ABC,$$

the first of the factors of 11 shown in (13) has norm 11, the second norm 121. Hence, there is only one equivalence class of associated primes of norm 11 in $K(\theta)$, as any integer in $K(\theta)$ which has norm 11 must divide 11, and, apart from unit factors, there is only one such integer.

We seek all the integers of $K(\theta)$ of norm 11 which are of the form $a+b\theta$. Since all primes of norm 11 in $K(\theta)$ are associated, any such prime must be an associate of $3-2\theta$. Let $\epsilon_0^m = u_m + v_m\theta + w_m\theta^2$ be a unit of $K(\theta)$ of norm 1. We require the coefficient of θ^2 in

$$(3 - 2\theta)(u_m + v_m\theta + w_m\theta^2)$$

to be zero. This yields

$$3w_m - 2v_m = 0. \tag{14}$$

We shall show that (14) is impossible for $m \neq 0$, which will imply Lemma 2. We now set

$$x_m + y_m\theta + z_m\theta^2 = (3 - 2\theta)(\epsilon_0^m). \tag{15}$$

Since

$$\epsilon_0^3 + 3\epsilon_0^2 + 3\epsilon_0 - 1 = 0, \tag{16}$$

i.e., $\epsilon_0^{m+3} + 3\epsilon_0^{m+2} + 3\epsilon_0^{m+1} - \epsilon_0^m = 0$, and $(1, \theta, \theta^2)$ is an integral basis for $K(\theta)$, it follows at once that

$$z_{m+3} + 3(z_{m+2} + z_{m+1}) - z_m = 0. \tag{17}$$

From (15) we have $x_m + y_m\theta + z_m\theta^2 = (3-2\theta)(u_m + v_m\theta + w_m\theta^2)$, and thus $z_m = 3w_m - 2v_m = 0$ by equation (14). Also (17) gives

$$z_{m+3} \equiv z_m \pmod 3. \tag{18}$$

Since $w_0 = v_0 = 0$, $z_0 \equiv 0 \pmod 3$.

Furthermore, since $w_1 = 0$, $v_1 = 1$, and $w_2 = 1$, $v_2 = -2$, we have

$$z_1 \equiv z_2 \equiv 1 \pmod 3,$$

which, by (18), implies that z_m can be zero only when $m \equiv 0 \pmod 3$. However, an easy calculation shows that

$$v_3 \equiv w_3 \equiv 0 \pmod 3, \quad \text{and} \quad 3w_3 - 2v_3 \equiv 3 \pmod 9.$$

Hence, by Theorem B, z_m is never zero for any $m \neq 0$, and Lemma 2 follows immediately.

LEMMA 3. *The equation $a^3 + 2b^3 = 33$ is impossible in rational integers* $[a, b]$.

Proof. First, we must find the number of equivalence classes of associated primes of norm 3 in $K(\theta)$. Since $x^3 - 2 \equiv (x-2)^3 \pmod 3$, the general theory of

algebraic numbers shows that 3 is a perfect cube in $K(\theta)$, apart from unit factors. We have, in fact,

$$3 = (\theta + 1)^3(\theta - 1),$$

and 3 is the cube of a prime of norm 3 times a unit factor, since $N(\theta - 1) = 1$. Hence, there is only one equivalence class of associated primes of norm 3 in $K(\theta)$, as any integer of norm 3 in $K(\theta)$ must divide 3, and, apart from unit factors, there is only one such integer.

We consider now the integer $1+2\theta^2$. Its norm is 33, and any other integer of norm 33 in $K(\theta)$ must be of the form $(1+2\theta^2)(\epsilon_0^m)$, as all primes of norm 11 in $K(\theta)$ are associated, as are all primes of norm 3.

As in the proof of Lemma 2, we put $\epsilon_0^m = u_m + v_m\theta + w_m\theta^2$, and

$$X_m + Y_m\theta + Z_m\theta^2 = (1 + 2\theta^2)(u_m + v_m\theta + w_m\theta^2).$$

Hence

$$Z_m = 2u_m + w_m. \tag{19}$$

Also, (16) yields

$$Z_{m+3} + 3(Z_{m+2} + Z_{m+1}) - Z_m = 0. \tag{20}$$

We seek all integers of $K(\theta)$ with norm 33 of the form $a+b\theta$. Hence, Z_m must be zero, and thus the congruence $Z_m \equiv 0 \pmod t$ must be solvable for every modulus t. We shall prove Lemma 3 by showing that the congruence

$$Z_m \equiv 0 \pmod{31} \tag{21}$$

is impossible.

To this end, we note that $\epsilon_0^0 \equiv 1 \pmod{31}$, and, after some calculation, we find

$$\epsilon_0^{10} = 521 - 62\theta - 279\theta^2, \quad \text{i.e.,} \quad \epsilon_0^{10} \equiv -6 \pmod{31}.$$

This implies

$$\epsilon_0^{20} \equiv 5 \pmod{31}, \quad \text{and} \quad \epsilon_0^{30} \equiv 1 \pmod{31}.$$

Hence, the Z_m must satisfy the conditions

$$Z_{m+10} \equiv -6Z_m \pmod{31}, \quad Z_{m+20} \equiv 5Z_m \pmod{31}, \quad \text{and} \quad Z_{m+30} \equiv Z_m \pmod{31}.$$

Thus to prove the impossibility of (21), it is sufficient to show that

$$Z_i \ (i = 0, 1, 2, \cdots, 9), \not\equiv 0 \pmod{31}.$$

By actual computation, using (19) and (20), we find the values of Z_0 to Z_9, modulo 31, to be 2, -2, 3, -1, -8, -1, -5, 10, 15 and 13, and none of these is zero. This proves Lemma 3.

Note: One may also prove Lemma 3 in exactly the same manner as above, by showing the impossibility of the congruence $Z_m \equiv 0 \pmod{11}$.

LEMMA 4. *For integral solutions* $[T, S]$ *of equation* (4), *subject to the conditions* (5), $d=1$.

Proof. Lemmas 2 and 3 show that the only integral solution of equations (6) is $[a, b]=[3, -2]$, and this solution clearly fulfills all the conditions (8). Hence, from (7), we get

$$11d^2 = 11, \quad \text{i.e.,} \quad d = 1.$$

Thus, the only integral solution of equation (4), subject to conditions (5), is $[T, S]=[3, -2]$, and therefore the only positive integral solution of (3) is $[N, n]=[1, 1]$. The proof is complete.

4. A summary, a conjecture, and some concluding remarks. We have shown that there are infinitely many positive integers n which possess first-power numerical centers and that the only positive integer possessing a second-power numerical center is 1. The results of part 3 suggest the following

CONJECTURE. *If* $k>1$, 1 *is the only positive integer which possesses a k-th-power numerical center.*

In a future article, we shall prove the validity of this conjecture for $k=3$, and shall conduct a partial investigation of it for $k=4$ and 5.

References

1. J. P. Adams, Puzzles for Everybody, Avon Publications, New York, 1955, p. 27.

2. R. Dedekind, Über die Anzahl der Idealklassen in reinen kubischen Zahlkörpern, Journ. für Math., 121 (1900) 40-123.

3. O. Hemer, On Some Diophantine Equations of the Type $y^2-f^2=x^3$, Math Scand., 4 (1956) 95–107.

4. T. Nagell, Solution complète de quelques équations cubiques à deux indéterminées, Journ. de Math., Serie IX, 4 (1925) 209–270.

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

Material for this department should be sent to J. H. Curtiss, University of Miami, Coral Gables, Florida 33146

TWO TERNARY ALGEBRAS AND THEIR ASSOCIATED LATTICES

L. H. MARTIN, Harvey Mudd College

This note is concerned with ternary algebras and their associated lattices. A background in lattice theory and a familiarity with ternary operations in lattices may be obtained in Birkhoff [1]. Papers by Birkhoff, Kiss, and Grau have discussed ternary algebras and associated distributive lattices [2, 3]. In these papers, the ternary operation, Θ, had the following realization in a general

distributive lattice:

$$\Theta(a, b, c) = (a \cup b) \cap (b \cup c) \cap (c \cup a) = (a \cap b) \cup (b \cap c) \cup (c \cap a).$$

This self-dual ternary operation gave rise to distributive lattice operations in a natural way.

In this discussion, ternary operations Ω will be considered which have the following dual realizations in their respective lattices:

$$\Omega_1(a, b, c) = (a \cup b) \cap (c \cup (a \cap b)),$$
$$\Omega_2(a, b, c) = (a \cap b) \cup (c \cap (a \cup b)).$$

These two operations are generalizations of the Θ-operation, for in a distributive lattice $\Omega_1(a, b, c) = \Theta(a, b, c) = \Omega_2(a, b, c)$. The remainder of this note will be concerned with two algebras, the λ and the μ, which lead to more general structures than distributive lattices.

Let L be a set containing distinguished elements 0 and I. Let λ be a ternary operation closed in L and satisfying the following relations for all $a, b, c \in L$:

$$\begin{aligned}
&\text{(i)} && \lambda(a, b, c) = \lambda(b, a, c)\\
&\text{(ii)} && \lambda(0, a, I) = a = \lambda(I, a, 0)\\
&\text{(iii)} && \lambda(a, \lambda(b, c, I), I) = \lambda(c, \lambda(b, a, I), I)\\
& && \lambda(a, \lambda(b, c, 0), 0) = \lambda(c, \lambda(b, a, 0), 0)\\
&\text{(iv)} && \lambda(a, \lambda(a, b, I), 0) = a = \lambda(a, \lambda(a, b, 0), I).
\end{aligned}$$

The system $\langle L, \lambda, 0, I\rangle$ will be called a λ-*ternary algebra*. The axioms are consistent since they are satisfied in any lattice with 0 and I if we set $\lambda(a, b, c) = \Omega_1(a, b, c)$ or $\Omega_2(a, b, c)$. The following two theorems are consequences of the axioms:

THEOREM 1. $\lambda(a, a, I) = a = \lambda(a, a, 0)$.

THEOREM 2. $\lambda(a, I, I) = I$; $\lambda(a, 0, 0) = 0$.

The proofs of the two theorems are similar, so only the proof of Theorem 2 will be given.

Proof of Theorem 2. It is necessary to show only that $\lambda(a, I, I) = I$ since the axioms occur in dual pairs.

$$\begin{aligned}
\lambda(a, I, I) &= \lambda(I, a, I) && \text{by (i)}\\
&= \lambda(I, \lambda(I, a, 0), I) && \text{(ii)}\\
\therefore\ \lambda(a, I, I) &= I && \text{(iv)}
\end{aligned}$$

We now define meet and join:

DEFINITION 1. $(a \cup b) = \lambda(a, b, I)$; $(a \cap b) = \lambda(a, b, 0)$.

With these definitions we have:

$a \cup a = a = a \cap a$		by Th. 1
$a \cup b = b \cup a$	$a \cap b = b \cap a$	(i)
$a \cup (b \cup c) = (a \cup b) \cup c$	$a \cap (b \cap c) = (a \cap b) \cap c$	(iii)
$a \cup (a \cap b) = a = a \cap (a \cup b)$		(iv)
$a \cap 0 = 0$	$a \cup I = I$	Th. 2
$a \cup 0 = a$	$a \cap I = a$	(ii)

We may now conclude the following important theorem:

THEOREM 3. *The system* $\langle L, \cup, \cap, 0, I\rangle$ *is a lattice with null element* 0 *and universe element* I. *Further, any lattice with* 0 *and* I *may be represented as a* λ-*ternary algebra.*

In other words, lattices with 0 and I may be characterized as algebraic systems having a single ternary operation.

Let M be a set containing distinguished elements 0 and I. Let μ be a ternary operation closed in M and satisfying the following relations for all $a, b, c \in M$:

$$\text{(i)}\quad \mu(a, b, c) = \mu(b, a, c)$$

$$\text{(ii)}\quad \mu(a, a, c) = a$$

$$\text{(iii)}\quad \mu(\mu(a, b, 0), \mu(c, \mu(a, b, I), 0), I) = \mu(a, b, c)$$
$$\mu(\mu(a, b, I), \mu(c, \mu(a, b, 0), I), 0) = \mu(a, b, c)$$

$$\text{(iv)}\quad \mu(a, \mu(b, c, I), I) = \mu(c, \mu(b, a, I), I)$$
$$\mu(a, \mu(b, c, 0), 0) = \mu(c, \mu(b, a, 0), 0);$$

the system $\langle M, \mu, 0, I\rangle$ will be called a μ-*ternary algebra*. Again the axioms are consistent for they have the Ω-realizations in modular lattices. Also, we have the following theorems:

THEOREM 4. $\mu(a, \mu(a, b, I), 0) = a = \mu(a, \mu(a, b, 0), I)$.

THEOREM 5. $\mu(0, a, I) = a = \mu(I, a, 0)$.

THEOREM 6. $\mu(a, 0, 0) = 0, \qquad \mu(a, I, I) = I.$

The proofs for the three theorems are similar, so only Theorem 4 is proven below:

Proof of Theorem 4. It is necessary to prove only that $\mu(a, \mu(a, b, 0), I) = a$ since the axioms occur in dual pairs.

$$
\begin{aligned}
a &= \mu(a, a, c) && \text{by (ii)}\\
&= \mu(\mu(a, a, 0), \mu(c, \mu(a, a, I), 0), I) && \text{(iii)}\\
&= \mu(a, \mu(c, a, 0), I) && \text{(ii)}\\
\therefore\ a &= \mu(a, \mu(a, c, 0), I) && \text{(i)}
\end{aligned}
$$

Again we make the following definition:

DEFINITION 2. $(a\cup b)=\mu(a, b, I)$ $\qquad$ $(a\cap b)=\mu(a, b, 0)$.

From the discussion of the λ-ternary algebra and the corresponding axioms, theorems, and definitions, it should be clear that the ordering relation determined by Definition 2 produces a lattice. Further, axiom (iii) says that in this lattice, $\langle M, \cup, \cap, 0, I\rangle$, we have for all $a, b, c\in M$:

$$\mu(a, b, c) = (a \cup b) \cap (c \cup (a \cap b)) = (a \cap b) \cup (c \cap (a \cup b)).$$

The last half of this relationship, by the Dedekind Theorem, is equivalent to asserting that the lattice is modular. Thus, we have:

THEOREM 7. *The μ-ternary algebra completely characterizes modular lattices with null and universe elements.*

Thus, we have shown that all lattices with null and universe elements can be characterized as algebraic systems with a single operation. Further, we have displayed a system, the μ-ternary algebra, in which modular lattice operations arise naturally from the algebraic operation and vice versa.

This work was supported by funds from the National Science Foundation, NSF-USE grant 21755.

References

1. G. Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Pub., 25 (1948) rev. ed.
2. ——— and S. A. Kiss, A ternary operation in distributive lattices, Bull. Amer. Math. Soc., 53 (1947) 749–752.
3. A. A. Grau, Ternary Boolean algebra, Bull. Amer. Math. Soc., 53 (1947) 567–572.

SOME REMARKS ON THE ISOGONAL AND CEVIAN TRANSFORMS. ALIGNMENTS OF REMARKABLE POINTS OF A TRIANGLE

A. VANDEGHEN, Liège, Belgium

1. Let ABC be a triangle of reference; α, β, γ the normal (or trilinear) coordinates of a point P; α', β', γ' the coordinates of its isogonal conjugate P'; α'', β'', γ'' the coordinates of its cevian (or isotomic) conjugate P'' [**1**, p. 69]. Then

$$\alpha\alpha' = \beta\beta' = \gamma\gamma', \qquad a^2\alpha\alpha'' = b^2\beta\beta'' = c^2\gamma\gamma''.$$

It follows that the isogonal and the cevian transforms of a line d, whose equation is $l\alpha+m\beta+n\gamma=0$, are conics circumscribed to ABC [2].

The equation of the line at infinity being $a\alpha+b\beta+c\gamma=0$, its isogonal transform is the circumcircle $a\beta'\gamma'+b\gamma'\alpha'+c\alpha'\beta'=0$ [2] and its cevian transform is Steiner's ellipse

$$\frac{\beta''\gamma''}{a}+\frac{\gamma''\alpha''}{b}+\frac{\alpha''\beta''}{c} = 0.$$

It is known that any circumscribed hyperbola passing through the orthocenter H is equilateral and that its center is on Euler's circle [3]. The isogonal conjugate of H is O (the circumcenter); let U be the cevian conjugate of H. Then:

(a) If d does not meet the circumcircle, its isogonal transform is an ellipse; if d is tangent to the circumcircle, its isogonal transform is a parabola; if d meets the circumcircle, its isogonal transform is a hyperbola; if d passes through O, its isogonal transform is an equilateral hyperbola.

(b) If d does not meet Steiner's ellipse, its cevian transform is an ellipse; if d is tangent to Steiner's ellipse, its cevian transform is a parabola; if d meets Steiner's ellipse, its cevian transform is a hyperbola; if d passes through U, its cevian transform is an equilateral hyperbola.

The propositions under (a) are given in [2].

2. (a) The isogonal transform of the line PP' meets this line in P and P'. Conversely, the points where d meets its isogonal transform are isogonal conjugates.

If P is self-conjugate (i.e., one of the tritangent centers), the line is tangent in P to its isogonal transform.

(b) The cevian transform of the line PP'' meets this line in P and P''. Conversely, the points where d meets its cevian transform are cevian conjugates.

If P is self-conjugate (i.e., the centroid G or one of the points where the parallels to the sides through the vertices meet), the line is tangent in P to its cevian transform.

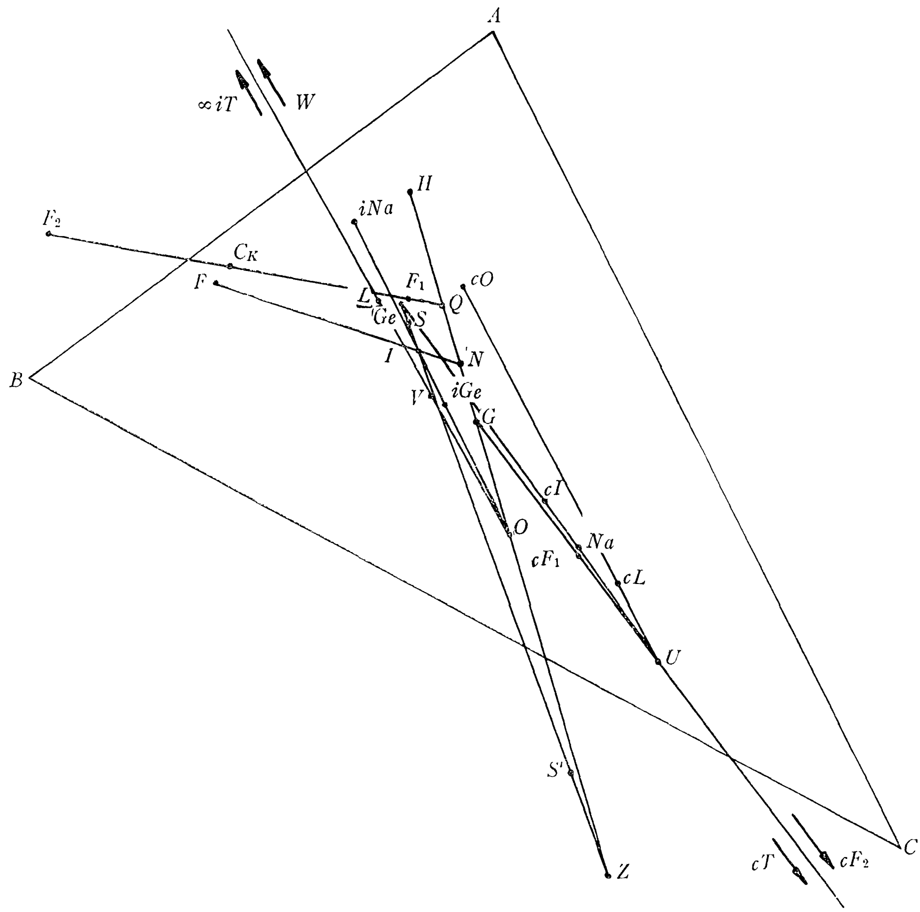
3. (a) Kiepert's hyperbola passes through A, B, C, H, G, F_1 and F_2 (isogonal centers or inner and outer Fermat's points), and T (Tarry's point) [2]. It is the isogonal transform of Apollonius' line which passes through O, L (Lemoine's point), the isodynamic points V and W (which are isogonal conjugates of Fermat's points); the point at infinity on this line is the isogonal conjugate of T.

Various properties of this hyperbola are given in [2] and [4]. Also its center is the midpoint of F_1F_2; the line F_1F_2 is the diameter conjugate to the direction of Euler's line, thus it intersects Euler's line in the midpoint of GH.

This hyperbola is also the cevian transform of its tangent at G, which passes through U and the cevian conjugates of F_1, F_2 and T.

(b) Jerabek's hyperbola passes through A, B, C, H, O, L and the isogonal conjugates of N (the nine-point center) and Z (de Longchamps' point) [2]. It is the isogonal transform of Euler's line. Thus the cevian conjugates of O and L are on a line through U.

(c) The equilateral hyperbola which is the isogonal transform of the line OI passes through A, B, C, H, I, Ge (Gergonne's point) and Na (Nagel's point);



List of the points in the figure:

H orthocenter
N center of nine-point circle
G centroid
Q midpoint of HG
O center of circumcircle
Z de Longchamps' point
S, S' centers of Soddy's circles
I center of incircle
Ge Gergonne's point
Na Nagel's point
F_1 Fermat's (or Steiner's) point, isogonal center
F_2 second Fermat's point, second isogonal center
C_K center of Kiepert's hyperbola
L Lemoine's point
V, W isodynamic points
T Tarry's point
F Feuerbach's point
U cevian conjugate of H
cP means cevian conjugate of point P
iP means isogonal conjugate of point P
HZ Euler's line
GeZ Soddy's line
OW Apollonius' line

its center is Feuerbach's point F (where the incircle and the nine-point circle are tangent) [2]. Thus the isogonal conjugates of Ge and Na are on OI; Ge and Na being cevian conjugates, these points, U, and the cevian conjugate of I, are on a line.

4. A circumscribed conic has a line d' as isogonal transform and a line d'' as cevian transform. Thus the product of the two transformations is a collineation; this results also from the relations

$$a^2\frac{\alpha''}{\alpha'} = b^2\frac{\beta''}{\beta'} = c^2\frac{\gamma''}{\gamma'}\cdot$$

This collineation transforms the sides into themselves.

The cevian transform of the circumcircle and the isogonal transform of Steiner's ellipse are reciprocal transverse lines of ABC; also, the second of these lines divides externally each side of ABC in lengths proportional to the squares of the adjacent sides: it is Lemoine's axis [1, p. 156].

5. It is interesting to gather in a figure various alignments of remarkable points of the triangle ABC, some of which are well known. The author would like readers to point out some other remarkable alignments.

Of course, the triangles that can be derived from ABC produce other alignments related to those of ABC. E.g., the triangles whose vertices are the points where the incircle and excircles are tangent to the sides: their Euler's lines pass through O. It has also been shown that the 3 external Soddy's lines meet in Z [5].

References

1. N. A. Court, College geometry, Barnes and Noble, New York, 1952.

2. J. Casey, A treatise on the analytical geometry of the point, line, circle, and conic sections, Hodges Figgis, Dublin; Longmans Green, London, 1893.

3. V. Falisse, Cours de géométrie analytique plane, Office de Publicité, Bruxelles, 1920.

4. P. J. Kelly and D. Merriell, Concentric polygons, this MONTHLY, 71 (1964) 37–41.

5. A. Vandeghen, Soddy's circles and the de Longchamps' point of a triangle, this MONTHLY, 71 (1964) 176–179.

HOW TO CREASE A SHEET OF PAPER

M. A. GOLDMAN, Norman, Oklahoma

1. Definition of a fold. Given a metric space (X, d) with topology associated with the metric d, given a subset $A\subset X$ which is metrized by the restriction of d tó $A\times A$ and given a mapping f: $A\to X$, assume there exists subsets $C\subset X$, $D\subset X$ with the following properties:

(1) $C=\overline{C}$, C is closed,

(2) $X-D=\overline{X-D}$, D is open,

(3) $A=C\cup\overline{D}$,

(4) $C\cap D=\varnothing$,

(5) $\overline{f(D)}\cap D=\varnothing$,

(6) $\forall p \in C, f(p) = p$,

(7) $\forall q, r \in \overline{D}, d(f(q), f(r)) = d(q, r)$.

Define the $C-D$ *fold of* A *in* (X, d), as $F \equiv C \cap \overline{D}$. If one can find sets $C, D \subset X$ with properties (1)–(7) one says that A *can be folded along* F. The mapping $f: A \to X$ is given the name of *a folding operation in* (X, d).

2. Example of a fold. As an example of a folding operation in a metric space let (X, d) be the euclidean plane (E^2, d), whose points are described by a cartesian coordinate system (x, y).

Let A be the closed rectangle $\{(x, y) \in E^2: 0 \leqq x \leqq 1, 0 \leqq y \leqq 2\}$.

Let C be the closed rectangle $\{(x, y) \in E^2: 0 \leqq x \leqq 1, 1 \leqq y \leqq 2\}$.

Let D be the open rectangle $\{(x, y) \in E^2: 0 < x < 1, 0 < y < 1\}$.

Define f by

$$\forall (x, y) \in C, f(x, y) = (x, y),$$
$$\forall (x, y) \in D, f(x, y) = (x, 2 - y),$$
$$\forall (x, y) \in \overline{D} - C, f(x, y) = (x, 2 - y).$$

F is the line segment $\{(x, y) \in E^2: 0 \leqq x \leqq 1, y = 1\}$.

3. Fold boundary points. Points of a fold F, are boundary points of both sets C and D. A point $x \in F$ lies in $C \cap \overline{D}$ so $x \in C$ and $x \in \overline{D}$. By (4), $C \cap D = \varnothing$ so $C \subset X - D \subset \overline{X - D}$, giving $x \in \overline{X - D}$. Thus $x \in \overline{D} \cap \overline{X - D}$. From (1), $x \in \overline{C}$. $D \subset X - C$ by (4) hence $\overline{D} \subset \overline{X - C}$ and $x \in \overline{D} \Rightarrow x \in \overline{X - C}$ giving $x \in \overline{C} \cap \overline{X - C}$.

4. Geodesic folds in E^2. In the special case that (X, d) is the euclidean plane (E^2, d), folds possess a geodesic property. If x_1, x_2, x_3 are given to be any three distinct points of an arbitrary $C-D$ fold of A in (E^2, d) then x_1, x_2, x_3 lie upon a line segment. The proof follows.

There exists an open sphere, S, of radius $\epsilon > 0$ centered about x_1 which does not contain x_2 or x_3. This is true whenever $0 < \epsilon < \frac{1}{3}$ minimum $d(x_1, x_j)$ for $j = 2, 3$. Since $x_1 \in \text{Bdry}(D)$ and $x_1 \in \overline{C}$ which implies $x_1 \notin D$, there exists a point $q \in D$ such that $q \in S$ and $q \neq x_1$. Thus q is distinct from x_1, x_2 and x_3. Because $x_1, x_2, x_3 \in C$, $f(x_1) = x_1$, $f(x_2) = x_2$, $f(x_3) = x_3$, from (6). Condition (7) implies that since x_1, x_2, x_3 and q are all in $\overline{D}$,

$$d(x_i, q) = d(f(x_i), f(q)) = d(x_i, f(q)) \quad \text{for } i = 1, 2, 3.$$

But since $q \in D$ and (5) requires $f(D) \cap D = \varnothing$ it follows that q and $f(q)$ are distinct points. Hence each x_i, for $i = 1, 2, 3$, is equidistant from the distinct points q and $f(q)$. In E^2 this implies that x_1, x_2 and x_3 each lie on the perpendicular bisector of the line segment included between q and $f(q)$ and thus lie on a common line.

5. Intuitive content of the definitions. The concepts of fold and folding operation are developed to characterize mathematically the essentially physical

problem of singly folding a flat sheet of paper flat on a table without tearing or otherwise stretching the paper. One must define what is done to the sheet when one folds it flat, before one can explain the straight crease which is empirically observed. The desired definition is given in the first section of the paper provided (X, d) is taken to be the euclidean plane. Straightness of the fold, which is the crease, follows.

The problem can be extended to metric spaces more general than (E^2, d) and has been formulated in language which allows an extension. The extended problem can be stated: For what metric spaces (X, d) is it true that given F, any fold in X and given any three points $x_1, x_2, x_3 \in F$, it follows that $d(x_i, x_j) + d(x_j, x_k) = d(x_i, x_k)$ for (i, j, k) some permutation of $(1, 2, 3)$? That is, in what metric spaces do folds possess a geodesic property?

Reference

1. B. Mendelson, Introduction to Topology, Allyn and Bacon, Boston, 1962, pp. 33–103.

A REMARK ON STIRLING'S FORMULA

A. J. MARIA, late of Brooklyn College

For Stirling's formula, given by $n! = \sqrt{2\pi}n^{n+1/2}e^{-n}e^{r_n}$, Herbert Robbins [1] proved by elementary means that

$$\frac{1}{12n+1} < r_n < \frac{1}{12n}, \qquad n = 1, 2, \cdots.$$

In the cited article the author showed that $r_n = \sum_{p=n}^{\infty} \epsilon_p$, where

$$\epsilon_p = \frac{1}{3(2p+1)^2} + \frac{1}{5(2p+1)^4} + \frac{1}{7(2p+1)^6} + \cdots,$$

and by observing that

$$\begin{aligned} \epsilon_p &> \frac{1}{3(2p+1)^2}\left\{1 + \frac{1}{3(2p+1)^2} + \frac{1}{(3(2p+1)^2)^2} + \cdots\right\} \\ &= \frac{1}{3(2p+1)^2}\,\frac{1}{1 - \dfrac{1}{3(2p+1)^2}} > \frac{1}{12}\left(\frac{1}{p + \dfrac{1}{12}} - \frac{1}{p + 1 + \dfrac{1}{12}}\right) \end{aligned}$$

he concluded that $1/(12n+1) < r_n$.

In this note we use a larger lower bound for ϵ_p and obtain by simple means the much better inequality

$$\frac{1}{12n + \dfrac{3}{2(2n+1)}} < r_n.$$

We shall now prove this inequality.

Proof. It is clear that

$$\epsilon_p > \frac{1}{3(2p+1)^2}\left\{1 + \frac{1}{2(2p+1)^2} + \frac{1}{2^2(2p+1)^4} + \cdots\right\},$$

since $2^{n-1} > (2n+1)/3$, $n \geqq 2$. Hence

$$\epsilon_p > \frac{1}{3(2p+1)^2}\left\{\frac{1}{1 - \dfrac{1}{2(2p+1)^2}}\right\} = \frac{1}{12p^2 + 12p + \frac{3}{2}}.$$

For each n we shall find $\alpha > 0$ so that

$$\frac{1}{12n^2 + 12n + \frac{3}{2}} - \frac{1}{12}\left(\frac{1}{n+\alpha} - \frac{1}{n+\alpha+1}\right)$$
$$= \frac{12\alpha^2 + \alpha(24n+12) - \frac{3}{2}}{(12n^2 + 12n + \frac{3}{2})12(n+\alpha)(n+\alpha+1)} = 0.$$

It follows that we must have $\alpha = [-2n - 1 + \sqrt{(2n+1)^2 + \frac{1}{2}}]/2$, and since $12\alpha^2 + \alpha(24p+12) - \frac{3}{2} > 0$, $p > n$, we have

$$\frac{1}{12p^2 + 12p + \frac{3}{2}} > \frac{1}{12}\left(\frac{1}{p+\alpha} - \frac{1}{p+\alpha+1}\right), \qquad p > n.$$

Therefore

$$r_n = \sum_{p=n}^{\infty} \epsilon_p > \frac{1}{12}\left(\frac{1}{n + \dfrac{-2n - 1 + \sqrt{(2n+1)^2 + \frac{1}{2}}}{2}}\right)$$
$$> \frac{1}{12\left(n + \dfrac{1}{8(2n+1)}\right)} = \frac{1}{12n + \dfrac{3}{2(2n+1)}}.$$

This completes the proof.

It is to be observed that for

$$\frac{1}{12n + \dfrac{3}{2(2n+1)}} < r_n < \frac{1}{12n}$$

we have

$$e^{r_n} = 1 + \frac{1}{12n} + \frac{1}{288n^2} + o(n^2),$$

while for $1/(12n+1) < r_n < 1/12n$ we can only conclude that $e^{r_n} = 1 + (1/12n) + o(n)$.

Reference

1. Herbert Robbins, A remark on Stirling's Formula, this MONTHLY, 62 (1955) 26–28.

STRONGLY CONNECTED SETS IN TOPOLOGY

NORMAN LEVINE, The Ohio State University

1. Strong connectivity.

DEFINITION 1. *A set A of a topological space X will be termed strongly connected (written henceforth as* s.c.*) iff $A \subset U$ or $A \subset V$ whenever $A \subset U \cup V$, U and V being open sets in X.*

THEOREM 1. *If A is* s.c., *then A is connected.*

Proof. If A is disconnected, then there exist open sets U and V for which

$$A = (A \cap U) \cup (A \cap V),\ A \cap U \neq \emptyset,\ A \cap V \neq \emptyset, \quad \text{and} \quad A \cap U \cap V = \emptyset.$$

Then $A \subset U \cup V$, but $A \not\subset U$ and $A \not\subset V$ which contradicts A being s.c.

Example 1. Let $X: a, b, c$ and $\tau: \phi, (a), (a, b), (a, c)$ and X. Now X is connected, but not s.c.

DEFINITION 2. *A set A will be termed weakly disconnected (written henceforth as* w.d.*) iff it is not* s.c.

Remark 1. From Theorem 1 it follows that disconnected implies w.d.

THEOREM 2. *A set A is* w.d. *iff there exist two nonempty disjoint sets E and F each closed in A.*

The proof is left as an exercise for the reader.

COROLLARY 1. *A set A is* w.d. *iff there exist two points x and y in A such that $A \cap c(x) \cap c(y) = \emptyset$, c denoting the closure operator.*

THEOREM 3. *Let (X, τ) be the topological space in example* 1 *and let (X^*, τ^*) be any topological space. Then X^* is* w.d. *iff there exists a continuous mapping $f: X^* \to X$ for which $\{b, c\} \subset f[X^*]$. If (X^*, τ^*) is connected, then X^* is* w.d. *iff there exists a continuous mapping $g: X^* \to X$ which is onto.*

THEOREM 4. *Let F be a closed subset of X and suppose (X, τ) is* s.c. *Then F is* s.c.

COROLLARY 2. *(X, τ) is* s.c. *iff every closed subset F is connected.*

THEOREM 5. *If $f: X \to Y$ is continuous and if A is* s.c. *in X, then $f[A]$ is* s.c. *in Y.*

The proofs of Theorems 3, 4 and 5 and Corollary 2 are left as exercises.

DEFINITION 3. *A topological space X is totally weakly disconnected* (t.w.d.) *iff singleton sets are the only nonempty* s.c. *sets.* (See Remark 2 in section 2.)

THEOREM 6. *A topological space X is T_1 iff it is* t.w.d.

Proof. Necessity: Singleton sets are clearly s.c. Now suppose A is a subset of X with two or more points. Let x be different from y in A. Then $\{x\}$ and $\{y\}$ are nonempty disjoint closed subsets of A and by Theorem 2, A is w.d.

Sufficiency. Take $x \in X$ and suppose $\{x\} \neq c(x)$. Let $y \in c(x) - \{x\}$ and let $A = \{x, y\}$. Then every open set which contains y also contains x and thus A is s.c. This is a contradiction.

THEOREM 7. *In a topological space (X, τ), every subset of X is* s.c. *iff τ is a chain, that is, if U and V are open, then $U \subset V$ or $V \subset U$.*

We omit the proof.

THEOREM 8. *Let $\{(X_\alpha, \tau_\alpha): \alpha \in \Delta\}$ be a nonempty family of nonempty spaces and let (X, τ) be the product space. Then (X, τ) is* s.c. *iff (X_α, τ_α) is* s.c. *for each $\alpha \in \Delta$.*

Proof. Necessity: This follows from Theorem 5 and the fact that projection maps are continuous and onto.

Sufficiency: Suppose that (X, τ) is not s.c. Then by Corollary 1, there exist points $x \neq y$ in X for which $c(x) \cap c(y) = \varnothing$. Then

$$\begin{aligned} \varnothing &= c(x) \cap c(y) \\ &= c\{\cap P_\alpha^{-1} P_\alpha(x)\} \cap c\{\cap P_\alpha^{-1} P_\alpha(y)\} \\ &= \cap P_\alpha^{-1} c_\alpha P_\alpha(x) \cap \cap P_\alpha^{-1} c_\alpha P_\alpha(y) \\ &= \cap P_\alpha^{-1}\{c_\alpha P_\alpha(x) \cap c_\alpha P_\alpha(y)\}. \end{aligned}$$

Thus there exists an $\alpha^* \in \Delta$ for which $c_{\alpha^*} P_{\alpha^*}(x) \cap c_{\alpha^*} P_{\alpha^*}(y) = \varnothing$. Then by Corollary 1, $(X_{\alpha^*}, \tau_{\alpha^*})$ is not s.c.

2. Strong local connectivity.

DEFINITION 4. *(X, τ) is termed strongly locally connected at x (written henceforth as* s.l.c.) *iff for $x \in O \in \tau$, there exists a* s.c. *open set G such that $x \in G \subset O$. (X, τ) is strongly locally connected iff it is* s.l.c. *at every point of X.*

COROLLARY 3. *If (X, τ) is* s.l.c., *then it is locally connected.*

This follows from Theorem 1.

Remark 1. s.l.c. does not imply s.c. (see example 1).

Remark 2. s.c. does not imply s.l.c. To see this let (R, τ) be the reals with the usual topology and let $R^* = R \cup \{\infty\}$ with $\tau^* = \tau \cup \{R^*\}$. Then (R^*, τ^*) is s.c., but not s.l.c. as the reader can easily verify. This example also serves as a "dispersion point space," that is, a strongly connected space which is rendered t.w.d. by the removal of a single point (see Definition 3).

Remark 3. Local connectedness does not imply s.l.c. The real line with the usual topology serves as an example.

LEMMA 1. *If X is* s.l.c. *and $f\colon X\to Y$ is continuous, open and onto, then Y is* s.l.c.

The proof follows from Theorem 5.

THEOREM 9. *Let $\{(X_\alpha, \tau_\alpha)\colon \alpha\in\Delta\}$ be a nonempty family of nonempty* s.c. *spaces and let (X, τ) be the product space. Then (X, τ) is* s.l.c. *iff (X_α, τ_α) is* s.l.c. *for each $\alpha\in\Delta$.*

Use Lemma 1 and Theorem 8.

Remark 4. We cannot drop s.c. from the hypotheses of Theorem 9. A discrete two-point space is s.l.c., but a product of such spaces may fail to be locally connected as is well known.

3. Weak components.

LEMMA 2. *Let $\mathcal{A}=\{A\}$ be a chain of* s.c. *sets in X. Then $\cup\{A: A\in\mathcal{A}\}$ is* s.c.

Proof. Let $\cup\{A: A\in\mathcal{A}\}\subset U\cup V$ where U and V are open and suppose that $\cup\{A: A\in\mathcal{A}\}$ is a subset of neither U nor V. There exists then an $x\in A_1\in\mathcal{A}$ such that $x\notin U$ and there exists a $y\in A_2\in\mathcal{A}$ such that $y\notin V$. Since $\mathcal{A}$ is a chain we may assume that $A_1\subset A_2$. Then $A_2\subset U\cup V$, but $A_2\not\subset U$ and $A_2\not\subset V$ which contradicts A_2 being s.c.

Remark 5. Example 1 shows that "chain" in Lemma 2 cannot be dropped.

DEFINITION 5. *We call A a weak component of X iff* (1) *A is* s.c. *and* (2) *A is maximal relative to* (1).

THEOREM 10. *Let A be a* s.c. *subset of a topological space X. Then there exists a weak component of X which contains A.*

Proof. Let $\mathcal{A}$ be the family of all s.c. sets which contain A. Then $\mathcal{A}$ is nonempty and by Lemma 2, is chain closed. By Zorn's lemma, $\mathcal{A}$ has a maximal element.

COROLLARY 3. *Each point x in a topological space X is contained in at least one weak component of X.*

Remark 6. In example 1, a is a member of two distinct weak components of X.

Remark 7. Weak components may be neither open nor closed. On the line, weak components are points and thus not open. In example 1, the weak components are not closed.

DEFINITION 6. *A set A is called an open weak component of X iff* (1) *A is open,* (2) *A is* s.c., *and* (3) *A is maximal relative to both* (1) *and* (2).

THEOREM 11. *If X is* s.l.c., *then each point is contained in at least one open weak component.*

Proof: Let x be a point of X and let $\mathfrak{A}$ be the family of all open s.c. sets which contain x. $\mathfrak{A}$ is nonempty since X is s.l.c. By Lemma 2 and the fact that a union of open sets is open, it follows that $\mathfrak{A}$ is chain closed. By Zorn's lemma, $\mathfrak{A}$ has a maximal element.

Example 2. Let $X=\{x_{ij}\}$, $i=1, 2, \cdots$ and $1\leqq j\leqq 2^{i-1}$. Let

$$\mathfrak{B} = \{x_{1i_1}, x_{2i_2}, \cdots, x_{si_s}\},$$

where $s\geqq 1$, $i_1=1$ and $2i_{j-1}-1\leqq i_j\leqq 2i_{j-1}$ for $j\geqq 2$. It is clear that $\mathfrak{B}$ is a base for a topology τ for X. Each point of X gets into an uncountable number of distinct open weak components of X. The proof is left to the reader.

THE DEGREE OF APPROXIMATION OF CERTAIN FUNCTIONS OF TWO VARIABLES BY A SUM OF FUNCTIONS OF ONE VARIABLE

T. J. RIVLIN, IBM, Yorktown Heights, AND R. J. SIBNER, Stanford University

Let $f(x, y)$ be continuous on the unit square, $S=I\times I$, where I is $[0, 1]$. Let $\|\cdot\|$ denote the sup norm on S and suppose that $g^*, h^*\in C(I)$ satisfy

$$m = \|f(x, y) - (g^*(x) + h^*(y))\| \leqq \|f(x, y) - (g(x) + h(y))\| \tag{1}$$

for all $g, h\in C(I)$, i.e. $g^*(x)+h^*(y)$ is a best approximation to $f(x, y)$ among all sums of continuous functions $g(x)+h(y)$. A duality theorem of Banach (cf. Shapiro [3]) tells us that

$$m = \sup |Lf|, \tag{2}$$

the supremum being taken over all linear functionals L of norm 1 which vanish on every continuous function of the form $g(x)+h(y)$.

Diliberto and Straus [1] showed that (2) holds if the supremum is taken over a more restricted set of linear functionals, the *permissible* functionals defined by

$$Lf = \frac{1}{2n}\sum_{k=1}^{2n} (-1)^k f(x_k, y_k), \tag{3}$$

where the points (x_k, y_k) are vertices of a *permissible line.* A permissible line, according to Diliberto and Straus, is a closed polygonal line lying entirely in S, each side of which is parallel to either the x or y axis. Given a permissible line of which $(x_1, y_1), \cdots, (x_{2n}, y_{2n})$ are vertices numbered by $x_{2j-1}=x_{2j}$; $y_{2j+1}=y_{2j}$, $j=1, 2, \cdots$, and the subscripts are taken modulo $2n$, the corresponding permissible functional given by (3) is defined. Note that a permissible functional is linear, has norm 1 and vanishes on functions of the form $g(x)+h(y)$.

The purpose of this note is to prove that for a restricted class of f, the value of m is easily determined. Namely,

THEOREM. *If $f_{xy}\geqq 0$ in S then*

$$m = \tfrac{1}{4}\iint_S f_{xy}d\,\Sigma = \tfrac{1}{4}[f(1, 1) - f(0, 1) + f(0, 0) - f(1, 0)]. \tag{4}$$

Proof. L^* defined by

$$L^*f = \tfrac{1}{4}[f(1, 1) - f(0, 1) + f(0, 0) - f(1, 0)]$$

is a permissible functional. Hence, to prove our theorem we need only show that any permissible functional L, defined by (3) satisfies

$$|Lf| \leqq L^*f.$$

It is no loss of generality to restrict ourselves, and we do, to permissible functionals L which satisfy $Lf \geqq 0$, since by reversing the labeling of the subscripts (i.e. the present sequence $1, 2, \cdots, 2n$ becomes $2n, 2n-1, \cdots, 1$), $-L$ is seen to be permissible and $|Lf| = |(-L)f|$.

We wish to prove, then, that

$$Lf \leqq L^*f. \tag{5}$$

Consider the functional L given by (3).

(a) If $y_{2k-1} < y_{2k}$ replace the terms $-f(x_{2k-1}, y_{2k-1}) + f(x_{2k}, y_{2k})$ in (3) by $-f(1, y_{2k-1}) + f(1, y_{2k})$ thus obtaining a new permissible functional, L'. Since

$$f(1, y_{2k}) - f(1, y_{2k-1}) - f(x_{2k}, y_{2k}) + f(x_{2k-1}, y_{2k-1}) = \int_{y_{2k-1}}^{y_{2k}} \int_{x_{2k}}^{1} f_{xy}\, dxdy \geqq 0$$

we see that $Lf \leqq L'f$.

(b) If $y_{2k-1} > y_{2k}$ replace terms $-f(x_{2k-1}, y_{2k-1}) + f(x_{2k}, y_{2k})$ in (3) by $-f(0, y_{2k-1}) + f(0, y_{2k})$ thus obtaining a new permissible functional L' such that $Lf \leqq L'f$.

For each $k = 1, 2, \cdots, n$ either (a) or (b) holds. If the functional is changed successively for each $k = 1, \cdots, n$, we obtain finally a permissible functional L'' with its defining vertices lying on the two vertical sides of S. Moreover, the $2n$ points are vertices of a permissible line and so can be grouped (possibly after deleting a vertex which occurs with indices of opposite parity) into s sets each of which consists of the 4 vertices of a rectangle, where $s \leqq n/2$, in such a fashion that the 4 vertices that make up a set contribute 4 terms to L'' which add up to

$$\iint_R f_{xy} d\,\Sigma \geqq 0,$$

where R is the rectangle defined by these vertices. That this can always be done is not hard to establish by mathematical induction on n.

Since $\iint_R f_{xy} d\Sigma \leqq \iint_S f_{xy} d\Sigma$, we conclude that

$$L''f \leqq \frac{s}{2n} \iint_S f_{xy} d\,\Sigma \leqq L^*f.$$

Since $Lf \leqq L''f$ the theorem is proved.

Remark 1. We have shown that if $f_{xy} \geqq 0$ then

$$\max |Lf| = L^*f.$$

Thus in this case a "rectangular" permissible functional achieves the supremum in (2). This is not the case when $f(x, y)$ is an arbitrary continuous function. For consider the piecewise linear function

$$h(t) = \begin{cases} 4t + 3, & -1 \leqq t \leqq -\frac{1}{2} \\ -2t, & |t| \leqq \frac{1}{2} \\ 4t - 3, & \frac{1}{2} \leqq t \leqq 1, \end{cases}$$

which is continuous for $-1 \leqq t \leqq 1$. Then $f(x, y) = h(x-y)$ is continuous on S. Consider the linear functional defined by

$$Lg = \tfrac{1}{6}[g(\tfrac{1}{2}, 0) - g(1, 0) + g(1, \tfrac{1}{2}) - g(0, \tfrac{1}{2}) + g(0, 1) - g(\tfrac{1}{2}, 1)].$$

Then $|Lf| = 1$, and hence $m \geqq 1$. But it is clear that $|L'f| < 1$ for every "rectangular" permissible functional L'.

Remark 2. A simple illustration of our Theorem is the function $f(x, y) = xy$ which cannot be approximated on S more closely than $\frac{1}{4}$ by $g(x) + h(y)$, as Golomb [2] notes.

The Theorem of this paper was suggested to us by Professor J. Moser and we wish to thank Professor Moser for several interesting discussions of this topic.

References

1. S. P. Diliberto, and E. G. Straus, On the approximation of a function of several variables by the sum of functions of fewer variables, Pacific J. Math., 1 (1951) 195–210.

2. M. Golomb, Approximation by functions of fewer variables, *On Numerical Approximation*, ed. R. E. Langer, University of Wisconsin Press, Madison, 1959.

3. H. S. Shapiro, Applications of normed linear spaces to function-theoretic extremal problems, *Lectures on Functions of a Complex Variable*, ed. W. Kaplan, et. al., University of Michigan Press, Ann Arbor, 1955.

ON THE PRODUCT OF QUOTIENT SPACES

C. J. HIMMELBERG, University of Kansas

Let $f_1: X_1 \to Y_1$, $f_2: X_2 \to Y_2$ be closed maps (map = continuous function) from the topological spaces X_1, X_2 onto the topological spaces Y_1, Y_2 respectively, and let $f = (f_1, f_2): X_1 \times X_2 \to Y_1 \times Y_2$ be the map defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for $(x_1, x_2) \in X_1 \times X_2$. Examples abound to show that f is not necessarily a closed map. In fact $Y_1 \times Y_2$ need not have the quotient topology, as defined for instance in [2 page 94], relative to f. For an example see [1 page 90, exercise 11]. The purpose of this note is to give some simple sufficient conditions that $Y_1 \times Y_2$ have the quotient topology.

THEOREM 1. *Suppose that f_i is a closed map from a topological space X_i onto a topological space Y_i such that $f_i^{-1}(y_i)$ is compact for each $y_i \in Y_i$, $i = 1, 2$. Then $f = (f_1, f_2)$ is a closed map.*

Proof. Note that $f^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2)$ for each $(y_1, y_2) \in Y_1 \times Y_2$. Let A be a closed subset of $X_1 \times X_2$ and let

$$V = \bigcup \{f^{-1}(y_1, y_2) \mid f^{-1}(y_1, y_2) \subset X_1 \times X_2 - A\}.$$

Then $Y_1 \times Y_2 - f(A) = f(V)$. We will show that $f(V)$ is open. Suppose that $(x_1, x_2) \in V$ and $f(x_1, x_2) = (y_1, y_2)$. Then $f^{-1}(y_1, y_2) = f_1^{-1}(y_1) \times f_2^{-1}(y_2)$ is a compact subset of $X_1 \times X_2 - A$ containing (x_1, x_2). By a theorem of Wallace [**2**, page 142] there exist open neighborhoods M_1, M_2 of $f_1^{-1}(y_1)$, $f_2^{-1}(y_2)$ respectively such that $M_1 \times M_2 \subset X_1 \times X_2 - A$. Since f_1, f_2 are closed maps, the sets

$$N_1 = \bigcup \{f_1^{-1}(y_1') \mid f_1^{-1}(y_1') \subset M_1\}, \qquad f_1(N_1)$$
$$N_2 = \bigcup \{f_2^{-1}(y_2') \mid f_2^{-1}(y_2') \subset M_2\}, \qquad f_2(N_2)$$

are all open, and of course $(x_1, x_2) \in N_1 \times N_2 \subset X_1 \times X_2 - A$. Moreover, $N_1 \times N_2 \subset V$, since, if $(x_1', x_2') \in N_1 \times N_2$ and $f(x_1', x_2') = (y_1', y_2')$, then

$$(x_1', x_2') \in f_1^{-1}(y_1') \times f_2^{-1}(y_2') \subset N_1 \times N_2 \subset X_1 \times X_2 - A.$$

Finally, since $f(N_1 \times N_2) = f_1(N_1) \times f_2(N_2)$ is an open neighborhood of (y_1, y_2), it follows that $f(V)$ is a neighborhood of (y_1, y_2).

THEOREM 2 (MORITA AND HANAI [**3**]). *Let f be a closed map from a T_1-space X onto a topological space Y such that* $\mathrm{Bd}\, f^{-1}(y)$ *is compact for each $y \in Y$. Then there is a closed subspace A of X such that* (a) *$g = f \circ \theta$ is a closed map from A onto Y, where $\theta: A \to X$ is the identity injection, and* (b) *$g^{-1}(y)$ is compact for each $y \in Y$.*

Proof. This theorem is contained in the proof of Theorem 1 in [**3**]. Its proof is so brief, however, that we include it (almost verbatim from [**3**]) for the sake of completeness. It is clear that, since f is closed, Y is a T_1-space. So, for each $y \in Y$, define an open subset $L(y)$ of X as follows:

$$L(y) = \begin{cases} \text{Interior}\, f^{-1}(y), & \text{if } \mathrm{Bd}\, f^{-1}(y) \neq \phi, \\ f^{-1}(y) - \{p_y\}, & \text{if } \mathrm{Bd}\, f^{-1}(y) = \phi, \end{cases}$$

where p_y is an arbitrary point of $f^{-1}(y)$. Set

$$L = \bigcup \{L(y) \mid y \in Y\}, \qquad A = X - L.$$

Then A is a closed subset of X. If $\theta: A \to X$ is the identity injection, then $g = f \circ \theta$ is a closed map from A onto Y such that

$$g^{-1}(y) = \begin{cases} \mathrm{Bd} f^{-1}(y), & \text{if } \mathrm{Bd}\, f^{-1}(y) \neq \phi, \\ \{p_{y},\} & \text{if } \mathrm{Bd}\, f^{-1}(y) = \phi. \end{cases}$$

Thus $g^{-1}(y)$ is compact for each $y \in Y$.

LEMMA. *Let f be a map from a topological space X onto a topological space Y. Then Y has the quotient topology relative to f if there exists a subspace A of X such that Y has the quotient topology relative to $f \circ \theta$, where θ is the identity injection of A into X.*

Proof. Trivial application of the definition of the quotient topology.

THEOREM 3. *Suppose that f_i is a closed map from a T_1-space X_i onto a topological space Y_i and that* $\mathrm{Bd}\, f_i^{-1}(y_i)$ *is compact, for each $y_i \in Y_i$, $i=1, 2$. Then $Y_1 \times Y_2$ has the quotient topology relative to (f_1, f_2).*

Proof. For $i=1, 2$, let A_i be a closed subset of X_i such that, if $\theta_i: A_i \to X_i$ is the identity injection, then $g_i = f_i \circ \theta_i$ is a closed map onto Y_i and $g_i^{-1}(y_i)$ is compact for each $y_i \in Y_i$. This is possible by Theorem 2. By Theorem 1, the map $g=(g_1, g_2): A_1 \times A_2 \to Y_1 \times Y_2$ is a closed map. Hence, by the Lemma, $Y_1 \times Y_2$ has the quotient topology relative to $f=(f_1, f_2)$.

In [3] and in [4] it is shown that if f is a closed map from a metrizable space X onto a topological space Y, then Y is metrizable iff $\mathrm{Bd}\, f^{-1}(y)$ is compact for each $y \in Y$. Hence we obtain the following corollary to Theorem 3.

COROLLARY. *If f_i is a closed map from a metrizable space X_i onto a metrizable space Y_i, $i=1, 2$, then $Y_1 \times Y_2$ has the quotient topology relative to (f_1, f_2).*

It should be observed that, in Theorem 1, we may not weaken the requirement that $f_i^{-1}(y_i)$ be compact to the requirement that $\mathrm{Bd}\, f_i^{-1}(y_i)$ be compact for $y_i \in Y_i$, $i=1, 2$. In other words, in Theorem 3 and the Corollary, (f_1, f_2) need not be a closed map. For instance, let X_1 be the real line, let $Y_1=\{0\}$, let $X_2=Y_2=[0, 1]$, let $f_1: X_1 \to Y_1$ be the constant map, and let $f_2: X_2 \to Y_2$ be the identity map. Then the sets $\mathrm{Bd}\, f_i^{-1}(y_i)$ are compact for each $y_i \in Y_i$, $i=1, 2$; but (f_1, f_2) takes the closed set $\{(n, 1/n) \mid n=1, 2, \cdots\}$ onto a nonclosed one. In fact, we may not even weaken the hypothesis of Theorem 1 by assuming $\mathrm{Bd}\, f_1^{-1}(y_1)$ compact for each $y_1 \in Y_1$ and $f_2^{-1}(y_2)$ compact for each $y_2 \in Y_2$, since, in the example above, $f_2^{-1}(y_2)$ is compact.

It is natural to ask whether Theorems 1 and 3 hold for infinite products, and it is not difficult to see that they do. The following extension of the theorem of Wallace used in the proof of Theorem 1 is needed.

THEOREM 4. *Let A_k be a compact subset of a topological space X_k for each $k \in K$, let $X=\pi\{X_k \mid k \in K\}$ have the product topology, and let U be a neighborhood of $A=\pi\{A_k \mid k \in K\}$ in X. Then, for each $k \in K$, there exists an open neighborhood M_k of A_k such that $A \subset \pi\{M_k \mid k \in K\} \subset U$, where $M_k = X_k$ for all but finitely many $k \in K$.*

Proof. If K has two elements we have the Wallace Theorem referred to above. An easy induction yields Theorem 4 for K finite. To prove the general case, let $V_1, \cdots, V_m$ be a finite open cover of A by basic open sets such that $V=\cup\{V_i \mid 1 \leqq i \leqq m\} \subset U$. Then each V_i is of the form $V_i=\pi\{V_{i,k} \mid k \in K\}$, where $V_{i,k}=X_k$ for all but finitely many k, say, except for $k \in L_i$ where L_i is a finite subset of K. Let $L=\cup\{L_i \mid 1 \leqq i \leqq m\}$. Then, if we do not distinguish between $\pi\{X_k(k \in K\}$ and $(\pi\{X_k \mid k \in L\}) \times (\pi\{X_k \mid k \in K-L\})$, we have

$$V = (\cup\{\pi\{V_{i,k} \mid k \in L\} \mid 1 \leqq i \leqq m\}) \times \pi\{X_k \mid k \in K-L\}.$$

Denote the first factor above by W. Since L is finite, W is an open neighborhood of $\pi\{A_k | k \in L\}$ in $\pi\{X_k | k \in L\}$. By the finite case, there exists for each $k \in L$ an open neighborhood M_k of A_k such that $\pi\{M_k | k \in L\} \subset W$. Hence $A \subset \pi\{M_k | k \in K\} \subset V \subset U$, where $M_k = X_k$ if $k \notin L$.

With little change in the proofs of Theorems 1 and 3, the reader may now prove the following

THEOREM 5. *Let f_k be a closed map from a topological space X_k onto a topological space Y_k for each $k \in K$; let $X = \pi\{X_k | k \in K\}$ and $Y = \pi\{Y_k | k \in K\}$ each have the product topology; and let $f: X \to Y$ be defined by $f(x)_k = f_k(x_k)$, where $x \in X$ and x_k is the k-th coordinate of x. Then*

a) *If $f_k^{-1}(y_k)$ is compact for each y_k in each Y_k, then f is a closed map.*

b) *If $\mathrm{Bd}\, f_k^{-1}(y_k)$ is compact for each y_k in each Y_k, and if each X_k is a T_1-space, then Y has the quotient topology relative to f.*

c) *If each X_k and each Y_k is metrizable, then Y has the quotient topology relative to f.*

The author wishes to thank the referee for several useful remarks, and in particular for the suggestion that Theorems 1 and 3 might be extended to arbitrary products as in Theorem 5.

This work was supported by the National Science Foundation Grant GP-209.

References

1. N. Bourbaki, Topologie Générale, Actualités Sci. Ind., Paris, 1142 (1951).

2. J. L. Kelley, General Topology, Van Nostrand, Princeton, N. J., 1955.

3. K. Morita and S. Hanai, Closed mappings and metric spaces, Proc. Japan Acad., 32 (1956) 10–14.

4. A. H. Stone, Metrizability of decomposition spaces, Proc. Amer. Math. Soc., 7 (1956) 690–700.

A NOTE ON ADDITIVE COMMUTATORS IN A RING

D. W. ROBINSON, Brigham Young University

For y an element of an arbitrary ring S, let Δ_y be the mapping on S given by $x\Delta_y = xy - yx$. Also, for θ a mapping on S let $K(\theta) = \{x \in S | x\theta = 0\}$. In this note we prove the following

THEOREM. *Let a and e be elements of a ring such that e is idempotent. Then for every positive integer n, $K(\Delta_a) \subseteq K(\Delta_e)$ implies $K(\Delta_a^n) \subseteq K(\Delta_e)$.*

The proof is in two parts.

(i) $$\text{If } K(\Delta_a) \subseteq K(\Delta_e), \text{ then } K(\Delta_a^n) \subseteq K(\Delta_e^n).$$

Indeed, let $K(\Delta_a) \subseteq K(\Delta_e)$. Since $a \in K(\Delta_a)$, it follows that $a \in K(\Delta_e)$. That is, a commutes with e. Hence, Δ_a commutes with Δ_e, which implies (i) by induction on n.

(ii) $$\text{If } e \text{ is idempotent, then } K(\Delta_e^n) = K(\Delta_e).$$

For if e is idempotent, then $e(x\Delta_e^2)=e((xe-ex)e-e(xe-ex))=-exe+ex$. Thus, $x\Delta_e^2=0$ implies $exe=ex$ and $x\Delta_e=xe-exe-exe+ex=x\Delta_e^2=0$. That is, $K(\Delta_e^2)\subseteq K(\Delta_e)$, which implies (ii) by induction on n.

The *raison d'être* of the theorem is the following important

COROLLARY. *Let A and B be linear transformations on a finite-dimensional vector space $\mathfrak{M}$ over a field Φ, and let $\mu(\lambda)$ be a polynomial over Φ. If $B\Delta_A^n=0$ for some n, then the Fitting components of $\mathfrak{M}$ relative to $\mu(A)$ are invariant under B.*

This corollary contains, in particular, both Lemma 1, p. 38 and Lemma 3, p. 40 of [1]. The proof is given as follows. Any linear transformation that commutes with $\mu(A)$ leaves invariant the Fitting components of $\mathfrak{M}$ relative to $\mu(A)$ and hence commutes with the projections on these components. That is, $K(\Delta_{\mu(A)})\subseteq K(\Delta_E)$, where E is a projection determined by the Fitting decomposition of $\mathfrak{M}$ relative to $\mu(A)$. Therefore, since $K(\Delta_A)\subseteq K(\Delta_{\mu(A)})$, it follows that $K(\Delta_A)\subseteq K(\Delta_E)$. The corollary is now an immediate consequence of the theorem.

The author expresses appreciation to the referee for his comments.

This paper was written while the author was on sabbatical leave at the California Institute of Technology, where he was supported in part by the National Science Foundation.

Reference

1. N. Jacobson, Lie algebras, Interscience tracts in pure and applied mathematics 10, Wiley, New York, 1962.

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

This department welcomes brief expository articles on topics closely related to classroom experience in courses that are normally available to undergraduate students, from the freshman year through early graduate work. Items of interest to teachers, such as pedagogical tactics, course improvement, new proofs and counterexamples, and fresh viewpoints in general, are invited. All material should be sent to Gertrude Ehrlich, Mathematics Department, University of Maryland, College Park, Md. 20740.

AN ALGEBRAIC METHOD IN DIFFERENTIAL EQUATIONS

MICHAEL GOLOMB, Purdue University

Many college curricula nowadays provide for an introduction to linear algebra and to linear differential equations in the same course or in nearly simultaneous courses. However, no existing textbook of differential equations makes sufficient use of linear algebra to simplify traditional methods and to expose the structural identity of seemingly unrelated problems. This note may serve to illustrate the use of matrix algebra in elementary differential equations.

The traditional "method of undetermined coefficients" applies to equations of the form

$$p(D)y(x) = \sum_{k=1}^{r} p_k(x)e^{\alpha_k x}, \tag{1}$$

where $p(D)$ is a polynomial (of degree $\geqq 1$) with constant coefficients in the differential operator $D=d/dx$, the $p_k(x)$ are polynomials in the independent variable x, and the α_k are distinct complex numbers. The method is an algorithm which does not give the solution in explicit form, and it is often presented without justification for its validity and limitation. We solve (1) by the inversion of a matrix.

We first consider

$$p(D)y(x) = x^n e^{\alpha x} \tag{2}$$

for a nonnegative integer n. Let $V_{n,\alpha}$ denote the $(n+1)$ dimensional vector space spanned by $e^{\alpha x}, xe^{\alpha x}, \cdots, x^n e^{\alpha x}$. D is a linear transformation of $V_{n,\alpha}$ into itself. Using the elements $x^k e^{\alpha x}/k!$ $(k=0, 1, \cdots, n)$ as a basis, we have the representation

$$D \leftrightarrow \begin{bmatrix} \alpha & 1 & 0 \cdots 0 \\ 0 & \alpha & 1 \cdots 0 \\ 0 & 0 & \alpha \cdots 0 \\ \cdot & \cdot & \cdot \quad\quad \cdot \\ 0 & 0 & 0 \cdots \alpha \end{bmatrix} \tag{3}$$

We write (3) as

$$D \leftrightarrow \alpha I + J, \tag{4}$$

where I is the identity matrix, and J is the matrix whose element in the ith row and jth column is $\delta_{i+1,j}$. J is nilpotent, $J^{n+1}=0$. From (4) it follows that the transformation $p(D)$ has the representation $p(\alpha I+J)$. The element $x^k e^{\alpha x}/k!$ of $V_{n,\alpha}$ is represented by the vector $\mathbf{e}_k$ whose ith component is $\delta_{i,k}$. If $y(x)$ is in $V_{n,\alpha}$ and $\mathbf{y}$ is its representing vector then the representation of (2) is

$$p(\alpha I + J)\mathbf{y} = n!\mathbf{e}_n. \tag{5}$$

It is readily verified that the matrix $p(\alpha I+J)$ has an inverse if and only if $p(\alpha)\neq 0$, and that it is given by

$$p^{-1}(\alpha I + J) = q_0(\alpha)I + \frac{1}{1!}q_1(\alpha)J + \cdots + \frac{1}{n!}q_n(\alpha)J^n, \tag{6}$$

where we write q_k for the kth derivative of $1/p$. Therefore, the solution of (5) is

$$
\begin{aligned}
\mathbf{y} &= n!p^{-1}(\alpha I + J)\mathbf{e}_n \\
&= n!\left[q_0(\alpha)\mathbf{e}_n + \frac{1}{1!}q_1(\alpha)\mathbf{e}_{n-1} + \cdots + \frac{1}{n!}q_n(\alpha)\mathbf{e}_0\right].
\end{aligned} \tag{7}
$$

This vector represents

$$
y(x) = \sum_{k=0}^{n} \binom{n}{k} q_k(\alpha)x^{n-k}e^{\alpha x} \tag{8}
$$

and this is the only solution of (2) in $V_{n,\alpha}$ if $p(\alpha)\neq 0$.

We now turn to the singular case $p(\alpha)=0$. Then (2) has no solution in $V_{n,\alpha}$. More generally, assume

$$
p(\alpha) = p'(\alpha) = \cdots = p^{(m-1)}(\alpha) = 0, \qquad p^{(m)}(\alpha) \neq 0. \tag{9}
$$

We show (2) has a solution in $V_{N,\alpha}$ where $N=n+m$. The reduced system (2) (right-hand term $=0$) has the representation

$$
p(\alpha I + J)\mathbf{y} = 0
$$

or

$$
\left[\frac{1}{m!}p^{(m)}(\alpha)J^m + \cdots + \frac{1}{N!}p^{(N)}(\alpha)J^N\right]\mathbf{y} = 0. \tag{10}
$$

It is easily seen that $\mathbf{y}$ is a solution of (10) if and only if the last $N-m+1$ components of $\mathbf{y}$ are 0. Thus, the nullspace of $p(\alpha I+J)$ is spanned by the m vectors $\mathbf{e}_0, \mathbf{e}_1, \cdots, \mathbf{e}_{m-1}$. This means that the solutions of $p(D)y=0$ (in $V_{N,\alpha}$) form the space $V_{m-1,\alpha}$, which is of course a well-known consequence of (9). In the matrix $p(\alpha I+J)$ the last m rows consist of zeros; therefore, in the adjoint matrix $p^*(\alpha I+J)$ the last m columns consist of zeros, and this means that the m vectors $\mathbf{e}_N, \mathbf{e}_{N-1}, \cdots, \mathbf{e}_{N-m+1}$ are in the nullspace of this matrix. By the Alternative Theorem for linear systems it follows that $p(\alpha I+J)\mathbf{y}=\mathbf{f}$ has a solution if and only if the last m components of the vector $\mathbf{f}$ are 0. Consequently, $p(D)y=f$ with f in $V_{N,\alpha}$ has a solution y in $V_{N,\alpha}$ if and only if f is in $V_{n,\alpha}$. Therefore, (2) has a solution in $V_{N,\alpha}$, but not in $V_{k,\alpha}$ if $k<N$. There are infinitely many solutions in $V_{N,\alpha}$, but, the solution is unique modulo $V_{m-1,\alpha}$. In particular, there is a unique solution $y_*(x)$ which is a linear combination of $x^m e^{\alpha x}, \cdots, x^N e^{\alpha x}$. We will now determine this solution.

The representation of (2) is

$$
\left[\frac{1}{m!}p^{(m)}(\alpha)J^m + \cdots + \frac{1}{N!}p^{(N)}(\alpha)J^N\right]\mathbf{y} = n!\mathbf{e}_n \tag{11}
$$

and if we write $\hat{p}(x)$ for $p(x)/(x-\alpha)^m$ then (11) becomes

$$
J^m\hat{p}(\alpha I + J)\mathbf{y} = n!\mathbf{e}_n. \tag{12}
$$

Since $\mathbf{e}_n = J^m \mathbf{e}_N$, (12) is satisfied if

$$\hat{p}(\alpha I + J)\mathbf{y} = n!\mathbf{e}_N. \tag{13}$$

Equation (12) is of the same form as (5), and its solution is

$$\begin{aligned}\mathbf{y} &= n!\hat{p}^{-1}(\alpha I + J)\mathbf{e}_N \\ &= n!\left[\hat{q}_0(\alpha)\mathbf{e}_N + \frac{1}{1!}\hat{q}_1(\alpha)\mathbf{e}_{N-1} + \cdots + \frac{1}{n!}\hat{q}_n(\alpha)\mathbf{e}_m\right],\end{aligned} \tag{14}$$

where $\hat{q}_k$ is the kth derivative of $1/\hat{p}$, and where we have omitted the part spanned by $\mathbf{e}_{m-1}, \cdots, \mathbf{e}_0$. Thus we have found the desired solution

$$y_*(x) = \frac{n!}{N!}\sum_{k=0}^{n}\binom{N}{k}\hat{q}_k(\alpha)x^{N-k}e^{\alpha x}. \tag{15}$$

We have proved the

THEOREM. *If $p(\alpha) \neq 0$ then equation* (2) *has a unique solution in $V_{n,\alpha}$ and it is given by* (8). *If $p(\alpha) = p'(\alpha) = \cdots = p^{(m-1)}(\alpha) = 0, p^{(m)}(\alpha) \neq 0$, then equation* (2) *has no solution in $V_{n+k,\alpha}$ for $k < m$, but infinitely many solutions in $V_{n+m,\alpha}$; the unique solution with zero component in $V_{m-1,\alpha}$ is given by* (15).

As an example, we consider

$$(D^3 + 1)y(x) = x^3e^{-x}.$$

Here $n = 3, \alpha = -1, p(D) = D^3 + 1, p(\alpha) = 0, p'(\alpha) \neq 0$, hence $m = 1, N = 4$. Furthermore, $\hat{p}(D) = D^2 - D + 1$, $\hat{q}_0(\alpha) = \frac{1}{3}$, $\hat{q}_1(\alpha) = \frac{1}{3}$, $\hat{q}_2(\alpha) = \frac{4}{9}$, $\hat{q}_3(\alpha) = \frac{2}{3}$.

$$y_*(x) = (\tfrac{2}{3}x + \tfrac{2}{3}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{12}x^4)e^{-x}.$$

If we deal with equation (1) and the degree of p_k is n_k then we look for a solution in the space V spanned by

$$e^{\alpha_1 x}, \cdots, x^{n_1}e^{\alpha_1 x}, \cdots, e^{\alpha_r x}, \cdots, x^{n_r}e^{\alpha_r x}.$$

The representation of D in V is the direct sum of representations in $V_{n_1,\alpha_1}, \cdots, V_{n_r,\alpha_r}$. Therefore, $1/p(D)$ is represented by the direct sum of the matrices $p^{-1}(\alpha_k I_k + J_k)$ (see (6)) if $p(\alpha_1)p(\alpha_2) \cdots p(\alpha_r) \neq 0$.

The method of solving differential equation (1) that we presented is based on the fact that V is an invariant subspace of the transformation D. Thus by choosing a basis of V, we obtained a matrix representation of $p(D)$, and inversion of $p(D)$ is reduced to an algebraic problem. The inverse matrix was easily found since the matrix representating $p(D)$ is in canonical form.

EVALUATION OF CERTAIN IMPROPER INTEGRALS BY RESIDUES

S. Melamed and H. Kaufman, McGill University

The treatment in elementary texts of the evaluation of integrals $\int_0^\infty f(x)dx$, f rational, by residues is unnecessarily restricted on two counts: (a) f is assumed even, (b) the region of integration is a half plane. In this note we point out that a certain class of functions is as easily handled as the even functions and that in many cases the half plane of integration can be replaced by a wedge of angle less than 180°. The work was motivated by problems in the texts of Hille [**1**; p. 251, No. 6] and Cartan [**2**; p. 115, No. 23(i)].

THEOREM. *Let f be a function satisfying the conditions*: (i) *f is meromorphic in the wedge $0<\theta<2\pi/n$ (n = positive integer $\geqq 2$) and analytic on the positive real axis*, (ii) *f has a finite number of singular points in this wedge*, (iii) *f is a function of z^n*, (iv) *$\int_{C_R} f(z)dz \to 0$ as $R\to\infty$ where C_R is the arc of a circle of radius R, centered at the origin and bounded by the rays $\theta=0$ and $\theta=2\pi/n$. Then*

$$\int_0^\infty f(x)dx = \frac{2\pi i}{1-\exp(2\pi i/n)} S = -\frac{\pi \exp(-\pi i/n)}{\sin(\pi/n)} S,$$

where S is the sum of the residues at the singular points of f lying in the above wedge.

Proof. We first remark that (iii) and the latter part of (i) imply that f has no singular points on the line $\theta=2\pi/n$. The proof follows the usual lines of integrating $f(z)$ around a suitable contour and applying the residue theorem. The contour in this case consists of a segment of a circle of radius R bounded by a circular arc C_R and the portions of the rays $\theta=0$ and $\theta=2\pi/n$ intercepted by this arc. As $R\to\infty$ the contribution along C_R goes to zero by (iv).

Example 1. $f(z)=1/(1+z^4)$. Here $n=4$, the only pole of f in the wedge $0<\theta<\pi/2$ is at $z=\exp(i\pi/4)$ with residue $(-1/4)\exp(i\pi/4)$. The theorem then leads to the result

$$\int_0^\infty dx/(1+x^4) = \pi/2\sqrt{2}.$$

It should be noted that this could also be considered an example with $n=2$. The ray $\theta=2\pi/n$ is then along the negative real axis, so that the contour becomes the standard semi-circle in the upper half plane. If the latter is used, however, then the residues at *two* poles must be evaluated. Similar remarks apply to any composite value of n.

Example 2. (Cartan.) $f(z)=1/(1+z^n)$, n = positive integer $\geqq 2$. The only pole of f in the wedge $0<\theta<2\pi/n$ is at $z=\exp(i\pi/n)$ with residue $-(1/n)\exp(i\pi/n)$. Thus

$$\int_0^\infty dx/(1+x^n) = \frac{-\pi\exp(-i\pi/n)}{\sin(\pi/n)}\left(-\frac{1}{n}\exp(i\pi/n)\right) = \frac{\pi/n}{\sin(\pi/n)}\cdot$$

With the understanding that the principal value of the integral is taken, the theorem is readily generalized to the case where f has simple poles on the positive real axis (hence by (iii) that it also has simple poles on the ray $\theta=2\pi/n$). If the contour is indented so that the poles on the bounding rays lie outside the contour, one obtains

$$PV\int_0^\infty f(x)dx = \frac{-\pi\exp(-\pi i/n)}{\sin(\pi/n)}\left(S+\frac{1}{2}S'\right),$$

where S is the sum of the residues at the poles in the wedge $0<\theta<2\pi/n$, and S' is the sum of the residues at the simple poles on the rays $\theta=0$, $\theta=2\pi/n$.

Example 3. $f(z) = 1/(a^3 - z^3)$, $a > 0$. Here $n = 3$, f has poles at $z = a$, $z=a\exp(2\pi i/3)$ on the rays $\theta=0$, $\theta=2\pi/3$, and has no poles within the wedge. The corresponding residues are $-1/(3a^2)$, $-(\exp(2\pi i/3))/(3a^2)$, whence

$$PV\int_0^\infty dx/(a^3 - x^3) = \frac{\pi}{3a^2}\cot(\pi/3) = \pi/(3\sqrt{3}a^2).$$

Another problem in Cartan is the evaluation of the integral $\int_0^\infty x^k/(1+x^n)dx$ $(n>1+k>0)$ by the same type of contour. The present results can be extended to cover the more general case $\int_0^\infty x^k f(x)dx$, where f is a meromorphic function of x^n.

References

1. E. Hille, Analytic Function Theory, vol. 1, Ginn & Co., Boston, Mass., 1959.
2. H. Cartan, Elementary Theory of Analytic Functions of One or Several Complex Variables, Addison-Wesley, Reading, Mass., 1963.

ON *L*- AND *R*-INTEGRALS

K. V. RAJESWARA RAO, Harvard University

In this note we give a proof—presumably new—of the following result on integrals of real functions.

THEOREM. *If f is a bounded real-valued function on the interval $[a, b]$ which is continuous except at points of a set S of Lebesgue measure zero, then f is both L- and R-integrable and*

$$L\int_a^b f = R\int_a^b f.$$

Here L stands for Lebesgue and R for Riemann. Our proof has the advantage that it works with equal facility whatever be the approach to the Lebesgue theory.

To proceed with the proof, let $\overline{F}$ and $\underline{F}$ be, respectively, the upper and lower

Riemann-Darboux *indefinite* integrals of f. We need to show that f is Lebesgue integrable and

$$\overline{F}(b) - \overline{F}(a) = L\int_a^b f = \underline{F}(b) - \underline{F}(a). \tag{1}$$

If $|f(x)| \leqq M$, $a \leqq x \leqq b$, then

$$|\overline{F}(x_1) - \overline{F}(x_2)| \leqq M(x_2 - x_1), \qquad a \leqq x_1 < x_2 \leqq b. \tag{2}$$

Further, the standard argument for the differentiability of the indefinite integral of a *continuous* function shows that

$$\overline{F}'(x) = f(x), \quad a < x < b, \quad x \notin S. \tag{3}$$

If we now set, for $x > b$, $\overline{F}(x) = \overline{F}(b)$, then the sequence of continuous (and hence Lebesgue integrable) functions

$$f_n(x) \equiv n[\overline{F}(x + 1/n) - \overline{F}(x)]$$

is, in view of (2), uniformly bounded on $[a, b]$, and in view of (3), converges almost everywhere to $f(x)$. Hence, by the Lebesgue dominated convergence theorem, f is Lebesgue integrable and

$$\lim_{n\to\infty} L\int_a^b f_n = L\int_a^b f. \tag{4}$$

But, by the translation-invariance of the L-integral,

$$L\int_a^b f_n = n\cdot\int_b^{b+1/n} \overline{F} - n\cdot\int_a^{a+1/n} \overline{F}; \tag{5}$$

here, the first term on the right is $\overline{F}(b)$, and the second, because of the (right) continuity of $\overline{F}$ at a tends, as $n\to\infty$, to $\overline{F}(a)$. Thus (4) and (5) yield the first equality of (1), and a similar argument applied to $\underline{F}$ completes the proof of the theorem.

REMARK. If one does not wish to take the translation-invariance of the L-integral as known, one can obtain (5) by the following elementary argument: both the functions

$$L\int_a^x \overline{F}(t + 1/n)\,dt \quad \text{and} \quad L\int_{a+1/n}^{x+1/n} \overline{F}(t)\,dt$$

are continuous on $[a, b]$, have the same derivative on (a, b) and the same value at $x = a$. Hence, by the mean value theorem of the differential calculus, they are equal for all x, $a \leqq x \leqq b$.

STRUCTURE THEOREM FOR OPEN SETS OF REAL NUMBERS

A. E. LABARRE, JR., Fresno State College

If G is a nonempty open set of real numbers, then G is the union of a countable number of disjoint open intervals.

Proof. We shall define an equivalence relation S on G, and obtain the component intervals as the equivalence classes determined by S.

Let $\overline{xy}$ denote the nonempty closed interval with endpoints x, y, and let

$$S = \{(x, y) \mid x, y \in G, \overline{xy} \subset G\}.$$

The binary relation S is an equivalence relation on G since

$$(1) \qquad (x, x) \in S \quad \text{for all } x \in G.$$

$$(2) \qquad (x, y) \in S \Rightarrow (y, x) \in S.$$

$$(3) \qquad (x, y) \in S \quad \text{and} \quad (y, z) \in S \Rightarrow (x, z) \in S.$$

The equivalence classes $C_x = \{y \mid \overline{xy} \subset G\}$, $x \in G$, form a partition of G, that is:

$$\bigcup_{x \in G} C_x = G \quad \text{and} \quad x, y \in G \quad \text{with} \quad C_x \neq C_y \Rightarrow C_x \cap C_y = \varnothing.$$

We show next that for arbitrary $x \in G$, C_x is an open interval. By definition, $C_x = \{y \mid \overline{xy} \subset G\}$. Let

$$a_x = \inf_{y \in C_x} \{y\}, \qquad b_x = \sup_{y \in C_x} \{y\}.$$

(We allow the infimum and supremum of a set to be infinite and adopt the usual convention that $-\infty < y < \infty$ for all real numbers y.) Then $a_x < b_x$. For, by definition of a_x and b_x, we have $a_x \leqq b_x$, and $a_x = b_x$ is impossible since G is open and $x \in G$ implies that there exists a neighborhood of x contained in G. Thus $I_x = \{y \mid a_x < y < b_x\}$ is an open interval and it is clear from the way in which a_x, b_x were obtained that $I_x \subseteq G$. Assuming that either a_x or b_x belongs to G leads to a contradiction (again we need only the definition of a_x, b_x and the assumption that G is open). Thus the equivalence classes determined by S are open intervals.

Finally, the proof that the equivalence classes (open intervals here) are countable in number can be given in the standard way by associating with each interval a single rational number belonging to that interval and showing that the function defined in this way has an inverse because the open intervals forming the domain of the function are pairwise disjoint.

The idea of the proof can be generalized to open sets in finite dimensional Euclidean spaces by replacing the definition of S with

$S = \{(x, y) \mid x, y \in G$ *and* G *contains an arc with end points* x, y *which is the homeomorphic image of the closed interval* $[0, 1]\}$.

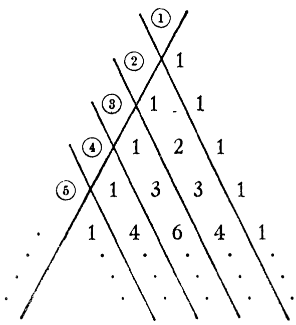
THE DIAGONALS OF PASCAL'S TRIANGLE

D. C. DUNCAN, Los Angeles

Every freshman knows that the nth row of Pascal's triangle gives the coefficients in the expansion of $(a+b)^n$. Much less widely known is the fact that the nth diagonal of the triangle gives the coefficients in the Maclaurin series

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k \quad \text{for } (1-x)^{-n},$$

as shown in the figure below. The author first observed this during his retirement after 47 years of teaching mathematics.



ON THE CARDINAL EQUALITY $2^{\mathfrak{m}} = \mathfrak{m}^{\mathfrak{m}}$

S. E. DICKSON, New Mexico State University

The purpose of this note is to give a short but instructive proof of the equality $2^{\mathfrak{m}} = \mathfrak{m}^{\mathfrak{m}}$ for any infinite cardinal $\mathfrak{m}$.

Clearly it suffices to show that $\mathfrak{m}^{\mathfrak{m}} \leqq 2^{\mathfrak{m}}$. To this end, let M be a set of cardinal $\mathfrak{m}$. We shall show directly that the set of functions from M into M can be placed into one-to-one correspondence with a subset of the set of functions from a set I of cardinal $\mathfrak{m}$ into the set $\{0, 1\}$ of two elements.

For each λ in M, let $I_\lambda = \{(\mu, \lambda) : \mu \in M\}$, so that the family $\{I_\lambda\}_{\lambda \in M}$ is disjoint and so that $I = \bigcup \{I_\lambda : \lambda \in M\}$ has cardinal $\mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$.

Now let f be any function from M into M. Define a function $\mathfrak{C}_f : I \to \{0, 1\}$ as follows: For $(\mu, \lambda) \in I$, let

$$\mathfrak{C}_f(\mu, \lambda) = \begin{cases} 1 & \text{if } \lambda = f(\mu), \\ 0 & \text{if } \lambda \neq f(\mu). \end{cases}$$

Given f, $\mathfrak{C}_f$ is clearly determined uniquely. But suppose also that g is a function from M into M and that $\mathfrak{C}_f = \mathfrak{C}_g$. Then for any $\mu \in M$, $1 = \mathfrak{C}_f(\mu, f(\mu)) = \mathfrak{C}_g(\mu, f(\mu))$,

so that $f(\mu) = g(\mu)$. Since μ was arbitrary in M, $f = g$, which shows that the correspondence is one-to-one. This concludes the proof.

The reader will note that the above proof can be summarized as follows: If M is a set of infinite cardinal $\mathfrak{m}$, the Cartesian product $M \times M$ has cardinal $\mathfrak{m}$. Moreover, any function from M into M determines a unique subset of $M \times M$, namely its graph. It is also clear that the graph determines the function, so that this correspondence is one-to-one. The subsets of $M \times M$, however, are in one-to-one correspondence with the functions from $M \times M$ into $\{0, 1\}$, the number of these being exactly $2^{\mathfrak{m}}$.

Reference

1. W. Sierpinski, Cardinal and Ordinal Numbers, Warsaw, 1958, pp. 153–154.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland
COLLABORATING EDITORS: JOHN D. BAUM, Oberlin College, and
JOHN A. BROWN, University of Delaware

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington, D. C. 20005.

UNIVERSITY PREPARATORY GEOMETRY PROGRAMS: A STUDY OF FOUR NATIONS

FRANCINE ABELES, Newark State College

Introduction. The rethinking of school mathematics is by no means confined to the United States alone. A similar movement is taking place throughout the major nations of Western Europe. There is much that is sound in the European programs that may be used to provide greater insight for further steps in our reform movement. It is the purpose of this report to present current developments in the national programs of Denmark, England and France, and in the states of the Federal Republic of West Germany.

The author has chosen to describe only the university oriented geometry programs in the mathematics streams for several reasons. As indicated in the Cambridge Report [2], the problem of "what shall be the proper direction for the reform in geometry?" is still an unanswered question in the United States. Secondly, the restriction to the university oriented programs provides the maximum information from which to draw guidelines for the articulation of secondary school and college mathematics. Finally, the programs described herein denote the flavor of the pre-university geometry background of future European mathematicians and scientists. Although it is the geometric content that is under study here, it should be understood that only in the existing English program is geometry taught as a separate subject.

12. Emanuel Sperner, Kongruenz und Bewegung, Mitteilungen der mathematischen Gesellschaft in Hamburg, IX (1959) 5–11.

13. Undervisningsministeriet, Det Nye Gymnasium, nr. 269, 1960, 45–47, 58–59, 88–98, 116–117.

14. University of London, Regulations and Syllabuses for the General Certificate of Education Examinations—Summer 1961 and January 1962, London, at the University Press, 42–48.

MATHEMATICAL CONTENT IN SOVIET TRAINING PROGRAMS FOR ELEMENTARY SCHOOL TEACHERS

B. R. VOGELI, Teachers College, Columbia University

The Soviet elementary school is comprised of four grades or levels. Children enter grade I at age seven and complete grade IV at age eleven. Upon completion of the elementary school program pupils enter the four-year middle school and then, if qualified, the two-year secondary school. The mathematics curriculum in grades I–IV is comparable in scope and content to the American elementary school program. Altogether, Soviet children receive 840 hours of instruction in mathematics in grades I–IV in comparison with an approximate total of 720 hours for pupils completing the six-year American elementary school.

Soviet teachers in grades I–IV are not mathematics specialists. Like American elementary school teachers, they teach all subjects at one grade level. The majority of Soviet elementary school teachers are prepared in normal schools [pedagogicheskikh uchilishch] operated for this purpose alone. Middle and secondary school teachers are prepared principally in pedagogical institutes. Recently, departments of elementary education have been established in pedagogical institutes for the purpose of preparing elementary school teachers.

Normal school curricula, like all other Soviet school programs, are dictated by the Ministry of Education. At present, two curricula—a two-year program and a four-year program—are mandatory for all normal schools. Prerequisites for admission to the two-year program include a secondary school education (10 years) and satisfactory performance on oral entrance examinations in Russian language, literature and arithmetic [1]. Admission requirements for the four-year program include a middle-school education (8 years) and satisfactory performance on examinations in Russian language and literature, arithmetic, algebra, and geometry [2].

Students completing the four-year curriculum accumulate 4551 hours of instruction in all areas. In addition, each student completes 1225 hours of various kinds of "pedagogical practice" with elementary school pupils. The two-year program consists of 1923 hours of instruction and 593 hours of pedagogical practice. In contrast, graduates of American four-year colleges and universities receive approximately 2000 hours of classroom instruction.

The four-year normal school curriculum is comprised of three cycles or sequences—the general education sequences, the professional sequence, and the practical experience sequence. All three sequences are in operation concurrently

throughout the four-year program. Table I is a compilation of the courses included in the general and professional sequences.

The two-year normal school program essentially is the four-year program with the general sequence deleted. Since two-year entrants already have completed analogous secondary school courses, this deletion is entirely reasonable. Certain reductions in the hours allocated topics in the two-year professional sequence occur; however, the content and sequence remain more or less intact. For example, two-year students study arithmetic and methods of teaching arithmetic for 271 hours rather than 317 as in the four-year curriculum.

Mathematics is assigned an important role in both the general and professional sequences. Approximately 23% of the general sequence and 13% of the professional sequence are devoted to mathematical studies including methods of teaching arithmetic. If one semester-hour credit in an American college or university is taken as the equivalent of sixteen class hours of instruction in a Soviet institution, the 405-hour mathematics program together with the 317-hour arithmetic course equal fifteen three-semester hour American courses.

The general sequence mathematics course is similar in scope and content to the mathematics program of the Soviet secondary school. Since all students enrolled in the four-year normal school program have completed only the middle school, this portion of the normal curriculum serves to complete their secondary education in mathematics. Standard secondary school mathematics texts are used by normal school students.

Like the secondary school mathematics program, the normal school program is comprised of two subsubjects—(a) algebra and elementary functions and (b) geometry. Table II is an abridged translation of the official syllabus for this portion of the general sequence.

The professional course "Arithmetic and methods of teaching arithmetic" is neither a review nor an extension of the Soviet middle school arithmetic program. Rather, the "course in arithmetic at normal schools introduces a series of topics of both theoretical and practical importance which are necessary for meeting graduation requirements and for the practical work of the elementary school teacher" [5]. Table III is an abridged translation of the official syllabus for the normal school arithmetic course.

Standard texts for the course in arithmetic and methods of teaching arithmetic are: B. A. Tulinov i Ia. F. Chekmarev, *Arifmetika dlia peduchilishch*, Izd. 7, Moscow: Uchpedgiz, 1963, and Ia. F. Chekmarev i V. T. Snegirev, *Metodika Arifmetiki dlia peduchilishch*, Izd. 12, (pererab.) Moscow: Uchpedgiz, 1962 [7].

In addition to these basic textbooks, problem texts, supplementary texts, and standard school textbooks for grades I–IV are consulted frequently.

The Tulinov-Chekmarev text begins with a discussion of the set concept and set operations. The concept of one-to-one correspondence is introduced and classified as an equivalence relation by reference to the usual axioms.

Each natural number is introduced as an expression for the abstract char-

acteristic shared by all members of a class of equivalent sets. Later the system of natural numbers is defined using the Peano axioms.

Numeration systems, both historical and in bases other than 10, are discussed extensively. The operations of addition and multiplication are introduced by reference to operations with sets. The properties of each operation are emphasized—commutativity, associativity, monotonicity, and so on. Subtraction and division are defined as the inverses, respectively, of addition and multiplication. Division by zero is discussed and excluded. Special attention is given the order of operations.

TABLE I

COURSES INCLUDED IN THE GENERAL AND PROFESSIONAL SEQUENCES OF THE SOVIET FOUR-YEAR NORMAL SCHOOL CURRICULUM [3]

	Hours per week per semester								*Total*
	I	II	III	IV	V	VI	VII	VIII	*Hours*
General Sequence									
1. History of the USSR	2	2	2	2	2	2			230
2. Modern history		1	2	2	1	1			137
3. Political science							3	3	96
4. Mathematics	4	3	2	3	2	2	2	4	405
5. Physics and astronomy	2	2	2	2	2	2	2	2	294
6. Chemistry and geology	3	3	3	2					215
7. Economic geography							4	4	128
8. Biology							2	4	96
9. Foreign language	2	2	2	2	1				175
10. Physical education	2	2	2	2	2	2			247
Professional Sequence									
11. Russian language Methods of teaching language and penmanship	4	4	2	3	3	3	3	3	464
12. Literature (general and children's)	4	2	3	2	3	2	3	3	394
13. Arithmetic and methods of teaching arithmetic	2	2	2	3	3	2	2	1	317
14. Natural science and natural science methods				2	2	3			135
15. Methods of teaching history						1			19
16. Anatomy, physiology, school hygiene	2	2							80
17. Psychology		4							92
18. Pedagogy			4	2	2	2	2	2	248
19. Singing and methods of teaching singing	2	2	2	2	2	2	1	2	278
20. Drawing and sculpture	2	2	1	2	2	2	2	2	277
21. Physical education methods				1	1	1	2		90
22. Supplementary academic material	2		1			1		4	134
Totals	33	33	31	32	28	28	28	34	4551

Theorems concerning operations with natural numbers are stated and proved using the "basic properties."

TABLE II

MATHEMATICS SYLLABUS: GENERAL SEQUENCE. FOUR-YEAR CURRICULUM [4]

Algebra and Elementary Functions	Geometry
First Semester	
1. Equations and inequalities of first degree (18)*	
2. Real numbers, quadratic equations (23)	
3. Powers with rational exponents, power functions (27)	
Second Semester	
4. Trigonometric functions of an arbitrary argument (39)	
	5. Metric relations in a triangle and in a circle (5)
	6. Geometric transformations in a plane (25)
Third Semester	
	7. Solution of triangles (21)
8. Trigonometric addition theorems and their corollaries (13)	
Fourth Semester	
9. Transformation of the sums and differences of trigonometric functions into products. Inverse transformations (8)	
10. Numerical sequences (17)	
11. Exponential and logarithmic functions (24)	
	12. Basic concepts of solid geometry, parallelism in space (17)
Fifth Semester	
	13. Perpendicularity in space. Dihedral and polyhedral angles (16)
	14. Polyhedra (18)
Sixth Semester	
	15. Surfaces and solids of revolution (18)
16. Functions and limits (12)	
17. The derivative and its use in investigating functions (8)	
Seventh Semester	
18. Derivatives of sine and cosine (20)	
19. The generalized concept of number. Complex numbers (12)	
Eighth Semester	
20. Solution of problems and review of algebra (18)	
	21. Measurement of volume (23)
	22. Solution of problems and review of geometry

* The numeral in parentheses following each topic indicates the number of class hours allocated to the topic.

TABLE III

ARITHMETIC AND METHODS OF TEACHING ARITHMETIC—FOUR-YEAR NORMAL SCHOOL CURRICULUM [6]

Arithmetic	Methods of Teaching Arithmetic
First Semester	
1. Natural numbers (34)	
Second Semester	
2. Multiplication of natural numbers (8) 3. Division of natural numbers (19) 4. Measurement (19)	
Third Semester	
5. Divisibility criteria (24) 6. Review (10)	
Fourth Semester	
7. Common fractions (24) 8. Decimal fractions (32) 9. Percent (10)	
Fifth Semester	
10. Approximate computation (8) 11. Ratio and proportion (9)	1. Problems of teaching arithmetic and the official arithmetic syllabus for elementary schools (3) 2. Methods of presenting arithmetic material (4) 3. Visual aids (2) 4. Organization of arithmetic instruction (3) 5. Evaluation in arithmetic (2) 6. Organization of instruction in arithmetic in two-room schools (2) 7. Methods of solving arithmetic problems (9) 8. Methods of oral and written examination (9) 9. The first decade (3) 10. The second decade (3)
Sixth Semester	
	11. Methods of teaching about whole numbers (15) 12. Methods of teaching the relationships among the components and the results of operations (2) 13. Methods of teaching denominate numbers (5) 14. Methods of teaching simple fractions (2) 15. Methods of teaching intuitive geometry (5) 16. Methods of teaching field measurement (4) 17. Methods of organizing extracurricular activities (3) 18. Short historical survey of the teaching of arithmetic (2)
Seventh Semester	
12. Solution of problems (20)	
Eighth Semester	
13. Review of the arithmetic course (16)	

THEOREM. *In order to divide a product by a given natural number, it is sufficient to divide one of the factors by the given number.*

Given: abc and n, natural numbers with $a \div n = q$.
To prove: $abc \div n = qbc$.

$(abc) \div n = (nq)bc \div n$	Given $a \div n = q$ or $a = nq$
$= nqbc \div n$	Associativity
$= qbcn \div n$	Commutativity
$= [(qbc)n] \div n$	Associativity
$= qbc$	definition of division [8].

The text contains standard number theoretic topics (divisibility criteria, the Euclidean Algorithm, GCD, LCM, proof of the fundamental theorem of arithmetic, primes, and composites, and so on), as well as more practical material (common and decimal fractions, percent, metric measure, approximate computation, ratio and proportion). Fractions are introduced as "fractional parts" as well as "answers to division problems involving two natural numbers" [9].

Work with common and decimal fractions is rather manipulative. Many rules for operating with fractions are stated as theorems with "proofs" given or required. The remainder of the text is a careful but dry presentation of approximate computation and the theory of ratio and proportion.

The Chekmarev-Snigirev methods text consists of the usual collection of teaching tips, visual aids and mathematical nonsense found in methods texts of any national origin.

Although the arithmetic syllabus is reasonably demanding and the standard texts at least semi-modern by American standards, the arithmetic knowledge of normal school graduates does not seem superior to that of American teachers. Soviet elementary school teachers observed by this writer stumbled over the same concepts and techniques that plague American teachers. Several explanations are possible. First of all, normal school entrants are not of particularly high academic caliber. Student responses in normal school classes attended by this writer were uniformly of lower quality than responses given by secondary school pupils in comparable mathematics courses. A second factor which certainly contributes to the ineffectiveness of the normal school mathematics program is the relatively low quality of instruction. Normal school classes attended by this writer seldom were as well executed as mathematics lessons in better Soviet secondary schools. While Soviet secondary school teachers frequently seemed to strive to modernize or "improve the scientific standards" of the school mathematics program, normal school instructors were unwilling to exploit the theoretical material already found in the official texts.

Both two-year and four-year curricula are concluded by a series of government certification examinations in political science, pedagogy, Russian language, and arithmetic. The arithmetic examination is oral and is based upon a list of questions distributed well in advance of the actual test. Normally, the examining board is comprised of teachers from the normal school, faculty members from

local institutes and universities and a representative of the Ministry of Education. Examinations witnessed by this writer were rather formal—even stuffy—with memorized responses rather than thoughtful explanations predominating.

It may be informative to compare the Soviet normal school program with the recommendations for the training of teachers of elementary school mathematics prepared by the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America. CUPM recommendations for the elementary level (Level I) include as a prerequisite "at least two years of college preparatory mathematics consisting of a year of algebra and a year of geometry or the same material in integrated courses" [**10**]. Since the Soviet middle school curriculum includes 285 hours of instruction in algebra and 246 hours of instruction in geometry, all Soviet normal school entrants clearly meet this initial requirement.

For the "college training" of elementary school teachers CUPM recommends the equivalent of the following courses [**11**]:

(A) A two-course sequence devoted to the structure of the real number system and its subsystems.

(B) A course devoted to the basic concepts of algebra.

(C) A course in informal geometry.

It is further noted that "the material in these courses might, in a sense, duplicate material studied in high school by the prospective teacher, but we urge that this material be covered again, this time from a more sophisticated, college-level point of view" [**12**].

It is clear from Table II that the Soviet four-year curriculum meets and exceeds CUPM requirements (B) and (C), although not all of the geometry program can be classified as informal. The four-year curriculum includes 239 hours of instruction in algebra and elementary functions and 166 hours of instruction in geometry. The Soviet two-year curriculum appears to fall short of the CUPM recommendations in algebra and geometry, however, students entering the two-year program already have completed comparable courses in the secondary school. Table III indicates that the arithmetic curricula of both the two- and four-year programs, although restricted to the structure of the system of positive rationals, are at least the equivalent of a two-course sequence in higher arithmetic.

Thus, both the Soviet two-year and four-year programs seem compatible with the CUPM level I recommendations. The point of view taken in Soviet mathematics courses for prospective elementary school teachers, however, may not be "more sophisticated" than secondary school courses as recommended by CUPM. Indeed, much of the Soviet curriculum is standard secondary school material. Perhaps the significant point is not that Soviet curricula fail to exceed secondary school standards, but rather that they insure mastery by all prospective elementary school teachers of secondary school mathematics.

References

1. For a complete description of this examination see: Pravila priema i programmy dlia postupaiushchikh v spednie spetzial'nye uchebnye zavedeniia SSSR v 1964 godu. Moscow: Izd. "Vysshaia shkola," 1964, pp. 9, 48–9.

2. *Ibid.*, pp. 17–24.

3. Uchebnye plany—*pedagogicheskikh uchilishch.* Moscow: Ministerstvo prosveschcheniia RSFSR, 1964, pp. 3–5.

4. Programmy shkol'nykh pedagogicheskikh uchilishch—Matematika. Moscow: Uchpedgiz, 1963.

5. Programmy shkol'nykh pedagogicheskikh uchilishch—Arifmetika. Metodika prepodavaniia arifmetiki. Moscow: Uchpedgiz, 1963, p. 5.

6. *Ibid.*, pp. 9–15, 30–36.

7. *Ibid.*, pp. 24, 42.

8. B. A. Tulinov i Ia. F. Chekmarev, *Arifmetika dlia peduchilishch.* Moscow: Uchpedgiz, 1963, p. 71.

9. *Ibid.*, p. 165.

10. Recommendations for the training of teachers of Mathematics, this MONTHLY, 67 (1960) 987.

11. *Ibid.*

12. *Ibid.*

A MEMORABLE TEACHER

E. T. PARKER, University of Illinois

Much is said and written about the factors, especially educational, leading to the development of research mathematicians. In looking back over the years, I feel certain that one course was crucial in my early constructive exposure to higher mathematics. (Naturally, no one's research career is fully determined by a single course; other professors contributed to my later development. Nonetheless, this course set my interests on the paths of that development.)

It was the summer of 1947. I was commencing graduate studies, having just completed a B.A. with honors in mathematics. The late Professor Claiborne G. Latimer visited Northwestern University that summer to teach a course in finite groups. Professor Latimer's style of presentation in that course exerted a vital influence on my career both in research and teaching.

Surprisingly little material was presented in that course of nine weeks' duration; the Sylow theorems and the basis theorem for abelian groups were not proved. Professor Latimer actually proved *nothing* for at least the first half of the course, and hardly so much as mentioned a formal definition. The reader may ask in astonishment, "What did the man do in five weeks of lecturing?"

He started by drawing a square with the vertices numbered 1, 2, 3, 4; then displaying the eight permutations of the vertices corresponding to rigid motions of the square. Each lecture was opened with a repetition thereof, followed soon by Professor Latimer's favorite phrase, in slow Southern drawl, "It might be interesting to see whether . . . ." Using only the set of rigid motions of the square, which Professor Latimer rarely designated as a *group*, he went through

the development for this barely nontrivial example of a long list of concepts: subgroup, normal subgroup, conjugate subgroups and elements, characteristic subgroup, center, commutator subgroup, automorphism, inner automorphism, homomorphic image, and thus virtually the whole range of basic ideas in finite group theory. His motivation glittered like a polished gem.

My certainty is growing each year that Professor Latimer's remarkable course left me with a lifetime enthusiasm for group theory and related topics in finite mathematics. Without this important early influence, it seems doubtful whether I would have achieved my major victory in research, part of the disproof of Euler's conjecture.

Furthermore, my experience with Professor Latimer's course has filled me with the conviction that a mathematics course should not be the transcription of a treatise from the professor's notes to the student's via the blackboard. Far better to play around leisurely with concepts than to present polished proofs of unmotivated theorems at top speed! I cannot successfully imitate Professor Latimer's inimitable style; but I am determined to follow in his footsteps in pedagogical viewpoint!

SCIENTIFIC MANPOWER COMMISSION

With reference to the various publications of the Scientific Manpower Commission mentioned on page 778 of the August–September 1965 issue of the MONTHLY, inquiries and orders for booklets should be directed to the Scientific Manpower Commission at 2101 Constitution Avenue, N. W., Washington, D. C. 20418 and *not* to the Executive Secretary of the Conference Board of the Mathematical Sciences.

PROBLEMS AND SOLUTIONS

Elementary Problems

Solutions of Elementary Problems should be sent to A. E. Livingston, Dept. of Math., University of Alberta, Edmonton, Alberta, Canada. To facilitate their consideration, solutions for Elementary Problems in this issue should be submitted on separate, signed sheets and should be mailed before April 30, 1966.

E 1835. *Proposed by Tudor Zamfirescu, Bucharest, Rumania*

A circumference is given in the plane. Show that the construction of an inscribed regular polygon with 13 sides is possible using a straightedge and an angle trisector.

E 1836. *Proposed by D. E. Daykin, University of Malaya, Kuala Lumpur*

Given a convergent series $\sum_{i=0}^{\infty} u_i$ of positive terms u_i, find the greatest lower bound g of $\sum_{i=0}^{\infty} \epsilon_i u_i$ over all sequences $\{\epsilon_i\}$, where ϵ_i is 0 or 1 and $\epsilon_{i-1}+\epsilon_i+\epsilon_{i+1}>0$ for $i\geqq 1$.

E 1837. *Proposed by Yasuhiko Ikebe, University of Texas*

Solve the difference equation in two positive integers, m and n,

$$f(m, n) = n[f(m-1, n-1) + f(m-1, n)],$$

under the condition $f(m, n)=0$ if $m<n$, and $f(m, 1)=1$ for every m.

E 1838. *Proposed by A. Oppenheim, University of Malaya, Kuala Lumpur*

Suppose that ABC is an acute-angled triangle: then

$$16 \prod \cos^2 A + 4 \sum \cos^2 B \cos^2 C \leqq 1, \tag{1}$$

$$4 \sum \cos^2 B \cos^2 C \leqq \sum \cos^2 A. \tag{2}$$

Equality occurs when ABC is equilateral or right-angled isosceles and in no other case.

E 1839. *Proposed by Simeon Reich, The Hebrew Reali Secondary School, Haifa, Israel*

In the triangle ABC, m_b and m_c (the medians to the sides b and c of the triangle) are perpendicular to each other. Prove cot B+cot $C\geqq 2/3$.

E 1840. *Proposed by Simeon Reich, the Hebrew Reali Secondary School, Haifa, Israel*

Prove:

$$\sum_{k=1}^{(x-1)/2} \cos \frac{2k\pi}{x} \Big/ \sum_{k=0}^{(x-3)/2} \sin \left[\frac{\pi}{x}(2k+1)\right] = -\cot \frac{\pi}{2x}\cdot$$

E 1841. *Proposed by D. S. Mitrinović, Belgrade University, Yugoslavia*

If a, b, c, $d(ad\neq bc)$ are complex numbers and t is a real variable, prove the following inequalities:

$$\frac{|a\bar{b}-b\bar{a}|}{|ad-bc|+|a\bar{d}-b\bar{c}|} \leqq \left|\frac{at+b}{ct+d}\right| \leqq \frac{|ad-bc|+|a\bar{d}-b\bar{c}|}{|c\bar{d}-d\bar{c}|}.$$

An algebraic proof is preferred. What will happen when $c\bar{d}-d\bar{c}=0$?

E 1842. *Proposed by Stanton Philipp, Long Beach State College, California*

Evaluate

$$\sum_{r=0}^{n} \binom{n}{r} \sum_{k=0}^{r} (-1)^k \frac{r!}{k!}\cdot$$

E 1843. *Proposed by A. K. Austin, The University, Sheffield, England*

Given integers a and b, define the function $f(x)=ax^2+bx$. Find all integers p, q, r such that $f(p)=f(q)+f(r)$.

E 1844. *Proposed by A. K. Austin, The University, Sheffield, England*

Do there exist points A, B, C such that for all points P in the plane of A, B, C, the distances AD, BD, CD are not all rational?

SOLUTIONS OF ELEMENTARY PROBLEMS

An Old Chestnut

E 1744 [1964, 1133]. *Proposed by Yasser Dakkah, S. S. Boys' School, Qalgilya, Jordan*

In the plane of a given triangle, locate a point whose distances from the vertices have the smallest possible sum of squares.

I. *Solution by Leon Bankoff, Los Angeles, California.* Geometric solutions to this well-known problem may be found in C. A. Durell, *Hints and Solutions to the Exercises in Modern Geometry*, Macmillan, London (1953) p. 28, problem 2; R. A. Johnson, *Modern Geometry*, Dover (reprint), pp. 174–175; W. J. M Clelland, *Geometry of the Circle*, Macmillan (1891) p. 32; and Richardson and Ramsey, *Modern Plane Geometry*, Macmillan (1900) pp. 135–136.

One such solution is based on the identity

$$PA^2 + PB^2 + PC^2 = MA^2 + MB^2 + MC^2 + 3PM^2,$$

where M is the median of triangle ABC.

II. *Solution by Cornelius Groenewoud, Buffalo, New York.* This problem is not new! Except for minor variations in wording, it appears in Franklin, *Methods of Advanced Calculus*, McGraw-Hill (1944), p. 77, problem 101; Smith, Salkover and Justice, *Calculus*, Wiley (1938) p. 458, problem 15; and Woods, *Advanced Calculus*, Ginn (1944) p. 131, problem 15. Further, it is posed for n points in space as problem 6 on page 298 of Buck, *Advanced Calculus*, McGraw-Hill (1956), and solved completely for n points in the plane in Brand, *Advanced Calculus*, Wiley (1955) p. 190, exercise 5.

III. *Solution by Kenneth Kramer, Student, Columbia University.* Let $X_i=(x_{i1}, x_{i2}, \cdots, x_{in})$ $(i=1, 2, \cdots, m)$ be vectors in real Euclidean n-space, and set $n\bar{x}_j=\sum_{i=1}^{m} x_{ij}$ $(j=1, 2, \cdots, n)$. If $X=(x_1, x_2, \cdots, x_n)$, then

$$S = \sum_{i=1}^{m}\sum_{j=1}^{n} (x_j - x_{ij})^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} (\bar{x}_j - x_{ij})^2 + m \sum_{j=1}^{n} (x_j - \bar{x}_j)^2,$$

so that S has its minimum value when $X=(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$, and

$$S_{\min} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^2 - m \sum_{j=1}^{n} \bar{x}_j^2.$$

Also solved by A. N. Aheart, R. G. Albert, M. A. Bershad, M. G. Beumer (Netherlands), D. A. Blaeuer, Andreas Blass, W. J. Blundon, Carolyn Brauer, R. F. Brinker, Arnold Brown, Jim Campbell, L. E. Carver, John Christopher, Allan Chuck and Peter Goldstein (jointly), R. J. Cormier, K. W. Crain, Huseyin Demir (Turkey), Bob De Vore, W. G. Dotson, Jr., Ragnar Dybvik (Norway), R. B. Eggleton (Australia), J. W. Ellis, Helene M. Fink, G. G. Ford, Philip Fung, Michael Goldberg, Hwa Hahn, D. M. Hancasky, J. R. Hanna, R. A. Jacobson, Agnis Kaugars, J. D. E. Konhauser, Hui-hsiang Kuo (Taiwan), George Lang, E. S. Langford, Murray Lieb, J. W. Losse, Robert Maas, D. C. B. Marsh, J. G. Moser, D. E. Moxness, J. B. Muskat, G. L. Musser, Robert Patenaude, George Purdy, Simeon Reich (Israel), L. A. Ringenberg, V. K. Rohatgi, M. S. R. K. Sastry, J. S. Scandale, Jr., P. A. Scheinok, G. P. Speck, Sidney Spital, Emile State, Sister M. Stephanie, M. J. Strauss, A. M. Vaidya, Simon Vatriquant (Belgium), Mary C. Vioni, R. A. Wiesen, F. D. Wilder, Charles Ziegenfus, and the proposer.

Blaeuer found E 1744 as a problem on page 451 of Granville, Smith, and Longley, *Elements of Differential and Integral Calculus*, Ginn (1934); Konhauser located it in M. S. R. Anjaneyulu, *Elements of Modern Pure Geometry*, p. 35, problem 13; and Ziegenfus calls attention to problem 13 on page 432 of Ford and Ford, *Calculus*, McGraw-Hill (1963).

Marbles in the Wine, Yet!

E 1745 [1964, 1133]. *Proposed by E. S. Langford, North American Aviation, Anaheim, California*

(1) Consider a wineglass in the shape of half a prolate spheroid. A marble is introduced into the glass. What is the radius of the smallest marble which will not touch the bottom?

(2) The similar problem for a paraboloid of revolution.

Solution by Hwa Hahn, Pennsylvania State University. Choose coordinate axes so that the wineglasses have equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{a^2}=1,\quad -b\leqq y\leqq 0\ \ (a<b);\qquad ax^2+az^2=y.$$

It is enough to consider the radii of curvature of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad -b\leqq y\leqq 0\ \ (a<b);\qquad ax^2=y.$$

The formula $R=(1+y'^2)^{3/2}/|y''|$ gives for these radii

$$[a^4+(b^2-a^2)x^2]^{3/2}/(a^4b)\quad\text{and}\quad(1+4a^2x^2)^{3/2}/(2a),$$

respectively, both of which are increasing functions of x. It is then clear that there is no smallest marble which will not touch the bottom of the glass, but there is a largest one which will, and its radius is given by evaluating the appropriate formula above at $x=0$, giving a^2/b and $1/(2a)$, respectively.

Also solved by John Beidler, J. A. Burslem, R. J. Cormier, M. S. Demos, Ragnar Dybvik (Norway), R. B. Eggleton (Australia), Fay Glasser, Michael Goldberg, Agnis Kaugars, J. D. E. Konhauser, Kenneth Kramer and Steven Minsker (jointly), D. C. B. Marsh, Robert Patenaude, J. S. Scandale, Jr., Robin Sibson (England), Sidney Spital, F. D. Wilder, Kenneth Yanosko, and the proposer.

Consecutive Divisors of Consecutive Integers

E 1746 [1964, 1133]. *Proposed by Michael Fried, University of Michigan.*

Given positive integers k and n, find all integers x such that $n|x$, $n+1|x+1$, $\cdots$, $n+k|x+k$.

Solution by Gary G. Ford, Student, University of Santa Clara, Santa Clara, California. Since $n|x$, we have $x=n+y$ for some nonnegative integer y ($x=0$ is clearly not a solution). Then $(n+i)|(x+i)$ ($i=0, 1, 2, \cdots, k$) implies that $(n+i)|\{(n+i)+y\}$ and, hence, that $n, n+1, \cdots, n+k$ divide y. Therefore, $x=n+tL$ for some nonnegative integer t, where L is the least common multiple of $n, n+1, \cdots, n+k$. That such an x is indeed a solution is evident.

Also solved by A. N. Aheart, R. G. Albert, Robert Bart, John Beidler, M. A. Bershad, M. G. Beumer (Netherlands), D. A. Blaeuer, Andreas Blass, W. J. Blundon, W. H. Bonney, Carolyn Brauer, J. A. Burslem, Mannis Charosh, John Christopher, R. J. Cormier, M. S. Demos, R. B. Eggleton (Australia), J. W. Ellis, John Ganci, P. K. Garlick, R. W. Gilmer, Jr., Michael Goldberg, Jerry Goodman, Cornelius Groenewoud, Martin Gugino, Hwa Hahn, D. M. Hancasky, B. A. Hausmann, S.J., R. A. Jacobson, Agnis Kaugars, E. S. Langford, Steve Ligh, C. C. Lindner, G. F. Lowerre, Robert Maas, Matthew Maddox and Humbert Norie (jointly), D. C. B. Marsh, Norman Miller, J. B. Muskat, G. L. Musser, Robert Patenaude, C. B. A. Peck, Harsh Pittie, Donald Quiring, S. M. Robinson, Robert Russell, P. A. Scheinok, Ralph Schreiber, Robin Sibson (England), D. L. Silverman, Mitchell Snyder, R. L. Syverson, A. M. Vaidya, Richard Vande Velde, S.J., Simon Vatriquant (Belgium), Emanuel Vegh, Gerald Werner, Kenneth Yanosko, and the proposer.

Charosh calls attention to the following problem which he proposed in the Mathematics Student Journal (January, 1962): A man and his grandson have the same birthday. For six consecutive birthdays the man is an integral number of times as old as his grandson. How old is each at the sixth of these birthdays?

$\sigma(kn-1)\equiv 0 \pmod{k}$

E 1747 [1964, 1133]. *Proposed by Joseph Arkin, Spring Valley, N. Y.*

If $\sigma(n)$ is the sum of the divisors of n, show that $\sigma(6q+5)\equiv 0 \pmod 6$ for all positive q. Is this an instance of a more general rule?

I. *Solution by Vincent C. Harris, San Diego State College, San Diego, California.* The result of the problem is a special case of the known fact that $\sigma(kq-1)\equiv 0 \pmod{k}$ for all nonnegative integers q if and only if $k>1$ and $k|24$ [K. G. Ramanathan, *Congruence properties of $\sigma(n)$, the sum of the divisors of n*, Math. Student, 11 (1943) 33–35, (Math. Rev., 6 (1945), 37), and H. Gupta, *Congruence properties of $\sigma(n)$*, Math. Student, 13 (1945) 25–29, (Math. Rev., 7 (1946) 273)].

Other references to this and similar problems: K. G. Ramanathan, *Congruence properties of Ramanujan's function $\tau(n)$*, Proc. Indian Acad. Sci., Sect. A, 19 (1944) 146–148, (Math. Rev., 6 (1945), 37); K. G. Ramanathan, *Congruence properties of $\sigma_a(n)$*, Math. Student, 13 (1945) 30, (Math. Rev., 7 (1946) 273); M. V. Subba Rao, *Congruence properties of $\sigma(n)$*, Math. Student, (1950) 17–18, (Math. Rev., 12 (1951) 675); V. C. Harris and M. V. Subba Rao, *Congruence properties of $\sigma_r(N)$*, Pac. Jour. Math., 12 (1962) 925–928.

II. *Solution by E. S. Langford, U. S. Naval Postgraduate School, Monterey, California.* The following two statements are equivalent for any positive integer n:

(1) $a^2 \equiv 1 \pmod{n}$ whenever $(a, n) = 1$,

(2) $\sigma(a) \equiv 0 \pmod{n}$ whenever $a \equiv -1 \pmod{n}$.

Furthermore, the only positive integers n for which (1) holds are the divisors of 24. (The problem at hand is of course the special case $n = 6$.)

The asserted equivalence is obvious for $n = 2$, so we henceforth assume that $n > 2$.

Assume first that (1) holds, and let a be such that $a \equiv -1 \pmod{n}$. Since $(a, n) = 1$, a is not a perfect square. (If $a = b^2$, then $(b, n) = 1$ and $-1 \equiv a = b^2 \equiv 1 \pmod{n}$ by (1).) Therefore a has an even number of divisors so that $\sigma(n) = \sum(d + a/d)$, where $d|a$ and $d < a^{1/2}$. If $d|a$, then $(d, n) = 1$, so that $d^2 \equiv 1 \pmod{n}$. Therefore $d(d + a/d) = d^2 + a \equiv 1 - 1 \equiv 0 \pmod{n}$. Since $(d, n) = 1$, this implies that $d + a/d \equiv 0 \pmod{n}$, and hence that $\sigma(a) \equiv 0 \pmod{n}$.

Conversely, suppose that n (>1) does not have property (1), so that there is a prime p for which $(p, n) = 1$ and $p^2 \not\equiv 1 \pmod{n}$. Then $px + ny = -1$ for appropriate integers x and y, and $(x, n) = 1$. By Dirichlet's Theorem, there is a prime $q \neq p$ for which $q = x \pm nt$ for some positive integer t. Thus, $-1 = p(x \pm nt) + n(y \mp pt) = pq + n(y \mp pt) \equiv pq \pmod{n}$. Suppose that $\sigma(pq) \equiv 0 \pmod{n}$. Then $0 \equiv p\sigma(pq) \equiv p(1 + p + q + pq) \equiv p(p + q) \equiv p^2 + pq \equiv p^2 - 1 \pmod{n}$, a contradiction.

It is well known that the only positive integers n for which (1) is true are the divisors of 24, but we give an argument for the sake of completeness. Recall the

THEOREM. *If $(c, n) = 1$ and $x^2 \equiv c \pmod{n}$ has a solution, then it has exactly $2^{\alpha+\beta}$ solutions where α is the number of distinct odd divisors of n and β is* 0, 1, *or* 2 *according as* $4 \nmid n$, $4 \| n$, *or* $8 | n$.

(See, for example, W. J. Le Veque, *Topics in Number Theory I*, Addison-Wesley, (1956) 65.) Suppose now that $n = 2^r \prod p^s$ where p denotes an odd divisor of n. Taking $c = 1$ in the Theorem and assuming that (1) is true, we must have $2^{\alpha+\beta} = \phi(n)$ and, hence, $2^\beta = 2^t \prod \{p^{s-1}(p-1)/2\}$, where t is the larger of $\alpha - 1$ and 0. It follows immediately that $s = 1$ and $1 = \prod\{(p-1)/2\}$ and then that $n | 24$.

(*Editor's comment:* Only one of the articles listed under Solution I was available to the Editor and it did not contain a resolution of E1747. Furthermore, the reviews there cited do not convey the complete nature of the proof. Hence, Solution II.)

III. *Solution by R. B. Eggleton, Avondale College, Cooranbong, N.S.W., Australia.* We deduce the result as a corollary to the

THEOREM. *For any prime p which divides $am + x$ to odd multiplicity,* $\sigma(am + x) \equiv 0 \pmod{p+1}$. *At least one such prime exists if x is a quadratic nonresidue modulo m.*

Proof. If the canonical factorization of $am+x$ is $\prod p^\alpha$, then $\sigma(am+x) = \prod \sum_{k=0}^{\alpha} p^k$. Since

$$\sum_{k=0}^{\alpha} p^k \equiv \begin{cases} 0 & (\text{mod } p+1) \quad \text{if } 2\nmid\alpha, \\ 1 & (\text{mod } p+1) \quad \text{if } 2\mid\alpha, \end{cases}$$

it follows immediately that $\sigma(am+x)\equiv 0 \pmod{p+1}$ when α is odd. Moreover, if x is a quadratic nonresidue modulo m, $am+x$ cannot be a perfect square . . . so there must be a prime factor p of $am+x$ for which α is odd.

For the problem at hand, $6q+5$ can only contain prime factors congruent to $\pm 1 \pmod 6$, and there must be one, say p, congruent to $-1 \pmod 6$, i.e., for which $6\mid(p+1)$. Since 5 is a quadratic nonresidue modulo 6, we have $\sigma(6q+5) \equiv 0 \pmod 6$.

Also solved by M. G. Beumer (Netherlands), Walter Bluger, Harry Felder III, R. W. Gilmer, Jr., Jerry Goodman, Hwa Hahn, Kenneth Kramer, Andrzej Makowski (Poland), D. C. B. Marsh, D. S. Martin, C. B. A. Peck, D. P. Roselle, David Singmaster, D. P. Tihansky and the proposer (all of whom supplied generalizations of one kind or another), as well as the following who solved the first part of the problem (and in some instances observed that "obvious" generalizations were not true): J. C. Abad, A. N. Aheart, M. A. Bershad, W. H. Bonney, D. I. A. Cohen, Huseyin Demir (Turkey), G. G. Ford, D. M. Hancasky, B. A. Hausmann, S.J., A. Himmelfarb, R. A. Jacobson, Agnis Kaugars, C. C. Lindner, Robert Maas, Leonora Mangine, G. L. Musser, Robert Patenaude, Harsh Pittie, Simeon Reich (Israel), Ralph Schreiber, Robin Sibson (England), D. L. Silverman, M. J. Strauss, Vis Upatisringa, and A. M. Vaidya.

Gilmer, Goodman, Kramer, Martin, Peck, and Singmaster supplied generalizations in the spirit of Solution II, while Beumer's was similar to Solution III. Other generalizations: If m has a prime factor p of odd multiplicity for which $p\equiv -1 \pmod 6$, then $\sigma(m)\equiv 0 \pmod 6$ (Bluger). If $(-1\mid n) = -1$ and $m\equiv -1 \pmod n$ is the product of primes $p\equiv \pm 1 \pmod n$, then $\sigma(m)\equiv 0 \pmod n$ (Felder, Marsh). If k is odd, $\sigma_k(6q+5)\equiv 0 \pmod 6$ for each integer q (Roselle). The others referred to various of the papers mentioned under Solution I.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be submitted on separate, signed sheets and should be mailed before June 30, 1966.

5340. *Proposed by Necdet Üçoluk, Purdue University*

Let A and B be two fixed points of the plane. If N is a variable point of the plane, say $u=\overline{NA}$, $v=\overline{NB}$, determine a geometric construction for the tangent line at the general point N on the locus of $uv^2=k$.

5341. *Proposed by George Purdy, The University, Reading, England*

It is known that

$$\tfrac{1}{2}\sum_{t=1}^{2} (-1)^t e^{(-1)^t x} = \sinh x.$$

Prove the integral analog:

$$\frac{1}{2}\int_1^2 (-1)^t e^{(-1)^t x}\,dt = \frac{\sinh x}{i\pi x}\,.$$

5342. *Proposed by R. B. Killgrove, the University of Wisconsin*

Suppose we have relations R_1, R_2, then we define $R=R_1\cap R_2$ as follows: xRy if and only if xR_1y and xR_2y. Now let S be the class of all total linear orderings of the positive integers, i.e., the relation L belongs to S if and only if (1) for any two positive integers x, y, we have xLy or yLx, (2) L is reflexive, (3) L is antisymmetric, (4) L is transitive. Now consider the relation D, "divides." Prove (i) D can be obtained in at least one way from intersecting certain members of S, and (ii) find the smallest (possibly an infinite cardinal) number of members of S needed to obtain D as their intersection. (Dedekind considered D as a partial ordering as an example of a lattice. Sometimes it is called Dedekind's ordering.)

5343. *Proposed by J. M. Gandhi, Maharani's College, Jaipur, India*

Show that

$$\sum_{t=0}^{\gamma}\frac{\gamma!}{t!}=(-1)^\gamma\begin{vmatrix} 0! & 1 & 0 & 0 & \cdots & 0 & 0\\ 1! & -\binom{1}{0} & \binom{1}{1} & 0 & \cdots & 0 & 0\\ 2! & \binom{2}{0} & -\binom{2}{1} & \binom{2}{2} & \cdots & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ (\gamma-1)! & (-1)^{\gamma-1}\binom{\gamma-1}{0} & \cdots & & & -\binom{\gamma-1}{\gamma-2} & \binom{\gamma-1}{\gamma-1}\\ \gamma! & (-1)^{\gamma}\binom{\gamma}{0} & (-1)^{\gamma-1}\binom{\gamma}{1} & & \cdots & \binom{\gamma}{\gamma-2} & -\binom{\gamma}{\gamma-1} \end{vmatrix}$$

5344. *Proposed by B. Fitzpatrick, Jr., Auburn University, Alabama*

In proving that a subset of E_n is n-dimensional if and only if it contains a nonempty open subset of E_n, Hurewicz and Wallman (*Dimension Theory*, p. 45) use the intuitively appealing statement that if X and Y are two countable subsets of E_n, "it is always possible to choose the coordinate system so that the axes are in general position (no parallel to a coordinate hyperplane contains more than one point x or one point y) since this condition requires the axes to avoid at most a countable number of directions." Verify the following proposition and show that it implies their statement.

If K is a countable subfield of the field of complex (real) numbers and n is a positive integer, $n>1$, then there exists an $n\times n$ unitary (orthogonal) matrix P such that for each $i=1,\cdots,n$, the elements of the ith row of P are linearly independent over K.

5345. *Proposed by Carl Evans, Union College, Barbourville, Ky.*

Let Π be a finite projective plane of order $n>2$. Show that there is a partition of the points of Π into two sets such that neither set contains all of the $n+1$ points on a line.

5346. *Proposed by Louis Comtet, Boulogne (Seine), France*

Let $s(n)=1+1/2+1/3+\cdots+1/n$; for each $x\geqq 1$, define $n(x)$ by: $s(n(x))\leqq x<s(n(x)+1)$. With the notations: $E(x)=\exp(x-\gamma)-1/2$, ($\gamma=0.57\cdots$ is the Euler constant), $\alpha(x)=1/(\exp(x-1)-1)$, $[u]$=integral part of u, and if $x\geqq 2$, prove that

$$[E(x)-\tfrac{3}{2}\alpha(x)]\leqq n(x)\leqq[E(x)+\tfrac{1}{12}\alpha(x)].$$

(N.B., if $x\geqq 2$, then $\frac{1}{12}\alpha(x)+\frac{3}{2}\alpha(x)<1$, and the formula gives at most two possible values for $n(x)$.)

5347. *Proposed by Sidney Spital, Polytechnic State College, Pomona, California*

Show that the coefficients in Adams-Bashforth integration

$$\gamma_n=(-1)^n\int_0^1\binom{-s}{n}ds$$

are given asymptotically by

$$\gamma_n\sim\frac{1}{\log n}+O\left(\frac{1}{\log^2 n}\right).$$

5348. *Proposed by Fred Galvin, St. Paul, Minnesota*

Show that there is a function f such that for any sequence $(x_1, x_2, \cdots)$ we have $x_n=f(x_{n+1}, x_{n+2}, \cdots)$ for all but finitely many n.

5349. *Proposed by S. P. Lloyd, Bell Telephone Laboratories, Murray Hill, N. J.*

With A a subset of a topological space X, let $\mathcal{K}(A)$ be the smallest class of subsets of X with the properties (i) A is in the class, (ii) for every member E of the class, both E' and E^- are members of the class. It is known that $\mathcal{K}(A)$ has at most 14 members and that there is a subset of the reals for which $\mathcal{K}(A)$ does have 14 members (J. L. Kelley, *General Topology*, p. 57). If $\mathfrak{F}_1(A)$ is the field of subsets of X generated by the members of $\mathcal{K}(A)$ show that $\mathfrak{F}_1(A)$ has at most 16,384 $(=2^{14})$ members, and give a subset of the reals for which $\mathfrak{F}_1(A)$ does have 16,384 members. Let $\mathfrak{F}_2(A)$ be the smallest field of subsets of X which has properties (i) and (ii) above. Give a subset of the closed unit interval for which $\mathfrak{F}_2(A)$ has infinitely many members.

SOLUTIONS OF ADVANCED PROBLEMS

Level Points of a Continuous Function

5245 [1964, 1137]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College*

Let f be a real function defined on $[0, 1]$. If the set of zeros of f is uncountable and nowhere dense, can f be continuous?

I. *Solution by Allen Rehm, Long Beach, Calif., and P. B. Kennedy, University of York, England and John Stout, Princeton University* (independently). The set E of zeros must be closed and its complement must be a countable disjoint union of open intervals (a, b). If $x \in (a, b)$, define $f(x) = \exp\{-(x-a)^{-1} + (b-x)^{-1}\}$; and $f(x) = 0$ for $x \in E$.

Then $f(x)$ is continuous, and in addition is infinitely differentiable. E may have the cardinality of the continuum and be nowhere dense, as in the instance of the Cantor set.

II. *Solution by John Zolnowsky, Freshman, South Dakota School of Mines and Technology.* The problem can be generalized as follows: Give an example of a continuous function $\phi: [0, 1] \to [0, 1]$ such that the set $\phi^{-1}(y)$, $y \in [0, 1]$ is closed, dense in itself, and nowhere dense.

In the solution we use the Hilbert space-filling curve. It has been proved that there is a function $\psi: [0, 1] \to [0, 1]^2$ which is continuous, onto, and such that for any interval $I = [p/4^n, (p+1)/4^n]$, where p is an integer, and $0 \leqq p < 4^n$, $\psi(I)$ is a closed square of sides $1/2^n$. (M. H. A. Newman, *Topology of Plane Point Sets*, p. 89.)

Let $L_y = \{(x, y): x \in [0, 1]\}$, $S_y = \psi^{-1}(L_y)$.

THEOREM. *Each S_y is closed, dense in itself, and nowhere dense.*

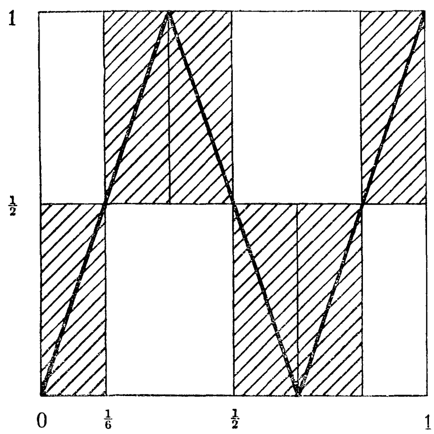
Proof. S_y is closed, since it is the inverse image of a closed set under a continuous function.

S_y is dense in itself. For, consider a point $t \in S_y$. To show that t is a limit point of S_y, it is necessary to show that for each $\epsilon > 0$, there exists a point $t' \in S_y$ such that $t' \neq t$ and $|t - t'| < \epsilon$. Choose n so that $1/2^n < \epsilon$. Let $I = [p/4^n, (p+1)/4^n]$ be the interval of length $1/4^n$ containing t. Write $R = \psi(I)$, so R is a closed square. Select $z \in R$ with $z \in L_y$ and $z \neq \psi(t)$. Then there is a point $t' \in I$ with $\psi(t') = z$, $t' = \psi^{-1}(L_y)$ and $|t - t'| \leqq 1/4^n < 1/2^n < \epsilon$.

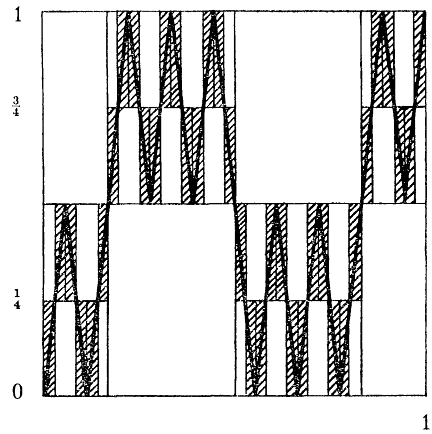
The sets S_y are nowhere dense, for suppose, on the contrary, that for some y, S_y is dense. Then, since S_y is closed, S_y must contain an interval (a, b). It follows that there is an interval I of the form $[p/4^n, (p+1)/4^n] \subset (a, b) \subset S_y$. But $\psi(I)$ is an entire closed square, and not a horizontal line, which is a contradiction.

Consider the function $\phi = \mathrm{pr}_2 \circ \psi$, where pr_2 is the projection on the second axis. ϕ is continuous, and clearly $\phi^{-1}(y) = S_y$. It is clear that the sets L_y are uncountable, and hence the sets $\phi^{-1}(y)$ are uncountable.

III. *Solution by R. F. Jolly, University of California, Riverside.* The following indicates the construction of a continuous function f, which intersects each horizontal line $y=c$, $0\leqq c\leqq 1$ in an uncountable nowhere dense set:



Step 1.



Step 2.

You may think of the function f as the uniform limit of a sequence of continuous functions (heavy lines in the figure) or its graph as the intersection of the members of a monotonic sequence of continua (each the union of finitely many shaded rectangles). Evidently each horizontal line which intersects the graph does so in a Cantor set.

Also solved by H. J. Al-Amiri, E. B. Anders, Roger Avelsgaard, Duane W. Bailey, K. F. Bailie, I. N. Baker (England), D. R. Barr, John Beidler, H. J. Biesterfeldt, Jr., Andreas Blass, W. H. Bonney, II, Robert Bowen, Joel Brawley, Jr., E. O. Buchman, George Cain, Jr., Jim Campbell, G. T. Cargo, H. J. Cohen, Louis Comtet (France), R. O. Davies (England), C. G. Denlinger, W. G. Dotson, Jr., Gregory Dropkin, Glen Haddock, Hwa Hahn, J. H. Halton & G. Rabinowitz, C. E. Hee, A. G. Heinicke, D. W. Hight, R. A. Jacobson, J. E. Joseph, H. D. Keesing, B. G. Klein, H. V. Kronk, E. S. Langford, H. C. Lauer, A. E. Livingston, O. P. Lossers (Netherlands), Coline Makepeace, M. D. Mavinkurve (India), Solomon Marcus (Rumania), W. J. Pervin, G. Piranian, J. R. Porter, Stephen Portnoy, Sylvester Reese (France), P. L. Renz, J. W. Roberts, I. D. Ruggles, Al Somayajulu, R. T. Sandberg, Klaus Schmitt, Robin Sibson (England), S. J. Sidney, G. P. Speck, D. A. Sprecher, F. W. Steutel (Netherlands), F. B. Strauss, M. V. Subbarao, B. R. Toskey, W. C. Waterhouse, R. E. Whitney & R. L. Duncan, J. C. Williams, S. V. Witt, Kenneth Yanosko, and two who omitted their names.

Compare E1715 [1965, 784].

Nondifferentiability of Inverse of a Continuous Modulus of Continuity

5246 [1964, 1137]. *Proposed by Solomon Marcus, University of Bucharest, Rumania*

Let f be continuous in $[a, b]$. For each $\epsilon>0$, let $\phi(\epsilon)$ be the greatest number η such that $|x'-x''|\leqq\eta$ implies that $|f(x')-f(x'')|\leqq\epsilon$. Does there exist such an f that ϕ is continuous and nondifferentiable in some point $\epsilon_0\in(0, +\infty)$?

I. *Solution by J. H. Halton, Brookhaven National Laboratory.* Consider any function having the form $f(x)=p+m(x-a)$ if $a\leqq x\leqq c$, $f(x)=q+n(x-c)$ if $c\leqq x\leqq b$, where $0<m<n$ and $q=p+m(c-a)$. Then $f(x)$ is continuous in $[a, b]$. If $\epsilon\leqq n(b-c)$, then clearly $\phi(\epsilon)=\epsilon/n$, so that $\phi'(\epsilon)=1/n$. If $\epsilon\geqq n(b-c)$, then $\phi(\epsilon)=\epsilon/m-(n-m)(b-c)/m$, so that $\phi'(\epsilon)=1/m$. Clearly $\phi(\epsilon)$ is continuous and nondifferentiable at $\epsilon_0=n(b-c)$.

II. *Solution by K. A. Post, Technological University, Eindhoven, Netherlands.* Let $f(x)=x+\sin x(x\in(-\infty, \infty))$. For fixed $\phi>0$ we have

$$f(x+\phi)-f(x)=\phi+2\cos(x+\tfrac{1}{2}\phi)\sin\tfrac{1}{2}\phi$$

which attains the maximal value $\epsilon(\phi)=\phi+2|\sin\frac{1}{2}\phi|$. Since $\epsilon(\phi)$ is continuous and monotonically increasing, its inverse function $\phi(\epsilon)$ is exactly the required function. Singularities are the points $\epsilon=2k\pi$ $(k=1, 2, \cdots)$.

Also solved by H. S. Al-Amiri, I. N. Baker (England), E. O. Buchman, G. T. Cargo, Roy O. Davies (England), Hwa Hahn, R. A. Jacobson, C. A. Long, M. D. Mavinkurve (India), G. Piranian, R. T. Sandberg, S. J. Sidney, G. P. Speck, Stuart Witt, and one who did not give his name.

Sidney raised the question as to whether an example is possible in which $\phi(\epsilon)$ does not possess one-sided derivatives. None of the contributed examples could serve.

Cosets of a Subgroup

5247 [1964, 1138]. *Proposed by T. J. Head, Iowa State University*

Let S be a subset of a group G. Define a relation in G by $x\sim_S y$ if and only if $Sx=Sy$. It is seen that $\sim_S$ is an equivalence relation. Prove that the associated equivalence classes are the right cosets of a subgroup of G.

I. *Solution by A. G. Heinicke, Dalhousie University.* Let Sa_i be the equivalence class containing the identity e. If $x\in Sa_i$, then $x=ex\in Sa_i\cap Sa_ix$, or $Sa_i=Sa_ix$. Thus, for any x and y in Sa_i, $xy^{-1}\in Sa_i$, since $Sa_ix=Sa_iy=Sa_i$, and therefore $exy^{-1}\in Sa_ixy^{-1}=Sa_i$. Sa_i is thus a subgroup of G.

II. *Solution by V. Ray Hancock, Emory and Henry College.* The relation $\sim_S$ is not only an equivalence relation, but is also a right congruence relation on the group G, because if x and y are any elements of G such that $x\sim_S y$ and if z is any element of G then $Sx=Sy$ and hence $S(xz)=S(yz)$ so that $xz\sim_S yz$. It is well known that the congruence class (modulo a right congruence relation on a group) containing the identity element is a subgroup whose right cosets form the congruence classes.

III. *Solution by Paul S. Schnare, Louisiana State University, New Orleans.* Let $H=\{x\in G: x\sim_S e\}$ where e is the identity. Claim: $Hg=\{y\in G: y\sim_S g\}$, the equivalence class of g. For, $y\in Hg$ iff $y=xg$ for some $x\sim_S e$ iff $Sy=Sxg=Sg$ iff $y\sim_S g$. Thus, $Hg\cap Hg'\neq\phi\Rightarrow Hg=Hg'$ for any $g, g'\in G$. In a solution to problem E 1712 [November, page 1022] it is shown that a subset H of a group G is a left coset of a subgroup of G if the left translates of H form a partition of G into dis-

joint subsets. Replacing "left" by "right" we see that the right translates of H are right cosets of a subgroup of G, viz. H.

This result can be obtained as a corollary of problem E 1712 [1965, 1022].

Also solved by Robert Bowen, Joel Brawley, Jr., Jim Campbell, David Carlson, Charles Chouteau, J. E. Connett, F. B. Crippen, Roy O. Davies (England), J. P. Delatore, M. J. DeLeon, W. G. Dotson, Jr., J. E. H. Elliott, L. E. English, M. P. Epstein, E. W. Ewing, Harley Flanders, R. W. Gilmer, Jr., H. A. Gindler, Ralph Greenberg, M. G. Greening (Australia), Israel Grossman, Hwa Hahn, Franklin Haimo, J. H. Halton, S. Heller, Paul Hill, Agatha Himmelfarb, R. A. Jacobson, Colonel Johnson, Jr., V. H. Keiser & J. P. Burling, John B. Kelly, S. C. King, B. S. Langford, Steve Ligh, C. C. Lindner, J. J. Martinez, M. D. Mavinkurve (India), José Morgado (Brazil), T. M. Morrisette, Joseph G. Moser, M. G. Murdeshwar, Stanton Philipp, Harsh Pittie, K. A. Post (Netherlands), Francis Sandomierski, S. J. Sidney, Jonathan Simon, Indranand Sinha, P. T. Smythe, John Stout, Jerome Teles, B. R. Toskey, W. C. Waterhouse, C. R. Williams, F. E. Willmore, Kenneth Yanosko, and the proposer.

Additivity and Multiplicitivity

5248 [1964, 1138]. *Proposed by A. A. Mullin, University of Illinois*

Let a natural number $n>1$ be factored into primes, $n=p_1^{\alpha_1}\cdots p_m^{\alpha_m}$ (p_i distinct) and let its own natural number exponents >1 be factored similarly. The process is to be repeated over and over again until it terminates with a unique "constellation" of prime numbers alone. E.g., the constellation for $192=2^{2\cdot 3}\cdot 3$, $10{,}000=2^{2^2}\cdot 5^{2^2}$. Recall that a number-theoretic function f is multiplicative when $f(a\cdot b)=f(a)\cdot f(b)$, $(a, b)=1$. Call a number-theoretic function g generalized multiplicative when $g(a\cdot b)=g(a)\cdot g(b)$ if the constellations for a and b have no prime in common. Prove (1) that every multiplicative number-theoretic function is generalized multiplicative and that there exists a generalized multiplicative function which is not multiplicative.

Let a number-theoretic function h be called additive when $h(a\cdot b)=h(a)+h(b)$, $(a, b)=1$. Call a number-theoretic function k generalized additive when $k(a\cdot b)=k(a)+k(b)$ if the constellations for a and b have no prime in common. Prove the additive analogue (2) of the multiplicative results (1).

I. *Solution by S. J. Sidney, Harvard University.* Let f be multiplicative. If the constellations for a and b have no prime in common, then certainly the same is true of their factorizations, so $f(a\cdot b)=f(a)\cdot f(b)$. Hence f is generalized multiplicative. The statement "additive implies generalized additive" is proved in exactly the same way.

Define $g(a)$ to be the product of all primes in the constellation of a, taken once only, regardless of how many times they appear in the constellation. Then g is clearly generalized multiplicative. But $g(9)=6$, $g(2)=2$, and $g(18)=6$, so $g(18)\neq g(9)\cdot g(2)$.

Similarly, if $k(a)$ is the sum of all primes in the constellation of a each taken once only, then k is generalized additive. But $k(9)=5$, $k(2)=2$, and $k(18)=5$; $k(18)\neq k(9)+k(2)$.

II. *Note by Harsh Pittie, Swarthmore College.* The concept of constellations being disjoint can be generalized thus: When a number has been factored into its constellation, we may talk about the primes appearing on various levels in it., i.e., exponents or exponents of exponents, etc. Call two numbers k-disjoint if their constellations are disjoint up to and including the kth level. Define a function to be k-multiplicative if it is multiplicative for all k-disjoint pairs. Then there exist functions which are k-multiplicative but not $(k-1)$-multiplicative. The argument proceeds as in a solution of the primary problem.

Also solved by Robert Bowen, A. A. Gioia & A. M. Vaidya, Ralph Greenberg, Hwa Hahn E. S. Langford, C. C. Lindner, A. E. Livingston, M. O. LeVan, Stanton Philipp, W. C. Waterhouse, and the proposer.

t-Dimensional Chessboard

5249 [1964, 1138]. *Proposed by W. A. Horn, RIAS and University of Maryland*

Let an $n\times n$ square S be divided up into 1×1 squares labeled a_j $(j=1, 2, \cdots, n^2)$. Define $a_j\sim a_k$ if $a_j=a_k$ or if a_j and a_k have any boundary point in common. Suppose that f is a function associating to each a_j some a_k satisfying the condition that if $a_j\sim a_i$ then $f(a_j)\sim f(a_i)$. Prove that there exists an a_r such that $f(a_r)\sim a_r$. Can this problem be extended to higher dimensions?

I. *Solution by Kenneth Yanosko and Richard Vande Velde, S.J., University of Chicago.* We give a constructive solution valid for t dimensions, t finite but arbitrary. "Squares" of the array A will be labeled by t coordinates, thus $\mathbf{a}=(a_1, a_2, \cdots, a_t)$, $\mathbf{b}=(b_1, b_2, \cdots, b_t)$ denote typical squares with a_i, $b_i \in \{1, 2, \cdots, n_i\}$, with n_i positive, not necessarily the same for all i.

We define a metric on A by $d(\mathbf{a}, \mathbf{a}')=\max |a_i-a_i'|$. Using this metric we have $\mathbf{a}\sim\mathbf{a}'$ iff $d(\mathbf{a}, \mathbf{a}')\leqq 1$, i.e., iff $|a_i-a_i'|\leqq 1$ for all i. We describe a procedure P, to find at least one $\mathbf{a}$ such that $\mathbf{a}\sim f(\mathbf{a})$. We prove that the search can be started anywhere and will terminate after a finite number of steps. We define the sequence recursively.

Given $\mathbf{a}^k$ and $f(\mathbf{a}^k)=\mathbf{b}^k$, we define $\mathbf{a}^{k+1}$ by:

$$\text{(P)} \qquad \overset{k+1}{a_i} = \begin{cases} \overset{k}{a_i} + 1 & \text{if } \overset{k}{a_i} < \overset{k}{b_i} \\ \overset{k}{a_i} & \text{if } \overset{k}{a_i} = \overset{k}{b_i} \\ \overset{k}{a_i} - 1 & \text{if } \overset{k}{a_i} > \overset{k}{b_i} \end{cases}$$

Given any initial square $\mathbf{a}^0$, this defines a unique sequence $\{\mathbf{a}^n\}$.

FACT 1. Given such a sequence generated by P, if at some stage k and for some coordinate i we have $f(\mathbf{a}^k)=\mathbf{b}^k$ and $|a_i^k-b_i^k|\leqq 1$ then the hypotheses imply $|a_i^m-b_i^m|\leqq 1$ for all $m\geqq k$. Therefore it remains to show that we can, for some value of n, force $|a_i^n-b_i^n|\leqq 1$ for each i.

FACT 2. If $|a_i^k - b_i^k| > 1$, then either

$$|a_i^{k+1} - b_i^{k+1}| < |a_i^k - b_i^k|, \text{ or} \tag{1}$$

(2) b_i^{k+1} is closer to the edge of the board, i.e., nearer 1 or n_i and

$$|a_i^{k+1} - b_i^{k+1}| = |a_i^k - b_i^k|.$$

Proof. Without loss of generality we can assume that $a_i^k > b_i^k$. Suppose that (1) does not hold; then, by P, $a_i^{k+1} = a_i^k - 1$ and, since f preserves adjacency, $b_i^{k+1} = b_i^k + e$, where $e = -1$, 0, or 1. But if $e = 0$ or $e = 1$, then (1) holds, contrary to assumption. Therefore $e = -1$. Hence b_i^{k+1} is one square closer to the edge.

Putting facts 1 and 2 together we see that P provides us with a sequence of squares $\{\mathbf{a}^n\}$ such that in every coordinate the distance between $\mathbf{a}^n$ and its image decreases to less than or equal to unity, and stays there; or P forces the images to the edge of the board, after which the distance between squares and their images must also decrease to unity or less. Hence we eventually obtain an $\mathbf{a}^N$ such that $d(\mathbf{a}^N, f(\mathbf{a}^N)) \leqq 1$, i.e., $\mathbf{a}^N \sim f(\mathbf{a}^N)$.

II. *Solution by the proposer.* The proof will be based on Brouwer's fixed point theorem. First split each a_j into triangles by drawing the diagonals. If p_j is the center point of a_j and $f(a_j) = a_k$, then define $F(p_j) = p_k$. For a corner point q_j of a_j, we define $F(q_j)$ as follows. Consider all squares a_k which contain q_j. There will be one, two, or four such squares. Call them a_{r_i}. By the hypotheses, $\bigcap_i f(a_{r_i}) \neq \emptyset$. Let $F(q_j)$ be any point in $\bigcap_i f(a_{r_i})$.

Having defined F on the center and corner points of each square, extend F linearly (in the obvious way) to the whole square. Then F is well defined, continuous, and $F(a_j) \subset f(a_j)$. By the Brouwer fixed point theorem, F has a fixed point x. If x is interior to a_j, then clearly $f(a_j) = a_j$. If x is on the boundary of some a_j, then a_j and $f(a_j)$ have a common boundary point, so that $f(a_j) \sim a_j$.

Clearly, this result can be extended to any finite number of dimensions. Also, note that there need be no square a_j such that $f(a_j) = a_j$, as may be seen by rotating a 2×2 square.

Also solved by Agatha Himmelfarb, John B. Kelly, C. B. A. Peck, Stanton Philipp, P. S. Schnare, and Ogi Tápal.

Peck and Philipp consider the problem as a king following another king on a chessboard. As in the proofs above, however, it is not necessary to start with a square array.

Maps of a Non-Compact Metric Space

5250 [1964, 1138]. *Proposed by M. D. Mavinkurve, Siddharth College, Bombay, India*

A known property of a compact metric space is that it cannot be mapped isometrically onto a proper part of itself. Is this property characteristic of compactness?

I. *Solution by H. S. Fox, undergraduate, University of Wisconsin-Milwaukee.* Denoting the natural numbers $\{1, 2, \cdots\}$ by N we may let E be the subset of the real numbers defined by $E = N \cup \{n - 1/(n+1) : n \in N\}$. It is rather obvious that this metric space is not compact but the only isometry of this space into itself is the identity. Thus the property does not characterize compactness.

II. *Solution by G. P. Speck, University of the Pacific.* Consider the noncompact set $X = (0, 1)$, with the usual topology. An isometry is a homeomorphism and the only proper subsets onto which X can be mapped homeomorphically are sets of the form: (a, b), where $a > 0$ and/or $b < 1$. Any such homeomorphism is obviously not an isometry.

Also solved by Robert Bowen, R. O. Davies (England), W. G. Dotson, Jr., Harley Flanders, R. F. Jolly, P. L. Renz, E. S. Langford, A. E. Livingston, Paul Manley & M. G. Murdeshwar, J. R. Porter, K. A. Post (Netherlands), S. M. Robinson, Robin Sibson (England), S. J. Sidney, W. C. Waterhouse, and the proposer.

Polynomial Equations with Unimodular Roots

5251 [1964, 1138]. *Proposed by L. A. Shepp, Bell Telephone Laboratories, Murray Hill, N. J.*

If $-1 \leqq \mu \leqq 1$, prove that each root of the equation $y^{n+1} - \mu y^n + \mu y - 1 = 0$ has modulus one.

I. *Solution by D. R. Hayes, University of Tennessee.* If $\mu = \pm 1$, or $n = 0$ or 1, then the problem is trivial; so we may assume in what follows that $\mu \neq \pm 1$, $n > 1$. The proof proceeds by contradiction. Suppose the equation has a solution with modulus not equal to 1. Since the reciprocal of this solution is also a solution, there is a solution $y = Re^{i\theta}$ with $R > 1$. Now $y^{n-1}(y - \mu) = y^{-1}(1 - \mu y)$, from which one obtains, upon squaring the modulus of both sides, that $R^{2n-2}|y - \mu|^2 = R^{-2}|1 - \mu y|^2$. Since $R > 1$, it follows from this last equation that $|y - \mu|^2 < R^{-2}|1 - \mu y|^2$ or that

$$R^2 - 2\mu R \cos\theta + \mu^2 < (1/R)^2 - 2\mu(1/R) \cos\theta + \mu^2.$$

Thus, $(R^2 - 1/R^2) < 2\mu \cos\theta(R - 1/R)$. This last inequality may be divided by the positive factor $R - R^{-1}$, yielding

$$R + 1/R < 2\mu \cos\theta \leqq 2,$$

which is impossible since $R > 1$.

II. *Solution by Robert Breusch, Amherst College.* More generally:

$$f(y) \equiv y^{n+1} - \mu y^{n-k+1} + \mu y^k - 1 = 0,$$

with k a positive integer $< \frac{1}{2}(n+1)$, has all its roots on $|y| = 1$, if $-1 \leqq \mu \leqq 1$.

For $\mu = \pm 1$, the statement follows from the factorization $f(y) = (y^k \mp 1)(y^{n-k+1} \pm 1)$. And for $|\mu| < 1$ and $y = e^{i\theta}$ (θ real) set

$$(1/2i) y^{-(n+1)/2} f(y) = g(\theta),$$

thus

$$g(\theta) = \sin[(n+1)\theta/2] - \mu \sin[(n-2k+1)\theta/2].$$

$g(\theta)$ has opposite signs at the endpoints of each of the intervals

$$[(2k-1)\pi/(n+1),\ (2k+1)\pi/(n+1)], \qquad k = 0, 1, \cdots, n.$$

Therefore $g(\theta)$ has $(n+1)$ zeros for $-\pi/(n+1) < \theta < 2\pi - \pi/(n+1)$, and thus $f(y)$ has $(n+1)$ zeros on $|y| = 1$.

III. *Solution by I. N. Baker, University of London, England.* Consider the more general equation $z^{n+1} - az^n + \bar{a}z - 1 = 0$, $|a| \leqq 1$, and put

$$z^n = (1 - \bar{a}z)/(z - a) = T(z). \tag{1}$$

If $|a| < 1$, and there is a value $|z| < 1$ solving this equation, we note (i) that $|z^n| < 1$ and (ii) that the bilinear mapping $T(z)$ satisfies $|T(z)| > 1$ for $|z| < 1$, thus obtaining a contradiction. Similarly $|z| > 1$ leads to a contradiction, and our result is proved for $|a| < 1$. The case $|a| = 1$ is proved directly.

IV. *Solution by C. B. A. Peck, State College, Penna.* For positive n and arbitrary μ, every solution is on the unit circle by a theorem due to M. Marden (*Geometry of the Zeros of a Polynomial in a Complex Variable*, Chap. 10), a special case of which may be stated as follows:

If $f(y) = a_n y^n + \cdots + a_0$ and $f^*(y) = y^n \bar{f}(1/y) = \bar{a}_0 y^n + \cdots + \bar{a}_n$ satisfy $a_n f^*(y) = \bar{a}_0 f(y)$, then all the zeros of f are on the unit circle.

In our case, $a_{n-k} = -\bar{a}_k$ and the condition is satisfied.

Also solved by R. A. Avelsgaard, J. L. Brenner & D. A. D'Esopo, J. W. Burrows, F. A. Butter, Jr., G. T. Cargo, F. W. Carroll, R. O. Davies (England), W. G. Dotson, Jr., D. C. Duncan, Mrs. A. C. Garstang, Michael Goldberg, Myron Goldstein, Agatha Himmelfarb, A. S. B. Holland & A. Meir, John B. Kelly, P. B. Kennedy, (England), R. C. Lyness (England), M. D. Mavinkurve (India), W. D. Munro, Stanton Philipp, P. S. Schnare, Robin Sibson (England), S. J. Sidney, Arnold Singer, Richard Sinkhorn, Sidney Spital, F. W. Steutel (Netherlands), Maynard Tomer, J. H. van Lint (Netherlands), David Zeitlin, and the proposer.

Complementary Sets of Integers

5252 [1964, 1138]. *Proposed by D. J. Newman, Yeshiva University*

A sequence $\{a_n\}$ of natural numbers is defined recursively as follows: $a_0 = 0$, a_{n+1} is the smallest natural number $> a_n$ and unequal to $a_k + k$ for any $k \leqq n$. The sequence begins 0, 1, 3, 4, 6, 8, 9, $\cdots$. Find an explicit expression for a_n as a function of n.

I. *Solution by Ralph E. Walde, University of California, Berkeley.* Clearly the sequence can be given the following equivalent definition: $\{a_n\}$ is the strictly increasing sequence of nonnegative integers such that if $A = \{a_n\}$, $B = \{a_n + n\}$, then (i) $A \cup B$ comprises all non-negative integers, and (ii) $A \cap B = \phi$.

B. M. Stewart (*Theory of Numbers*, p. 52) proves the following theorem:

Let b and c be two positive irrational numbers such that $1/b+1/c=1$, then the two series $\{[bn]\}$ and $\{[cn]\}$ for $n=1, 2, \cdots$, represent all positive integers without repetition (the brackets indicate the greatest integer function).

If $c=b+1$, then $b=\frac{1}{2}(1+\sqrt{5})$. Since $[(b+1)n]=[bn]+n$, it is seen that $[bn]$ is the required representation for a_n.

II. *Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands.* The pairs (a_k, a_k+k) are the safe combinations in what is known as Wythoff's game (cf. *Nieuw Archief voor Wiskunde*, (2), VII (1907), pp. 199–202.) We have

$$a_k = [\tfrac{1}{2}k(1 + \sqrt{5})].$$

For a proof cf., e.g. W. Ahrens, *Mathematische Unterhaltungen und Spiele*, Leipzig, 1910.

Also solved by M. G. Beumer (Netherlands), M. T. L. Bizley (England), W. J. Blundon, W. H. Bonney, Robert Breusch, E. O. Buchman, L. Carlitz, F. W. Carroll, Michael Goldberg, S. W. Golomb, M. G. Greening (Australia), Sidney Heller, Al Jebber, John B. Kelly, A. H. Kruse, Morris Rothe, F. W. Steutel (Netherlands), and the proposer.

Using Fibonacci procedures, Bizley and Steutel give the identities

$$a_{a_n} = a_n + n - 1, \qquad a_{a_n+n} = 2a_n + n.$$

Greening, in addition, obtains

$$a_{a_n+n-1} = 2a_n + n - 2, \qquad b_{u_{2n}} = u_{2n+2},$$
$$a_{u_{2n}} = u_{2n+1}, \qquad a_{u_{2n+1}} = u_{2n+2} - 1,$$

where $\{u_n\}$ is the Fibonacci sequence.

Beumer offers the following references:

1. Th. Skolem, Math. Scandinavica, 5 (1957), 57–68.
2. W. A. Wythoff, Nieuw Archief voor Wiskunde.
3. Ky Fan, The Dunkel Memorial Problem Book, Problem 4399, p. 57.

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, University of California, Berkeley, and
E. P. VANCE, Oberlin College

Materials intended for review should be sent directly as follows: Books: R. A. Rosenbaum, Wesleyan University, Middletown, Conn. 06457. Programmed Materials: K. O. May, University of California, Berkeley, Calif. 94704. Films: E. P. Vance, Oberlin College, Oberlin, Ohio 44074.

Boolean Algebra. By R. L. Goodstein. Pergamon Press, Inc., New York, 1963, 140 pp. \$1.95.

This little paperback is a completely self-contained study of the theory of Boolean algebras from an elementary standpoint. It is entirely abstract, and

Elements of Numerical Analysis. By Peter Henrici. Wiley, New York, 1964. xv+328 pp. $8.00.

A. S. Householder's *Principles of Numerical Analysis* (McGraw-Hill, 1953) was the first textbook, written with high-speed digital computation in mind, which was suited to the teaching of numerical analysis in a mathematics department. Though many books on the mathematics of computation have since appeared, the instructor who seeks a given level of course, a given type of emphasis, and a given style of presentation will often fail to find a textbook meeting reasonable standards of accuracy and rigor. This book, whose author is both an expert numerical analyst and expert writer, is welcome indeed.

The book can be used as a basis for either a one or two semester course in numerical analysis. The material, designed to be given in conjunction with a three unit course in computer programming, requires a background of calculus and differential equations. Further background on complex numbers, polynomials and difference equations is given in the first three chapters. For the most part, the rest of the book is divided into two main sections: Solution of Equations, and Interpolation and Approximation. A brief concluding section touches upon the propagation of rounding error. Each section consists of a succession of "algorithms," or statements of numerical procedures, accompanied by theorems about the algorithms, numerous illustrations, and problem sets. In all, there are some 300 problems of varying difficulty.

In addition to the usual material on iterative methods of solving equations, the first section discusses Steffensen's iteration, iterative methods for systems of (two) nonlinear equations, the theory of linear difference equations and its application to the study of Bernoulli's method, and the quotient-difference algorithm. Part two contains Lagrange's interpolation formula, Aitken's scheme, and a rather complete study of interpolation with tabular differences and its applications. These include numerical differentiation and integration and the numerical solution of ordinary differential equations. Noteworthy features are a discussion of Romberg integration and consistent use of knowledge of the asymptotic behavior of errors to improve the accuracy of approximations.

In view of the author's definition of numerical analysis as "the theory of constructive methods in mathematical analysis," the exclusion of numerical methods in linear algebra is natural, though perhaps inconvenient for those who want a broader view of numerical analysis in a single course. Failure to mention divided differences is more surprising. But the one major shortcoming is that many of the more exciting algorithms (Romberg, Q-D, Runge-Kutta, Steffensen iteration for systems) have theoretical foundations lying beyond the scope of the text. In a treatise, or in a textbook on numerical methods for that matter, one's choice of algorithms is properly based on their effectiveness, elegance, or popularity. In numerical analysis, however, it is the computing that is ancillary, not the theory. Though it is often expedient to sacrifice mathematical discipline when computing, the reviewer doubts that it is advisable in a mathematics textbook.

Professor Henrici's book is fresh, up-to-date and authoritative. As a text it succeeds in presenting an interesting and important sector of numerical analysis, and also in strengthening the student's grasp of some of the basic notions of mathematics.

NATHANIEL MACON, Institute for Defense Analyses

Elementary Geometry from an Advanced Standpoint. By Edwin Moise. Addison-Wesley, Reading, Mass., 1963. 419 pp. $8.75.

The author has two objectives. The first is to reteach high school geometry with a degree of precision and modernity which is, to say the least, uncommon at the school level. The second is to treat some of the more delicate questions of logical structure.

It is likely that the first objective will be accomplished, and the second significantly served, by a course covering the first eighteen chapters, a little more than half the book. These chapters deal with all the topics of the usual high school course and with quite a bit besides, with the beginnings of non-euclidean geometry, for example. The geometry is essentially G. D. Birkhoff's metric geometry, and it is very nicely compared, in Chapter 8, with the synthetic. All of this material is well presented and requires no more of an able student than hard work and a sound background in elementary algebra. Indeed, these chapters could be used in the schools. The second half of the book, on the other hand, is considerably more demanding. Nearly all of it is aimed at establishing, in Chapters 25 and 26, the consistency of the plane geometry postulates and the independence of the parallel postulate. Here the student will need some knowledge of the analytic properties of the reals. He probably will also need a good deal more guidance; for not only are the topics more difficult, but the discussion frequently is less clear than in the first half.

It follows that the later chapters restrict use of the book as a whole to no less than the able and reasonably advanced undergraduate. While such a student may find parts of the first half pedestrian, he will certainly find much in the second to stimulate him. He will find in general an abundance of well-written, sound mathematics; completeness; a tightly structured presentation; and, on the deficiency side, neither a bibliography nor an adequate number of exercises. Yet all of these considerations are less important to an estimate of the book's value than that one feature which most distinguishes it among college geometry texts: the emphasis on a painstaking and rigorous development from first principles. The author's implied argument that in an undergraduate geometry course it is better to concentrate on cleaning up elementary geometry than to develop a large number of advanced topics is, in my opinion, a persuasive one. If some of the materials now being tried in the schools prove successful and are widely accepted, the argument will lose at least part of its force. In the meantime, for those who agree, this would seem to be the logical choice of text.

G. P. JOHNSON, The University of the South

The book is marred also by inadequate attention to mechanical detail or proofreading. There are more simple errata than one should expect, in wrong numbers and symbols, incorrect figures and wrong answers to problems. And there is no index, though there is a useful glossary.

Nevertheless, a good book, which serves its purpose. In addition to attaining an understanding of some important mathematical ideas an observant reader may also get some notion of a good mathematician at work.

ROBERT SINGLETON, Wesleyan University

Combinatorial Geometry in the Plane. By Hugo Hadwiger and Hans Debrunner, translated and supplemented by Victor Klee. Holt, Rinehart and Winston, New York, 1964. 113 pp. $3.75.

This fine little book will probably prove valuable primarily as a vehicle for undergraduate independent study. It presents, as problems, approximately one hundred theorems, many of which were unproved until quite recently, on convexity, coverings, graphs, and other topics all restricted to the plane. There is also discussion of extension to higher dimensions, and of open problems. The discussion surrounding the theorems and their organization makes for a coherent development and provides some help toward obtaining proofs. Proofs in fact appear, many rather sketchily, at the end of the book. Students will need a substantial knowledge of analysis and some of college level geometry; and they must be prepared to cope with a book which will provide not a great deal more than an outline, albeit an excellent one, for their work.

G. P. JOHNSON, The University of the South

A Survey of Matrix Theory and Matrix Inequalities. By Marvin Marcus and Henryk Minc. Allyn and Bacon, Boston, 1964. xvi+180 pp. $8.75.

This is an extremely useful survey of classical results in matrix theory and a number of recent developments in the areas of matrix inequalities, location of characteristic roots, and nonnegative matrices. It contains a detailed and useful bibliography of recent research papers. The book will be of great service to anyone working in the field of matrix theory, and also to mathematical physicists, engineers, and others who need a ready source of particular results.

RICHARD BELLMAN, University of Southern California

Cours D'Analyse de l'École Polytechnique, Tome III, Théorie des Equations. By J. Favard. Gauthier-Villars, Paris, 1963. 542 pp.

This third volume of Cours d'Analyse is devoted to partial differential equations, integral equations and calculus of variations. Each chapter ends with a set of exercises and supplementary notes. The presentation is standard but the level of advancement is high, as witness the inclusion of a discussion of

Plateau's problem. It is somewhat curious that the chapter on integral equations comes after the chapter on differential equations so that the author cannot avail himself of this important tool in places where its intervention is decisive. As a partial compensation the author does discuss, but rather briefly, Green's functions in the chapter on integral equations.

While the book corresponds to no course given in American colleges and universities many teachers might find it useful to have Favard's book as a source of topics and problems.

M. KAC, The Rockefeller Institute

Vector Calculus. By B. W. Lindgren. Macmillan, New York, 1964. viii+183 pp. $6.50.

This book presents a brief account of the calculus of three-dimensional vector space. The approach is basically heuristic with fine points receiving scant attention. Chapter 1, Vector Algebra, has the advantage of many illustrations. The ideas and terminology of linear algebra are avoided. (Two 3-by-3 matrices which are inverses of each other are called "reciprocal sets" of ordered triples of vectors in $V_3(R)$.) The remaining chapter headings are: Functions of a Single Variable; Angular Velocity; Functions of Position; Green's, Stokes' and Related Theorems; and Curvilinear Coordinates. The book seems appropriate for a course taken by weaker students who have completed a "cookbook" calculus and have had no exposure to linear algebra. Although 179 exercises may seem enough for a text of 176 pages this reviewer would like to have seen even more.

STEPHEN HOFFMAN, Trinity College

Mathematical Methods of Physics. By Jon Mathews and R. L. Walker. Benjamin, New York and Amsterdam, 1964. 475 pp. $12.50.

The book evolved from the notes which have been used in the course at CALTECH for first-year physics graduate students. There is little effort to teach physics in the text. The emphasis is on mathematical techniques. The authors state that there is deliberate nonuniformity in the depth of presentation. Sixteen chapters contain classical topics: ordinary differential equations, series, integral transforms, complex variables, matrices, special functions, partial differential equations, eigenfunctions, Green's functions, perturbation, simple integral equations, numerical methods, elements of probability, statistics, tensors, differential geometry and elements of groups. The book can be used as a text for physics seniors or first year engineering graduate students. But one may wonder why there are omitted recently developed strong techniques like integral operators, algebraic techniques used in system analysis, etc. There are included many examples and an ample bibliography.

M. Z. v. KRZYWOBLOCKI, Michigan State University

Local Analytic Geometry. By Shreeram Shankar Abhyankar. Academic Press, New York, 1964. xv+484 pp. $18.00.

Lest the unsuspecting reader feel that he has already learned analytic geometry before his calculus courses, and that "local analytic geometry" is some variant thereof, it is worth noting that this text is about functions of several complex variables. It is written with the algebraist's hand, so that this fact may tend to be lost among the rings, complete valued fields, epimorphisms, etc. It is well written, perhaps a bit tightly, and is definitely a specialized book for the expert. The more algebraic geometry and algebra one knows, the easier it will be to read it.

MELVIN HAUSNER, New York University

BRIEF MENTION

Applied Mathematics for Engineers and Scientists, 2nd ed. By Sergei A. Schelkunoff. Van Nostrand, Princeton, N. J. 1965. 496 pp. $9.50.

First edition, 1948. Three new chapters have been added.

Art and Geometry. William M. Ivins. Dover, New York, 1965. 113 pp. $1.00.

Reprint of the work first published by Harvard University Press in 1946.

The Current Interpretation of Wave Mechanics. By Louis De Broglie. American Elsevier, New York, 1964. 95 pp. $6.00.

Proceedings of the Colloquium on Abelian Groups. L. Fuchs and T. Schmidt, editors. Akademiai Kiado, Budapest, 1964. 162 pp. $6.50.

The fifteen research papers in this volume are those given at the Colloquium in Tihany, September 1963, but not published in full length elsewhere, and a few other closely connected papers.

Introduction to ALGOL. By R. Baumann, M. Feliciano, F. L. Bauer, and K. Samelson. Prentice-Hall, Englewood Cliffs, N. J., 1964. 176 pp. $6.75.

A primer for nonspecialists, with emphasis on practical uses.

Tables of the Principal Unitary Representations of Fedorov Groups. Mathematical Tables Series, Vol. 34. D. K. Faddeyev. Pergamon Press, Inc., New York, 1964. 155 pp. $10.00.

Modern Higher Algebra, 5th ed. By George Salmon. Chelsea, New York, 1964. 376 pp. $4.95.

The Theory of Substitutions 2nd ed. By Eugen Netto. Translated by F. N. Cole. Chelsea, New York, 1964. 305 pp. $3.95.

Number Theory, 2nd ed. By Trygve Nagell. Chelsea, New York, 1964. 309 pp. $5.50.

Reprint of first edition (1951) with corrections and minor changes.

Advances in Game Theory. Annals of Mathematics Studies, 52. Edited by M. Dresher, L. S. Shapley and A. W. Tucker. Princeton University Press, Princeton, 1964. 679 pp. $8.50.

Seventy-nine research papers on various aspects of game theory.

Advanced Calculus, 2nd ed. By C. R. Buck. McGraw-Hill, New York, 1965. 525 pp. $9.75.

Revised edition of a text first published in 1956

Tables of the Normal Probability Integral; The Normal Density and its Normalized Derivatives. By N. V. Smirnov. Translated by D. E. Brown, Pergamon Press, Inc. New York, 1965. 125 pp. $7.50.

Tables of the Function $W(z)=e^{-z^2}\int_0^z e^{x^2}dx$ *in the Complex Domain.* By K. A. Karpov. Pergamon Press, Inc., New York, 1965. 515 pp. $19.75.

Tables of the Function $F(z)=\int_0^z e^{x^2}\,dx$ *in the Complex Domain.* By K. A. Karpov. Pergamon Press, Inc., New York, 1964. 490 pp. $20.00.

Theory of Graphs and its Applications. Miroslav Fiedler, editor. Academic Press, New York, 1965. 234 pp. $10.00.

Proceedings of the Symposium held in Smolenice, Czechoslovakia, in June, 1963.

Recent Advances in Matrix Theory. Hans Schneider, editor. University of Wisconsin Press, Madison, 1964. 142 pp. $4.00.

Proceedings of an Advanced Seminar conducted by the Mathematics Research Center, U. S. Army, at the University of Wisconsin, October, 1963. Six mathematicians were invited to speak on subjects of their choice.

The Concept of a Riemann Surface. By Hermann Weyl. Addison-Wesley, Reading, Mass., 1964. 190 pp. $12.50.

Translation of the third (1955) edition of a classic first published in 1913.

Boolean Algebras, 2nd ed. By Roman Sikorski. Academic Press, New York, 1964. 237 pp. $9.50.

Readings in Mathematical Psychology, Vol. II. Duncan Luce, Robert Bush, and Eugene Galanter, editors. Wiley, New York, 1965. 544 pp. $8.95.

This volume contains thirty-three papers in six areas: computers, language, social interaction, sensory processes, preference and utility, and Bayesian statistics.

Introduction to Mathematical Statistics, 2nd ed. By R. V. Hogg and A. T. Craig. Macmillan, New York, 1965. 382 pp. $8.50.

Some discussions are expanded and a number of new topics are added.

Tables of Elliptical Integrals, Part I. By V. M. Velyakov, R. I. Kravtsova, and M. G. Rappoport. Translated by P. Basu. Pergamon Press, Inc. New York, 1965. 647 pp. $20.00.

Nonlinear Integral Equations. P. M. Anselone, editor. U. of Wisconsin Press, Madison, Wis., 1964. 379 pp. $6.50.

Proceedings of an advanced seminar conducted by the Mathematics Research Center, U. S. Army, at the University of Wisconsin in April, 1963.

Tables of the Legendre Functions, Part 1. By M. I. Zhurina and L. N. Karmazina. Pergamon Press, Inc., New York, 1964. 309 pp. $2.80.

Advances in Computers. By F. L. Alt and M. Rubinoff. Academic Press, New York, 1964. 395 pp. $14.00.

This volume contains survey articles on Computers in Election Night Broadcasting, Automatic Programming in Eastern Europe, Artificial Intelligence and Self-Organization, Automatic Optical Design, Digital Computers in Nuclear Reactor Design, and Procedure Oriented Languages.

Handbook of Mathematical Tables and Formulas, 4th ed. By R. S. Burington. McGraw-Hill, New York, 1965. 421 pp. $4.50.

Elphyma Tables. By E. Ingelstam and S. Sjoberg. Wiley, New York, 1964. 99 pp. \$3.50. Tables, formulas and nomograms within Electricity, Physics, and Mathematics.

Philosophy of Science. The Delaware Seminar, Vol 2. William L. Reese, general editor. Interscience, Wiley, New York, 1964. 550 pp. \$14.50.

This volume contains contributions by a dozen and a half philosophers and scientists under the headings: Scientific Explanation, Prediction, and Theories; Space and Time; Particles, Fields, and Quantum Mechanics; Induction and Measurement; Science and Man; and Cosmology.

Similarity Analysis of Boundary Value Problems in Engineering. By Arthur G. Hansen. Prentice-Hall, Englewood Cliffs, N. J., 1964. 114 pp. \$6.75.

Asymptotic Solutions of Differential Equations and their Applications. Calvin A. Wilcox, editor. Wiley, New York, 1964. 249 pp. \$4.95.

Proceedings of a Symposium held at the Mathematics Research Center, U. S. Army, at the University of Wisconsin, in May, 1964.

Studies in Subjective Probability, Henry E. Kyburg and Howard E. Smokler, editors. Wiley, New York, 1964. 205 pp. \$4.75.

This volume contains an introduction by the editors, articles by John Venn (1888), Emile Borel (1924), Frank P. Ramsey (1926), Bruno de Finetti (1937), Bernard O. Koopman (1940), and Leonard J. Savage (1961), and an extensive bibliography.

An Introduction to Transport Theory. By G. M. Wing. Wiley, New York, 1963. 169 pp. \$7.95.

It is the author's purpose to provide the physicist, chemist, and engineer with an insight into the state of transport theory and to introduce the mathematicians to the subject.

Computing Methods in Optimization Problems. By A. V. Balakrishnan and L. W. Neustadt, editors. Academic Press, New York, 1964. 328 pp. \$6.50.

Proceedings of a conference held at UCLA in January, 1964 to disseminate information about recent research and computing experience.

The Complex j-Plane. Complex Angular Momentum in Nonrelativistic Scattering Theory. By R. G. Newton. Benjamin, New York, 1964. 235 pp. \$9.00.

Aufgaben und Lehrsätze aus der Analysis. By G. Polya and G. Szegö. Springer-Verlag, Berlin, 1964. Zweiter Band, 407 pp. DM 34.00. Erster Band, 338 pp. DM 34.00.

CHARTER FLIGHTS TO 1966 INTERNATIONAL CONGRESS

Arrangements have been made so that MAA members can participate in the charter flights announced by the American Mathematical Society for attendance at the International Congress of Mathematicians to be held in Moscow, August 16–26, 1966. For details see page 655 of the August issue and page 769 of the November issue of the NOTICES of the AMS, or write to the American Mathematical Society, P.O. Box 6248, Providence, R.I. 02904, for further information. Applications should be made immediately.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Glen Haddock, Arkansas College, represented the Association at the inauguration of Clifton L. Ganus, Jr. as President of Harding College on September 18, 1965.

Professor Caroline A. Lester, State University of New York at Albany, represented the Association at the inauguration of Joseph C. Palamountain, Jr. as President of Skidmore College on September 25, 1965.

Professor L. E. Mehlenbacher, University of Detroit, represented the Association at the inauguration of William R. Keast as President of Wayne State University on October 28, 1965.

Professor C. N. Mills, Sioux Falls College, represented the Association at the inauguration of Charles L. Balcer as President of Augustana College on September 18, 1965.

Professor L. J. Montzingo, Seattle Pacific College, represented the Association at the inauguration of the Very Rev. John A. Fitterer as President of Seattle University on October 13, 1965.

University of Arizona: Assistant Professor S. G. Tellman, University of Washington, Dr. G. D. Taylor, University of Michigan, and Dr. N. M. Wigley, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, have been appointed Assistant Professors; Visiting Professor Edgar Karst, University of Oklahoma, has been appointed Lecturer.

California Institute of Technology: Assistant Professor Galen Seever, University of California, Los Angeles, and Dr. Keith Phillips, University of Washington, have been appointed Assistant Professors; Associate Professor Adriano Garsia has been promoted to Professor; Professor W. A. J. Luxemburg will be on leave of absence from June 20, 1965 to January 1, 1966.

University of California, Berkeley: Associate Professor R. S. Lehman has accepted the position of Vice Chairman of the Department of Mathematics; Assistant Professor G. E. Bredon has been promoted to Associate Professor; Mr. Alfred Gray has been promoted to Assistant Professor and awarded an NSF Postdoctoral Fellowship for the spring semester of 1966; Dr. F. W. Warner, III, has been promoted to Assistant Professor; Professor W. G. Bade has been appointed Research Professor in the Miller Institute for Basic Research for the academic year 1965–66; Professor D. H. Lehmer will be on sabbatical leave for the academic year 1965–66 and will carry on research in Berkeley and at the Australian National University in Canberra; Professor M. H. Protter is on sabbatical leave for the academic year 1965–66 but will remain in residence during the academic year; Professor Abraham Seidenberg will be on sabbatical leave for the spring semester of 1966 and will carry on research in Rome, Italy.

Carleton University: Associate Professors P. R. Beesack and D. K. Dale have been promoted to Professors.

Carnegie Institute of Technology: Associate Professor Oswald Wyler, University of New Mexico, has been appointed Professor; Assistant Professor S. P. Franklin, University of Florida, has been appointed Assistant Professor; Associate Professor R. A. Moore has been promoted to Associate Head of the Department of Mathematics.

Case Institute of Technology: Professor Melvin Henriksen, Purdue University, has been appointed Head of the Department of Mathematics; Assistant Professor L. R. Bragg has been promoted to Associate Professor.

Central Missouri State College: Associate Professor C. E. Kelley has been promoted to Professor; Mr. Larry Dilley has been promoted to Assistant Professor.

Clemson University: Professor Andrew Sobczyk, Southern Illinois University, has been appointed Professor; Dr. Kenzo Seo, Colorado State University, has been appointed Associate Professor; Drs. J. V. Brawley, Jr. and T. G. Proctor, North Carolina State of the University of North Carolina, have been appointed Assistant Professors.

University of Colorado: Dr. Stanislaw Ulam, Los Alamos Scientific Laboratory, Los Alamos, New Mexico, has been appointed Professor; Dr. James Moser, University of Colorado, has been appointed Assistant Professor; Dr. Collin Hightower, U. S. Army, Washington, D. C., has been appointed Acting Assistant Professor; Associate Professor Arne Magnus has been promoted to Professor; Assistant Professor Donald Monk has been promoted to Associate Professor.

Cornell College: Mr. D. J. Samuelson, University of California, Berkeley, has been appointed Assistant Professor; Assistant Professor L. E. Claborn has been promoted to Associate Professor.

Georgetown University: Dr. E. R. Bobo, University of Virginia, has been appointed Assistant Professor; Associate Professor Anne E. Scheerer has accepted a Phillip Foundation Internship for the academic year 1965–66 at Boston University.

University of Georgia: Assistant Professor C. B. Schaufele, Louisiana State University, has been appointed Assistant Professor; Assistant Professor R. W. Heath has been promoted to Associate Professor.

Idaho State University: Drs. R. W. Deming, New Mexico State University, and V. H. Keiser, University of Colorado, have been appointed Assistant Professors; Associate Professor John Hilzman has been appointed Chairman of the Department of Mathematics.

Institute for Advanced Study: The following have been appointed Members in Mathematics: Associate Professor P. T. Church, Syracuse University; Dr. M. M. Cohen, University of Michigan; Professor V. F. Cowling, Rutgers, The State University; Assistant Professor R. G. Douglas, University of Michigan; Professor Murray Gerstenhaber, University of Pennsylvania; Dr. G. J. Janusz, University of Oregon; Assistant Professor B. F. Jones, Jr., Rice University; and Mr. M. V. Mielke, Indiana University.

State University of Iowa: Assistant Professor F. J. Kosier, University of Wisconsin, has been appointed Associate Professor; Assistant Professor Marilyn J. Zweng, University of California at Santa Barbara, and Dr. P. E. Waltman, Sandia Corporation, Albuquerque, New Mexico, have been appointed Assistant Professors; Assistant Professors George Burke and M. A. Geraghty have been promoted to Associate Professors.

Kansas State University: Professor C. H. Cunkle, Clarkson School of Technology, has been appointed Professor; Assistant Professor R. L. Yates has been promoted to Associate Professor; Associate Professor T. A. Mossman retired on June 1, 1965 with the title of Associate Professor Emerita.

Le Moyne College, Syracuse: Assistant Professor E. F. Baumgartner, Bates College, has been appointed Assistant Professor; Mrs. Polly Burkhead has been promoted to Assistant Professor; Assistant Professor T. S. Frank has been promoted to Associate Professor.

Michigan State University: Professor P. H. Doyle, Virginia Polytechnic Institute, has been appointed Professor; Assistant Professor G. D. Anderson, Eastern Michigan University, Dr. E. C. Ingraham, University of Oregon, and Dr. J. J. Masterson, Purdue University, have been appointed Assistant Professors; Assistant Professors J. R. Reay, Western Washington State College, and W. E. Bonnice, University of New Hampshire, have been appointed Visiting Assistant Professors; Associate Professor M. L. Tomber has been promoted to Professor; Assistant Professor W. T. Sledd has been promoted to Associate Professor.

University of Michigan: Drs. J. H. Smith, J. A. Smoller, and Harold Stark have been promoted to Assistant Professors; Professor R. V. Churchill retired May 1965 with the title of Professor Emeritus.

University of New Mexico: Associate Professor Abraham Hillman, University of Santa Clara, has been appointed Associate Professor; Dr. Richard Metzler, Wayne State University, has been appointed Assistant Professor; Assistant Professors Merle Mitchell, Heinz Renggli, and Arthur Steger have been promoted to Associate Professors; Professor F. C. Gentry retired in April 1965.

North Central College: Dr. R. P. Polivka, International Business Machines, Poughkeepsie, New York, has been appointed Visiting Associate Professor; Mr. D. E. Johnson has been promoted to Assistant Professor.

Northeast Louisiana State College: Dr. Lonnie Bennett, Oklahoma State University, and Mrs. Emma Mixon, Grayson, Louisiana, have been appointed Assistant Professors; Assistant Professor Edward Anders has been promoted to Associate Professor; Assistant Professor D. R. Bedgood has been appointed Head of the Department of Mathematics.

University of Omaha: Associate Professor H. L. Hunzeker has been promoted to Professor; Assistant Professor Benjamin Stern retired in June 1965.

San Diego State College: Associate Professor G. A. Becker has been promoted to Professor; Assistant Professor R. B. Killgrove has been promoted to Associate Professor.

University of Toledo: Dr. W. S. Hatcher, University of Neuchatel, Switzerland, has been appointed Associate Professor; Dr. H. W. Vayo, Harvard Medical School, has been appointed Assistant Professor; Assistant Professor Budmon Davis has been promoted to Associate Professor.

University of Utah: Dr. J. O. Sather, U. S. Army Mathematics Research Center, University of Wisconsin, and Dr. L. J. Grim, University of Minnesota, have been appointed Assistant Professors; Professor W. R. Scott, University of Kansas, has been appointed Professor.

University of Windsor: Assistant Professor Richard Courter, Dalhousie University, has been appointed Associate Professor; Rev. D. T. Faught has been promoted to Professor.

University of Wisconsin, Milwaukee: Visiting Assistant Professor H. C. Howard, University of Maryland, has been appointed Associate Professor; Assistant Professor H. J. Biesterfeldt, Jr., South Dakota School of Mines and Technology, Dr. R. L. Gantos, Michigan State University, and Dr. J. L. Stebbins, Wayne State University, have been appointed Assistant Professors; Associate Professor E. H. Feller has been promoted to Professor; Assistant Professor R. J. Mihalek has been promoted to Associate Professor; Associate Professor W. J. Pervin has been appointed Chairman of the Department of Mathematics.

Dr. Bernice L. Auslander, Newton Centre, Massachusetts, has been appointed Instructor at Wellesley College.

Professor I. A. Barnett, University of North Carolina, has been appointed Visiting Professor at Fairleigh Dickinson University.

Assistant Professor T. A. Bick, Hobart and William Smith Colleges, has been promoted to Associate Professor.

Dr. Sidney Birnbaum, Martin Company, Denver, Colorado, has been appointed Associate Professor at the University of South Carolina.

Professor A. T. Brauer, University of North Carolina, has been appointed Visiting Professor at Wake Forest College.

Assistant Professor C. A. Brown, The Citadel, has been promoted to Associate Professor.

Associate Professor George Crocker, University of Southern Mississippi, is on sabbatical leave at Auburn University for the academic year 1965–66.

Mr. L. A. Davis, Jr., Agricultural, Mechanical and Normal College, has been promoted to Assistant Professor.

Mr. Floyd Deardorff, York, Pennsylvania, has been appointed Assistant Professor at Monmouth College, New Jersey.

Mr. D. A. DePasse, Pepperdine College, has been promoted to Assistant Professor.

Professor E. A. Franz, Culver-Stockton College, has been appointed Professor at Illinois College.

Associate Professor Francisco Garriga, University of Puerto Rico, has been promoted to Professor.

Assistant Professor A. R. Goddard, Tarleton State College, has been promoted to Associate Professor.

Mr. H. P. Guillotte, Rhode Island College, has been promoted to Assistant Professor.

Dr. B. H. Harris, Southwest Baptist College, has been promoted to Professor.

Associate Professor M. J. Hellman, Rutgers, The State University, Newark, has been promoted to Professor.

Dr. Dagmar R. Henney, University of Maryland, has been appointed Assistant Professor at George Washington University.

Dr. P. F. Hultquist, Ball Brothers Research Corporation, Boulder, Colorado, has been appointed Professor of Applied Mathematics and Electrical Engineering at the Colorado Springs Center of the University of Colorado.

Mr. M. A. Jacobs, Fairleigh Dickinson University, has been appointed Instructor.

Assistant Professor B. A. Jensen, Nebraska Wesleyan University, has been promoted to Associate Professor.

Associate Professor C. Kassimatis, University of Windsor, has been appointed Professor at Wayne State University.

Dr. Joseph Lewittes, Harvard University, has been appointed Assistant Professor at the Belfer Graduate School of Science at Yeshiva University.

Professor E. D. McCarthy, University of Detroit, retired on July 1, 1965.

Assistant Professor John Mack, Ohio University, has been appointed Associate Professor at the University of Kentucky.

Dr. R. M. Mathsen, University of Nebraska, has been appointed Assistant Professor at Concordia College.

Associate Professor Elsie Muller, Morningside College, has been promoted to Professor and appointed Chairman of the Department of Mathematics.

Dr. Charles Patt, Yeshiva University, has been promoted to Assistant Professor.

Mr. J. F. Porter, Hiram College, has been promoted to Assistant Professor.

Dr. Stanley Preiser, United Nuclear Corporation, White Plains, New York, has been appointed Assistant Professor at the Polytechnic Institute of Brooklyn.

Assistant Professor C. D. Robinson, Arizona State University, has been appointed Associate Professor at the University of Mississippi.

Associate Professor J. M. Shapiro, Ohio State University, has been promoted to Professor.

Professor Emeritus C. A. Shook, Lehigh University, has been appointed Visiting Lecturer at Muhlenberg College.

Dr. S. T. Stern, State University College at Buffalo, has been promoted to Professor.

Assistant Professor E. A. Tanis, University of Nebraska, has been appointed Associate Professor at Hope College.

Dr. Daniel B. J. Tomiuk, Cité Universitaire de Paris, Paris, France, has been appointed Associate Professor at the University of Ottawa.

Assistant Professor J. W. Toole, University of Maine, has been promoted to Associate Professor.

Assistant Professor Charles Vanden Eynden, University of Arizona, has been appointed Assistant Professor at Miami University.

Mr. William Wetherbee, Bowling Green State University, has been appointed Associate Professor at Mansfield State College.

Dr. R. K. Williams, Vanderbilt University, has been appointed Assistant Professor at Southern Methodist University.

Professor R. L. Flanders, Oklahoma State University, died on July 8, 1965. He was a member of the Association for 42 years.

Professor Evan Johnson, Jr., Pennsylvania State University, died on July 13, 1965. He was a member of the Association for 23 years.

Professor R. H. Marquis, Ohio University, died on July 3, 1965. He was a member of the Association for 30 years.

Associate Professor George E. Schweigert, University of Pennsylvania, died on July 13, 1965. He was a member of the Association for 7 years.

Professor Emeritus T. McN. Simpson, Jr., Randolph-Macon College, died on July 7, 1965. He was a charter member of the Association.

NEWS ITEM

The Stanford Competitive Examination in Mathematics is being discontinued as of the spring of 1965. Over the past twenty-one years students in most of the high schools in the far Western states participated in this annual competition.

This competitive examination, which was initiated and organized by Professors G. Polya and G. Szego, was one of the pioneering efforts to use such competition in high schools as a means of stimulating interest in mathematics and in problem-solving in particular, and to discover and encourage students of unusual ability.

The competition was administered by the Department of Mathematics of Stanford University which records its appreciation of the generous cooperation of the many high school teachers and administrators who gave freely of their time to conduct examination sessions.

THE AFRICAN-AMERICAN INSTITUTE

The African-American Institute, 345 East 46th Street, New York, N. Y. 10017, has immediate openings for high school teachers of mathematics, with preference for those able to teach at least one science as well. Teachers will fill two-year AAI contracts either in Nkumbi International College, north central Zambia, or Kurasini International Education Center, Dar es Salaam, Tanzania. Maintained by AAI, both schools are primarily for refugee students from southern Africa. Certification and at least three years' teaching experience required. Master's degree preferred. Appointees receive round-trip transportation for themselves and dependents, overseas allowances, free housing, various fringe benefits in addition to salary, which ranges from \$6,500 to \$9,400, depending upon candidates' qualifications and prior earnings. Application forms should be requested from AAI Personnel Assistant at the address above.

CORNELL UNIVERSITY

On July 1, 1965, Cornell University established an intercollege Department of Computer Science in the Colleges of Engineering and Arts and Sciences. To aid the creation of this department and further its growth, the Alfred P. Sloan Foundation has awarded Cornell University a grant of one million dollars. The fields of study and research now represented include programming languages and systems, numerical analysis, data processing and information retrieval, and automata theory and theory of computation. The Department of Computer Science is authorized to grant the Ph.D. and M.S. degrees in Computer Science.

Further information can be obtained by writing to Professor J. Hartmanis, Department of Computer Science, Upson Hall, Cornell University, Ithaca, New York.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-eighth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado School of Mines, Golden, Colorado, on Friday and Saturday, May 7 and 8, 1965. The Rocky Mountain Section of SIAM participated.

INDEX TO VOLUME 72, 1965

THE AMERICAN MATHEMATICAL MONTHLY

GENERAL MATHEMATICAL PAPERS

ALGEBRA AND THEORY OF NUMBERS

ANALYSIS

GEOMETRY AND TOPOLOGY

PROBABILITY AND STATISTICS

OTHER APPLICATIONS

PROFESSIONAL MATTERS

MATHEMATICAL NOTES

EDITED BY J. H. CURTISS, University of Miami

CLASSROOM NOTES

EDITED BY GERTRUDE EHRLICH, University of Maryland

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN R. MAYOR, AAAS and University of Maryland

COLLABORATING EDITORS: JOHN D. BAUM, Oberlin College and JOHN A. BROWN, University of Delaware

BRIEF COMMENTS

PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Bloomfield College

COLLABORATING EDITORS: J. BARLAZ, Rutgers—The State University; L. CARLITZ, Duke University; H. S. M. COXETER, University of Toronto; H. EVES, University of Maine, A. E. LIVINGSTON, University of Alberta; and A. WILANSKY, Lehigh University.

PROBLEMS PROPOSED

PROBLEMS SOLVED

SOLUTIONS

Numbers in **boldface** type refer to problems, those in lightface, to pages.

MISCELLANEOUS

RECENT PUBLICATIONS AND PRESENTATIONS

EDITED BY R. A. ROSENBAUM, Wesleyan University
COLLABORATING EDITORS: K. O. MAY, University of California, Berkeley
and E. P. VANCE, Oberlin College

BRIEF MENTION

104–106, 568–570, 810–811, 1156–1158

Names of authors are in ordinary type, those of reviewers in capitals.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

PERSONAL ITEMS

GENERAL INFORMATION

NECROLOGY

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

PERSONAL INFORMATION

The following persons presented papers at meetings of the Association and its Sections: